

# VARIETIES OF TOPOLOGICAL GROUPS AND LEFT ADJOINT FUNCTORS

Dedicated to the memory of Hanna Neumann

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## 1. Introduction

In [6] and [2] Markov and Graev introduced their respective concepts of a free topological group. Graev's concept is more general in the sense that every Markov free topological group is a Graev free topological group. In fact, if  $FG(X)$  is the Graev free topological group on a topological space  $X$ , then it is the Markov free topological group  $FM(Y)$  on some space  $Y$  if and only if  $X$  is disconnected. This, however, does not say how  $FG(X)$  and  $FM(X)$  are related.

We show that  $FM(X)$  is isomorphic to the coproduct (in the category of all topological groups) of  $FG(X)$  and the discrete group  $Z$  of integers. This result is analagous to one announced by Ward [16] namely that the Markov free abelian topological group is isomorphic to the direct product of the Graev free abelian topological group and  $Z$ . Both results are special cases of the following:

If  $\mathfrak{B}$  is a (non-indiscrete) variety of topological groups [7] and  $F(X, \mathfrak{B})$  and  $G(X, \mathfrak{B})$  are respectively the Markov and Graev free topological groups of  $\mathfrak{B}$  on  $X$ , then  $F(X, \mathfrak{B})$  is isomorphic to the coproduct in  $\mathfrak{B}$  of  $G(X, \mathfrak{B})$  and a one-generator Markov free topological group of  $\mathfrak{B}$ .

As an immediate consequence of this we see that topological spaces with isomorphic Graev free topological groups have isomorphic Markov free topological groups. Another consequence is that every  $G(X, \mathfrak{B})$  is projective in  $\mathfrak{B}$ . We also use the above result to answer a question of Nummela [14] on Markov free topological groups.

In [8] we introduced the concept of a  $\beta$ -variety as a variety for which the Markov free topological groups have some pleasant properties. We show that we obtain the same class of varieties if we based the definition of a  $\beta$ -variety on Graev free topological groups. This is noteworthy, since recent work [12] has

shown (for example) that varieties generated by connected locally compact groups are  $\beta$ -varieties.

## 2. Definitions and results

**DEFINITION.** A non-empty class  $\mathfrak{B}$  of (not necessarily Hausdorff) topological groups is said to be a *variety of topological groups* if it is closed under the operations of taking subgroups, quotient groups, arbitrary cartesian products and isomorphic images.

**DEFINITION.** Let  $\mathfrak{B}$  be a variety,  $X$  a topological space and  $F(X, \mathfrak{B})$  a member of  $\mathfrak{B}$ . Then  $F(X, \mathfrak{B})$  is said to be a *Markov free topological group of  $\mathfrak{B}$  on  $X$*  if it has the properties:

- (a) there exists a mapping  $\eta: X \rightarrow F(X, \mathfrak{B})$  such that  $\eta: X \rightarrow \eta(X)$  is a homeomorphism,
- (b) for any continuous mapping  $\gamma$  of  $X$  into any member  $H$  of  $\mathfrak{B}$ , there exists a unique continuous homomorphism  $\Gamma$  of  $F(X, \mathfrak{B})$  into  $H$  such that  $\Gamma\eta = \gamma$ .

**DEFINITION.** Let  $\mathfrak{B}$  be a variety,  $X$  a topological space and  $G(X, \mathfrak{B})$  a member of  $\mathfrak{B}$ . Distinguish in  $X$  and arbitrary point  $e$ . Then  $G(X, \mathfrak{B})$  is said to be a *Graev free topological group of  $\mathfrak{B}$  on  $X$*  if it has the properties:

- (a) there exists a mapping  $\eta: X \rightarrow G(X, \mathfrak{B})$  such that  $\eta: X \rightarrow \eta(X)$  is a homeomorphism and  $\eta(e)$  is the identity element of  $G(X, \mathfrak{B})$ ,
- (b) for any continuous mapping  $\gamma$  of  $X$  into any member  $H$  of  $\mathfrak{B}$  such that  $\gamma(e)$  is the identity element of  $H$ , there exists a unique continuous homomorphism  $\Gamma$  of  $G(X, \mathfrak{B})$  into  $H$  such that  $\Gamma\eta = \gamma$ .

