

VECTOR BUNDLES OVER QUATERNIONIC KÄHLER MANIFOLDS

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(Received February 23, 1987)

Introduction. On vector bundles over oriented 4-dimensional Riemannian manifolds, the notion of self-dual and anti-self-dual connections plays an important role in the geometry of 4-dimensional Yang-Mills theory (see Atiyah, Hitchin and Singer [A-H-S]).

On the other hand, in his differential-geometric study of stable holomorphic vector bundles, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds. Let E be a vector bundle over a quaternionic Kähler manifold M , and $p: Z \rightarrow M$ the corresponding twistor space defined by Salamon [S1]. Now the purpose of the present paper is to give a quaternionic Kähler analogue of self-dual and anti-self-dual connections, and then to construct a natural correspondence between E 's with such connections and the set of Einstein-Hermitian vector bundles over Z .

Let \mathbf{H} be the skew field of quaternions. Then the $Sp(n) \cdot Sp(1)$ -module $\wedge^2 \mathbf{H}^n$ is a direct sum $N'_2 \oplus N''_2 \oplus L_2$ of its irreducible submodules N'_2, N''_2, L_2 , where N'_2 (resp. L_2) is the submodule of the elements fixed by $Sp(n)$ (resp. $Sp(1)$) and for $n = 1$, we have $N''_2 = \{0\}$. Hence, the vector bundle $\wedge^2 T^*M$ is written as a direct sum $A'_2 \oplus A''_2 \oplus B_2$ of its holonomy-invariant subbundles in such a way that A'_2, A''_2, B_2 correspond respectively to N'_2, N''_2, L_2 . Now, a connection for E is called an A'_2 -connection (resp. B_2 -connection) if the corresponding curvature is an $\text{End}(E)$ -valued A'_2 -form (resp. B_2 -form). Then we have:

THEOREM (0.1). *All A'_2 -connections and also all B_2 -connections are Yang-Mills connections.*

Furthermore, for E with a B_2 -connection we can associate an E -valued elliptic complex (cf. (3.2)) similar to those of Salamon [S2]. Such complexes allow us to analyze the space of infinitesimal deformations of B_2 -connections (see Theorem (3.5)).

For our quaternionic Kähler manifold M , a pair (E, D_E) of a vector bundle E over M and a B_2 -connection D_E on E is called a *Hermitian pair* on M if D_E is a Hermitian connection on E . On the other hand, a pair (F, D_F) of a holomorphic vector bundle over Z and a Hermitian $(1, 0)$ -

connection D_F on F is called an *excellent pair* on Z if the following conditions are satisfied:

(a) F with the corresponding Hermitian metric h_F restricts to a flat bundle on each fibre of $p: Z \rightarrow M$. (Hence the real structure $\tau: Z \rightarrow Z$ (cf. Nitta and Takeuchi [N-T]) naturally lifts to a bundle automorphism $\tau': F \rightarrow F$.)

(b) Let $\sigma: F \rightarrow F^*$ be the bundle map defined by $F_z \ni f \mapsto \sigma(f) \in F_{\tau(z)}^*$ ($z \in Z$), where $\sigma(f)(g) := h_F(g, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then σ is an anti-holomorphic bundle automorphism. We then have the following generalization of a result of Penrose's type (cf. Atiyah, Hitchin and Singer [A-H-S]; see also Salamon [S2], Berard-Bergery and Ochiai [B-O]):

THEOREM (0.2). *Let \mathcal{H} (resp. $\tilde{\mathcal{H}}$) be the set of all Hermitian pairs (resp. all excellent pairs) on M (resp. Z). Then*

$$\mathcal{H} \ni (E, D_E) \mapsto (p^*E, p^*D_E) \in \tilde{\mathcal{H}}$$

defines a bijective correspondence between \mathcal{H} and $\tilde{\mathcal{H}}$.

In particular, if M has positive scalar curvature, then every excellent pair (F, D_F) on Z is a Ricci-flat Einstein-Hermitian vector bundle.

Finally, I would like to express my sincere gratitude to Professors H. Ozeki and M. Takeuchi for valuable suggestions. Special thanks are due also to Professors I. Enoki and T. Mabuchi for constant encouragement.

1. Notation, convention and preliminaries. In this section, we give a quick review of the basic facts on quaternionic Kähler manifolds (for more details see Salamon [S1], Nitta and Takeuchi [N-T]).

