mathematics

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Various Types of q-Differential Equations of Higher Order for q-Euler and q-Genocchi Polynomials

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Abstract: One finds several q-differential equations of a higher order for q-Euler polynomials and q-Genocchi polynomials. Additionally, we have a few q-differential equations of a higher order, which are mixed with q-Euler numbers and q-Genocchi polynomials. Moreover, we investigate some symmetric q-differential equations of a higher order by applying symmetric properties of q-Euler polynomials and q-Genocchi polynomials.

Keywords: q-Euler polynomials; q-Genocchi polynomials; q-differential equation of higher order; symmetric property

MSC: 12H20; 35G05; 81Q15

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Citation: Ryoo, C.-S.; Kang, J.-Y. Various Types of q-Differential Equations of Higher Order for q-Euler and q-Genocchi Polynomials. *Mathematics* **2022**, *10*, 1181. https://doi.org/10.3390/math10071181

Academic Editor: Dumitru Baleanu

Received: 23 February 2022
Accepted: 30 March 2022
Published: 5 April 2022

1. Introduction

The Bernoulli equation is written as

$$\frac{dy}{dx} + p(x)y - g(x)y^m = 0, \tag{1}$$

where m is any real number, and $p(x)$ and $g(x)$ are continuous functions; see [1].

The above Bernoulli equation is one of the equations that can convert nonlinear equations into linear equations. The equation was first discussed in a work by Jacob Bernoulli in 1695, after whom it is named. For example, this Bernoulli equation can solve problems modeled by nonlinear differential equations and also solve equations about the population expressed in logistic equations or Verhulst equations.

In [2,3], we note

$$\sum_{n=0}^{\infty} \mathcal{E}_n \frac{t^n}{n!} = \frac{2}{e^t+1}, \quad \sum_{n=0}^{\infty} \mathcal{E}_n(x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{tx},$$

where $\mathcal{E}_n$ is the Euler numbers and $\mathcal{E}_n(x)$ is the Euler polynomials. If $m = 0$ in Equation (1), then the Bernoulli equation has the solution which is the generating function of the Euler polynomials. The equation is as follows.

$$\frac{d}{dx}\mathcal{E}_{n-1}(x) + \frac{\mathcal{E}_0(1)+2x}{\mathcal{E}_1(1)}\mathcal{E}_{n-1}(x) + \frac{2}{\mathcal{E}_1(1)}\mathcal{E}_n(x) = 0, \tag{2}$$

where $\mathcal{E}_n(x)$ is the Euler polynomials.

In q-calculus, we consider the first order of the q-Bernoulli equation $D_q y + p(x)y - g(x)y^m = 0$. When $m = 0$ in Equation (1), the q-Euler polynomials is the solution of the following q-differential equation of the first order.

$$D_q\mathcal{E}_{n-1,q}(x) + \frac{q(\mathcal{E}_{0,q}(1)+2qx)}{\mathcal{E}_{1,q}(1)}\mathcal{E}_{n-1,q}(x) + \frac{2}{q^{n-1}\mathcal{E}_{1,q}(1)}\mathcal{E}_{n,q}(qx) = 0, \tag{3}$$

where D_q is the derivative in q-calculus and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomials. We note that Equation (3) becomes Equation (2) when $q \to 1$.

Through Equation (3), the goal of this paper is to find out the form of differential equations of a higher order. We also find several symmetric properties of differential equations of a higher order, the structure of differential equations of a higher order, the properties of polynomials at $q \to 1$, and so on. To introduce q-Euler polynomials and q-Genocchi polynomials, we will summarize the definitions and make the arrangements required in this paper as follows.

The q-number, which is important in q-calculus, was first introduced by Jackson, see [4,5]. From the discovery of the q-number, various useful results were considered and studied in q-series, q-special functions, quantum algebras, q-discrete distribution, q-differential equation, q-calculus, and so on; see [2,6–15]. Here, we would like to briefly review several significant concepts of q-calculus, which we need for this paper.

Let $n, q \in \mathbb{R}$ with $q \neq 1$. The number

$$[n]_q = \frac{1 - q^n}{1 - q}$$

is called the q-number. We note that $\lim_{q \to 1}[n]_q = n$. In particular, for $k \in \mathbb{Z}$, $[k]_q$ is called the q-integer.

The q-Gaussian binomial coefficients are defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{cases} 0 & \text{if } r > m \\ \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & \text{if } r \leq m \end{cases},$$

where m and r are non-negative integers. For $r = 0$, the value is 1 since the numerator and the denominator are both empty products. One notes $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[0]_q! = 1$.

Definition 1. *The q-derivative of a function f with respect to x is defined by*

$$D_{q,x}f(x) := D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad \textit{for} \quad x \neq 0,$$

and $D_q f(0) = f'(0)$.

One can prove that f is differentiable at zero, and it is clear that $D_q x^n = [n]_q x^{n-1}$. Let us point out that $D_{q,x}^{(k)} f(x)$ converges to $f^{(k)}(x)$ as q goes to 1. From Definition 1, we can see some formulae for the q-derivative.

Theorem 1. *From Definition 1, we note that*

$$\begin{aligned} &\textit{(i)} \quad D_q(f(x)g(x)) = q(x)D_q f(x)_f(qx)D_q g(x) \\ &\qquad\qquad\qquad\quad = f(x)D_q g(x) + g(qx)D_q f(x), \\ &\textit{(ii)} \quad D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)} \\ &\qquad\qquad\qquad\quad = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}, \\ &\textit{(iii)} \quad \textit{for any constants } a \textit{ and } b, \\ &\qquad D_q(af(x) + bg(x)) = aD_q f(x) + bD_q g(x). \end{aligned}$$

Definition 2. *Let z be any complex numbers with $|z| < 1$. We introduce the following two series, called q-exponential functions*

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \qquad E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

We note that $\lim_{q\to 1} e_q(z) = e^z$.

Theorem 2 ([15]). *From Definition 2, we note that*

$$\begin{aligned} &(i) \quad e_q(x)e_q(y) = e_q(x+y), \quad \text{if} \quad yx = qxy. \\ &(ii) \quad e_q(x)E_q(-x) = 1. \\ &(iii) \quad e_{q^{-1}}(x) = E_q(x). \end{aligned}$$

Due to the above two types of q-exponential functions, Euler, Bernoulli, and Genocchi polynomials are defined as new types of polynomials, and many mathematicians have studied their properties; see [2,3,12,16–19]. In addition, it is studied in various fields, such as the structure of approximations of polynomials and their relevance to fractals by using computers; see [17,20,21]. The definition of each polynomial used in this paper can be confirmed in Definitions 3 and 4.

Definition 3. *The q-Euler numbers and polynomials can be expressed as*

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q} \frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1}, \quad \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1} e_q(tx), \ \ \textit{respectively}.$$

When $q \to 1$ in Definition 3, we can find the Euler numbers and polynomials.

Definition 4. *The q-Genocchi numbers and polynomials can be expressed as*

$$\sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}, \quad \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1} e_q(tx), \ \ \textit{respectively}.$$

When $q \to 1$ in Definition 4, we can find the Genocchi numbers and polynomials as

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}, \quad \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{tx}.$$

Based on the previous content, our purpose is to find various q-differential equations of higher order that contain q-Euler polynomials and q-Genocchi polynomials as solutions of the equation of a higher order. In Section 2, we find a q-differential equation of higher order that has q-Euler polynomials as the solution and check its associated properties. In Section 3, not only are we able to find a q-differential equation of a higher order that is the solution of q-Genocchi polynomials, but we can also address a q-differential equation of a higher order in combination with the q-Euler number or polynomials. Various properties can be identified based on these equations of a higher order.