**DEFINITION.** Let  $\mathfrak{B}$  be a variety and  $\{G_i: i \in I\}$  a family of members of  $\mathfrak{B}$ . Then the topological group  $F$  in  $\mathfrak{B}$  is said to be a  $\mathfrak{B}$ -product of  $\{G_i: i \in I\}$ , denoted by  $\prod_{\mathfrak{B}} G_i$ , if it has the properties:

- (a) for each  $i \in I$ , there exists a mapping  $\eta_i: G_i \rightarrow F$  such that  $\eta_i: G_i \rightarrow \eta_i(G_i)$  is an isomorphism,
- (b) if for each  $i \in I$ ,  $\gamma_i$  is a continuous homomorphism of  $G_i$  into any member  $H$  of  $\mathfrak{B}$  then there exists a unique continuous homomorphism  $\Gamma$  of  $F$  into  $H$  such that  $\Gamma\eta_i = \gamma_i$  for each  $i$ .

Every variety  $\mathfrak{B}$  defines a complete category [1]: the objects are the members of  $\mathfrak{B}$  and the morphisms are the continuous homomorphisms between members of  $\mathfrak{B}$ . ( $\mathfrak{B}$  has products and equalizers.) The forgetful functor  $S: \mathfrak{B} \rightarrow \text{Top}$  (the category of topological spaces) preserves products and equalizers and is therefore continuous (preserves limits). The solution set condition is satisfied and Freyd's adjoint functor theorem shows that  $S$  has a left adjoint  $F: \text{Top} \rightarrow \mathfrak{B}$ . If  $\eta: X \rightarrow SFX$  is the front adjunction, then for every continuous map  $\gamma: X \rightarrow SH$ , where  $H \in \mathfrak{B}$ ,

there exists a unique continuous homomorphism  $\Gamma$  of  $FX$  into  $H$  such that  $S(\Gamma)\eta = \gamma$ . Further, it is clear that  $\eta: X \rightarrow \eta(X)$  is a homeomorphism if and only if  $V$  has a member with a subspace homeomorphic to  $X$ . Thus we have:

**THEOREM 1.** *Let  $\mathfrak{B}$  be a variety and  $X$  a topological space. Then  $F(X, \mathfrak{B})$  exists if and only if  $\mathfrak{B}$  has a member with a subspace homeomorphic to  $X$ .*

It is obvious that if  $F(X, \mathfrak{B})$  exists, then it is unique, up to isomorphism. Noting that the class of groups of groups which, with some topology, appear in  $\mathfrak{B}$  is a variety of groups [13] it is shown in [7] that  $F(X, \mathfrak{B})$  is the free group on the set  $\eta(X)$  of the underlying variety of groups. In particular,  $\eta(X)$  generates  $F(X, \mathfrak{B})$  algebraically. This distinguishes varietal categories from other categories of topological groups. For example, if  $C$  is the category of compact groups, then the forgetful functor  $S: C \rightarrow \text{Top}$  has a left adjoint  $F: \text{Top} \rightarrow C$ . However for  $X \in \text{Top}$  and  $\eta$  the front adjunction:  $X \rightarrow SFX$ , it is *not* true that  $\eta(X)$  generates  $FX$  algebraically. Rather, the subgroup generated by  $\eta(X)$  is dense in  $FX$ . (For further comments see [11].)

Let us denote by  $\text{Top}_0$  the category of pointed spaces; the objects are  $(X, x_0)$  with  $x_0 \in X \in \text{Top}$  and the morphisms are base point preserving continuous maps. There is a forgetful functor  $S_0: \mathfrak{B} \rightarrow \text{Top}_0$ , since all groups are pointed at the identity 1 and morphisms preserve identities. By Freyd's theorem  $S_0$  has a left adjoint  $G: \text{Top}_0 \rightarrow \mathfrak{B}$ . If  $\eta_0: (X, x_0) \rightarrow (S_0G(X, x_0), 1)$  is the front adjunction, then for any continuous map  $\gamma$  of  $X$  into  $S_0H$ , where  $H \in \mathfrak{B}$  and  $\gamma(x_0)$  is the identity in  $H$ , there exists a unique continuous homomorphism  $\Gamma$  of  $G(X, x_0)$  into  $H$  such that  $S(\Gamma)\eta_0 = \gamma$ . Using arguments similar to those used for  $F(X, \mathfrak{B})$  in [7], we can show that  $G(X, x_0)$  is the free group on the set  $\{x: x \in \eta_0(X) - x_0\}$  of the underlying variety of groups. So  $\eta_0(X)$  generates  $G(X, x_0)$  algebraically. Again we see that  $\eta_0$  maps  $X$  homeomorphically into  $\eta_0(X)$  if and only if there exists a member  $H$  of  $\mathfrak{B}$  having a subspace homeomorphic to  $X$ . [This statement is stronger than we might expect but this is because we are dealing with categories of topological groups. So if  $\theta: X \rightarrow H$  is such that  $\theta: X \rightarrow \theta(X)$  is a homeomorphism, then by defining  $\theta_0: X \rightarrow H$  by  $\theta_0(x) = \theta(x)\theta(x_0)^{-1}$  for each  $x \in X$ , we see that  $\theta_0: X \rightarrow \theta_0(X)$  is a homeomorphism and  $\theta_0(x_0)$  is the identity element of  $H$ .

