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Research Article

Various Topologies Generated from E_j-Neighbourhoods via Ideals

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One of the considerable subjects in mathematics is the study of topology. Deducing topology from arbitrary binary relations has enticed the attention of many researchers. So, we devote this article to generate some kinds of topologies from ideals and E_j-neighborhoods which are induced from any binary relation. We define new types of approximations and accuracy measures from these topologies and then compare them with their counterparts induced directly from E_j-neighborhoods and ideals. Also, we show that the approximations and accuracy measures given, herein, are better than those introduced in some previous studies under any arbitrary relation.

1. Introduction

Rough set theory [1, 2]is one of the followed methods to handle the vagueness (uncertainty) of the information systems data and imperfect knowledge. In this theory, each subset is associated with two crisp sets (called lower and upper approximations) generated from an equivalence relation. To extend the applications of rough set theory, many authors have replaced an equivalence relation by different kinds of relations.

The interaction between topological and rough set theory is due to Skowron [3] and Wiweger [4] who first discussed the role of topological aspects in rough sets. Then, a combination of rough set theory and topological theory became the main goal of many studies [5–11]. This interaction also included some generalizations of topology such as minimal structure [12]. We refer the reader to [13] to see the main contributions which investigated the relationships between topology and rough set theory.

Ideals in a topological space have been taken into account by Kuratowski [14] and defined as a nonempty collection $\mathcal{I}$ of subsets of a universe which is closed under finite union and subsets. Kandil et al. [15] applied the notion of ideals with $\langle r \rangle$-neighborhoods to generalize Pawlak's approximations. They showed that their results decrease the boundary region in comparison with Pawlak's method [2], Allam's method [16], and Yao's method [17].

Recently, some new types of neighborhood systems have been introduced and studied. Among them, E_j-neighborhood systems [18] and C_j-neighborhood systems [19] are studied. In [18], Al-Shami et al. exploited E_j-neighborhoods to establish new rough and topological approximations. They compared between them and showed that the accuracy measure obtained from rough approximations is better than their counterparts obtained from topological approximations.

Through this study, we first construct new topological spaces using the ideas of E_j-neighborhood and ideals to minimize the boundary region and maximize the accuracy measure of a set compared with the approaches introduced in [18]. Second, we establish new rough approximations that improved the topological approximations.

2. Preliminaries

In this section, we recollect several substantial features and outcomes of rough set theory, especially those regarding to some sorts of neighborhood systems.

Definition 1

(1) [1] A binary relation $\mathscr{R}$ on U (i.e., $\mathscr{R} \subseteq U \times U$) is said to be

 (i) Equivalence if it is reflexive (i.e., $(z, z) \in \mathscr{R}$ for each $z \in U$), symmetric (i.e., $(y, z) \in \mathscr{R}$ if $(z, y) \in \mathscr{R}$), and transitive (i.e., $(y, w) \in \mathscr{R}$, whenever $(y, z) \in \mathscr{R}$ and $(z, w) \in \mathscr{R}$)
 (ii) Tolerance if it is reflexive and symmetric
 (iii) Preorder (quasiorder or dominance) if it is reflexive and transitive
 (iv) Partial order if it is a antisymmetric (i.e., $y = z$, whenever $(y, z) \in \mathscr{R}$ and $(z, y) \in \mathscr{R}$) preorder
 (v) Diagonal if $\mathscr{R} = \{(z, z)\colon z \in U\}$
 (vi) Serial if every $z \in U$, there exists $w \in U$, such that $z\mathscr{R}w$

(2) [2] For an equivalence relation $\mathscr{R}$ on U and a subset $M \subseteq U$, the two related sets $\underline{\mathscr{R}}(M) = \bigcup\{G \in U/\mathscr{R}\colon G \subseteq M\}$ and $\overline{\mathscr{R}}(M) = \bigcup\{G \in U/\mathscr{R}\colon M \cap G \neq \varnothing\}$ are called lower approximation and upper approximation of M, respectively.

Definition 2 (see [16, 17, 20]). Let $\mathscr{R}$ be a binary relation on U. The j-neighborhoods of $y \in U$ ($N_j(y)$, in short) are defined for each $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ as follows:

(1) r-neighborhood: $N_r(y) = \{z \in U\colon y\mathscr{R}z\}$
(2) l-neighborhood: $N_l(y) = \{z \in U\colon y\mathscr{R}z\}$
(3) i-neighborhood: $N_i(y) = N_r(y) \cap N_l(y)$
(4) u-neighborhood: $N_u(y) = N_r(y) \cup N_l(y)$
(5) $\langle r\rangle$-neighborhood: $N_{\langle r\rangle}(y) = \cap\{N_r(z)\colon y \in N_r(z)\}$ provided that there exists $N_r(z)$ containing y. Otherwise, $N_{\langle r\rangle}(y) = \varnothing$.
(6) $\langle l\rangle$-neighborhood: $N_{\langle l\rangle}(y) = \cap\{N_l(z)\colon y \in N_l(z)\}$ provided that there exists $N_l(z)$ containing y. Otherwise, $N_{\langle l\rangle}(y) = \varnothing$.
(7) $\langle i\rangle$-neighborhood: $N_{\langle i\rangle}(y) = N_{\langle r\rangle}(y) \cap N_{\langle l\rangle}(y)$
(8) $\langle u\rangle$-neighborhood: $N_{\langle u\rangle}(y) = N_{\langle r\rangle}(y) \cup N_{\langle l\rangle}(y)$

Henceforth, for all the following results, we will deal with all the values of j, $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, unless otherwise noted.

Definition 3 (see [20]). Let $\mathscr{R}$ be a binary relation on U and $\psi_j\colon U \longrightarrow P(U)$ be a mapping which assigns for each z in U its j-neighborhood in $P(U)$. Then, (U, R, ψ_j) is called a j-neighborhood space (briefly, N_jS).

Proposition 1 (see [18]). *Let $(U, \mathscr{R}, \psi_j)$ be N_jS and $y \in U$. Then,*

(1) If R is a reflexive relation, then $N_{\langle j\rangle}(y) \subseteq N_j(y)$, for all $j \in \{r, l, i, u\}$
(2) If R is a transitive relation, then $N_j(y) \subseteq N_{\langle j\rangle}(y)$, for all $j \in \{r, l, i, u\}$
(3) If R is a symmetric relation, then

$$\begin{aligned} N_r(y) &= N_l(y) = N_i(y) = N_u(y), \\ N_{\langle r\rangle}(y) &= N_{\langle l\rangle}(y) = N_{\langle i\rangle}(y) = N_{\langle u\rangle}(y). \end{aligned} \tag{1}$$

Theorem 1 (see [20]). *Let $(U, \mathscr{R}, \psi_j)$ be N_jS. Then, for each j, the collection $\top_j = \{M \subseteq U\colon \forall y \in M, N_j(y) \subseteq M\}$ is a topology on U.*

Definition 4 (see [18]). Let $(U, \mathscr{R}, \psi_j)$ be N_jS. Then, a set $M \subseteq U$ is called j-open set if $M \in \top_j$, and its complement is called j-closed set. The family Γ_j of all j-closed sets of a j-neighborhood space is defined by $\Gamma_j = \{F \subseteq U\colon F^c \in \top_j\}$, where F^c is the complement of F.