2. Several q-Differential Equations of Higher Order and Properties of q-Euler Polynomials

In this section,we show that the q-Euler polynomials are solutions to some q-differential equations of a higher order. Moreover, we introduce a special q-differential equation of a higher order which is related to a symmetric property for q-Euler polynomials.

Let

$$F_q(t,x) := \frac{2}{e_q(t)+1} e_q(tx) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{[n]_q!}. \tag{4}$$

From Definition 3, we find the q-Euler numbers in Equation (4). From Table 1, we also can see a few q-Euler numbers $\mathcal{E}_{n,q}$ and polynomials $\mathcal{E}_{n,q}(x)$ as follows.

Table 1. q-Euler numbers and polynomials.

n	$\mathcal{E}_{n,q}$	$\mathcal{E}_{n,q}(x)$
0	1	1
1	$\frac{1}{2}$	$x-\frac{1}{2}$
2	$\frac{1}{4}[2]_q-\frac{1}{2}$	$x^2-\frac{1}{2}[2]_qx+\frac{1}{4}[2]_q-\frac{1}{2}$
3	$-\frac{1}{8}[3]_q[2]_q+\frac{1}{2}[3]_q-\frac{1}{2}$	$x^3-\frac{1}{2}[3]_qx^2+\left(\frac{1}{4}[3]_q[2]_q-\frac{1}{2}[3]_q\right)x-\frac{1}{8}[3]_q[2]_q+\frac{1}{2}[3]_q-\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

Lemma 1. *For* $0<q<1$*, we have*

$$\text{(i)}\quad \mathcal{E}_{n-k,q}(x)=\frac{[n-k]_q!}{[n]_q!}D_{q,x}^{(k)}\mathcal{E}_{n,q}(x),$$

$$\text{(ii)}\quad \mathcal{E}_{n-k,q}(q^{-1}x)=\frac{q^k[n-k]_q!}{[n]_q!}D_{q,x}^{(k)}\mathcal{E}_{n,q}(q^{-1}x).$$

Proof. (i) We will use induction to show the lemma. Applying q-derivative in (4), we have

$$D_{q,x}^{(1)}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_q!}=\frac{2}{e_q(t)+1}D_{q,x}^{(1)}e_q(tx)=\sum_{n=0}^{\infty}[n]_q\mathcal{E}_{n-1,q}(x)\frac{t^n}{[n]_q!}.$$

From the above equation, we find a relation such as

$$D_{q,x}^{(1)}\mathcal{E}_{n,q}(x)=[n]_q\mathcal{E}_{n-1,q}(x).$$

In a similar way, we have

$$D_{q,x}^{(2)}\mathcal{E}_{n,q}(x)=[n]_q[n-1]_q\mathcal{E}_{n-2,q}(x).$$

Therefore, we can find

$$D_{q,x}^{(k)}\mathcal{E}_{n,q}(x)=[n]_q[n-1]_q\cdots[n-(k-1)]_q\mathcal{E}_{n-k,q}(x),$$

which is the desired result.

(ii) We omit the proof of (ii) in Lemma 1 because we can derive the required result if we use a similar method in the proof of (i) in Lemma 1. □

Theorem 3. *The q-Euler polynomials $\mathcal{E}_{n,q}(x)$ is a solution of the following q-differential equation of a higher order.*

$$\frac{\mathcal{E}_{n-1,q}(1)}{2[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n-1,q}(x)+\frac{q\mathcal{E}_{n-2,q}(1)}{2[n-2]_q!}D_{q,x}^{(n-2)}\mathcal{E}_{n-1,q}(x)+\cdots+\frac{q^{n-3}\mathcal{E}_{2,q}(1)}{2[2]_q!}D_{q,x}^{(2)}\mathcal{E}_{n-1,q}(x)$$
$$+\frac{q^{n-2}\mathcal{E}_{1,q}(1)}{2}D_{q,x}^{(1)}\mathcal{E}_{n-1,q}(x)+\frac{(q^{-1}\mathcal{E}_{0,q}(1)+2x)q^n}{2}\mathcal{E}_{n-1,q}(x)+\mathcal{E}_{n,q}(qx)=0.$$

Proof. Using the q-derivative, we can note

$$D_{q,t}\left(\frac{2}{e_q(t)+1}\right)=-\frac{2e_q(t)}{(e_q(t)+1)(e_q(qt)+1)}. \tag{5}$$

We consider the q-derivative after substituting qx instead of x in (4). From Equation (5), we have

$$\begin{aligned} D_{q,t}F_q(t,qx) &= D_{q,t}\left(\frac{2}{e_q(t)+1}e_q(qtx)\right) \\ &= e_q(qtx)\left(\frac{-2e_q(t)}{(e_q(t)+1)(e_q(qt)+1)}\right) + \frac{2qx}{e_q(qt)+1}e_q(qtx) \\ &= F_q(qt,x)\left(qx - \frac{1}{e_q(t)+1}e_q(t)\right) \\ &= \sum_{n=0}^{\infty}\left(q^{n+1}x\mathcal{E}_{n,q}(x) - 2^{-1}\sum_{k=0}^{n}[n]_q q^{n-k}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}. \end{aligned} \tag{6}$$

To make the calculations easier, we multiply t in Equation (6). Then, we find

$$\begin{aligned} &tD_{q,t}F_q(t,qx) \\ &= \sum_{n=0}^{\infty}[n]_q\left(q^n x\mathcal{E}_{n-1,q}(x) - 2^{-1}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(x)\right)\frac{t^n}{[n]_q!}. \end{aligned} \tag{7}$$

Additionally, we can obtain the following equation from (4).

$$tD_{q,t}F_q(t,qx) = \sum_{n=0}^{\infty}[n]_q\mathcal{E}_{n,q}(qx)\frac{t^n}{[n]_q!}. \tag{8}$$

By comparing the coefficients of Equations (7) and (8), we find

$$2^{-1}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(x) = q^n x\mathcal{E}_{n-1,q}(x) - \mathcal{E}_{n,q}(qx). \tag{9}$$

From (i) in Lemma 1, we consider the following equation.

$$\mathcal{E}_{n-k-1,q}(x) = \frac{[n-k-1]_q!}{[n-1]_q!}D_{q,x}^{(k)}\mathcal{E}_{n-1,q}(x). \tag{10}$$

Substituting the right hand side of (10) to the left hand side of (9), we obtain

$$2^{-1}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(x) = 2^{-1}\sum_{k=0}^{n-1}\frac{q^{n-k-1}\mathcal{E}_{k,q}(1)}{[k_q!]}D_{q,x}^{(k)}\mathcal{E}_{n-1,q}(x). \tag{11}$$

Using Equations (9) and (11), we complete the required result. □

We can see Corollaries 1 and 2 when $q \to 1$ in Theorem 3.

Corollary 1. *The Euler polynomial $\mathcal{E}_n(x)$ is a solution of the following differential equation of a higher order.*

$$\begin{aligned} &\frac{\mathcal{E}_{n-1}(1)}{2(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\mathcal{E}_{n-1}(x) + \frac{\mathcal{E}_{n-2}(1)}{2(n-2)!}\frac{d^{n-2}}{dx^{n-2}}\mathcal{E}_{n-1}(x) + \cdots + \frac{\mathcal{E}_2(1)}{4}\frac{d^2}{dx^2}\mathcal{E}_{n-1}(x) \\ &+ \frac{\mathcal{E}_1(1)}{2}\frac{d}{dx}\mathcal{E}_{n-1}(x) + \frac{\mathcal{E}_0(1)+2x}{2}\mathcal{E}_{n-1}(x) + \mathcal{E}_n(x) = 0. \end{aligned}$$

Corollary 2. *In (9), one holds*

$$2\mathcal{E}_n(x) = 2x\mathcal{E}_{n-1}(x) - \sum_{k=0}^{n-1}\binom{n-1}{k}\mathcal{E}_{n-k-1}(1)\mathcal{E}_k(x),$$

where $\mathcal{E}_n(x)$ is the Euler polynomials.