**THEOREM 2.** *If  $\mathfrak{B}$  is a variety and  $X$  is a topological space, then  $G(X, \mathfrak{B})$  exists if and only if some member of  $\mathfrak{B}$  has a subspace homeomorphic to  $X$ . Further, if  $G(X, \mathfrak{B})$  exists then it is unique, up to isomorphism. (In particular, it is independent of the choice of base point.)*

**PROOF.** Our above discussion leaves only the last sentence to be verified. We show that if  $x_1$  and  $x_2$  are in  $X$ , then there is an isomorphism  $\tau: G(X, x_1) \rightarrow G(X, x_2)$ . (We continue the above notation, with  $\eta_i: (X, x_i) \rightarrow S_0G(X, x_i)$ ,  $i = 1, 2$ .)

Define a continuous map  $\gamma_1: X \rightarrow S_0G(X, x_2)$  by  $\gamma_1(x) = \eta_2(x) \eta_2(x_1)^{-1}$ , for

all  $x \in X$ . Then  $\gamma_1(x_1)$  is the identity element of  $G(X, x_2)$ . Therefore, there exists a unique continuous homomorphism  $\tau: G(X, x_1) \rightarrow G(X, x_2)$  such that  $S_0(\tau)\eta_1 = \gamma_1$ . Similarly we can define a continuous map  $\gamma_2: X \rightarrow S_0G(X, x_1)$  by  $\gamma_2(x) = \eta_1(x)\eta_1(x_2)^{-1}$  for all  $x \in X$ , and there exists a unique continuous homomorphism  $\tau': G(X, x_2) \rightarrow G(X, x_1)$  such that  $S_0(\tau')\eta_2 = \gamma_2$ .

It is easily verified that for each  $x \in X$ ,  $\tau'\tau(\eta_1(x)) = \eta_1(x)$ . Since  $\eta_1(x)$  generates  $G(X, x_1)$  algebraically,  $\tau'\tau$  acts identically on  $G(X, x_1)$ . Similarly  $\tau\tau'$  acts identically on  $G(X, x_2)$ . Thus  $\tau$  is an isomorphism of  $G(X, x_1)$  onto  $G(X, x_2)$  and the proof is complete.

Now we note that the forgetful functor  $T: \text{Top}_0 \rightarrow \text{Top}$  has a left adjoint, namely the functor  $P$  with  $PX = (X \cup \{*\}, *)$ , the space obtained by adjoining an isolated base point. Since  $TS_0: \mathfrak{B} \rightarrow \text{Top}$  is just the forgetful functor  $S$ , then  $F$  is naturally isomorphic to  $GP$ . Thus we have:

**THEOREM 3.** *If  $\mathfrak{B}$  is a variety, then every Markov free topological group of  $\mathfrak{B}$  is a Graev free topological group of  $\mathfrak{B}$ . More precisely each  $F(X, \mathfrak{B})$  is isomorphic to  $G(Y, \mathfrak{B})$ , where  $Y$  is the disjoint union of  $X$  and a single point.*

Our next theorem answers the question: When is a Graev free topological group a Markov free topological group?

