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VECTOR BUNDLES OF LOW RANK ON A MULTIPROJECTIVE SPACE

DAMIAN M. MAINGI

In this paper we construct vector bundles over a multiprojective space and study their properties. We first set out to establish the existence of monads on a multiprojective space $\mathbb{P}^n \times \mathbb{P}^m$, for all integers $n,m > 0$. Then, we study the vector bundles associated to these monads.

1. Introduction

The aim the paper is to contribute to a deeper understanding of indecomposable vector bundles on algebraic varieties. In this context, one of the most interesting problems deals with the existence of indecomposable rank r vector bundles on algebraic varieties. First observe that the difficulty in constructing non-splitting vector bundles on algebraic varieties increases when the difference between the rank of the bundle and the dimension of the variety increases. Indeed, the most interesting problem is to find indecomposable vector bundles of low rank comparing with the dimension of the ambient space. In this context we have the famous Hartshorne's conjecture concerning the non-existence of indecomposable rank 2 vector bundles on n-dimensional projective spaces for $n \geq 7$. This conjecture has been one of the main motivations for a great activity in the study of low rank vector bundles on projective spaces. On 3-dimensional projective

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spaces there are plenty of examples of indecomposable rank 2 vector bundles. But, in spite of many efforts, in the last thirty years very few indecomposable rank two vector bundles on n-dimensional projective spaces, $n > 3$ are known e.g. [2, 8, 9]. More precisely, in positive characteristic $p \neq 2$, there are no known indecomposable rank 2 vector bundles on $\mathbb{P}^n$ for $n > 4$. In positive characteristic, N. Mohan Kumar [9] constructed many indecomposable rank 2 bundles on $\mathbb{P}^4$. In characteristic zero, the Horrocks-Mumford bundle [5] is essentially the only known indecomposable rank 2 bundle on $\mathbb{P}^4$. We will construct vector bundles of low rank via monads and analyze their properties.

Monads appear in many contexts within algebraic geometry and they are very useful in construction of vector bundles with prescribed invariants like rank, determinants, chern class etc. Monads were first introduced by Horrocks who showed that all vector bundles E on $\mathbb{P}^3$ could be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \xrightarrow{A} \oplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \xrightarrow{B} \oplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0$$

where A and B are matrices whose entries are homogeneous polynomials of degrees $b_j - a_i$ and $c_n - b_j$ respectively for some integers i, j, n.

The paper consists of 4 sections. In the first two sections we give the necessary terms to put a reader into context. In the third section we establish the existence of monads by explicit construction of the maximal rank maps on a multiprojective space. In the fourth section we have the main result i.e. we construct indecomposable low rank vector bundles on a multiprojective space. Throughout the paper when we talk of X we mean a nonsingular projective variety over an algebraically closed field k of characteristic zero.

2. Preliminaries

Definition 2.1. Let X be a nonsingular projective variety. A *monad* on X is a complex of vector bundles:

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow 0$$

with α injective and β surjective.

Definition 2.2. A monad as defined above has a display

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{A} \arrow[r] \arrow[d, equal] & \mathscr{K} \arrow[r] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{A} \arrow[r, "\alpha"'] & \mathscr{B} \arrow[r] \arrow[d, "\beta"'] & \mathscr{Q} \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{C} \arrow[r, equal] \arrow[d] & \mathscr{C} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

From which we have:

(i) $\mathscr{K} = kernel(\beta)$

(ii) $\mathscr{Q} = cokernel(\beta)$

(iii) $\mathscr{E} = ker(\beta)/im(\alpha)$

(iv) rank($\mathscr{E}$) = rank($\mathscr{B}$) - rank($\mathscr{A}$) - rank($\mathscr{C}$)

(v) $c_t(\mathscr{E}) = c_t(\mathscr{B}) \cdot c_t(\mathscr{A})^{-1} \cdot c_t(\mathscr{C})^{-1}$ and particularly $c_1(\mathscr{E}) = c_1(\mathscr{B}) - c_1(\mathscr{A}) - c_1(\mathscr{C})$.