Theorem 4. *The q-Euler polynomial $\mathcal{E}_{n,q}(x)$ satisfies the following q-differential equation of a higher order.*

$$\sum_{l=0}^{n-1}\frac{\mathcal{E}_{l,q}}{2[n-l-1]_q![l]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n-1,q}(x)+\sum_{l=0}^{n-2}\frac{q\mathcal{E}_{l,q}}{2[n-l-2]_q![l]_q!}D_{q,x}^{(n-2)}\mathcal{E}_{n-1,q}(x)+\cdots$$
$$+\sum_{l=0}^{2}\frac{q^{n-3}\mathcal{E}_{l,q}}{2[2-l]_q![l]_q!}D_{q,x}^{(2)}\mathcal{E}_{n-1,q}(x)+\sum_{l=0}^{1}\frac{q^{n-2}\mathcal{E}_{l,q}}{2[1-l]_q![l]_q!}D_{q,x}^{(1)}\mathcal{E}_{n-1,q}(x)$$
$$+\frac{(q^{-1}\mathcal{E}_{0,q}-2x)q^n}{2}\mathcal{E}_{n-1,q}(x)+\mathcal{E}_{n,q}(qx)=0,$$

where $\mathcal{E}_{n,q}$ is the q-Euler numbers.

Proof. To find the q-differential equation of higher order including q-Euler numbers, we transform Equation (6) as follows.

$$D_{q,t}F_q(t,qx)=\sum_{n=0}^{\infty}\left(q^{n+1}x\mathcal{E}_{n,q}(x)-2^{-1}\sum_{k=0}^{n}\sum_{l=0}^{k}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\l\end{bmatrix}_q q^{n-k}\mathcal{E}_{l,q}\mathcal{E}_{n-k-1,q}(x)\right)\frac{t^n}{[n]_q!}.$$

From the similar method in Theorem 3, we find

$$2^{-1}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\begin{bmatrix}n-1\\k\end{bmatrix}_q\begin{bmatrix}k\\l\end{bmatrix}_q q^{n-k-1}\mathcal{E}_{l,q}\mathcal{E}_{n-k-1,q}(x)=q^n x\mathcal{E}_{n-1,q}(x)-\mathcal{E}_{n,q}(qx). \tag{12}$$

Using (i) in Lemma 1 in the left hand side of (12), we can find the desired result. □

From Theorem 4, we can find Corollaries 3 and 4 when $q\to 1$.

Corollary 3. *The Euler polynomials $\mathcal{E}_n(x)$ satisfy the following differential equation of a higher order.*

$$\sum_{l=0}^{n-1}\frac{\mathcal{E}_l}{2(n-l-1)!l!}\frac{d^{n-1}}{dx^{n-1}}\mathcal{E}_{n-1}(x)+\sum_{l=0}^{n-2}\frac{\mathcal{E}_l}{2(n-l-2)!l!}\frac{d^{n-2}}{dx^{n-2}}\mathcal{E}_{n-1}(x)+\cdots$$
$$+\sum_{l=0}^{2}\frac{\mathcal{E}_l}{2(2-l)!l!}\frac{d^2}{dx^2}\mathcal{E}_{n-1}(x)+\sum_{l=0}^{1}\frac{\mathcal{E}_l}{2(1-l)!l!}\frac{d}{dx}\mathcal{E}_{n-1}(x)$$
$$+\frac{\mathcal{E}_0-2x}{2}\mathcal{E}_{n-1}(x)+\mathcal{E}_n(x)=0,$$

where $\mathcal{E}_n$ is the Euler numbers.

Corollary 4. *In Equation (12), the following holds:*

$$2^{-1}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\binom{n-1}{k}\binom{k}{l}\mathcal{E}_l\mathcal{E}_{n-k-1}(x)=x\mathcal{E}_{n-1}(x)-\mathcal{E}_n(x),$$

where $\mathcal{E}_n$ is the Euler numbers and $\mathcal{E}_n(x)$ is the Euler polynomials.

From Theorems 3 and 4, we obtain Corollary 5.

Corollary 5. *Let $0<q<1$. Then, one holds*

$$\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(x)=\sum_{k=0}^{n-1}\sum_{l=0}^{k}\begin{bmatrix}n-1\\k\end{bmatrix}_q\begin{bmatrix}k\\l\end{bmatrix}_q q^{n-k-1}\mathcal{E}_{l,q}\mathcal{E}_{n-k-1,q}(x).$$

Theorem 5. *The q-Euler polynomials $\mathcal{E}_{n,q}(x)$ are a solution of the following q-differential equation of a higher order.*

$$\frac{\mathcal{E}_{n-1,q}(1)}{2[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n-1,q}(q^{-1}x)+\frac{\mathcal{E}_{n-2,q}(1)}{2[n-2]_q!}D_{q,x}^{(n-2)}\mathcal{E}_{n-1,q}(q^{-1}x)+\cdots$$
$$+\frac{\mathcal{E}_{3,q}(1)}{2[3]_q!}D_{q,x}^{(3)}\mathcal{E}_{n-1,q}(q^{-1}x)+\frac{\mathcal{E}_{2,q}(1)}{2[2]_q!}D_{q,x}^{(2)}\mathcal{E}_{n-1,q}(q^{-1}x)+\frac{\mathcal{E}_{1,q}(1)}{2}D_{q,x}^{(1)}\mathcal{E}_{n-1,q}(q^{-1}x)$$
$$+\frac{(\mathcal{E}_{0,q}(1)-2x)}{2}\mathcal{E}_{n-1,q}(q^{-1}x)+q^{1-n}\mathcal{E}_{n,q}(x)=0.$$

Proof. Using the q-derivative in Equation (4), we have

$$tD_{q,t}F_q(qt,x)=\sum_{n=0}^{\infty}[n]_q q^n\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{13}$$

Replacing qt instead of t and applying the q-derivative in (4), we also find

$$tD_{q,t}F_q(qt,x) = \sum_{n=0}^{\infty}[n]_q\left(q^{2n-1}x\mathcal{E}_{n-1,q}(q^{-1}x)-2^{-1}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{2n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(q^{-1}x)\right)\frac{t^n}{[n]_q!}. \tag{14}$$

Comparing the coefficients of Equations (12) and (13), we obtain

$$2^{-1}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{2n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(q^{-1}x)=q^{2n-1}x\mathcal{E}_{n-1,q}(q^{-1}x)-q^n\mathcal{E}_{n,q}(x). \tag{15}$$

From (ii) in Lemma 1, we note

$$\mathcal{E}_{n-k-1,q}(q^{-1}x)=\frac{q^k[n-k-1]_q!}{[n-1]_q!}D_{q,x}^{(k)}\mathcal{E}_{n-1,q}(q^{-1}x). \tag{16}$$

Using (16) in the left hand side of (15), we derive

$$\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{2n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(q^{-1}x)=\sum_{k=0}^{n-1}\frac{q^{2n-1}\mathcal{E}_{k,q}(1)}{[k]_1!}D_{q,x}^{(k)}\mathcal{E}_{n-1,q}(q^{-1}x). \tag{17}$$

We can find a equation combining the right hand side of (17) and (15), which shows the required result. □

We find Corollary 6 when $q\to 1$ in Theorem 5.