**THEOREM 4.** *Let  $X$  be a topological space and  $\mathfrak{B}$  a variety such that  $G(X, \mathfrak{B})$  exists.*

(i) *If  $X$  is connected then  $G(X, \mathfrak{B})$  is connected. Consequently, if  $\mathfrak{B}$  contains any non-indiscrete group then  $G(X, \mathfrak{B})$  is not a Markov free topological group of  $\mathfrak{B}$ .*

(ii) *If  $X$  is disconnected, then there exists a topological space  $K$  such that  $G(X, \mathfrak{B})$  is isomorphic to  $F(K, \mathfrak{B})$ .*

**PROOF.** Let  $\eta_0: X \rightarrow S_0G(X, \mathfrak{B})$ , as before.

(i) Since  $\eta_0(X)$  is connected and contains the identity element 1 of  $G(X, \mathfrak{B})$ , the component of 1 contains  $\eta_0(X)$  and hence is the whole group  $G(X, \mathfrak{B})$ .

(ii) If 1 is an isolated point of  $\eta_0(X)$ , then by the comments preceding Theorem 3,  $G(X, \mathfrak{B})$  is isomorphic to  $F(X - 1, \mathfrak{B})$ . That  $G(X, \mathfrak{B})$  is not a Markov free topological group of  $\mathfrak{B}$ , for  $\mathfrak{B}$  non-indiscrete, follows from Theorem 6.1 of [9].

Now assume only that  $X$  is disconnected. Then  $\eta_0(X) = X_1 \cup X_2$  where  $X_1$  and  $X_2$  are open subsets of  $\eta_0(X)$ . Let  $1 \in X_1$  and  $f$  be any element of  $X_2$ . Put  $Y = \{1\} \cup fX_1 \cup X_2$ . It is clear that  $G(X, \mathfrak{B})$  is  $G(Y, \mathfrak{B})$ . To complete the proof we only have to show that 1 is an isolated point of  $Y$ .

Since  $G(X, \mathfrak{B})$  is not indiscrete there is an element  $a \in G(X, \mathfrak{B})$  such that  $a \notin \text{cl.}\{1\}$ . Then  $1 \in A \subset U$ , where  $A$  is closed in  $G(X, \mathfrak{B})$ ,  $U$  is open in  $G(X, \mathfrak{B})$  and  $a \notin U$ . Define a mapping  $\gamma$  of  $X$  into  $G(X, \mathfrak{B})$  by  $\gamma(x) = 1$  if  $\eta_0(x) \in X_1$  and  $\gamma(x) = a$  if  $\eta_0(x) \in X_2$ . Since  $\gamma$  is continuous there exists a continuous homomorphism  $\Gamma$  of  $G(X, \mathfrak{B})$  into itself such that  $\Gamma\eta_0 = \gamma$ . Clearly  $\Gamma(fX_1) = \Gamma(X_2) = a$  while  $\Gamma(1) = 1$ .

So  $\Gamma^{-1}(A) \cap Y = \Gamma^{-1}(U) \cap Y = 1$ . Hence 1 is an isolated point of  $Y$ . Thus the proof is complete.

In [10] the following is proved:

**THEOREM 5.** *Let  $\mathfrak{B}$  be a variety and  $\{G_i: i \in I\}$  a family of members of  $\mathfrak{B}$ . Then  $\coprod_{\mathfrak{B}}$  exists and is unique, up to isomorphism.*

We note that a  $\mathfrak{B}$ -product is simply a coproduct in the category  $\mathfrak{B}$ . Looking at coproducts in our other categories we have: if  $\{X_i: i \in I\}$  is a family in  $\text{Top}$ , then the coproduct  $\coprod X_i$  is the disjoint union of the  $X_i$ . If  $\{(X_i, x_i); i \in I\}$  is a family in  $\text{Top}_0$ , then the coproduct  $\coprod (X_i, x_i)$  is the space obtained from the coproduct in  $\text{Top}$  by identifying all points  $x_i$  (when considered in the coproduct) to a single point  $x_0$ . Now we use the well known fact that left adjoints preserve colimits—in particular, coproducts. So we have

$$(1) \quad F(\coprod X_i) = \coprod_{\mathfrak{B}} F(X_i)$$

$$(2) \quad G(\coprod (X_i, x_i)) = \coprod_{\mathfrak{B}} G(X_i, x_i)$$

Then (2) gives:

**THEOREM 6.** *Let  $\mathfrak{B}$  be a variety and  $\{X_i: i \in I\}$  a family of topological spaces. In each  $X_i$  distinguish a point  $e_i$  and let  $Y$  be the free union of the  $X_i$  with all the  $e_i$  identified. If  $G(Y, \mathfrak{B})$  exists, then it is isomorphic to  $\coprod_{\mathfrak{B}} G(X_i, \mathfrak{B})$ .*

**PROOF.** By Theorem 2, the existence of  $G(Y, V)$  implies the existence of  $G(X_i, \mathfrak{B})$ , for each  $i \in I$ . The result is then an immediate consequence of statement (2) above.