Definition 2.3 ([7])**.** Let X be a nonsingular projective variety, let $\mathscr{L}$ be a very ample invertible sheaf, and V, W, U be finite dimensional k-vector spaces. A linear monad on X is a short complex of sheaves,

$$0 \longrightarrow V \otimes \mathscr{L}^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathscr{L} \longrightarrow 0$$

where $\alpha \in Hom(V,W) \otimes H^0\mathscr{L}$ is injective and $\beta \in Hom(W,U) \otimes H^0\mathscr{L}$ is surjective.

Definition 2.4 ([7])**.** A torsion free sheaf E on X is said to be a *linear sheaf* on X if it can be represented as the cohomology sheaf of a linear monad i.e. $E = ker(\beta)/im(\alpha)$, moreover $rank(E) = w - u - v$, where $w = \dim W$, $v = \dim V$ and $u = \dim U$.

Note 2.5. Let X be a non-singular irreducible projective variety of dimension d and let H be an ample line bundle on X. For a torsion free sheaf F on X one sets:
$\deg_H F := c_1(F) \cdot H^{d-1}$, $\mu_H(F) := \frac{c_1(F)H^{d-1}}{rk(F)}$ and $P_F(m) := \chi(F \otimes \mathcal{O}_X(mH))$.

Definition 2.6. Let X be an algebraic variety and let E be a torsion free sheaf on X. Then E is $\mathscr{L}$-stable if every subsheaf $F \hookrightarrow E$ satisfies $\mu_{\mathscr{L}}(F) < \mu_{\mathscr{L}}(E)$, where $\mathscr{L}$ is an ample invertible sheaf.

Let us now turn our attention to our main ambient variety $X = \mathbb{P}^n \times \mathbb{P}^m$. We denote by $\langle h,t \rangle$ the generators of $\mathrm{Pic}(X)$.
Denote by $\mathcal{O}_X(a,b) := p_1{}^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes p_2{}^*\mathcal{O}_{\mathbb{P}^m}(b)$, where p_1 and p_2 are natural projections from X to $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively.
For any line bundle $\mathscr{L} = \mathcal{O}_X(a,b)$ on X and a vector bundle E, we will write $E(a,b) = E \otimes \mathcal{O}_X(a,b)$ and $(a,b) := a[h \times \mathbb{P}^n] + b[\mathbb{P}^n \times t]$ to represent its corresponding divisor. The normalization of E on X with respect to $\mathscr{L}$ is defined as follows: Set $d = \deg_{\mathscr{L}}(\mathcal{O}_X(1,0))$, since $\deg_{\mathscr{L}}(E(-k_E,0)) = \deg_{\mathscr{L}}(E) - 2k \cdot rank(E)$ there's a unique integer $k_E := \lceil \mu_{\mathscr{L}}(E)/d \rceil$ such that $1 - d.rank(E) \leq \deg_{\mathscr{L}}(E(-k_E,0)) \leq 0$. The twisted bundle $E_{\mathscr{L}-norm} := E(-k_E,0)$ is called the $\mathscr{L}$-normalization of E. Finally we define the linear functional $\delta_{\mathscr{L}}$ on $\mathbb{Z}^2$ as $\delta_{\mathscr{L}}(p_1,p_2) := deg_{\mathscr{L}}\mathcal{O}_X(p_1,p_2)$
The following is a generalization of the Hoppe Criterion of stability.

Proposition 2.7 ([6])**.** *Let X be a polycyclic variety with Picard number 2, let $\mathscr{L}$ be an ample line bundle and let E be a rank $r > 1$ holomorphic vector bundle over X.*
If $H^0((\bigwedge^q E)_{\mathscr{L}-norm}(p_1,p_2)) = 0$ for $1 \leq q \leq r-1$ and every $(p_1,p_2) \in \mathbb{Z}^2$ such that $\delta_{\mathscr{L}} \leq 0$ then E is $\mathscr{L}$-stable.