Corollary 6. *The Euler polynomials $\mathcal{E}_n(x)$ are a solution of the following differential equation of a higher order.*

$$\frac{\mathcal{E}_{n-1}(1)}{2(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\mathcal{E}_{n-1}(x)+\frac{\mathcal{E}_{n-2}(1)}{2(n-2)!}\frac{d^{n-2}}{dx^{n-2}}\mathcal{E}_{n-1}(x)+\cdots+\frac{\mathcal{E}_3(1)}{2(3)!}\frac{d^3}{dx^3}\mathcal{E}_{n-1}(x)$$
$$+\frac{\mathcal{E}_2(1)}{4}\frac{d^2}{dx^2}\mathcal{E}_{n-1}(x)+\frac{\mathcal{E}_1(1)}{2}\frac{d}{dx}\mathcal{E}_{n-1}(x)+\frac{(\mathcal{E}_0(1)-2x)}{2}\mathcal{E}_{n-1}(x)+\mathcal{E}_n(x)=0.$$

Theorem 6. *The q-Euler polynomials $\mathcal{E}_{n,q}(x)$ are a solution of the following q-differential equation of a higher order.*

$$\sum_{l=0}^{n-1}\frac{\mathcal{E}_{l,q}}{2[n-l-1]_q![l]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n-1,q}(q^{-1}x)+\sum_{l=0}^{n-2}\frac{\mathcal{E}_{l,q}}{2[n-l-2]_q![l]_q!}D_{q,x}^{(n-2)}\mathcal{E}_{n-1,q}(q^{-1}x)+\cdots$$
$$+\sum_{l=0}^{2}\frac{\mathcal{E}_{l,q}}{2[2-l]_q![l]_q!}D_{q,x}^{(2)}\mathcal{E}_{n-1,q}(q^{-1}x)+\sum_{l=0}^{1}\frac{\mathcal{E}_{l,q}}{2[1-l]_q![l]_q!}D_{q,x}^{(1)}\mathcal{E}_{n-1,q}(q^{-1}x)$$
$$+\frac{(\mathcal{E}_{0,q}-2x)}{2}\mathcal{E}_{n-1,q}(q^{-1}x)+q^{1-n}\mathcal{E}_{n,q}(x)=0.$$

Proof. Substituting q^{2n}, $q^{-1}x$ instead of q^n and x, respectively, in Corollary 3, we have

$$\begin{aligned}&\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{2n-k-1}\mathcal{E}_{k,q}(1)\mathcal{E}_{n-k-1,q}(q^{-1}x)\\&=\sum_{k=0}^{n-1}\sum_{l=0}^{k}\begin{bmatrix}n-1\\k\end{bmatrix}_q\begin{bmatrix}k\\l\end{bmatrix}_q q^{2n-k-1}\mathcal{E}_{l,q}\mathcal{E}_{n-k-1,q}(q^{-1}x).\end{aligned} \tag{18}$$

Replacing (18) instead of (14), we derive

$$2^{-1}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\begin{bmatrix}n-1\\k\end{bmatrix}_q\begin{bmatrix}k\\l\end{bmatrix}_q q^{-k}\mathcal{E}_{l,q}\mathcal{E}_{n-k-1,q}(q^{-1}x)=x\mathcal{E}_{n-1,q}(q^{-1}x)-q^{1-n}\mathcal{E}_{n,q}(x). \tag{19}$$

From (16), the left hand side of (19) is changed as

$$2^{-1}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\begin{bmatrix}n-1\\k\end{bmatrix}_q\begin{bmatrix}k\\l\end{bmatrix}_q q^{-k}\mathcal{E}_{l,q}\mathcal{E}_{n-k-1,q}(q^{-1}x)=2^{-1}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{\mathcal{E}_{l,q}}{[k-l]_q![l]_q!}D_{q,x}^{(k)}\mathcal{E}_{n-1,q}(q^{-1}x).$$

Therefore, we have

$$2^{-1}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{\mathcal{E}_{l,q}}{[k-l]_q![l]_q!}D_{q,x}^{(k)}\mathcal{E}_{n-1,q}(q^{-1}x)=x\mathcal{E}_{n-1,q}(q^{-1}x)-q^{1-n}\mathcal{E}_{n,q}(x),$$

which obtains the desired result by using the similar method in Theorem 4. □

Here, we have Corollary 7 when $q\to 1$ in Theorem 6.

Corollary 7. *The Euler polynomials $\mathcal{E}_n(x)$ are a solution of the following differential equation of a higher order.*

$$\begin{aligned}&\sum_{l=0}^{n-1}\frac{\mathcal{E}_l}{2(n-l-1)!l!}\frac{d^{n-1}}{dx^{n-1}}\mathcal{E}_{n-1}(x)+\sum_{l=0}^{n-2}\frac{\mathcal{E}_l}{2(n-l-2!l!}\frac{d^{n-2}}{dx^{n-2}}\mathcal{E}_{n-1}(x)+\cdots\\&+\sum_{l=0}^{2}\frac{\mathcal{E}_l}{2(2-l)!l!}\frac{d^2}{dx^2}\mathcal{E}_{n-1}(x)+\sum_{l=0}^{1}\frac{\mathcal{E}_l}{2(1-l)!l!}\frac{d}{dx}\mathcal{E}_{n-1}(x)\\&+\frac{(\mathcal{E}_0-2x)}{2}\mathcal{E}_{n-1}(x)+\mathcal{E}_n(x)=0.\end{aligned}$$

Theorem 7. *Let $ab\neq 0$, $x, X\in\mathbb{R}$ and $0<q<1$. Then, we have*

$$\begin{aligned}&\frac{\mathcal{E}_{n,q}(bx)}{[n]_q!}D_{q,X}^{(n)}\mathcal{E}_{n,q}(aX)+\frac{b\mathcal{E}_{n-1,q}(bx)}{[n-1]_q!}D_{q,X}^{(n-1)}\mathcal{E}_{n,q}(aX)+\cdots+\frac{b^{n-2}\mathcal{E}_{2,q}(bx)}{[2]_q!}D_{q,X}^{(2)}\mathcal{E}_{n,q}(aX)\\&+b^{n-1}\mathcal{E}_{1,q}(bx)D_{q,X}^{(1)}\mathcal{E}_{n,q}(aX)+b^n\mathcal{E}_{0,q}(bx)\mathcal{E}_{n,q}(aX)\\&=\frac{\mathcal{E}_{n,q}(ax)}{[n]_q!}D_{q,X}^{(n)}\mathcal{E}_{n,q}(bX)+\frac{a\mathcal{E}_{n-1,q}(ax)}{[n-1]_q!}D_{q,X}^{(n-1)}\mathcal{E}_{n,q}(bX)+\cdots+\frac{a^{n-2}\mathcal{E}_{2,q}(ax)}{[2]_q!}D_{q,X}^{(2)}\mathcal{E}_{n,q}(bX)\\&+a^{n-1}\mathcal{E}_{1,q}(ax)D_{q,X}^{(1)}\mathcal{E}_{n,q}(bX)+a^n\mathcal{E}_{0,q}(ax)\mathcal{E}_{n,q}(bX).\end{aligned}$$

Proof. To find the q-differential equation of a higher order using a symmetric property of q-Euler polynomials, we can construct form A, such as

$$A:=\frac{4e_q(abtx)e_q(abtX)}{(e_q(at)+1)(e_q(bt)+1)}.$$

Using the generating function of q-Euler polynomials and Cauchy products, form A is transformed as

$$A = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k} \mathcal{E}_{k,q}(bx) \mathcal{E}_{n-k,q}(aX) \right) \frac{t^n}{[n]_q!} \tag{20}$$

and

$$A = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k a^{n-k} \mathcal{E}_{k,q}(ax) \mathcal{E}_{n-k,q}(bX) \right) \frac{t^n}{[n]_q!}. \tag{21}$$

Applying the coefficient comparison method on Equations (20) and (21), we find a symmetric property such as

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k} \mathcal{E}_{k,q}(bx) \mathcal{E}_{n-k,q}(aX) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k a^{n-k} \mathcal{E}_{k,q}(ax) \mathcal{E}_{n-k,q}(bX). \tag{22}$$