Definition 5 (see [18]). Let $\mathscr{R}$ be an arbitrary binary relation on U. The E_j-neighborhood of $y \in U$ ($E_j(y)$, in short) is defined for each j as follows:

(1) $E_r(y) = \{z \in U\colon N_r(z) \cap N_r(y) \neq \varnothing\}$
(2) $E_l(y) = \{z \in U\colon N_l(z) \cap N_l(y) \neq \varnothing\}$
(3) $E_i(y) = E_r(y) \cap E_l(y)$
(4) $E_u(y) = E_r(y) \cup E_l(y)$
(5) $E_{\langle r\rangle}(y) = \{z \in U\colon N_{\langle r\rangle}(z) \cap N_{\langle r\rangle}(y) \neq \varnothing\}$
(6) $E_{\langle l\rangle}(y) = \{z \in U\colon N_{\langle l\rangle}(z) \cap N_{\langle l\rangle}(y) \neq \varnothing\}$
(7) $E_{\langle i\rangle}(y) = E_{\langle r\rangle}(y) \cap E_{\langle l\rangle}(y)$
(8) $E_{\langle u\rangle}(y) = E_{\langle r\rangle}(y) \cup E_{\langle l\rangle}(y)$

Theorem 2 (see [18]). *Let $\mathscr{R}$ be an arbitrary binary relation on U and $y \in U$. Then, E_j-neighborhoods have the following properties:*

(1) $y \in E_j(z)$ iff $z \in E_j(y)$
(2) If $\mathscr{R}$ is reflexive, then $E_{\langle j\rangle}(y) \subseteq E_j(y)$ and $\mathcal{N}_j(y) \subseteq E_j(y)$, for all j
(3) If $\mathscr{R}$ is symmetric, then $E_r(y) = E_l(y) = E_i(y) = E_u(y)$ and $E_{\langle r\rangle}(y) = E_{\langle l\rangle}(y) = E_{\langle i\rangle}(y) = E_{\langle u\rangle}(y)$
(4) If $\mathscr{R}$ is transitive, then $E_j(y) \subseteq E_{\langle j\rangle}(y)$, for all $j \in \{r, l, i, u\}$
(5) If $\mathscr{R}$ is symmetric and transitive, then $E_j(y) = \mathcal{N}_j(y)$ and $E_j(y) \subseteq E_j(z)$ (if $y \in E_j(z)$), for each j

(6) If $\mathscr{R}$ is preorder, then $E_j(y) = E_{\langle j \rangle}(y)$, for all $j \in \{r, l, i, u\}$

(7) If $\mathscr{R}$ is an equivalence relation, then for each j, all $E_j(y)$ are identical, $E_j(y) = \mathscr{N}_j(y)$, and $E_j(y) = E_j(z)$ iff $y \in E_j(z)$

In [18], Al-Shami et al. formulated the concepts of E_j-lower and E_j-upper approximations and E_j-accuracy measure of a subset M in terms of E_j-neighborhoods as follows:

Definition 6 (see [18]). Let M be a subset of an $N_jS(U, \mathscr{R}, \psi_j)$. Then, the E_j-lower and E_j-upper approximations and E_j-accuracy measure of a subset M are

(1) $\mathscr{R}_j^-(M) = \{z \in U: E_j(z) \subseteq M\}$

(2) $\mathscr{R}_j^+(M) = \{z \in U: E_j(z) \cap M \neq \varnothing\}$

(3) $\mu_j(M) = (|\mathscr{R}_j^-(M) \cap M| / |\mathscr{R}_j^+(M) \cup M|)$, where $|M| \neq 0$

Theorem 3 (see [18]). *Let $(U, \mathscr{R}, \psi_j)$ be N_jS and $M, M \subseteq U$. Then, for each j, the following properties hold:*

(i) $\varnothing \subseteq \mathscr{R}_j^-(\varnothing)$ and $\mathscr{R}_j^+(U) \subseteq U$

(ii) $\mathscr{R}_j^-(U) = U$ and $\mathscr{R}_j^+(\varnothing) = \varnothing$

(iii) If $M \subseteq M$, then $\mathscr{R}_j^-(M) \subseteq \mathscr{R}_j^-(M)$ and $\mathscr{R}_j^+(M) \subseteq \mathscr{R}_j^+(M)$

(iv) $\mathscr{R}_j^-(M \cap M) = \mathscr{R}_j^-(M) \cap \mathscr{R}_j^-(M)$ and $\mathscr{R}_j^+(M \cup M) = \mathscr{R}_j^+(M) \cup \mathscr{R}_j^+(M)$

(v) $\mathscr{R}_j^-(M^c) = (\mathscr{R}_j^+(M))^c$ and $\mathscr{R}_j^+(M^c) = (\mathscr{R}_j^-(M))^c$

(vi) $\mathscr{R}_j^-(M) \cup \mathscr{R}_j^-(M) \subseteq \mathscr{R}_j^-(M \cup M)$ and $\mathscr{R}_j^+(M \cap M) \subseteq \mathscr{R}_j^+(M) \cap \mathscr{R}_j^+(M)$

(vii) Generally, $\mathscr{R}_j^-(\mathscr{R}_j^-(M)) \neq \mathscr{R}_j^-(M)$ and $\mathscr{R}_j^+(\mathscr{R}_j^+(M)) \neq \mathscr{R}_j^+(M)$

Also, Al-Shami et al. [18] employed E_j-neighborhoods to generate various topologies and studied their basic characteristics. Furthermore, they introduced the concepts of E_j-lower and E_j-upper approximations and E_j-accuracy measure of a subset M induced from these topologies.

Theorem 4 (see [18]). *Let $(U, \mathscr{R}, \psi_j)$ be N_jS. For each j, the collection $\top_{E_j} = \{M \subseteq U: \forall y \in M, E_j(y) \subseteq M\}$ is a topology on U.*

Definition 7 (see [18]). Let $(U, \mathscr{R}, \psi_j)$ be N_jS. A set $M \subseteq U$ is called E_j-open set if $M \in \top_{E_j}$, and its complement is called E_j-closed set. The family $\perp_{E_j}$ of all E_j-closed sets of a j-neighborhood space is defined by $\perp_{E_j} = \{F \subseteq U: F^c \in \top_{E_j}\}$.

Theorem 5 (see [18]). *Let $\mathscr{R}$ be an arbitrary binary relation on U and $y \in U$. If $\mathscr{R}$ is an equivalence relation, then $\top_{E_r} = \top_{E_l} = \top_{E_i} = \top_{E_u} = \top_{E_{\langle r \rangle}} = \top_{E_{\langle l \rangle}} = \top_{E_{\langle i \rangle}} = \top_{E_{\langle u \rangle}}$.*

Definition 8 (see [18]). Let $\top_{E_j}$ be a topology generated by E_j-neighborhoods. If $M \subseteq U$ and $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$, then the E_j-lower and E_j-upper approximations and E_j-accuracy of M are defined, respectively, as

(1) $L_j^{\ominus}(M) = \cup \{O \in \top_{E_j}: O \subseteq M\} = \mathrm{int}_{E_j}(M)$, where $\mathrm{int}_{E_j}(M)$ represents the interior points of M w.r.t. $\top_{E_j}$

(2) $U_j^{\oplus}(M) = \cap \{F \in \mathscr{R}_{E_j}: M \subseteq F\} = \mathrm{cl}_{E_j}(M)$, where $\mathrm{cl}_{E_j}(M)$ represents closure points of M w.r.t. $\top_{E_j}$

(3) $\mu_j(M) = (|L_j^{\ominus}(M)| / |U_j^{\oplus}(M)|)$, where $|U_j^{\oplus}(M)| \neq 0$

Theorem 6 (see [21]). *Let $(U, \mathscr{R}, \psi_j)$ be N_jS and $\mathscr{I}$ be an ideal on U. If $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$, then the collection $\top_j^{\mathscr{I}} = \{M \subseteq U: \forall y \in M, N_j(y) - M \in \mathscr{I}\}$ is a topology on U.*

3. Sorts of Approximations Based on E_j-Neighbourhoods and Ideals

Al-Shami et al. [18] constructed approximations relying on various topologies that are induced from the four types of E_j-neighbourhoods. In this portion, we shall generalize these topologies by using ideals and deduce new rough approximations based on E_j-neighbourhoods and ideals. We explain the relationships between these approximations and provide illustrative examples.

3.1. Various Topologies Generated from E_j-Neighbourhoods via Ideals.

In this part, we employ E_j-neighborhoods and ideal $\mathscr{I}$ to generate various topologies $\zeta_j^{\mathscr{I}}$ that are finer than the previous one generated by E_j-neighborhoods due to [18] for any relation.

First, we are going to offer a method of generating some topologies by using E_j-neighborhoods and ideal $\mathscr{I}$.

Theorem 7. *Let $(U, \mathscr{R}, \psi_j)$ be N_jS. For each j, the collection $\zeta_j^{\mathscr{I}} = \{M \subseteq U: \forall y \in M, E_j(y) - M \in \mathscr{I}\}$ is a topology on U.*

Proof. Let $M_\alpha \in \zeta_j^{\mathscr{I}}$, $\alpha \in \Delta$, and $z \in \cup_{\alpha \in \Delta} M_\alpha$, then there exists $\alpha_0 \in \Delta$ s.t. $z \in M_{\alpha_0}$. Hence, $[E_j(z) - M_{\alpha_0}] \in \mathscr{I}$. Since $-(\cup_{\alpha \in \Delta} M_\alpha) \subseteq - M_{\alpha_0}$, then $[E_j(z) - (\cup_{\alpha \in \Delta} M_\alpha)] \in \mathscr{I}$, i.e., $\cup_{\alpha \in \Delta} M_\alpha \in \zeta_j^{\mathscr{I}}$.