Definition 2.8. A vector bundle E is said to be decomposable if it is isomorphic to a direct sum $E_1 \oplus E_2$ of two non-zero vector bundles, otherwise E is indecomposable.

Definition 2.9. A vector bundle E on X is said to be simple if its only endomorphisms are the constants i.e. $\mathrm{Hom}(E,E) = k$ which is equivalent to $h^0(X, E \otimes E^*) = 1$.

Remark 2.10. a. A simple vector bundle is necessarily indecomposable ([2]).
b. If a vector bundle is stable then it is simple.

Proposition 2.11 ([1])**.** *Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles. Then we have the following exact sequences involving exterior and symmetric powers:*

$$(a)\; 0 \longrightarrow \bigwedge^q E \longrightarrow \bigwedge^q F \longrightarrow \bigwedge^{q-1} F \otimes G \longrightarrow \cdots \longrightarrow F \otimes S^{q-1}G \longrightarrow S^q G \longrightarrow 0$$

$$(b)\ 0 \longrightarrow S^q E \longrightarrow S^{q-1}E \otimes F \longrightarrow \cdots \longrightarrow E \otimes \bigwedge^{q-1} F \longrightarrow \bigwedge^{q} F \longrightarrow \bigwedge^{q} G \longrightarrow 0$$

Proposition 2.12 (Künneth Formula)**.** *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ then it holds:*

$$H^r(X, \mathcal{O}_X(c,d)) \cong \bigoplus_{p+q=r} H^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(c)) \otimes H^q(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$$

for integers p,q,r,c,d.

Lemma 2.13. *If $q_1 + q_2 > 0$ then $h^p(\mathcal{O}_X(-q_1,-q_2)^{\oplus k}) = 0$ where $X = \mathbb{P}^n \times \mathbb{P}^m$ and for $0 \le p < \dim(X) - 1$, k a non negative integer*

Proof. By the Künneth and Bott formulae. □

Lemma 2.14 ([6])**.** *Let A and B be vector bundles canonically pulled back from A' on $\mathbb{P}^n$ and B' on $\mathbb{P}^m$ then*

$$H^q(\bigwedge^{s}(A \otimes B)) = \sum_{k_1+\cdots+k_q} \left\{ \bigoplus_{i=1}^{s} \left[\Sigma_{j=0}^{s} \Sigma_{m=0}^{k_i} H^m(\wedge^j(A)) \otimes H^{k_i-m}(\wedge^{s-j}(B)) \right] \right\}.$$

3. Existence of Monads

The goal of this section is to prove the existence of two types of monads. One will be constructed using an ample line bundle as a building piece and the other using a different kind of line bundle as a building piece. We start by recalling the existence and classification of linear monads on $\mathbb{P}^n$ given by Fløystad in [3].

Lemma 3.1. *Let $k \ge 1$. There exists monads on $\mathbb{P}^k$ whose maps are matrices of linear forms,*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^k}^a(-1) \xrightarrow{\ \alpha\ } \mathcal{O}_{\mathbb{P}^k}^b \xrightarrow{\ \beta\ } \mathcal{O}_{\mathbb{P}^k}^c(1) \longrightarrow 0$$

if and only if at least one of the following is fulfilled;
(1) $b \ge 2c + k - 1$, $b \ge a + c$ and
(2) $b \ge a + c + k$

Proof. [3]. □

In the following Theorem we get our first existence result.