(1.1) Let $H^{(m)}$ denote the standard $Sp(m)$ -module $\mathbf{H}^m (= \mathbf{C}^{2m})$ of complex dimension $2m$, where $\mathbf{H} = \mathbf{R} + i\mathbf{R} + j\mathbf{R} + k\mathbf{R} (= \mathbf{C} + j\mathbf{C})$. $Sp(m) = \{S \in GL(m, \mathbf{H}) \mid S \cdot {}^t\bar{S} = I\}$ is imbedded in $GL(2m, \mathbf{C})$ by

$$Sp(m) \ni A + jB \mapsto \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \in GL(2m, \mathbf{C})$$

where $A, B \in GL(m, \mathbf{C})$. Then the multiplication on \mathbf{H}^m by j from the right naturally induces a $Sp(m)$ -equivariant anti-linear map $j^{(m)}: H^{(m)} \rightarrow H^{(m)}$ with $(j^{(m)})^2 = -\text{id}$. We now define a non-degenerate skew-symmetric bilinear form $\omega^{(m)}$ on \mathbf{H}^m by

$$\omega^{(m)}(h, h') := -\langle h, j^{(m)}h' \rangle \quad (h, h' \in \mathbf{H}^m),$$

where \langle , \rangle is the standard Hermitian inner product on $\mathbf{C}^{2m} (= \mathbf{H}^m)$. This $\omega^{(m)}$ can be regarded as an $Sp(m)$ -invariant bilinear form on $H^{(m)}$ such

that

$$(1.1.1) \quad \omega^{(m)}(j^{(m)}h, j^{(m)}h') = (\omega^{(m)}(h, h'))^{-1} \quad (h, h' \in H^{(m)}).$$

Let $Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1) / Z_2$. Then $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ is naturally a $Sp(n) \cdot Sp(1)$ -module of complex dimension $4n$ with a real structure $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \ni a \mapsto \bar{a} \in H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ defined by

$$(1.1.2) \quad (h \otimes h')^{-1} := j^{(n)}h \otimes j^{(1)}h' \quad (h \in H^{(n)}, h' \in H^{(1)}).$$

We consider the corresponding real form $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$ of $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$. Then the symmetric bilinear form $\omega^{(n)} \otimes \omega^{(1)} \in S^2((H^{(n)})^* \otimes (H^{(1)})^*)$ induces an inner product $\langle \cdot, \cdot \rangle$ on $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$.

(1.2) Recall that a $4n$ -dimensional Riemannian manifold (M, g_M) is called a quaternionic Kähler manifold, if its linear holonomy group is contained in $Sp(n) \cdot Sp(1) (\subset SO(4n))$ with the additional condition for $n = 1$ that g_M is a self-dual Einstein metric. *Throughout this paper, we fix once for all a quaternionic Kähler manifold (M, g_M) .* By the well-known reduction theorem (see, for instance, Kobayashi and Nomizu [K-N]), the frame bundle of the tangent bundle TM is reduced to a principal $Sp(n) \cdot Sp(1)$ -bundle P . Then TM can be regarded as the vector bundle

$$(1.2.1) \quad P \times_{Sp(n) \cdot Sp(1)} (H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$$

associated to the $Sp(n) \cdot Sp(1)$ -module $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$. The inner product $\langle \cdot, \cdot \rangle$ on $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$ induces a Riemannian metric g on TM , which coincides with g_M up to constant multiple. Without loss of generality, we may assume $g = g_M$.

(1.3) Let $Sp(n)$ act trivially on \mathbb{C}^2 . Then the standard $Sp(1)$ -action on \mathbb{C}^2 naturally induces an $Sp(n) \times Sp(1)$ -action (resp. $Sp(n) \cdot Sp(1)$ -action) on \mathbb{C}^2 (resp. $P^1\mathbb{C}$). Associated to these actions, we have:

$$\begin{aligned} \hat{p}: V (:= P \times_{Sp(n) \times Sp(1)} \mathbb{C}^2) &\rightarrow M \\ (\text{resp. } p: Z (:= P \times_{Sp(n) \cdot Sp(1)} P^1\mathbb{C}) &\rightarrow M), \end{aligned}$$

which is a "locally defined" vector bundle (resp. a globally defined fibre bundle). Here, the bundle Z is nothing but $P(V) := V - \{\text{zero section}\} / \mathbb{C}^*$, and is called the twistor space of M (see Salamon [S1; p. 147]). Then Z is a complex manifold with a natural real structure τ as follows:

(1.3.1) By the connection on V induced from that of P , we have a decomposition of $T(V - \{\text{zero section}\})$ into the subbundles S^h and S^v corresponding respectively to horizontal and vertical distributions. Let y be an arbitrary point of $V - \{\text{zero section}\}$, and put $x := \hat{p}(y)$. Via the projection \hat{p} , the fibre $(S^h)_y$ of S^h over y is regarded as the tangent

space $T_x M$ at x . Then by the identification of $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ with $(T_x M)^{\mathbb{C}}$ (cf. (1.2.1)), the space $H^{(n)} \otimes \mathbb{C}y$ defines a \mathbb{C} -linear subspace of $(T_x M)^{\mathbb{C}}$, denoted also by $H^{(n)} \otimes \mathbb{C}y$. Furthermore, let $(H^{(n)} \otimes \mathbb{C}y)'$ be the subspace of $(T_x^* M)^{\mathbb{C}}$ corresponding to $H^{(n)} \otimes \mathbb{C}y$ via the natural isomorphism $(T_x^* M)^{\mathbb{C}} \cong (T_x M)^{\mathbb{C}}$ induced by g_M . Now we define the complex structure of $T_y V$ by specifying the subspace $\wedge_y^{1,0}$ of $(1, 0)$ -forms in $(T_y^* V)^{\mathbb{C}}$ as follows:

$$\wedge_y^{1,0} = (\wedge_y^{1,0})^h \oplus (\wedge_y^{1,0})^v,$$

where $(\wedge_y^{1,0})^h := \hat{p}^*((H^{(n)} \otimes \mathbb{C}y)')$, and $(\wedge_y^{1,0})^v$ is the subspace of $(1, 0)$ -forms in $T_y \mathbb{C}^2$ by the identification of V_x with \mathbb{C}^2 . Then this induces a complex structure on Z .

(1.3.2) The map $j^{(1)}: H^{(1)} \rightarrow H^{(1)}$ naturally defines an antilinear bundle automorphism $\hat{c}: V \rightarrow V$, which induces a real structure τ on Z .

(1.3.3) Recall that M always has a constant scalar curvature (denoted by t). Let g_F be the Fubini-Study metric for $\mathbb{P}^1 \mathbb{C} (= (\mathbb{C} + j\mathbb{C} - \{0\})/\mathbb{C}^*)$. If $t \neq 0$, then for some nonzero real constant c_t ,

$$g_Z := p^* g_M + c_t g_F$$

defines a pseudo-Kählerian metric on Z , i.e., the corresponding $(1, 1)$ -form on Z is a nondegenerate d -closed $(1, 1)$ -form.

2. A'_2 -connections and B_2 -connections. We shall here give fundamental properties of the A'_2 -connections and B_2 -connections defined in the Introduction.

(2.1) Let $(H^{(m)})^*$ be the dual $Sp(m)$ -module of $H^{(m)}$. Then in view of $\wedge^2(H^{(1)})^* = \mathbb{C}\omega^{(1)}$, we have

$$\wedge^2((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*) = (\wedge^2(H^{(n)})^* \otimes_{\mathbb{C}} S^2(H^{(1)})^*) \oplus (S^2(H^{(n)})^* \otimes_{\mathbb{C}} \mathbb{C}\omega^{(1)}).$$

Furthermore, the $Sp(n)$ -module $\wedge^2(H^{(n)})^*$ is written as a direct sum $\mathbb{C}\omega^{(n)} + \wedge_0^2(H^{(n)})^*$ of its submodules, where $\wedge_0^2(H^{(n)})^*$ is the orthogonal complement of $\mathbb{C}\omega^{(n)}$ in $\wedge^2(H^{(n)})^*$. Hence,

$$(2.1.1) \quad \wedge^2((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*) = N_2'^{\mathbb{C}} \oplus N_2''^{\mathbb{C}} \oplus L_2^{\mathbb{C}},$$