Let $M_1, M_2 \in \zeta_j^{\mathscr{I}}$ and $z \in M_1 \cap M_2$. Then, $[E_j(z) - M_1] \in \mathscr{I}$ and $[E_j(z) - M_2] \in \mathscr{I}$. According to properties of $\mathscr{I}$, $[E_j(z) - M_1] \cup [E_j(z) - M_1] \in \mathscr{I}$. Hence, $[E_j(z) - (M_1 \cap M_2)] \in \mathscr{I}$. It follows that $M_1 \cap M_2 \in E_j^{\mathscr{I}}$.

Easily, $\varnothing, U \in E_j^{\mathscr{I}}, \forall j$. Consequently, $\zeta_j^{\mathscr{I}}$ is a topology on U. □

Definition 9. Let $(U, \mathscr{R}, \psi_j)$ be N_jS and $\mathscr{I}$ be an ideal on U. A set $M \subseteq U$ is called $\zeta_j^{\mathscr{I}}$-open set if $M \in \zeta_j^{\mathscr{I}}$, and the complement of $\zeta_j^{\mathscr{I}}$-open set is called $\zeta_j^{\mathscr{I}}$-closed set. The family $\Pi_j^{\mathscr{I}}$ of all $\zeta_j^{\mathscr{I}}$-closed sets is defined by $\Pi_j^{\mathscr{I}} = \{F \subseteq U: F^c \in \zeta_j^{\mathscr{I}}\}$.

Theorem 8. *Let $(U, \mathscr{R}, \psi_j)$ be N_jS and $\mathscr{I}$ be an ideal on U. Then,*

(1) $\top_{E_j} \subseteq \zeta_j^{\mathcal{J}}$
(2) If $\mathcal{R}$ is a reflexive relation and $j \in \{r, l, i, u\}$, then $\zeta_j^{\mathcal{J}} \subseteq \zeta_{\langle j \rangle}^{\mathcal{J}}$
(3) If $\mathcal{R}$ is a symmetric, then $\zeta_r^{\mathcal{J}} = \zeta_l^{\mathcal{J}} = \zeta_i^{\mathcal{J}} = \zeta_u^{\mathcal{J}}$ and $\zeta_{\langle r \rangle}^{\mathcal{J}} = \zeta_{\langle l \rangle}^{\mathcal{J}} = \zeta_{\langle i \rangle}^{\mathcal{J}} = \zeta_{\langle u \rangle}^{\mathcal{J}}$
(4) If $\mathcal{R}$ is a transitive relation and $j \in \{r, l, i, u\}$, then $\zeta_{\langle j \rangle}^{\mathcal{J}} \subseteq \zeta_j^{\mathcal{J}}$
(5) If $\mathcal{R}$ is a preorder relation and $j \in \{r, l, i, u\}$, then $\zeta_{\langle j \rangle}^{\mathcal{J}} = \zeta_j^{\mathcal{J}}$
(6) If $\mathcal{R}$ is an equivalence relation, then for each j, all $\zeta_j^{\mathcal{J}}$ are identical, $\top_j^{\mathcal{J}} = \zeta_j^{\mathcal{J}}$.

Proof. In view of Theorem 2, then the proof is obvious.

The next proposition shows that the relation between the topologies $\zeta_j^{\mathcal{J}}$ and E_j-neighborhoods is reversible for each j. □

Proposition 2. *Let $(U, \mathcal{R}, \psi_j)$ be N_jS and $\mathcal{J}$ be an ideal on U. Then, the following results hold.*

(1) $\zeta_u^{\mathcal{J}} \subseteq \zeta_r^{\mathcal{J}} \cap \zeta_l^{\mathcal{J}} \subseteq \zeta_r^{\mathcal{J}} \cup \zeta_l^{\mathcal{J}} \subseteq \zeta_i^{\mathcal{J}}$
(2) $\zeta_{\langle u \rangle}^{\mathcal{J}} \subseteq \zeta_{\langle r \rangle}^{\mathcal{J}} \cap \zeta_{\langle l \rangle}^{\mathcal{J}} \subseteq \zeta_{\langle r \rangle}^{\mathcal{J}} \cup \zeta_{\langle l \rangle}^{\mathcal{J}} \subseteq \zeta_{\langle i \rangle}^{\mathcal{J}}$

Proof. Since $E_i \subseteq E_v \subseteq E_u$ and $E_{\langle i \rangle} \subseteq E_{\langle v \rangle} \subseteq E_{\langle u \rangle}$, where $v = r$ or l, then the proof is obvious. □

Example 1. Let $U = \{w_1, w_2, w_3, w_4\}$ and $\mathcal{R} = \{(w_1, w_1), (w_1, w_3), (w_1, w_4), (w_2, w_4), (w_4, w_2), (w_4, w_4)\}$ be a binary relation on U. In Tables 1 and 2 , we calculate N_j-neighborhoods and E_j-neighborhoods for each element of U.

Lemma 1. *Let $y \in U$ and $j \in \{r, l, i, u\}$. Then, for any binary relation $\mathcal{R}$ on U, $\mathcal{N}_{\langle j \rangle}(y) \subseteq E_{\langle j \rangle}(y)$.*

Proof. Obvious. □

Lemma 2. *Let $\mathcal{I}, \mathcal{J}$ be two ideals on $N_jS(U, \mathcal{R}, \psi_j)$. If $\mathcal{I} \subseteq \mathcal{J}$, then $\zeta_j^{\mathcal{I}} \subseteq \zeta_j^{\mathcal{J}}$ for each j.*

Proof. Straightforward.

The following example shows that the inclusion in Proposition 2 and Lemma 2 cannot be replaced by the equality relation. □

Example 2. Continued from Example 1.

$$\begin{aligned} \top_{E_{\langle i \rangle}} &= \top_{E_{\langle l \rangle}} = \top_{E_i} = \top_{E_r} = \{\emptyset, U, \{w_3\}, \{w_1, w_2, w_4\}\}, \\ \top_{E_{\langle u \rangle}} &= \top_{E_{\langle r \rangle}} = \top_{E_u} = \top_{E_l} = \{\emptyset, U\}. \end{aligned} \tag{2}$$

If $\mathcal{I} = \{\emptyset, \{w_1\}\}$, then

$$\begin{aligned} \zeta_r^{\mathcal{I}} &= \{\emptyset, U, \{w_3\}, \{w_2, w_4\}, \{w_1, w_2, w_4\}, \{w_2, w_3, w_4\}\}, \\ \zeta_l^{\mathcal{I}} &= \{\emptyset, U, \{w_2, w_3, w_4\}\}, \\ \zeta_i^{\mathcal{I}} &= \{\emptyset, U, \{w_3\}, \{w_2, w_4\}, \{w_1, w_2, w_4\}, \{w_2, w_3, w_4\}\}, \\ \zeta_u^{\mathcal{I}} &= \{\emptyset, U, \{w_2, w_3, w_4\}\}, \\ \zeta_{\langle r \rangle}^{\mathcal{I}} &= \zeta_{\langle u \rangle}^{\mathcal{I}} = \{\emptyset, U, \{w_2, w_3, w_4\}\}, \\ \zeta_{\langle l \rangle}^{\mathcal{I}} &= \zeta_{\langle i \rangle}^{\mathcal{I}} = \{\emptyset, U, \{w_3\}, \{w_2, w_4\}, \{w_1, w_2, w_4\}, \{w_2, w_3, w_4\}\}. \end{aligned} \tag{3}$$

If $\mathcal{J} = \{\emptyset, \{w_1\}, \{w_4\}, \{w_1, w_4\}\}$, then

$$\begin{aligned} \zeta_r^{\mathcal{J}} &= \{\emptyset, U, \{w_2\}, \{w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_2, w_4\}, \{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, \{w_2, w_3, w_4\}\}, \\ \zeta_l^{\mathcal{J}} &= \{\emptyset, U, \{w_2\}, \{w_3\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2, w_3\}, \{w_2, w_3, w_4\}\}, \\ \zeta_i^{\mathcal{J}} &= \{\emptyset, U, \{w_1\}, \{w_2\}, \{w_3\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_2, w_4\}, \{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, \{w_2, w_3, w_4\}\}, \\ \zeta_u^{\mathcal{J}} &= \{\emptyset, U, \{w_2\}, \{w_3\}, \{w_2, w_3\}, \{w_1, w_2, w_3\}, \{w_2, w_3, w_4\}\}, \\ \zeta_{\langle r \rangle}^{\mathcal{J}} &= \zeta_{\langle u \rangle}^{\mathcal{J}} = \{\emptyset, U, \{w_2, w_3\}, \{w_1, w_2, w_3\}, \{w_2, w_3, w_4\}\}, \\ \zeta_{\langle l \rangle}^{\mathcal{J}} &= \zeta_{\langle i \rangle}^{\mathcal{J}} = \{\emptyset, U, \{w_2\}, \{w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_2, w_4\}, \{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, \{w_2, w_3, w_4\}\}. \end{aligned} \tag{4}$$

Table 1: N_j-neighborhoods of each element in U.