Theorem 3.2. *Let $X=\mathbb{P}^n\times\mathbb{P}^m$ and let $\mathscr{L}=\mathscr{O}_X(\rho,\sigma)$ be an ample line bundle on X. Denote by $N=h^0(\mathscr{O}_X(\rho,\sigma))-1$. Let α,β,γ be positive integers such that at least one of the following conditions holds*
(1) $\beta\geq 2\gamma+N-1$, and $\beta\geq\alpha+\gamma$,
(2) $\beta\geq\alpha+\gamma+N$.
Then, there exists a linear monad on X of the form

$$0\longrightarrow\mathscr{O}_X^{\alpha}(-\rho,-\sigma)\xrightarrow{A}\mathscr{O}_X^{\beta}\xrightarrow{B}\mathscr{O}_X^{\gamma}(\rho,\sigma)\longrightarrow 0$$

Proof. For any ample invertible sheaf $\mathscr{L}=\mathscr{O}_X(\rho,\sigma)$ we have an embedding

$$i^*:X=\mathbb{P}^n\times\mathbb{P}^m\hookrightarrow\mathbb{P}(H^0(X,\mathscr{O}_X(\rho,\sigma)))\cong\mathbb{P}^{\binom{n+\rho}{\rho}\cdot\binom{m+\sigma}{\sigma}-1}$$

such that $i^*(\mathscr{O}_X(1))\simeq\mathscr{L}$.
Suppose that one of the conditions is satisfied. Then, by Fløystad[3], there exists a linear monad

$$0\longrightarrow\mathscr{O}_{\mathbb{P}^N}^{\alpha}(-1)\xrightarrow{A}\mathscr{O}_{\mathbb{P}^N}^{\beta}\xrightarrow{B}\mathscr{O}_{\mathbb{P}^N}^{\gamma}(1)\longrightarrow 0$$

on $\mathbb{P}^N$.
Notice that

$$\begin{aligned}A\in\operatorname{Hom}(\mathscr{O}_{\mathbb{P}^N}^{\alpha}(-1),\mathscr{O}_{\mathbb{P}^N}^{\beta})&\cong H^0(\mathbb{P}^N,\mathscr{O}_{\mathbb{P}^N}(1)^{\alpha\beta})\\&\cong H^0(X,\mathscr{L})^{\alpha\beta}\cong\operatorname{Hom}_X({\mathscr{L}^{-1}}^{\alpha},\mathscr{O}_X^{\beta})\\B\in\operatorname{Hom}(\mathscr{O}_{\mathbb{P}^N}^{\beta},\mathscr{O}_{\mathbb{P}^N}^{\gamma}(1))&\cong H^0(\mathbb{P}^N,\mathscr{O}_{\mathbb{P}^N}(1)^{\beta\gamma})\cong\operatorname{Hom}_X(\mathscr{O}_X^{\beta},\mathscr{L}^{\gamma}).\end{aligned}$$

Thus, A and B induce a monad on X:

$$0\longrightarrow{\mathscr{L}^{-1}}^{\alpha}\xrightarrow{\bar{A}}\mathscr{O}_X^{\beta}\xrightarrow{\bar{B}}\mathscr{L}^{\gamma}\longrightarrow 0$$

which proves what we want. □

Note that for small cases it is not difficult to construct the matrices A and B above explicitly. For instance, it can be done in the following case.

Corollary 3.3. *Let $X=\mathbb{P}^n\times\mathbb{P}^m$ and $\mathscr{L}=\mathscr{O}_X(1,1)$. Denote by $N=h^0(\mathscr{O}_X(1,1))-1$. Let α,β,γ be positive integers such that at least one of the following conditions holds*
(1) $\beta\geq 2\gamma+N-1$, and $\beta\geq\alpha+\gamma$,
(2) $\beta\geq\alpha+\gamma+N$.
Then, there exists a linear monad on X of the form

$$0\longrightarrow\mathscr{O}_X^{\alpha}(-1,-1)\xrightarrow{A}\mathscr{O}_X^{\beta}\xrightarrow{B}\mathscr{O}_X^{\gamma}(1,1)\longrightarrow 0$$