We could state a similar theorem for Markov free topological groups. However, we first prove a lemma which allows us to prove a stronger version.

**DEFINITION.** Let  $G$  be a topological group and  $X$  a subspace of  $G$  which generates  $G$  algebraically. Then  $G$  is said to be a *relatively free topological group with free generating space  $X$* , if every continuous mapping of  $X$  into  $G$  can be extended to a continuous endomorphism of  $G$ . (See [7])

Clearly each  $F(X, \mathfrak{B})$  is a relatively free topological group with free generating space  $\eta(X)$ .

**LEMMA.** *Let  $G$  be a relatively free topological group with free generating space  $X$ . If  $G$  is not indiscrete, then the identity element 1 is an isolated point of  $Y = X \cup \{1\}$ .*

**PROOF.** Since  $G$  is not indiscrete, there is a  $g \in G$  such that  $g \notin \text{cl}\{1\}$ . Then  $g \in A \subset U$ , where  $A$  is a closed subset of  $G, U$  is an open subset of  $G$  and  $1 \notin U$ . Define a mapping  $\gamma$  of  $X$  into  $G$  by  $\gamma(X) = g$ . Since  $\gamma$  is continuous, there exists a continuous homomorphism  $\Gamma$  of  $G$  into itself such that  $\Gamma|X = \gamma$ . Now  $\Gamma(X) = g$

and  $\Gamma(1) = 1$ . So  $\Gamma^{-1}(A) \cap Y = \Gamma^{-1}(U) \cap Y = X$ . Hence 1 is an isolated point of  $Y$ .

Our next theorem says somewhat more than Theorem 3.10 of [10].

**THEOREM 7.** *Let  $\mathfrak{B}$  be a non-indiscrete variety and  $\{X_i : i \in I\}$  a family of topological spaces. If  $F(X_i, \mathfrak{B})$  exists for each  $i \in I$ , then  $\coprod_{\mathfrak{B}} F(X_i, \mathfrak{B})$  is isomorphic to  $F(Y, \mathfrak{B})$ , where  $Y$  is the free union of the  $X_i$ .*

**PROOF.** Using the above lemma we see that  $Y$  is homeomorphic to a subspace of  $\prod_{i \in I} F(X_i, \mathfrak{B})$ . Therefore, by Theorem 1,  $F(Y, \mathfrak{B})$  exists. It is now clear from earlier statement (1) that  $F(Y, \mathfrak{B})$  is isomorphic to  $\coprod_{\mathfrak{B}} F(X_i, \mathfrak{B})$ .

Now let  $X$  and  $Y$  be topological spaces with  $x \in X$  and  $X \coprod Y$  their free union. Then  $(X \coprod Y, x) = (X, x) \coprod (Y \cup \{x\}, x) = (X, x) \coprod PY$ . Hence we have

$$G(X \coprod Y, x) = G(X, x) \coprod GPY.$$

Since the functor  $GP$  is naturally isomorphic to  $F$ , we have

$$(3) \quad G(X \coprod Y, x) = G(X, x) \coprod FY.$$

**THEOREM 8.** *Let  $\mathfrak{B}$  be a non-indiscrete variety and  $X$  and  $Y$  topological spaces such that  $F(X, \mathfrak{B})$  and  $F(Y, \mathfrak{B})$  exist. If  $Z$  is the free union of  $X$  and  $Y$ , then  $G(Z, \mathfrak{B})$  is isomorphic to both  $F(X, \mathfrak{B}) \coprod_{\mathfrak{B}} G(Y, \mathfrak{B})$  and  $G(X, \mathfrak{B}) \coprod_{\mathfrak{B}} F(Y, \mathfrak{B})$ .*

**PROOF.** It is shown in Theorem 7 that if  $F(X, \mathfrak{B})$  and  $F(Y, \mathfrak{B})$  exist, then so does  $F(Z, \mathfrak{B})$ . Therefore, by Theorem 2,  $G(Z, \mathfrak{B})$  exists. The result is then an immediate consequence of the above statement (3).