$N_j(U)$	w_1	w_2	w_3	w_4
N_r	$\{w_1, w_3, w_4\}$	$\{w_4\}$	$\varnothing$	$\{w_2, w_4\}$
N_l	$\{w_1\}$	$\{w_4\}$	$\{w_1\}$	$\{w_1, w_2, w_4\}$
N_i	$\{w_1\}$	$\{w_4\}$	$\varnothing$	$\{w_2, w_4\}$
N_u	$\{w_1, w_3, w_4\}$	$\{w_4\}$	$\{w_1\}$	$\{w_1, w_2, w_4\}$
$N_{\langle r\rangle}$	$\{w_1, w_3, w_4\}$	$\{w_2, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_4\}$
$N_{\langle l\rangle}$	$\{w_1\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	$\{w_4\}$
$N_{\langle i\rangle}$	$\{w_1\}$	$\{w_2, w_4\}$	$\varnothing$	$\{w_4\}$
$N_{\langle u\rangle}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_4\}$

Table 2: E_j-neighborhoods of each element in U.

$E_j(U)$	w_1	w_2	w_3	w_4
E_r	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	$\{w_1, w_2, w_4\}$
E_l	$\{w_1, w_3, w_4\}$	$\{w_2, w_4\}$	$\{w_1, w_3, w_4\}$	U
E_i	$\{w_1, w_4\}$	$\{w_2, w_4\}$	$\varnothing$	$\{w_1, w_2, w_4\}$
E_u	U	$\{w_1, w_2, w_4\}$	$\{w_1, w_3, w_4\}$	U
$E_{\langle r\rangle}$	U	U	U	U
$E_{\langle l\rangle}$	$\{w_1, w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	$\{w_2, w_4\}$
$E_{\langle i\rangle}$	$\{w_1, w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	$\{w_2, w_4\}$
$E_{\langle u\rangle}$	U	U	U	U

3.2. Generalized Rough Approximations Based on Various Topologies Generated from E_j-Neighbourhoods via Ideals. Herein, we will construct some kinds of rough approximations using the topologies generated from E_j-neighborhoods and ideals and give some properties of them.

Definition 10. Let $\zeta_j^{\mathcal{J}}$ be a topology generated by E_j-neighborhoods and ideal $\mathcal{J}$. Then, E_j-lower and E_j-upper approximations and E_j-accuracy of a subset $M \subseteq U$ are defined, respectively, for each j as

(1) $L_j^{\mathcal{J}\ominus}(M) = \mathrm{int}_{E_j}^{\mathcal{J}}(M)$, where $\mathrm{int}_{E_j}^{\mathcal{J}}(M)$ represents interior of M w.r.t. $\zeta_j^{\mathcal{J}}$

(2) $U_j^{\mathcal{J}\oplus}(M) = \mathrm{cl}_{E_j}^{\mathcal{J}}(M)$, where $\mathrm{cl}_{E_j}^{\mathcal{J}}(M)$ represents closure of M w.r.t. $\zeta_j^{\mathcal{J}}$

(3) $\sigma_j^{\mathcal{J}}(M) = (|L_j^{\mathcal{J}\ominus}(M)|/|U_j^{\mathcal{J}\oplus}(M)|)$, where $|U_j^{\mathcal{J}\oplus}(M)| \neq 0$

Henceforth, $(U, \mathcal{R}, \mathcal{J}, \psi_j)$ is $\mathcal{N}_j S$ with ideal $\mathcal{J}$ on U and denoted by $E_j^{\mathcal{J}} S$.

Several fundamental properties of $L_j^{\mathcal{J}\ominus}(M)$ and $U_j^{\mathcal{J}\oplus}(M)$ are listed in the next proposition.

Proposition 3. *Let $M, \acute{M}$ be subsets of $E_j^{\mathcal{J}} S(U, \mathcal{R}, \mathcal{J}, \psi_j)$. Then, the following properties hold for each j.*

(1) $L_j^{\mathcal{J}\ominus}(M) = (U_j^{\mathcal{J}\oplus}(M^c))^c$

(2) $L_j^{\mathcal{J}\ominus}(U) = U$

(3) If $M \subseteq \acute{M}$, then $L_j^{\mathcal{J}\ominus}(M) \subseteq L_j^{\mathcal{J}\ominus}(\acute{M})$

(4) $L_j^{\mathcal{J}\ominus}(M \cap \acute{M}) = L_j^{\mathcal{J}\ominus}(M) \cap L_j^{\mathcal{J}\ominus}(\acute{M})$

(5) $L_j^{\mathcal{J}\ominus}(M \cup \acute{M}) \supseteq L_j^{\mathcal{J}\ominus}(M) \cup L_j^{\mathcal{J}\ominus}(\acute{M})$

(6) $L_j^{\mathcal{J}\ominus}(\varnothing) = \varnothing$

(7) $L_j^{\mathcal{J}\ominus}(M) \subseteq M$

(8) $L_j^{\mathcal{J}\ominus}(M) = L_j^{\mathcal{J}\ominus}(L_j^{\mathcal{J}\ominus}(M))$

(9) $U_j^{\mathcal{J}\oplus}(M) = (L_j^{\mathcal{J}\ominus}(M^c))^c$

(10) $U_j^{\mathcal{J}\oplus}(\varnothing) = \varnothing$

(11) If $M \subseteq \acute{M}$, then $U_j^{\mathcal{J}\oplus}(M) \subseteq U_j^{\mathcal{J}\oplus}(\acute{M})$

(12) $U_j^{\mathcal{J}\oplus}(M \cup \acute{M}) = U_j^{\mathcal{J}\oplus}(M) \cup U_j^{\mathcal{J}\oplus}(\acute{M})$

(13) $U_j^{\mathcal{J}\oplus}(M \cap \acute{M}) \subseteq U_j^{\mathcal{J}\oplus}(M) \cap U_j^{\mathcal{J}}(\acute{M})$

(14) $U_j^{\mathcal{J}\oplus}(U) = U$

(15) $M \subseteq U_j^{\mathcal{J}\oplus}(M)$

(16) $U_j^{\mathcal{J}\oplus}(M) = U_j^{\mathcal{J}\oplus}(U_j^{\mathcal{J}\oplus}(M))$

According to Example 2, Table 3 explains generalized rough approximations based on various topologies generated from E_j-neighbourhoods via ideals.

Table 4 demonstrates that the accuracy measure for $j = i$ is the highest from the cases $j \in \{r, l, i, u\}$, and the accuracy measures for $j = \langle i\rangle, \langle l\rangle$ are the highest from the case $j \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. However, we can find another example illustrating that the accuracy measure for $j = \langle i\rangle$ is the highest from the case $j \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

In the following remark, the inclusion relation of parts 3, 5, 7, 11, 13, and 15 in Proposition 3 cannot be replaced by the equality relation.

Remark 1. Example 2 and Table 3 show that

(1) If $M = \{w_1\}$ and $\acute{M} = \{w_2\}$, then $L_r^{\mathcal{J}}(M) \subseteq L_r^{\mathcal{J}}(\acute{M})$ and $U_j^{\mathcal{J}\oplus}(M) \subseteq U_j^{\mathcal{J}\oplus}(\acute{M})$, whereas $M \nsubseteq \acute{M}$.

(2) If $M = \{w_2\}$, $\acute{M} = \{w_4\}$, then $L_i^{\mathcal{J}}(M) \cup L_i^{\mathcal{J}}(\acute{M}) = \{w_2\}$ and $L_i^{\mathcal{J}}(M \cup \acute{M}) = \{w_2, w_4\}$. Hence, $L_i^{\mathcal{J}}(M \cup \acute{M}) \nsubseteq L_i^{\mathcal{J}}(M) \cup L_i^{\mathcal{J}}(\acute{M})$.

(3) If $M = \{w_1\}$, then $M \nsubseteq L_{\langle i\rangle}^{\mathcal{J}}(M)$.