where $N_2'^{\mathbb{C}} := \mathbb{C}\omega^{(n)} \otimes_{\mathbb{C}} S^2(H^{(1)})^*$, $N_2''^{\mathbb{C}} := \wedge_0^2(H^{(n)})^* \otimes_{\mathbb{C}} S^2(H^{(1)})^*$ and $L_2^{\mathbb{C}} := S^2(H^{(n)})^* \otimes_{\mathbb{C}} \mathbb{C}\omega^{(1)}$. Note that the $Sp(n) \cdot Sp(1)$ -modules $N_2'^{\mathbb{C}}$, $N_2''^{\mathbb{C}}$, $L_2^{\mathbb{C}}$ respectively admit real forms N_2' , N_2'' , L_2 fixed by the real structure induced from the one in (1.1.2). We have the identification $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong (H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*$ by the metric $\langle\langle \cdot, \cdot \rangle\rangle$ (cf. (1.1)). Together with $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong H^n \otimes_{\mathbb{R}} \mathbb{C}$, the above (2.1.1) induces the decomposition of its real form:

$$\wedge^2 H^n = N'_2 \oplus N''_2 \oplus L_2,$$

which is nothing but the decomposition in the Introduction now for our principal $Sp(n) \cdot Sp(1)$ -bundle P , the bundle T^*M is regarded as the vector bundle associated to the $Sp(n) \cdot Sp(1)$ -module $((H^{(n)})^* \otimes_C (H^{(1)*}))_R = H^n$. Hence, $\wedge^2 T^*M$ is a direct sum $A'_2 \oplus A''_2 \oplus B_2$ of its subbundles A'_2, A''_2, B_2 corresponding respectively to the $Sp(n) \cdot Sp(1)$ -modules N'_2, N''_2, L_2 (cf. Introduction).

(2.2) Fix an arbitrary point x of M . Note that each point z on the fibre Z_x defines an almost complex structure J_z on T_x^*M (cf. (1.3.1)). We then have the corresponding space $\wedge^{1,1}(T_x^*M, J_z)$ of $(1, 1)$ -forms of (T_x^*M, J_z) . Choose a point $y (\neq 0)$ of V such that its natural image (denoted by $[y]$) is z . In view of (1.3.1), the space $\wedge^{1,1}(T_x^*M, J_z)$ in $\wedge^2(T_x^*M)^c$ is associated to the C -linear subspace $(H^{(n)} \otimes_C Cy)' \wedge ((H^{(n)} \otimes_C Cy)')^-$ in the $Sp(n) \cdot Sp(1)$ -module $(H^{(n)} \otimes_C H^{(1)*}) \wedge (H^{(n)} \otimes_C H^{(1)*})$. Since $j^{(n)}$ preserves $H^{(n)}$, we have (cf. (1.1.2)):

$$\begin{aligned} (H^{(n)} \otimes_C Cy) \wedge ((H^{(n)} \otimes_C Cy)')^- &= (H^{(n)} \otimes_C Cy) \wedge (H^{(n)} \otimes_C Cj^{(1)}y) \\ &= (\wedge^2 H^{(n)} \otimes_C C(y \otimes j^{(1)}y + j^{(1)}y \otimes y)) \oplus (S^2 H^{(n)} \otimes_C C(y \wedge j^{(1)}y)). \end{aligned}$$

The space $C(y \wedge j^{(1)}y)$ (where $y \wedge j^{(1)}y = (y \otimes j^{(1)}y - j^{(1)}y \otimes y)/2$) in $H^{(1)} \otimes_C H^{(1)}$ corresponds to $C\omega^{(1)}$ in $(H^{(1)*} \otimes_C (H^{(1)*})^*)$ via the natural isomorphism $H^{(1)} \otimes_C H^{(1)} \cong (H^{(1)*} \otimes_C (H^{(1)*})^*)$ induced by the nondegenerate bilinear form $\omega^{(1)}$. Furthermore,

$$\cap_y C(y \otimes j^{(1)}y + j^{(1)}y \otimes y) = \{0\},$$

where \cap_y always denotes the intersection taken over all y in $V_x - \{0\}$. Thus,

$$\cap_y (H^{(n)} \otimes_C Cy)' \wedge \overline{(H^{(n)} \otimes_C Cy)'} = S^2(H^{(n)*}) \otimes_C C\omega^{(1)} = L_2 \quad (\text{cf. Introduction}),$$

and we obtain:

LEMMA (2.3). *The fibre $(B_2)_x$ of B_2 over x is given by*

$$(B_2)_x = \cap_y \wedge^{1,1}(T_x^*M, J_{[y]}).$$

We next give a typical example of an A'_2 -connection and also a B_2 -connection.