Proof. Suppose $\beta = 2\gamma + N - 1$, and $\beta = \alpha + \gamma$, that is equality on the first condition. Then a monad of the form:

$$0 \longrightarrow \mathcal{O}_X^{\alpha}(-1,-1) \xrightarrow{A} \mathcal{O}_X^{\beta} \xrightarrow{B} \mathcal{O}_X^{\gamma}(1,1) \longrightarrow 0$$

exists where

$$B := \left(\begin{array}{cccc|cccc} x_0y_0\cdots & x_iy_j & & & x_{i+1}y_{j+1}\cdots & x_ay_b & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & x_0y_0\cdots & x_iy_j & & & x_{i+1}y_{j+1}\cdots & x_ay_b \end{array}\right)$$

an r by $2r+m+n$ matrix and

$$A := \left(\begin{array}{cccc} x_{i+1}y_{j+1}\cdots & x_ay_b & & \\ & \ddots & \ddots & \\ & & x_{i+1}y_{j+1}\cdots & x_ay_b \\ \hline -x_0y_0\cdots & -x_iy_j & & \\ & \ddots & \ddots & \\ & & -x_0y_0\cdots & -x_iy_j \end{array}\right)$$

a $2r+m+n$ by $r+n+m$ matrix. $\square$

Now we are going to consider another kind of monad.

Lemma 3.4. *Given four matrices f_1, f_2, g_1 and g_2 with f_1 and f_2 of order k by $n+k$, and g_1 and g_2 of order $n+k$ by k, as shown;*

$$f_1 = \begin{pmatrix} & & y_n^a\cdots y_0^{a+nk} \\ & \iddots & \iddots \\ y_n^a\cdots y_0^{a+nk} & & \end{pmatrix}, \quad f_2 = \begin{pmatrix} & & x_n^a\cdots x_0^{a+nk} \\ & \iddots & \iddots \\ x_n^a\cdots x_0^{a+nk} & & \end{pmatrix}$$

$$g_1 = \begin{pmatrix} x_0^{a+nk} & & \\ \vdots & \ddots & x_0^{a+nk} \\ x_n^a & \ddots & \vdots \\ & & x_n^a \end{pmatrix}, \quad g_2 = \begin{pmatrix} y_0^{a+nk} & & \\ \vdots & \ddots & y_0^{a+nk} \\ y_n^a & \ddots & \vdots \\ & & y_n^a \end{pmatrix},$$

for any non-negative integer a, we define two matrices f and g as follows

$$f = \begin{pmatrix} f_1 & -f_2 \end{pmatrix} \quad \textit{and} \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Then we have:
(i) $f \cdot g = 0$ and
(ii) The matrices f and g have maximal rank

Proof. (i) Since $f_1 \cdot g_1 = f_2 \cdot g_2$, then we have that

$$f \cdot g = \begin{pmatrix} f_1 & -f_2 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1 g_1 & -f_2 g_2 \end{pmatrix} = 0$$

(ii) Notice that the rank of the two matrices drops if and only if all $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ are zeros. Hence maximal rank. $\square$

Using the matrices given in the above lemma we are going to construct a monad.

Theorem 3.5. *Let n and k be positive integers. Then there exists a rank $2n$ vector bundle on X which is the cohomology of a linear monad on $X = \mathbb{P}^n \times \mathbb{P}^n$ of the form;*

$$M_\bullet : 0 \longrightarrow \mathcal{O}_X(-1,-1)^k \xrightarrow{f} \mathcal{O}_X(0,-1)^{n+k} \oplus \mathcal{O}_X(-1,0)^{n+k} \xrightarrow{g} \mathcal{O}_X(1,1)^k \longrightarrow 0$$

Proof. The maps f and g in the monad are the matrices given in Lemma 3.4.
Notice that
$f \in \mathrm{Hom}(\mathcal{O}_X(-1,-1)^k, \mathcal{O}_X(0,-1)^{n+k} \oplus \mathcal{O}_X(-1,0)^{n+k})$ and
$g \in \mathrm{Hom}(\mathcal{O}_X(0,-1)^{n+k} \oplus \mathcal{O}_X(-1,0)^{n+k}, \mathcal{O}_X(1,1)^k)$.
Hence by the above lemma they define the desired monad. $\square$