**COROLLARY 1.** *Let  $\mathfrak{B}$  be a non-indiscrete variety and  $X$  a topological space. If  $F(X, \mathfrak{B})$  exists, then it is isomorphic to  $G(X, \mathfrak{B}) \coprod_{\mathfrak{B}} F(Y, \mathfrak{B})$ , where  $Y$  is a one-point topological space. (cf. [15].)*

**REMARK 1.** The above corollary is of most interest when  $\mathfrak{B}$  contains the discrete group  $Z$  of integers, in which case  $F(Y, \mathfrak{B})$  is isomorphic to  $Z$ .

**COROLLARY 2.** *If  $\mathfrak{B}$  is a variety and  $X$  and  $Y$  are topological spaces such that  $G(X, \mathfrak{B})$  and  $G(Y, \mathfrak{B})$  are isomorphic then  $F(X, \mathfrak{B})$  and  $F(Y, \mathfrak{B})$  are isomorphic.*

**REMARK 2.** As an application of our work we answer a question of Nummela [13].

Let  $X$  be a compact group,  $FG(X)$  the Graev free topological group on  $X$  (in the variety of all topological groups) and  $\sigma: FG(X) \rightarrow X$  the canonical quotient morphism. Nummela shows that if  $H$  is the kernel of  $\sigma$ , then  $H$  is a Graev free topological group. (Consequently, every compact group has ‘‘projective dimension’’ one.)

He asks if the above proposition is true with  $FG(X)$  replaced by  $FM(X)$ , the

Markov free topological group on  $X$ , and “Graev” replaced by “Markov”. The answer is in the affirmative.

We note that  $FM(X) \cong FG(X) \amalg Z$  and if  $\sigma'$  is the canonical quotient morphism from  $FM(X)$  onto  $X$ , then the kernel of  $\sigma'$  is  $H \amalg Z$  (where  $H$  is as above) and consequently is a Markov free topological group, since  $H$  is a Graev free topological group.

REMARK 3. Projective topological groups have been studied in [8], [10], [3], [4], [5], [13], [14] and [15]. For our purposes here, we say the topological group  $P \in \mathfrak{B}$  is projective in  $\mathfrak{B}$  if  $P$  is a retract of  $F(X, \mathfrak{B})$ , for some topological space  $X$ . We point out that Corollary 1 implies that for any non-indiscrete variety  $\mathfrak{B}$ ,  $G(X, \mathfrak{B})$  is a retract of  $F(X, \mathfrak{B})$ . Thus we see that for any variety  $\mathfrak{B}$ ,  $G(X, \mathfrak{B})$  is projective in  $\mathfrak{B}$ .

DEFINITION. A variety  $\mathfrak{B}$  is said to be a  $\beta$ -variety if for each Tychonoff space  $X$ ,  $F(X, \mathfrak{B})$  exists and is Hausdorff.

For comments on  $\beta$ -varieties see [8] and [12].

THEOREM 9. A variety  $\mathfrak{B}$  is a  $\beta$ -variety if and only if  $G(X, \mathfrak{B})$  exists and is Hausdorff for each Tychonoff space  $X$ .

PROOF. Let  $\mathfrak{B}$  be a  $\beta$ -variety and  $X$  a Tychonoff space. Then  $F(X, \mathfrak{B})$  exists and is Hausdorff. Corollary 1 implies that  $G(X, \mathfrak{B})$  exists and is isomorphic to a sub-group of  $F(X, \mathfrak{B})$ . Therefore  $G(X, \mathfrak{B})$  is Hausdorff.

Conversely let  $\mathfrak{B}$  be a variety such that  $G(X, \mathfrak{B})$  exists and is Hausdorff for each Tychonoff space  $X$ . Let  $Y$  be the disjoint union of  $X$  and  $\{a\}$ . Then  $Y$  is a Tychonoff space. Consequently  $G(Y, \mathfrak{B})$  exists and is Hausdorff. However, by Theorem 3,  $G(Y, \mathfrak{B})$  is isomorphic to  $F(X, \mathfrak{B})$ ; that is,  $F(X, \mathfrak{B})$  exists and is Hausdorff.

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