(4) If $M = \{w_1, w_2, w_3\}$ and $\acute{M} = \{w_1, w_3, w_4\}$, then $U_j^{\mathcal{J}\oplus}(M) \cap U_j^{\mathcal{J}\oplus}(\acute{M}) = \{w_1, w_3, w_4\}$ and $U_j^{\mathcal{J}\oplus}(M \cap \acute{M}) = \{w_1, w_3\}$. Hence, $U_j^{\mathcal{J}\oplus}(M) \cap U_j^{\mathcal{J}\oplus}(\acute{M}) \nsubseteq U_j^{\mathcal{J}\oplus}(M \cap \acute{M})$.

(5) If $M = \{w_2\}$, then $U_{\langle l\rangle}^{\mathcal{J}\oplus}(M) \nsubseteq M$.

Definition 11. Let $(U, \mathcal{R}, \mathcal{J}, \psi_j)$ be $E_j^{\mathcal{J}} S$ on U. A subset M of U is called

(1) Totally $\mathcal{J}_j$ definable, if $L_j^{\mathcal{J}\ominus}(M) = M = U_j^{\mathcal{J}\oplus}(M)$

(2) Internally $\mathcal{J}_j$ definable, if $L_j^{\mathcal{J}\ominus}(M) = M$ and $U_j^{\mathcal{J}\oplus}(M) \neq M$

(3) Externally $\mathcal{J}_j$ definable, if $L_j^{\mathcal{J}\ominus}(M) \neq M$ and $U_j^{\mathcal{J}\oplus}(M) = M$

(4) $\mathcal{J}_j$ rough, if $L_j^{\mathcal{J}\ominus}(M) \neq M$ and $U_j^{\mathcal{J}\oplus}(M) \neq M$

Remark 2. Example 2 and Table 3 show that $\{w_3\}$ is a totally $\mathcal{J}_r$-definable set, $\{w_1, w_2\}$ is an internally $\mathcal{J}_r$-definable set, $\{w_1, w_4\}$ is an externally $\mathcal{J}_r$-definable set, and $\{w_1, w_2\}$ is a $\mathcal{J}_l$-rough set.

TABLE 3: The j approximations for each $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

$M \subseteq U$	$L_r^{\mathcal{J}}(M)$	$U_r^{\mathcal{J}\oplus}(M)$	$L_l^{\mathcal{J}}(M)$	$U_l^{\mathcal{J}\oplus}(M)$	$L_i^{\mathcal{J}}(M)$	$U_i^{\mathcal{J}\oplus}(M)$	$L_u^{\mathcal{J}}(M)$	$U_u^{\mathcal{J}\oplus}(M)$	$L_{\langle r\rangle}^{\mathcal{J}}(M)$	$U_{\langle r\rangle}^{\mathcal{J}\oplus}(M)$	$L_{\langle l\rangle}^{\mathcal{J}}(M)$	$U_{\langle l\rangle}^{\mathcal{J}\oplus}(M)$	$L_{\langle i\rangle}^{\mathcal{J}}(M)$	$U_{\langle i\rangle}^{\mathcal{J}\oplus}(M)$	$L_{\langle u\rangle}^{\mathcal{J}}(M)$	$U_{\langle u\rangle}^{\mathcal{J}\oplus}(M)$
$\{w_1\}$	$\varnothing$	$\{w_1\}$	$\varnothing$	$\{w_1\}$	$\{w_1\}$	$\{w_1\}$	$\varnothing$	$\{w_1\}$	$\varnothing$	$\{w_1\}$	$\varnothing$	$\{w_1\}$	$\varnothing$	$\{w_1\}$	$\varnothing$	$\{w_1\}$
$\{w_2\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_2, w_4\}$	$\{w_2\}$	$\{w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	$\{w_1, w_2, w_4\}$
$\{w_3\}$	$\{w_3\}$	$\{w_3\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_3\}$	$\{w_3\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\{w_3\}$	$\{w_3\}$	$\{w_3\}$	$\varnothing$	U
$\{w_4\}$	$\varnothing$	$\{w_4\}$	$\varnothing$	$\{w_4\}$	$\varnothing$	$\{w_4\}$	$\varnothing$	$\{w_4\}$	$\varnothing$	$\{w_4\}$	$\varnothing$	$\{w_4\}$	$\varnothing$	$\{w_4\}$	$\varnothing$	$\{w_4\}$
$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U	$\{w_1, w_2\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U
$\{w_1, w_3\}$	$\{w_3\}$	$\{w_1, x\}$	$\{w_1, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_3\}$	$\{w_1, w_3\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\{w_1, w_3\}$	$\{w_3\}$	$\{w_1, w_3\}$	$\varnothing$	U
$\{w_1, w_4\}$	$\varnothing$	$\{w_1, w_4\}$	$\varnothing$	$\{w_1, w_4\}$	$\{w_1\}$	$\{w_1, w_4\}$	$\varnothing$	$\{w_1, w_4\}$	$\varnothing$	$\{w_1, w_4\}$	$\varnothing$	$\{w_1, w_4\}$	$\varnothing$	$\{w_1, w_4\}$	$\varnothing$	$\{w_1, w_4\}$
$\{w_2, w_3\}$	$\{w_2, w_3\}$	U	$\{w_2, w_3\}$	U	$\{w_2, w_3\}$	$\{w_2, w_3, w_4\}$	$\{w_2, w_3\}$	U	$\{w_2, w_3\}$	U	$\{w_2, w_3\}$	U	$\{w_2, w_3\}$	U	$\{w_2, w_3\}$	U
$\{w_2, w_4\}$	$\{w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_2, w_4\}$	$\{w_2, w_4\}$	$\{w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U	$\{w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U
$\{w_3, w_4\}$	$\{w_3\}$	$\{w_3, w_4\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_3\}$	$\{w_3, w_4\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\{w_3, w_4\}$	$\{w_3\}$	$\{w_3, w_4\}$	$\varnothing$	U
$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_3\}$	U
$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U
$\{w_1, w_3, w_4\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U
$\{w_2, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	U	$\{w_2, w_3, w_4\}$	U	$\{w_2, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	U	$\{w_2, w_3, w_4\}$	U	$\{w_2, w_3, w_4\}$	U	$\{w_2, w_3, w_4\}$	U	$\{w_2, w_3, w_4\}$	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U	U	U

Table 4: Comparison between $\sigma_j^{\mathcal{I}}$-accuracy measures for $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

$M \subseteq U$	$\sigma_r^{\mathcal{I}}(M)$	$\sigma_l^{\mathcal{I}}(M)$	$\sigma_i^{\mathcal{I}}(M)$	$\sigma_u^{\mathcal{I}}(M)$	$\sigma_{\langle r\rangle}^{\mathcal{I}}(M)$	$\sigma_{\langle l\rangle}^{\mathcal{I}}(M)$	$\sigma_{\langle i\rangle}^{\mathcal{I}}(M)$	$\sigma_{\langle u\rangle}^{\mathcal{I}}(M)$
$\{w_1\}$	0	0	1	0	0	0	0	0
$\{w_2\}$	(1/3)	(1/2)	(1/2)	(1/3)	0	(1/3)	(1/3)	0
$\{w_3\}$	1	(1/3)	1	(1/3)	0	1	1	0
$\{w_4\}$	0	0	0	0	0	0	0	0
$\{w_1, w_2\}$	(2/3)	(1/3)	(2/3)	(1/3)	0	(2/3)	(2/3)	0
$\{w_1, w_3\}$	(1/2)	(2/3)	1	(1/3)	0	(1/2)	(1/2)	0
$\{w_1, w_4\}$	0	0	(1/2)	0	0	0	0	0
$\{w_2, w_3\}$	(1/2)	(1/2)	(2/3)	(1/2)	(1/2)	(1/2)	(1/2)	(1/2)
$\{w_2, w_4\}$	(2/3)	(1/2)	1	(1/3)	0	(2/3)	(2/3)	0
$\{w_3, w_4\}$	(1/2)	(1/3)	(1/2)	(1/3)	0	(1/2)	(1/2)	0
$\{w_1, w_2, w_3\}$	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)
$\{w_1, w_2, w_4\}$	1	(1/3)	1	(1/3)	0	1	1	0
$\{w_1, w_3, w_4\}$	(1/3)	(2/3)	(2/3)	(1/3)	0	(1/3)	(1/3)	0
$\{w_2, w_3, w_4\}$	(3/4)	(3/4)	1	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)
U	1	1	1	1	1	1	1	1

The following theorem presents the relationships between the current approximations given in Definition 10 and the previous one which was mentioned in Definition 8.