From (ii) in Lemma 1, we can remark

$$\mathcal{E}_{n-k,q}(\alpha X) = \frac{[n-k]_q!}{\alpha^k [n]_q!} D_{q,X}^{(k)} \mathcal{E}_{n,q}(\alpha X), \quad \text{for } \alpha \neq 0. \tag{23}$$

Using (23) in the both sides of (22), we obtain

$$\sum_{k=0}^{n} \frac{b^{n-k}}{[k]_q!} \mathcal{E}_{k,q}(bx) D_{q,X}^{(k)} \mathcal{E}_{n,q}(aX) = \sum_{k=0}^{n} \frac{a^{n-k}}{[k]_q!} \mathcal{E}_{k,q}(ax) D_{q,X}^{(k)} \mathcal{E}_{n,q}(bX).$$

From the above equation, we express the required result and complete the proof of Theorem 7. □

Corollary 8. *Setting $a = 1$ in Theorem 7, we have*

$$\begin{aligned}
&\frac{\mathcal{E}_{n,q}(bx)}{[n]_q!} D_{q,X}^{(n)} \mathcal{E}_{n,q}(X) + \frac{b\mathcal{E}_{n-1,q}(bx)}{[n-1]_q!} D_{q,X}^{(n-1)} \mathcal{E}_{n,q}(X) + \cdots + \frac{b^{n-2}\mathcal{E}_{2,q}(bx)}{[2]_q!} D_{q,X}^{(2)} \mathcal{E}_{n,q}(X) \\
&+ b^{n-1} \mathcal{E}_{1,q}(bx) D_{q,X}^{(1)} \mathcal{E}_{n,q}(X) + b^n \mathcal{E}_{0,q}(bx) \mathcal{E}_{n,q}(X) \\
&= \frac{\mathcal{E}_{n,q}(x)}{[n]_q!} D_{q,X}^{(n)} \mathcal{E}_{n,q}(bX) + \frac{\mathcal{E}_{n-1,q}(x)}{[n-1]_q!} D_{q,X}^{(n-1)} \mathcal{E}_{n,q}(bX) + \cdots + \frac{\mathcal{E}_{2,q}(x)}{[2]_q!} D_{q,X}^{(2)} \mathcal{E}_{n,q}(bX) \\
&+ \mathcal{E}_{1,q}(x) D_{q,X}^{(1)} \mathcal{E}_{n,q}(bX) + \mathcal{E}_{0,q}(x) \mathcal{E}_{n,q}(bX).
\end{aligned}$$

Corollary 9. *Let $ab \neq 0$, $0 < q < 1$ and $q \to 1$ in Theorem 7. Then, the following holds*

$$\begin{aligned}
&\frac{\mathcal{E}_n(bx)}{n!} \frac{d^n}{dX^n} \mathcal{E}_n(aX) + \frac{b\mathcal{E}_{n-1}(bx)}{(n-1)!} \frac{d^{n-1}}{dX^{n-1}} \mathcal{E}_n(aX) + \cdots + \frac{b^{n-2}\mathcal{E}_2(bx)}{2!} \frac{d^2}{dX^2} \mathcal{E}_n(aX) \\
&+ b^{n-1} \mathcal{E}_1(bx) \frac{d}{dX} \mathcal{E}_n(aX) + b^n \mathcal{E}_0(bx) \mathcal{E}_n(aX) \\
&= \frac{\mathcal{E}_n(ax)}{n!} \frac{d^n}{dX^n} \mathcal{E}_n(bX) + \frac{a\mathcal{E}_{n-1}(ax)}{(n-1)!} \frac{d^{n-1}}{dX^{n-1}} \mathcal{E}_n(bX) + \cdots + \frac{a^{n-2}\mathcal{E}_2(ax)}{2!} \frac{d^2}{dX^2} \mathcal{E}_n(bX) \\
&+ a^{n-1} \mathcal{E}_1(ax) \frac{d}{dX} \mathcal{E}_n(bX) + a^n \mathcal{E}_0(ax) \mathcal{E}_n(bX).
\end{aligned}$$

3. Some q-Differential Equations of Higher Order Related to q-Genocchi Polynomials

In this section, we find several q-differential equations of higher order for q-Genocchi polynomials. We also obtain q-differential equations of a higher order of q-Genocchi polynomials including q-Euler numbers. From the symmetric property of q-Genocchi polynomials, we derive q-differential equations of a higher order of mixed q-Genocchi and q-Euler polynomials.

Let

$$\mathcal{G}_q(t,x)=\frac{2t}{e_q(t)+1}e_q(tx)=\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{24}$$

Theorem 8. *The q-Genocchi polynomials $G_{n,q}(x)$ is a solution of the following q-differential equation of a higher order.*

$$\begin{aligned}&\frac{q^{-1}G_{n,q}}{2[n]_q!}D_{q,x}^{(n)}G_{n,q}(x)+\frac{G_{n-1,q}}{2[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(x)+\cdots\\&+\frac{q^{n-3}G_{2,q}}{2[2]_q!}D_{q,x}^{(2)}G_{n,q}(x)+\frac{q^{n-2}G_{1,q}}{2}D_{q,x}^{(1)}G_{n,q}(x)+\frac{(G_{0,q}+2)q^{n-1}}{2}G_{n,q}(x)\\&-[n]_qG_{n,q}(qx)+(x-q^{-2})[n]_qq^nG_{n-1,q}(x)=0,\end{aligned}$$

where $G_{n,q}$ are the q-Genocchi numbers.

Proof. Using the q-derivative and multiplying t in Equation (24), we have

$$tD_{q,t}\mathcal{G}_q(t,qx)=\sum_{n=0}^{\infty}[n]_qG_{n,q}(qx)\frac{t^n}{[n]_q!}. \tag{25}$$

Applying the q-derivative in the generating function of q-Genocchi polynomials, we find

$$\begin{aligned}D_{q,t}\mathcal{G}_q(t,qx)&=\mathcal{G}_q(qt,x)\left(\frac{1}{q(e_q(t)+1)}+(1-t)q^{-1}t^{-1}+qx\right)\\&=\sum_{n=0}^{\infty}q^nG_{n,q}(x)\frac{t^n}{[n]_q!}\left((2q)^{-1}\sum_{n=0}^{\infty}G_{n,q}\frac{t^{n-1}}{[n]_q!}+q^{-1}t^{-1}-q^{-1}+qx\right).\end{aligned} \tag{26}$$

Multiplying t in (26), we have

$$\begin{aligned}&tD_{q,t}\mathcal{G}_q(t,qx)\\&=\sum_{n=0}^{\infty}\left(2^{-1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{n-k-1}G_{k,q}G_{n-k,q}(x)+q^{n-1}G_{n,q}(x)\right)\frac{t^n}{[n]_q!}\\&+\sum_{n=0}^{\infty}(x-q^{-2})[n]_qq^nG_{n-1,q}(x)\frac{t^n}{[n]_q!}\end{aligned} \tag{27}$$

From (25) and (27), we obtain

$$\begin{aligned}&[n]_qG_{n,q}(qx)-q^{n-1}G_{n,q}(x)-(x-q^{-2})[n]_qq^nG_{n-1,q}(x)\\&=2^{-1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{n-k-1}G_{k,q}G_{n-k,q}(x).\end{aligned} \tag{28}$$

By using mathematical induction to find the relation of $G_{n,q}(x)$ and $D_{q,x}G_{n,q}(x)$, we note that

$$G_{n-k,q}(x)=\frac{[n-k]_q!}{[n]_q!}D_{q,x}^{(k)}G_{n,q}(x). \tag{29}$$

Substituting the right side of Equation (29) instead of $G_{n-k,q}(x)$ in (28), we find

$$\begin{aligned}&[n]_qG_{n,q}(qx)-q^{n-1}G_{n,q}(x)-(x-q^{-2})[n]_qq^nG_{n-1,q}(x)\\&=\sum_{k=0}^{n}\frac{q^{n-k-1}G_{k,q}}{2[k]_q!}D_{q,x}^{(k)}G_{n,q}(x).\end{aligned} \tag{30}$$

From the above Equation (30), we can obtain the desired result. □

Corollary 10. *Let $q \to 1$ in Equation (28). Then, one holds*

$$(n-1)G_n(x) - nxG_{n-1}(x) = 2^{-1}\sum_{k=0}^{n}\binom{n}{k}G_k G_{n-k}(x),$$

where G_n is the Genocchi numbers and $G_n(x)$ is the Genocchi polynomials.