4. Main Results

Theorem 4.1. *Any vector bundle E, given as the cohomology of a monad of the form*

$$M_\bullet : 0 \longrightarrow \mathcal{O}_X(-1,-1)^k \xrightarrow{f} \mathcal{O}_X(0,-1)^{n+k} \oplus \mathcal{O}_X(-1,0)^{n+k} \xrightarrow{g} \mathcal{O}_X(1,1)^k \longrightarrow 0$$

where f and g are as defined on the previous page, is simple.

Proof. First of all consider the display of the monad $M_\bullet$,

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \mathcal{O}_X(-1,-1)^k \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & \mathcal{O}_X(0,-1)^{n+k} \oplus \mathcal{O}_X(-1,0)^{n+k} \arrow[r] & \mathcal{O}_X(1,1)^k \arrow[r] & 0 \\
 & E \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

where K is the Kernel of the map g. To prove that E is simple first, we are going to prove that K is stable for some ample line bundle $\mathscr{L}$.
Consider the ample line bundle $\mathscr{L} = \mathcal{O}_X(1,1) = \mathcal{O}(L)$. Its class in $\mathrm{Pic}(X) = \langle [h\times\mathbb{P}^n], [\mathbb{P}^n\times t]\rangle$ corresponds to the class $1\cdot[h\times\mathbb{P}^n]+1\cdot[\mathbb{P}^n\times t]$, where h and t are hyperplanes of $\mathbb{P}^n$ with the intersection product induced by $h^n = 1 = t^n$ and $h^{n+1} = 0 = t^{n+1}$.
Now from the display we get

$$\begin{aligned} c_1(K) &= c_1(\mathcal{O}_X(0,-1)^{n+k}\oplus\mathcal{O}_X(-1,0)^{n+k}) - c_1(\mathcal{O}_X(1,1)^k) \\ &= (n+k)(0,-1)+(n+k)(-1,0)-k(1,1) \\ &= (-n-2k,-n-2k). \end{aligned}$$

Hence since $L^{2n} > 0$, the degree of K is

$$\begin{aligned} \deg_{\mathscr{L}} K &= (-n-2k)([h\times\mathbb{P}^n]+[\mathbb{P}^n\times t])\cdot(1\cdot[h\times\mathbb{P}^n]+1\cdot[\mathbb{P}^n\times t])^{2n-1} \\ &= -(n+2k)L^{2n} < 0. \end{aligned}$$

Since $\deg_{\mathscr{L}} K < 0$, then $(\bigwedge^q K)_{\mathscr{L}-norm} = (\bigwedge^q K)$ and it suffices by the generalized Hoppe Criterion (Proposition 2.7), to prove that $h^0(\bigwedge^q K(-p_1,-p_2)) = 0$ for all $1\le q\le rank(K)-1$ and $p_1+p_2\ge 0$.
On twisting the horizontal exact sequence of the display of the monad by $\mathcal{O}_X(-p_1,-p_2)$ we get,

$$\begin{aligned} 0\to K(-p_1,-p_2)\to \mathcal{O}_X(-p_1,-1-p_2)^{n+k}\oplus\mathcal{O}_X(-1-p_1,-p_2)^{n+k} \\ \to \mathcal{O}_X(1-p_1,1-p_2)^k\to 0 \end{aligned}$$

and taking the wedge powers of the sequence (see Proposition 2.11 (a)) we have

$$\begin{aligned} 0\to \bigwedge^q K(-p_1,-p_2)\to \bigwedge^q(\mathcal{O}_X(-p_1,-1-p_2)^{n+k}\oplus\mathcal{O}_X(-1-p_1,-p_2)^{n+k}) \\ \to \bigwedge^{q-1}(\mathcal{O}_X(1-2p_1,-2p_2)^{n+2k}\oplus\mathcal{O}_X(-2p_1,1-2p_2)^{n+2k})\to\cdots \end{aligned}$$