Theorem 9. *Let* $(U, \mathcal{R}, \mathcal{I}, \psi_j)$ *be* $E_j^{\mathcal{I}}S$ *and* $M \subseteq U$. *Then, for each* j,

(1) $L_j^{\ominus}(M) \subseteq L_j^{\mathcal{I}\ominus}(M)$

(2) $U_j^{\mathcal{I}\oplus}(M) \subseteq U_j^{\oplus}(M)$

(3) $\mu_j(M) \leq \sigma_l^{\mathcal{I}}(M)$

Proof. It follows from the fact that $\top_{E_j} \subseteq \zeta_j^{\mathcal{I}}$ for each j.

According to Theorem 2, the topologies $\top_j^{\mathcal{I}}$ and $\zeta_j^{\mathcal{I}}$ coincide for each j when $\mathcal{R}$ is a symmetric and transitive relation. This means that our approximations given in Definition 10 and those given in [21] are identical under a symmetric and transitive relation. On the other hand, there is a limitation in our approaches as the next result demonstrates. □

Theorem 10. *Let* $(U, \mathcal{R}, \mathcal{I}, \psi_j)$ *be* $E_j^{\mathcal{I}}S$, *such that* $\mathcal{R}$ *is a reflexive relation. Then,* $\zeta_j^{\mathcal{I}} \subseteq \top_j^{\mathcal{I}}$ *for each* j.

Proof. Let $M \in \zeta_j^{\mathcal{I}}$. Then, $\mathcal{N}_j(y) - M \in \mathcal{I}$. Since $E_j(y) \subseteq \mathcal{N}_j(y)$ for all j under a symmetric and transitive relation (item 2 of Theorem 2) and $\mathcal{I}$ is an ideal, then $E_j(y) - M \in \mathcal{I}$. Hence, $\zeta_j^{\mathcal{I}} \subseteq \top_j^{\mathcal{I}}$. □

4. Classifications of Approximations and Regions in Terms of E_j-Neighborhoods and Ideals

By using ideals and E_j-neighborhoods, we deduce another method to define $E_j^{\mathcal{I}}$-lower and $E_j^{\mathcal{I}}$-upper approximations, $E_j^{\mathcal{I}}$-boundary region, and $E_j^{\mathcal{I}}$-accuracy measure of a subset M. Then, these kinds of approximations are compared with those in Section 3.2. Also, we clarify the relationships among them with the aid of several examples and show their essential properties.

Definition 12. Let M be a subset of $E_j^{\mathcal{I}}S(U, \mathcal{R}, \mathcal{I}, \psi_j)$. Then, the ${}^{\mathcal{I}}\mathcal{R}_j^-$-lower and ${}^{\mathcal{I}}\mathcal{R}_j^+$-upper approximations of a subset M are defined as

(1) ${}^{\mathcal{I}}\mathcal{R}_j^-(M) = \{z \in U\colon E_j(z) - M \in \mathcal{I}\}$

(2) ${}^{\mathcal{I}}\mathcal{R}_j^+(M) = \{z \in U\colon E_j(z) \cap M \notin \mathcal{I}\}$

(3) $\nu_j^{\mathcal{I}}(M) = (|{}^{\mathcal{I}}\mathcal{R}_j^-(M) \cap M| / |{}^{\mathcal{I}}\mathcal{R}_j^+(M) \cup M|)$, $|{}^{\mathcal{I}}\mathcal{R}_j^+(M)| \neq 0$

It should be noted that when $\mathcal{I} = \{\varnothing\}$ in Definition 12, then the present approximations ${}^{\mathcal{I}}\mathcal{R}_j^-(M)$ and ${}^{\mathcal{I}}\mathcal{R}_j^+(M)$ coincide with the previous ones ${}^{\mathcal{I}}\mathcal{R}_j^-(M)$ and $\mathcal{R}_j^+(M)$ in Definition 6. So, the current work is considered as a generalization of Al-Shami et al. work [18]. Now, we shall study the properties of ${}^{\mathcal{I}}\mathcal{R}_j^-(M)$ and ${}^{\mathcal{I}}\mathcal{R}_j^+(M)$ as it is presented in the following results.

Proposition 4. *Let* M *be a subset of* $E_j^{\mathcal{I}}S(U, \mathcal{R}, \mathcal{I}, \psi_j)$. *Then, the following properties hold for each* j.

(i) ${}^{\mathcal{I}}\mathcal{R}_u^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_r^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_i^-(M)$ *and* ${}^{\mathcal{I}}\mathcal{R}_u^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_l^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_i^-(M)$

(ii) ${}^{\mathcal{I}}\mathcal{R}_i^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_r^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_u^+(M)$ *and* ${}^{\mathcal{I}}\mathcal{R}_i^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_l^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_u^+(M)$

(iii) $\nu_u^{\mathcal{I}}(M) \leq \nu_r^{\mathcal{I}}(M) \leq \nu_i^{\mathcal{I}}(M)$ *and* $\nu_u^{\mathcal{I}}(M) \leq \nu_l^{\mathcal{I}}(M) \leq \nu_i^{\mathcal{I}}(M)$

(iv) ${}^{\mathcal{I}}\mathcal{R}_{\langle u\rangle}^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle i\rangle}^-(M)$ *and* ${}^{\mathcal{I}}\mathcal{R}_{\langle u\rangle}^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle i\rangle}^-(M)$

(v) ${}^{\mathcal{I}}\mathcal{R}_{\langle i\rangle}^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle u\rangle}^+(M)$ *and* ${}^{\mathcal{I}}\mathcal{R}_{\langle i\rangle}^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_{\langle u\rangle}^+(M)$

(vi) $\nu_{\langle u\rangle}^{\mathcal{I}}(M) \leq \nu_{\langle r\rangle}^{\mathcal{I}}(M) \leq \nu_{\langle i\rangle}^{\mathcal{I}}(M)$ *and* $\nu_{\langle u\rangle}^{\mathcal{I}}(M) \leq \nu_{\langle l\rangle}^{\mathcal{I}}(M) \leq \nu_{\langle i\rangle}^{\mathcal{I}}(M)$

Proof. Straightforward. □

Theorem 11. *Let* M *and* M' *be subsets of* $E_j^{\mathcal{I}}S(U, \mathcal{R}, \mathcal{I}, \psi_j)$. *Then, the following properties hold for each* j.

(i) ${}^{\mathcal{I}}\mathcal{R}_j^-(U) = U$

(ii) If $M \subseteq M'$, then ${}^{\mathcal{I}}\mathcal{R}_j^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_j^-(M')$
(iii) ${}^{\mathcal{I}}\mathcal{R}_j^-(M \cap M') = {}^{\mathcal{I}}\mathcal{R}_j^-(M) \cap {}^{\mathcal{I}}\mathcal{R}_j^-(M')$
(iv) ${}^{\mathcal{I}}\mathcal{R}_j^-(M^c) = ({}^{\mathcal{I}}\mathcal{R}_j^+(M))^c$
(v) If $M^c \in \mathcal{I}$, then ${}^{\mathcal{I}}\mathcal{R}_j^-(M) = U$
(vi) ${}^{\mathcal{I}}\mathcal{R}_j^+(\varnothing) = \varnothing$
(vii) If $M \subseteq M'$, then ${}^{\mathcal{I}}\mathcal{R}_j^+(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_j^+(M')$
(viii) ${}^{\mathcal{I}}\mathcal{R}_j^+(M \cup M') = {}^{\mathcal{I}}\mathcal{R}_j^+(M) \cup {}^{\mathcal{I}}\mathcal{R}_j^+(M')$
(ix) ${}^{\mathcal{I}}\mathcal{R}_j^+(M^c) = ({}^{\mathcal{I}}\mathcal{R}^-(M))^c$
(x) If $M \in \mathcal{I}$, then ${}^{\mathcal{I}}\mathcal{R}_j^+(M) = \varnothing$

Proof. We prove only (i), (ii), (iii), (iv), and (v), and the rest of proof is similar.