Theorem 9. *The q-Genocchi polynomials $G_{n,q}(x)$ satisfies the following q-differential equation of a higher order*

$$\begin{aligned}&\frac{q^{-1}\mathcal{E}_{n-1,q}}{2[n-1]_q!}D_{q,x}^{(n)}G_{n,q}(x) + \frac{\mathcal{E}_{n-2,q}}{2[n-2]_q!}D_{q,x}^{(n-1)}G_{n,q}(x) + \frac{q\mathcal{E}_{n-3,q}}{2[n-3]_q!}D_{q,x}^{(n-2)}G_{n,q}(x) + \cdots \\ &+ \frac{q^{n-4}\mathcal{E}_{2,q}}{2[2]_q!}D_{q,x}^{(3)}G_{n,q}(x) + \frac{q^{n-3}\mathcal{E}_{1,q}}{2}D_{q,x}^{(2)}G_{n,q}(x) + \frac{q^{n-2}\mathcal{E}_{0,q}}{2}D_{q,x}^{(1)}G_{n,q}(x) \\ &+ q^{n-1}G_{n,q}(x) + [n]_q q^n(x-q^{-2})G_{n-1,q}(x) - [n]_q G_{n,q}(qx) = 0\end{aligned}$$

where $\mathcal{E}_{n,q}$ is the q-Euler numbers.

Proof. To find the desired result, we note a relation between q-Euler numbers and q-Genocchi numbers as

$$G_{n,q} = [n]_q\mathcal{E}_{n-1,q}, \qquad \text{where} \quad n \geq 1. \tag{31}$$

Using (31), we can express the other form of (28) as follows.

$$\begin{aligned}&[n]_q G_{n,q}(qx) - q^{n-1}G_{n,q}(x) - (x-q^{-2})[n]_q q^n G_{n-1,q}(x) \\ &= 2^{-1}\sum_{k=1}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q q^{n-k-1}[k]_q\mathcal{E}_{k-1,q}G_{n-k,q}(x).\end{aligned} \tag{32}$$

Applying Equation (29) in (32), we have

$$\begin{aligned}&[n]_q G_{n,q}(qx) - q^{n-1}G_{n,q}(x) - (x-q^{-2})[n]_q q^n G_{n-1,q}(x) \\ &= 2^{-1}\sum_{k=1}^{n}\frac{q^{n-k-1}\mathcal{E}_{k-1,q}}{[k-1]_q!}D_{q,x}^{(k)}G_{n,q}(x),\end{aligned}$$

which obtains the required result, immediately. □

Corollary 11. *From Theorems 8 and 9, the following holds.*

$$\sum_{k=0}^{n}\frac{q^{n-k-1}G_{k,q}}{[k]_q!}D_{q,x}^{(k)}G_{n,q}(x) = \sum_{k=1}^{n}\frac{q^{n-k-1}\mathcal{E}_{k-1,q}}{[k-1]_q!}D_{q,x}^{(k)}G_{n,q}(x).$$

Theorem 10. *The q-Genocchi polynomials $G_{n,q}(x)$ is a solution of the following q-differential equation of a higher order.*

$$\begin{aligned}&\frac{G_{n,q}}{2[n]_q!}D_{q,x}^{(n)}G_{n,q}(q^{-1}x) + \frac{G_{n-1,q}}{2[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(q^{-1}x) + \frac{G_{n-2,q}}{2[n-2]_q!}D_{q,x}^{(n-2)}G_{n,q}(q^{-1}x) + \cdots \\ &+ \frac{G_{3,q}}{2[3]_q!}D_{q,x}^{(3)}G_{n,q}(q^{-1}x) + \frac{G_{2,q}}{2[2]_q!}D_{q,x}^{(2)}G_{n,q}(q^{-1}x) + \frac{G_{1,q}}{2}D_{q,x}^{(1)}G_{n,q}(q^{-1}x) \\ &+ \frac{(G_{0,q}+2)}{2}G_{n,q}(q^{-1}x) - [n]_q\Big(q^{1-n}G_{n,q}(x) - (x-q^{-1})G_{n-1,q}(q^{-1}x)\Big) = 0,\end{aligned}$$

where $G_{n,q}$ is the q-Genocchi numbers.

Proof. We remark the following relation between $G_{n,q}(x)$ and $D_{q,x}^{(k)}G_{n,q}(x)$ by mathematical induction.

$$G_{n-k,q}(q^{-1}x) = \frac{q^k[n-k]_q!}{[n]_q!}D_{q,x}^{(k)}G_{n,q}(q^{-1}x). \tag{33}$$

Using the q-derivative in Equation (24), we find

$$tD_{q,t}\mathcal{G}_q(qt,x) = \sum_{n=0}^{\infty}[n]_q q^n G_{n,q}(x)\frac{t^n}{[n]_q!}, \tag{34}$$

and

$$\begin{aligned}
&tD_{q,t}\mathcal{G}_q(qt,x)\\
&=\sum_{n=0}^{\infty}2^{-1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{2n-k-1}G_{k,q}G_{n-k,q}(q^{-1}x)\frac{t^n}{[n]_q!}\\
&+\sum_{n=0}^{\infty}\Big(q^{2n-1}G_{n,q}(q^{-1}x)+(qx-1)[n]_q q^{2(n-1)}G_{n-1,q}(q^{-1}x)\Big)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{35}$$

Comparing the coefficients of the right hand sides on (34) and (35), we obtain

$$\begin{aligned}
&[n]_q q^n G_{n,q}(x) - q^{2n-1}G_{n,q}(q^{-1}x) - (qx-1)[n]_q q^{2(n-1)}G_{n-1,q}(q^{-1}x)\\
&=2^{-1}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{2n-k-1}G_{k,q}G_{n-k,q}(q^{-1}x).
\end{aligned} \tag{36}$$

Replacing $G_{n-k,q}(q^{-1}x)$ of (36) with (33), we have

$$\begin{aligned}
&[n]_q q^n G_{n,q}(x) - q^{2n-1}G_{n,q}(q^{-1}x) - (qx-1)[n]_q q^{2(n-1)}G_{n-1,q}(q^{-1}x)\\
&=2^{-1}\sum_{k=0}^{n}\frac{G_{k,q}}{[k]_q!}D_{q,x}^{(k)}G_{n,q}(q^{-1}x).
\end{aligned} \tag{37}$$

From Equation (37), we finish the proof of Theorem 10. □

Theorem 11. *The q-Genocchi polynomials $G_{n,q}(x)$ satisfy the following q-differential equation of a higher order.*

$$\begin{aligned}
&\frac{q^{n-2}\mathcal{E}_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n-1,q}(q^{-1}x)+\frac{q^{n-2}\mathcal{E}_{n-2,q}}{[n-2]_q!}D_{q,x}^{(n-2)}G_{n-1,q}(q^{-1}x)+\cdots\\
&+\frac{q^{n-2}\mathcal{E}_{3,q}}{[3]_q!}D_{q,x}^{(3)}G_{n-1,q}(q^{-1}x)+\frac{q^{n-2}\mathcal{E}_{2,q}}{[2]_q!}D_{q,x}^{(2)}G_{n-1,q}(q^{-1}x)+q^{n-2}\mathcal{E}_{1,q}D_{q,x}^{(1)}G_{n-1,q}(q^{-1}x)\\
&+(\mathcal{E}_{0,q}+2(qx-1))q^{n-2}G_{n-1,q}(q^{-1}x)+2\Big([n]_q^{-1}q^{n-1}G_{n,q}(q^{-1}x)-G_{n,q}(x)\Big)=0,
\end{aligned}$$

where $\mathcal{E}_{n,q}$ is the q-Euler numbers.