Taking cohomology we have the injection:
$0\longrightarrow H^0(X,\bigwedge^q K(-p_1,-p_2))\hookrightarrow H^0(X,\bigwedge^q(\mathcal{O}_X(-p_1,-1-p_2)^{n+k}$
$\oplus\,\mathcal{O}_X(-1-p_1,-p_2)^{n+k}))$.
Now $H^0(X,\bigwedge^q(\mathcal{O}_X(-p_1,-1-p_2)^{n+k}\oplus\mathcal{O}_X(-1-p_1,-p_2)^{n+k}))=0$
by Lemma 2.14 which in turn implies that $h^0(\bigwedge^q K(-p_1,-p_2)) = 0$ and thus K is stable.
Let us now prove that E is simple.
The first step is to take the dual of the vertical short exact sequence to get

$$0\longrightarrow E^*\longrightarrow K^*\longrightarrow \mathcal{O}_X(1,1)^k\longrightarrow 0$$

Tensoring by E we get

$$0 \longrightarrow E\otimes E^* \longrightarrow E\otimes K^* \longrightarrow E(1,1)^k \longrightarrow 0$$

which implies that $h^0(X,E\otimes E^*) \leq h^0(X,E\otimes K^*)$.
Now we dualize the horizontal sequence to get

$$0 \longrightarrow \mathcal{O}_X(-1,-1)^k \longrightarrow \mathcal{O}_X(0,1)^{n+k} \oplus \mathcal{O}_X(1,0)^{n+k} \xrightarrow{g} K^* \longrightarrow 0.$$

Now twisting by $\mathcal{O}_X(-1,-1)$ and taking cohomology (for ease we will write $H^i(\mathcal{O}_X(\alpha,\beta)^k)$ instead of $H^i(X,\mathcal{O}_X(\alpha,\beta)^k)$) and get

$$\begin{aligned}0\longrightarrow H^0(\mathcal{O}_X(-2,-2)^k)\longrightarrow H^0(\mathcal{O}_X(-1,0)^{n+k})\oplus H^0(\mathcal{O}_X(0,-1)^{n+k})\longrightarrow\\ \longrightarrow H^0(K^*(-1,-1))\longrightarrow H^1(\mathcal{O}_X(-2,-2)^k)\longrightarrow\\ \longrightarrow H^1(\mathcal{O}_X(-1,0)^{n+k})\oplus H^1(\mathcal{O}_X(0,-1)^{n+k})\longrightarrow H^1(K^*(-1,-1))\longrightarrow\\ \longrightarrow H^2(\mathcal{O}_X(-2,-2)^k)\end{aligned}$$

from which we deduce $H^0(X,K^*(-1,-1)) = 0$ and $H^1(X,K^*(-1,-1)) = 0$.

Next, tensor the vertical sequence by K^* to get

$$0 \longrightarrow K^*(-1,-1)^k \longrightarrow K^*\otimes K \longrightarrow K^*\otimes E \longrightarrow 0$$

and taking cohomology we have

$$0 \to H^0(K^*(-1,-1)^k) \to H^0(K^*\otimes K) \to H^0(K^*\otimes E) \to H^1(K^*(-1,-1)^k)$$

But $H^1(K^*(-1,-1)^k = 0$ for $n > 1$.

This implies that $h^0(X,K^*\otimes K) = h^0(X,K^*\otimes E)$. Since K is stable, it is simple.
So, altogether we have: $1 \leq h^0(X,E\otimes E^*) \leq h^0(X,E\otimes K^*) = h^0(X,K^*\otimes K) = 1$ and therefore E is simple. □

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DAMIAN M. MAINGI
School of Mathematics
College of Biological and Physiological Sciences
Chiromo Campus-Riverside Drive
University of Nairobi
P.O Box 30197, 00100 Nairobi-Kenya
e-mail: dmaingi@uonbi.ac.ke