(i) For each $z \in U$, we have $E_j(z) - U = \varnothing \in \mathcal{I}$; then, ${}^{\mathcal{I}}\mathcal{R}_j^-(U) = U$.
(ii) $z \in {}^{\mathcal{I}}\mathcal{R}_j^-(M)$, then $E_j(z) - M \in \mathcal{I}$. Since $M \subseteq M'$, $E_j(z) - M' \in \mathcal{I}$, i.e., $z \in {}^{\mathcal{I}}\mathcal{R}_j^-(M')$. So, ${}^{\mathcal{I}}\mathcal{R}_j^-(M) \subseteq {}^{\mathcal{I}}\mathcal{R}_j^-(M')$.
(iii) It follows from (ii) that ${}^{\mathcal{I}}\mathcal{R}_j^-(M \cap M') \subseteq {}^{\mathcal{I}}\mathcal{R}_j^-(M) \cap {}^{\mathcal{I}}\mathcal{R}_j^-(M')$. Conversely, let $z \in {}^{\mathcal{I}}\mathcal{R}_j^-(M) \cap {}^{\mathcal{I}}\mathcal{R}_j^-(M')$. Then, $z \in {}^{\mathcal{I}}\mathcal{R}_j^-(M)$ and $z \in {}^{\mathcal{I}}\mathcal{R}_j^-(M')$. Therefore, $E_j(z) - M \in \mathcal{I}$ and $E_j(z) - M' \in \mathcal{I}$. Thus, $E_j(z) - (M \cap M') \in \mathcal{I}$. Hence, $z \in {}^{\mathcal{I}}\mathcal{R}_j^-(M \cap M')$, as required.
(iv) Since $z \in {}^{\mathcal{I}}\mathcal{R}_j^-(M^c)$ iff $E_j(z) - M^c \in \mathcal{I}$, iff $E_j(z) \cap M \in \mathcal{I}$, iff $z \notin {}^{\mathcal{I}}\mathcal{R}_j^+(M)$, iff $z \in ({}^{\mathcal{I}}\mathcal{R}_j^+(M))^c$. Then, the result holds.
(v) Obvious

In view of (ii) and (vii) of Theorem 11, the next corollary is obvious. □

Corollary 1. *Let M and M' be subsets of $E_j^{\mathcal{I}}S(U, \mathcal{R}, \mathcal{I}, \psi_j)$. Then, the following properties hold for each j.*

(i) ${}^{\mathcal{I}}\mathcal{R}_j^-(M) \cup {}^{\mathcal{I}}\mathcal{R}_j^-(M') \subseteq {}^{\mathcal{I}}\mathcal{R}_j^-(M \cup M')$
(ii) ${}^{\mathcal{I}}\mathcal{R}_j^+(M \cap M') \subseteq {}^{\mathcal{I}}\mathcal{R}_j^+(M) \cap {}^{\mathcal{I}}\mathcal{R}_j^+(M')$

Example 3. In view of example 1, Table 5 offers $j$approximations for each j.

According to Table 5, we construct Table 6 which represents the $\nu_j^{\mathcal{I}}$-accuracy measure for every subsets of U.

To elucidate that the converses of the items (ii), (v), (vii), and (x) of Theorem 11 and Corollary 1 are not always credible, we display the next example.

Example 4. In view of Example 1 and Table 5, note the following.

(i) For $M = \{w_3\}$ and $M' = \{w_2, w_4\}$, we have ${}^{\mathcal{I}}\mathcal{R}_r^-(M) = \{w_3\}$ and ${}^{\mathcal{I}}\mathcal{R}_r^-(M') = U$. But, $M \nsubseteq M'$.
(ii) For $M = \{w_2, w_4\}$, ${}^{\mathcal{I}}\mathcal{R}_r^-(M) = U$. But, $M^c \notin \mathcal{I}$.
(iii) For $M = \{w_1\}$, we find $M' = \{w_2\}$, we have ${}^{\mathcal{I}}\mathcal{R}_u^+(M) = \varnothing$ and ${}^{\mathcal{I}}\mathcal{R}_u^+(M') = \{w_1, w_2, w_4\}$. But, $M' \nsubseteq M$.
(iv) For $M = \{w_3\}$, ${}^{\mathcal{I}}\mathcal{R}_r^+(M) = \varnothing$. But, $M \notin \mathcal{I}$.
(v) For $M = \{w_2\}$ and $M' = \{w_4\}$, we have ${}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^-(M) = \{w_1, w_3\}$, ${}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^-(M') = \{w_1, w_3\}$, and ${}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^-(S \cup M) = U$. Then, ${}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^-(M) \cup {}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^-(M) \neq {}^{\mathcal{I}}\mathcal{R}_{\langle l\rangle}^-(S \cup M)$.
(vi) For $M = \{w_1, w_3\}$ and $M' = \{w_1, w_4\}$, we have ${}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^+(M) = U$, ${}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^+(M') = U$, and ${}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^+(S \cap M) = \varnothing$. Then, ${}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^+(M) \cap {}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^+(M) \neq {}^{\mathcal{I}}\mathcal{R}_{\langle r\rangle}^+(S \cap M)$.

Note that some basic properties of rough sets with respect to ${}^{\mathcal{I}}\mathcal{R}_j^-$-lower and ${}^{\mathcal{I}}\mathcal{R}_j^+$-upper approximations may evaporate. In what follows, we mention those missing properties.

(i) ${}^{\mathcal{I}}\mathcal{R}_j^-(M) \subseteq M$
(ii) ${}^{\mathcal{I}}\mathcal{R}_j^-(\varnothing) = \varnothing$
(iii) ${}^{\mathcal{I}}\mathcal{R}_j^-({}^{\mathcal{I}}\mathcal{R}_j^-(M)) = {}^{\mathcal{I}}\mathcal{R}_j^-(M)$
(iv) $M \subseteq {}^{\mathcal{I}}\mathcal{R}_j^+(M)$
(v) ${}^{\mathcal{I}}\mathcal{R}_j^+(U) = U$
(vi) ${}^{\mathcal{I}}\mathcal{R}_j^+({}^{\mathcal{I}}\mathcal{R}_j^+(M)) = {}^{\mathcal{I}}\mathcal{R}_j^+(M)$

The next example supports assertions of the above note.

Example 5. In view of Example 1 and Table 5, note the following.

(i) If $M = \{w_1\}$, then ${}^{\mathcal{I}}\mathcal{R}_i^-(M) = \{w_3\}$. Hence, ${}^{\mathcal{I}}\mathcal{R}_i^-(M) \nsubseteq M$.
(ii) ${}^{\mathcal{I}}\mathcal{R}_r^-(\varnothing) = \{w_3\} \neq \varnothing$.
(iii) If $M = \{w_2, w_4\}$, then ${}^{\mathcal{I}}\mathcal{R}_u^-(M) = \{w_2\}$ and ${}^{\mathcal{I}}\mathcal{R}_u^-({}^{\mathcal{I}}\mathcal{R}_u^-(M)) = \varnothing$. Hence, ${}^{\mathcal{I}}\mathcal{R}_u^-({}^{\mathcal{I}}\mathcal{R}_u^-(M)) \neq {}^{\mathcal{I}}\mathcal{R}_u^-(M)$.
(iv) If $M = \{w_1\}$, then ${}^{\mathcal{I}}\mathcal{R}_l^+(M) = \varnothing$. Hence, $M \nsubseteq {}^{\mathcal{I}}\mathcal{R}_l^+(M)$.
(v) ${}^{\mathcal{I}}\mathcal{R}_i^+(U) = \{w_1, w_2, w_4\} \neq U$.
(vi) If $M = \{w_2\}$, then ${}^{\mathcal{I}}\mathcal{R}_u^+(M) = \{w_1, w_2, w_4\}$ and ${}^{\mathcal{I}}\mathcal{R}_j^+({}^{\mathcal{I}}\mathcal{R}_j^+(M)) = U$. Hence, ${}^{\mathcal{I}}\mathcal{R}_j^+({}^{\mathcal{I}}\mathcal{R}_j^+(M)) \neq {}^{\mathcal{I}}\mathcal{R}_j^+(M)$.

Lemma 3. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on $(U, \mathcal{R}, \psi_j)$ and $M \subseteq U$. If $\mathcal{I} \subseteq \mathcal{J}$, then the following statements hold for each j:*

(1) ${}^{\mathcal{I}}\mathcal{R}_j^-(M) \subseteq \underline{\mathcal{R}}^{j\mathcal{J}}(M)$
(2) ${}^{\mathcal{I}}\mathcal{R}_j^+(M) \supseteq \overline{\mathcal{R}}_j^{\mathcal{J}}(M)$
(3) $\nu_j^{\mathcal{I}}(M) \leq \nu_j^{\mathcal{J}}(M)$

Proof. Straightforward.