Proof. In the process of the proof of Theorem 10, we consider substituting (31) on (35) to replace q-Euler numbers. We omit the proof of Theorem 11 since we use a similar pattern of proof in Theorem 10. □

Corollary 12. *For $q\to 1$ in Theorem 11, the Genocchi polynomials $G_n(x)$ satisfy the following differential equation of a higher order.*

$$\begin{aligned}
&\frac{\mathcal{E}_{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}G_{n-1}(x)+\frac{\mathcal{E}_{n-2}}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}}G_{n-1}(x)+\cdots+\frac{\mathcal{E}_3}{3!}\frac{d^3}{dx^3}G_{n-1}(x)\\
&+\frac{\mathcal{E}_2}{2!}\frac{d^2}{dx^2}G_{n-1}(x)+\mathcal{E}_1\frac{d}{dx}G_{n-1}(x)+(\mathcal{E}_0+2(x-1))G_{n-1}(x)+2\Big(n^{-1}-1\Big)G_n(x)=0,
\end{aligned}$$

where $\mathcal{E}_n$ is the Euler numbers.

Theorem 12. *Let $0 < q < 1$. Then, we find*

$$\begin{aligned}
&\frac{G_{n,q}(bx)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(aX) + \frac{bG_{n-1,q}(bx)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(aX) + \cdots \\
&+ \frac{b^{n-2}G_{2,q}(bx)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(aX) + b^{n-1}G_{1,q}(bx)D_{q,X}^{(1)}G_{n,q}(aX) + b^nG_{0,q}(bx)G_{n,q}(aX) \\
&= \frac{G_{n,q}(ax)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(bX) + \frac{aG_{n-1,q}(ax)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(bX) + \cdots \\
&+ \frac{a^{n-2}G_{2,q}(ax)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(bX) + a^{n-1}G_{1,q}(ax)D_{q,X}^{(1)}G_{n,q}(bX) + a^nG_{0,q}(ax)G_{n,q}(bX).
\end{aligned}$$

Proof. We present the form B_1 to find a symmetric property for q-Genocchi polynomials as follows.

$$\begin{aligned}
B_1 :&= \frac{4abt^2e_q(abtx)e_q(abtX)}{(e_q(at)+1)(e_q(bt)+1)} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q a^kb^{n-k}G_{k,q}(bx)G_{n-k,q}(aX)\right)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q b^ka^{n-k}G_{k,q}(ax)G_{n-k,q}(bX)\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

From the above equation, we obtain a symmetric property such as

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q a^kb^{n-k}G_{k,q}(bx)G_{n-k,q}(aX) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q b^ka^{n-k}G_{k,q}(ax)G_{n-k,q}(aX). \tag{38}$$

To replace $G_{n-k,q}(aX)$, $G_{n-k,q}(bX)$ with $D_{q,X}^{(k)}G_{n,q}(aX)$, $D_{q,X}^{(k)}G_{n,q}(bX)$, respectively in (38), we find

$$G_{n-k,q}(\beta X) = \frac{[n-k]_q!}{\beta^k[n]_q!}D_{q,X}^{(k)}G_{n,q}(\beta X) \quad \text{where} \quad \beta \neq 0. \tag{39}$$

Applying Equation (39) in (38), we have

$$\sum_{k=0}^{n}\frac{b^{n-k}}{[k]_q!}G_{k,q}(bx)D_{q,X}^{(k)}G_{n,q}(aX) = \sum_{k=0}^{n}\frac{a^{n-k}}{[k]_q!}G_{k,q}(ax)D_{q,X}^{(k)}G_{n,q}(bX).$$

From the above equation, we complete the proof of Theorem 12. □

Corollary 13. *Putting $a = 1$ in Theorem 12, the following holds*

$$\begin{aligned}
&\frac{G_{n,q}(bx)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(X) + \frac{bG_{n-1,q}(bx)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(X) + \cdots + \frac{b^{n-2}G_{2,q}(bx)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(X) \\
&+ b^{n-1}G_{1,q}(bx)D_{q,X}^{(1)}G_{n,q}(X) + b^nG_{0,q}(bx)G_{n,q}(X) \\
&= \frac{G_{n,q}(x)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(bX) + \frac{G_{n-1,q}(x)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(bX) + \cdots + \frac{G_{2,q}(x)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(bX) \\
&+ G_{1,q}(x)D_{q,X}^{(1)}G_{n,q}(bX) + G_{0,q}(x)G_{n,q}(bX).
\end{aligned}$$

Corollary 14. *Let $q \to 1$ in Theorem 12. Then, one holds*

$$\frac{G_n(bx)}{n!}\frac{d^n}{dX^n}G_n(aX)+\frac{bG_{n-1}(bx)}{(n-1)!}\frac{d^{n-1}}{dX^{n-1}}G_n(aX)+\cdots+\frac{b^{n-2}G_2(bx)}{2!}\frac{d^2}{dX^2}G_n(aX)$$
$$+b^{n-1}G_1(bx)\frac{d}{dX}G_n(aX)+b^nG_0(bx)G_n(aX)$$
$$=\frac{G_n(ax)}{n!}\frac{d^n}{dX^n}G_n(bX)+\frac{aG_{n-1}(ax)}{(n-1)!}\frac{d^{n-1}}{dX^{n-1}}G_n(bX)+\cdots+\frac{a^{n-2}G_2(ax)}{2!}\frac{d^2}{dX^2}G_n(bX)$$
$$+a^{n-1}G_1(ax)\frac{d}{dX}G_n(bX)+a^nG_0(ax)G_n(bX).$$

Theorem 13. *Let $0<q<1$ and $ab\neq 0$. Then, we derive*

$$\frac{b^{-1}\mathcal{E}_{n,q}(bx)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(aX)+\frac{\mathcal{E}_{n-1,q}(bx)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(aX)+\cdots$$
$$+\frac{b^{n-3}\mathcal{E}_{2,q}(bx)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(aX)+b^{n-2}\mathcal{E}_{1,q}(bx)D_{q,X}^{(1)}G_{n,q}(aX)+b^{n-1}\mathcal{E}_{0,q}(bx)G_{n,q}(aX)$$
$$=\frac{a^{-1}\mathcal{E}_{n,q}(ax)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(bX)+\frac{\mathcal{E}_{n-1,q}(ax)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(bX)+\cdots$$
$$+\frac{a^{n-3}\mathcal{E}_{2,q}(ax)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(bX)+a^{n-2}\mathcal{E}_{1,q}(ax)D_{q,X}^{(1)}G_{n,q}(bX)+a^{n-1}\mathcal{E}_{0,q}(ax)G_{n,q}(bX).$$

Proof. To use the other symmetric property, we construct form B_2 as

$$B_2:=\frac{4abte_q(abtx)e_q(abtX)}{(e_q(at)+1)(e_q(bt)+1)}$$
$$=\frac{2a}{e_q(at)+1}e_q(abtx)\frac{2bt}{e_q(bt)+1}e_q(abtX)$$
$$=\frac{2b}{e_q(bt)+1}e_q(abtx)\frac{2at}{e_q(at)+1}e_q(abtX).$$

From the above equation, we find a mixed symmetric property, combining q-Euler polynomials and q-Genocchi polynomials as

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a^{k+1}b^{n-k}\mathcal{E}_{k,q}(bx)G_{n-k,q}(aX)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q b^{k+1}a^{n-k}\mathcal{E}_{k,q}(ax)G_{n-k,q}(aX). \tag{40}$$

Using Equation (39), we transform (40) as

$$\sum_{k=0}^{n}\frac{b^{n-k-1}}{[k]_q!}\mathcal{E}_{k,q}(bx)D_{q,X}^{(k)}G_{n,q}(aX)=\sum_{k=0}^{n}\frac{a^{n-k-1}}{[k]_q!}\mathcal{E}_{k,q}(ax)D_{q,X}^{(k)}G_{n,q}(bX),$$

which obtains the result that is desired, at once. □

Corollary 15. *Putting $a=1$ in Theorem 13, the following holds*

$$\frac{b^{-1}\mathcal{E}_{n,q}(bx)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(X)+\frac{\mathcal{E}_{n-1,q}(bx)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(X)+\cdots+\frac{b^{n-3}\mathcal{E}_{2,q}(bx)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(X)$$
$$+b^{n-2}\mathcal{E}_{1,q}(bx)D_{q,X}^{(1)}G_{n,q}(X)+b^{n-1}\mathcal{E}_{0,q}(bx)G_{n,q}(X)$$
$$=\frac{\mathcal{E}_{n,q}(x)}{[n]_q!}D_{q,X}^{(n)}G_{n,q}(bX)+\frac{\mathcal{E}_{n-1,q}(x)}{[n-1]_q!}D_{q,X}^{(n-1)}G_{n,q}(bX)+\cdots+\frac{\mathcal{E}_{2,q}(x)}{[2]_q!}D_{q,X}^{(2)}G_{n,q}(bX)$$
$$+\mathcal{E}_{1,q}(x)D_{q,X}^{(1)}G_{n,q}(bX)+\mathcal{E}_{0,q}(x)G_{n,q}(bX).$$

Theorem 14. *Let $0 < q < 1$ and $ab \neq 0$. Then, we obtain*

$$\begin{aligned}
&\frac{b^{-1}\mathcal{E}_{n,q}}{[n]_q!}D_{q,x}^{(n)}G_{n,q}(ax) + \frac{\mathcal{E}_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(ax) + \cdots + \frac{b^{n-3}\mathcal{E}_{2,q}}{[2]_q!}D_{q,x}^{(2)}G_{n,q}(ax)\\
&+ b^{n-2}\mathcal{E}_{1,q}D_{q,x}^{(1)}G_{n,q}(ax) + b^{n-1}\mathcal{E}_{0,q}G_{n,q}(ax)\\
&= \frac{a^{-1}\mathcal{E}_{n,q}}{[n]_q!}D_{q,x}^{(n)}G_{n,q}(bx) + \frac{\mathcal{E}_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(bx) + \cdots + \frac{a^{n-3}\mathcal{E}_{2,q}}{[2]_q!}D_{q,x}^{(2)}G_{n,q}(bx)\\
&+ a^{n-2}\mathcal{E}_{1,q}D_{q,x}^{(1)}G_{n,q}(bx) + a^{n-1}\mathcal{E}_{0,q}G_{n,q}(bx).
\end{aligned}$$

Proof. Consider form B_3 as

$$B_3 := \frac{4abte_q(abtx)}{(e_q(at)+1)(e_q(bt)+1)}.$$

From form B_3, we can find a symmetric property which is related to q-Euler numbers and q-Genocchi polynomials. Additionally, we have a relation between $G_{n-k,q}(\beta X)$ and $D_{q,X}^{(k)}G_{n,q}(\beta X)$ in Equation (39). Therefore, we omit the proof of Theorem 14, because we can apply the proof technique from Theorem 13. □

Corollary 16. *Setting $a = 1$ in Theorem 14, one holds*

$$\begin{aligned}
&\frac{b^{-1}\mathcal{E}_{n,q}}{[n]_q!}D_{q,x}^{(n)}G_{n,q}(x) + \frac{\mathcal{E}_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(x) + \cdots + \frac{b^{n-3}\mathcal{E}_{2,q}}{[2]_q!}D_{q,x}^{(2)}G_{n,q}(x)\\
&+ b^{n-2}\mathcal{E}_{1,q}D_{q,x}^{(1)}G_{n,q}(x) + b^{n-1}\mathcal{E}_{0,q}G_{n,q}(x)\\
&= \frac{\mathcal{E}_{n,q}}{[n]_q!}D_{q,x}^{(n)}G_{n,q}(bx) + \frac{\mathcal{E}_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}G_{n,q}(bx) + \cdots + \frac{\mathcal{E}_{2,q}}{[2]_q!}D_{q,x}^{(2)}G_{n,q}(bx)\\
&+ \mathcal{E}_{1,q}D_{q,x}^{(1)}G_{n,q}(bx) + \mathcal{E}_{0,q}G_{n,q}(bx).
\end{aligned}$$

Corollary 17. *From form B_3 of Theorem 14, we have*

$$\begin{aligned}
&\frac{bG_{n,q}}{[n]_q!}D_{q,x}^{(n)}\mathcal{E}_{n,q}(ax) + \frac{b^2G_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n,q}(ax) + \cdots + \frac{b^{n-1}G_{2,q}}{[2]_q!}D_{q,x}^{(2)}\mathcal{E}_{n,q}(ax)\\
&+ b^nG_{1,q}D_{q,x}^{(1)}\mathcal{E}_{n,q}(ax) + b^{n+1}G_{0,q}\mathcal{E}_{n,q}(ax)\\
&= \frac{aG_{n,q}}{[n]_q!}D_{q,x}^{(n)}\mathcal{E}_{n,q}(bx) + \frac{a^2G_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n,q}(bx) + \cdots + \frac{a^{n-1}G_{2,q}}{[2]_q!}D_{q,x}^{(2)}\mathcal{E}_{n,q}(bx)\\
&+ a^nG_{1,q}D_{q,x}^{(1)}\mathcal{E}_{n,q}(bx) + a^{n+1}G_{0,q}\mathcal{E}_{n,q}(bx).
\end{aligned}$$

Corollary 18. *Setting $a = 1$ in Corollary 17, the following holds:*

$$\begin{aligned}
&\frac{bG_{n,q}}{[n]_q!}D_{q,x}^{(n)}\mathcal{E}_{n,q}(x) + \frac{b^2G_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n,q}(x) + \cdots + \frac{b^{n-1}G_{2,q}}{[2]_q!}D_{q,x}^{(2)}\mathcal{E}_{n,q}(x)\\
&+ b^nG_{1,q}D_{q,x}^{(1)}\mathcal{E}_{n,q}(x) + b^{n+1}G_{0,q}\mathcal{E}_{n,q}(x)\\
&= \frac{G_{n,q}}{[n]_q!}D_{q,x}^{(n)}\mathcal{E}_{n,q}(bx) + \frac{G_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n,q}(bx) + \cdots + \frac{G_{2,q}}{[2]_q!}D_{q,x}^{(2)}\mathcal{E}_{n,q}(bx)\\
&+ G_{1,q}D_{q,x}^{(1)}\mathcal{E}_{n,q}(bx) + G_{0,q}\mathcal{E}_{n,q}(bx).
\end{aligned}$$

4. Conclusions

In this paper, we find several q-differential equations for q-Euler polynomials and q-Genocchi polynomials. This work obtains symmetric properties of the q-differential equation, which is related to q-Euler and q-Genocchi polynomials. Based on this paper, we believe that many readers can generate and visualize new concepts for special polynomials.

Author Contributions: Conceptualization, J.-Y.K.; Formal analysis, J.-Y.K.; Investigation, C.-S.R.; Project administration, C.-S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data presented in this study is available on request from the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

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