The next theorem shows the relationships between the approximations ${}^{\mathcal{I}}\mathcal{R}_j^-(M)$ and ${}^{\mathcal{I}}\mathcal{R}_j^+(M)$ of the current approximations in Definition 6 and $\mathcal{R}_j^-(M)$ and $\mathcal{R}_j^+(M)$ of Definition 6 for each j. □

Table 5: The j approximations for each $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

$M \subseteq U$	$^{\mathscr{I}}\mathscr{R}_r^-(M)$	$^{\mathscr{I}}\mathscr{R}_r^+(M)$	$^{\mathscr{I}}\mathscr{R}_l^-(M)$	$^{\mathscr{I}}\mathscr{R}_l^+(M)$	$^{\mathscr{I}}\mathscr{R}_i^-(M)$	$^{\mathscr{I}}\mathscr{R}_i^+(M)$	$^{\mathscr{I}}\mathscr{R}_u^-(M)$	$^{\mathscr{I}}\mathscr{R}_u^+(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle r\rangle}^-(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle r\rangle}^+(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle l\rangle}^-(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle l\rangle}^-(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle i\rangle}^-(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle i\rangle}^+(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle u\rangle}^-(M)$	$^{\mathscr{I}}\mathscr{R}_{\langle r\rangle}^+(M)$
$\{w_1\}$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_1, w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$
$\{w_2\}$	U	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_2, w_4\}$	U	$\{w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	$\varnothing$	U
$\{w_3\}$	$\{w_3\}$	$\varnothing$	$\{w_1, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_3\}$	$\varnothing$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	U
$\{w_4\}$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_1, w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$
$\{w_1, w_2\}$	U	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_2, w_4\}$	U	$\{w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	$\varnothing$	U
$\{w_1, w_3\}$	$\{w_3\}$	$\varnothing$	$\{w_1, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_3\}$	$\varnothing$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	U
$\{w_1, w_4\}$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_1, w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$
$\{w_2, w_3\}$	U	$\{w_1, w_2, w_4\}$	U	U	U	$\{w_2, w_4\}$	U	U	U	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	U	U
$\{w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_2, w_4\}$	U	$\{w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	$\varnothing$	U
$\{w_3, w_4\}$	$\{w_3\}$	$\varnothing$	$\{w_1, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_3\}$	$\varnothing$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	U
$\{w_1, w_2, w_3\}$	U	$\{w_1, w_2, w_4\}$	U	U	U	$\{w_2, w_4\}$	U	U	U	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	U	U
$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	$\{w_2\}$	$\{w_2, w_4\}$	U	$\{w_2, w_4\}$	$\{w_2\}$	$\{w_1, w_2, w_4\}$	$\varnothing$	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	$\varnothing$	U
$\{w_1, w_3, w_4\}$	$\{w_3\}$	$\varnothing$	$\{w_1, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_3\}$	$\varnothing$	$\{w_3\}$	$\{w_1, w_3, w_4\}$	$\varnothing$	U	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	U
$\{w_2, w_3, w_4\}$	U	$\{w_1, w_2, w_4\}$	U	U	U	$\{w_2, w_4\}$	U	U	U	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	U	U
U	U	$\{w_1, w_2, w_4\}$	U	U	U	$\{w_2, w_4\}$	U	U	U	U	U	$\{w_1, w_2, w_4\}$	U	$\{w_1, w_2, w_4\}$	U	U
$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_1, w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\{w_3\}$	$\varnothing$	$\varnothing$	$\varnothing$

Table 6: Comparison between $\nu_j^{\mathcal{J}}$-accuracy measures for $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

$M \subseteq U$	$\nu_r^{\mathcal{J}}(M)$	$\nu_l^{\mathcal{J}}(M)$	$\nu_i^{\mathcal{J}}(M)$	$\nu_u^{\mathcal{J}}(M)$	$\nu_{\langle r\rangle}^{\mathcal{J}}(M)$	$\nu_{\langle l\rangle}^{\mathcal{J}}(M)$	$\nu_{\langle i\rangle}^{\mathcal{J}}(M)$	$\nu_{\langle u\rangle}^{\mathcal{J}}(M)$
$\{w_1\}$	0	0	1	0	0	0	0	0
$\{w_2\}$	(1/3)	(1/2)	(1/2)	(1/3)	0	(1/3)	(1/3)	0
$\{w_3\}$	1	(1/3)	1	(1/3)	0	1	1	0
$\{w_4\}$	0	0	0	0	0	0	0	0
$\{w_1, w_2\}$	(2/3)	(1/3)	(2/3)	(1/3)	0	(2/3)	(2/3)	0
$\{w_1, w_3\}$	(1/2)	(2/3)	1	(1/3)	0	(1/2)	(1/2)	0
$\{w_1, w_4\}$	0	0	(1/2)	0	0	0	0	0
$\{w_2, w_3\}$	(1/2)	(1/2)	(2/3)	(1/2)	(1/2)	(1/2)	(1/2)	(1/2)
$\{w_2, w_4\}$	(2/3)	(1/2)	1	(1/3)	0	(2/3)	(2/3)	0
$\{w_3, w_4\}$	(1/2)	(1/3)	(1/2)	(1/3)	0	(1/2)	(1/2)	0
$\{w_1, w_2, w_3\}$	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)
$\{w_1, w_2, w_4\}$	1	(1/3)	1	(1/3)	0	1	1	0
$\{w_1, w_3, w_4\}$	(1/3)	(2/3)	(2/3)	(1/3)	0	(1/3)	(1/3)	0
$\{w_2, w_3, w_4\}$	(3/4)	(3/4)	1	(3/4)	(3/4)	(3/4)	(3/4)	(3/4)
U	1	1	1	1	1	1	1	1

Theorem 12. *Let $(U, \mathcal{R}, \psi_j, \mathcal{J})$ and $M \subseteq U$. Then, the following statements hold for each j:*

(1) $\mathcal{R}_j^-(M) \subseteq {}^{\mathcal{J}}\mathcal{R}_j^-(M)$

(2) ${}^{\mathcal{J}}\mathcal{R}_j^+(M) \subseteq \mathcal{R}_j^+(M)$

(3) $\mu_j(M) \leq \nu_j^{\mathcal{J}}(M)$

Proof. Direct to prove.

To preserve some Pawalk's properties which are loss in the approximations given in Definition 12, we formulate new approximations and accuracy measures in the following. □

Definition 13. Let M be a subset of $E_j^{\mathcal{J}}S(U, \mathcal{R}, \mathcal{J}, \psi_j)$. Then, the $\mathcal{R}_{j\diamond}$-lower and $\mathcal{R}_j^{\diamond}$-upper approximations and $\nu_j^{\mathcal{J}}$-accuracy measure of a subset M are

(1) $\mathcal{R}_{j\diamond}(M) = {}^{\mathcal{J}}\mathcal{R}_j^-(M) \cap M$

(2) $\mathcal{R}_j^{\diamond}(M) = {}^{\mathcal{J}}\mathcal{R}_j^+(M) \cup M$

(3) $\nu_{j\diamond}^{\mathcal{J}}(M) = (|\mathcal{R}_{j\diamond}(M)| / |\mathcal{R}_j^{\diamond}(M)|)$, $|\mathcal{R}_j^{\diamond}(M)| \neq 0$.

Remark 3. It should be noted that the current approximations $\mathcal{R}_{j\diamond}(M)$ and $\mathcal{R}_j^{\diamond}(M)$ in Definition 13 have the same properties of the current approximations ${}^{\mathcal{J}}\mathcal{R}_j^-(M)$ and ${}^{\mathcal{J}}\mathcal{R}_j^+(M)$, which are stated in Theorem 11 and Corollary 1. Additionally, it satisfies the following properties:

(1) $\mathcal{R}_{j\diamond}(M) \subseteq M \subseteq \mathcal{R}_j^{\diamond}(M)$

(2) $\mathcal{R}_{j\diamond}(\emptyset) = \emptyset$, $\mathcal{R}_j^{\diamond}(U) = U$.

5. Conclusion and Future Work

Rough set is a powerful mathematical to deal with uncertainty. Approximation operators are the core concepts in rough set content; they have topological properties similar to all/some properties of the interior and closure operators. Neighborhood systems are one of the followed methods to study rough approximations using topological interior and closure operators.

In this study, we have initiated different types of topologies from ideals and E_j-neighborhoods induced from any binary relations. We have applied these topologies to study new kinds of approximations and accuracy measures. Then, we have compared between them and their counterparts induced directly from E_j-neighborhoods and ideals. Also, we have illustrated the advantages of our approaches to obtain higher accuracy measures than those proposed in [18]. Some limitations of our approaches have been investigated.

In the upcoming works, we will study new types of topologies and approximations induced from other neighborhoods and ideals. Also, we will investigate the concepts and results presented, herein, in fuzzy and rough sets contents.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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