EXAMPLE (2.4). If $n \geq 2$, the induced connection on the locally defined vector bundle

$$V := P \times_{Sp(n) \times Sp(1)} H^{(1)} \quad (\text{resp. } W := P \times_{Sp(n) \times Sp(1)} H^{(n)})$$

is an A'_2 -connection (resp. B_2 -connection). See Salamon [S1; p. 150] for

related computations of curvatures.

Recall that a connection ∇ is called a *Yang-Mills connection* if the corresponding curvature R^∇ satisfies $d^\nabla * R^\nabla = 0$. We shall finally show:

THEOREM (2.5). *All A'_2 -connections and also all B_2 -connections are Yang-Mills connections.*

COROLLARY (2.6). *The Riemannian connection on TM is a Yang-Mills connection.*

PROOF OF (2.6). By (1.2), (2.4) and (2.5), we obtain (2.6).

PROOF OF (2.5). Fix an arbitrary point x_0 of M . It then suffices to show $(d^\nabla * R^\nabla)(x_0) = 0$. We may take a local section s to P over a neighbourhood U of x_0 such that the corresponding differential at the point x_0 transforms the tangent space $T_{x_0}M$ to a horizontal space at $s(x_0)$ in the tangent space $T_{s(x_0)}P$. Let (u^1, \dots, u^{4n}) be the local frame of $T^*M|_U$ associated to s . Then all covariant derivatives of u^i 's ($1 \leq i \leq 4n$) at the point x_0 is zero. Moreover in terms of the frame (u^1, \dots, u^{4n}) , we can identify $T^*M|_U$ with $U \times \mathbf{R}^{4n}$ ($U \times \mathbf{H}^n$). Note that ∇ on E naturally induces a connection (denoted by the same ∇) on $\text{End}(E)$.

(i) We first assume that ∇ is an A'_2 -connection on E . Recall that the rank 3 subbundle A'_2 of $\wedge^2 T^*M$ corresponds to the $Sp(n) \cdot Sp(1)$ -submodule N'_2 of $\wedge^2 \mathbf{H}^n$, where N'_2 is the irreducible submodule of the elements fixed by $Sp(n)$ (cf. Introduction). Let I, J and K be

$$\begin{aligned}
 I &= \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+2} + u^{4k+3} \wedge u^{4k+4}), \\
 J &= \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+3} + u^{4k+4} \wedge u^{4k+2}), \\
 K &= \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+4} + u^{4k+2} \wedge u^{4k+3}).
 \end{aligned}$$

Then it is easy to check that $A'_{2|U}$ is spanned by the sections I, J and K . Therefore, the curvature form R^∇ is written on U as

$$R^\nabla = a \otimes I + b \otimes J + c \otimes K,$$

where a, b and c are smooth sections to $\text{End}(E)$ over U . Let (u_{11}, \dots, u_{4n}) be the base for $TM|_U$ dual to (u^1, \dots, u^{4n}) defined by $u^i(u_j) = \delta_{ij}$. Then by the first Bianchi identity,

$$\begin{aligned}
 0 &= d^\nabla(R^\nabla)(x_0) \\
 &= \sum_{i=1}^{4n} \{ (\nabla_i a) u^i(x_0) \wedge I(x_0) + (\nabla_i b) u^i(x_0) \wedge J(x_0) + (\nabla_i c) u^i(x_0) \wedge K(x_0) \},
 \end{aligned}$$

where ∇_i denotes $\nabla_{u_i(x_0)}$. Consequently,

$$\nabla_i a = \nabla_i b = \nabla_i c = 0, \text{ for } 1 \leq i \leq 4n \text{ if } n \geq 2.$$

Therefore, $(d^\nabla * R^\nabla)(x_0) = 0$.

(ii) We next assume that ∇ is a B_2 -connection on E . Since the vector subbundle B_2 (of rank $n(2n + 1)$) of $\wedge^2 T^*M$ corresponds to the irreducible $Sp(n) \cdot Sp(1)$ -submodule L_2 of the elements in $\wedge^2 \mathbf{H}^n$ fixed by $Sp(1)$, the subbundle $B_{2|U}$ is spanned by

$$I_s, J_s, K_s, D_{pq}, E_{pq}, F_{pq}, G_{pq}, \quad (0 \leq s \leq n - 1, 0 \leq p < q \leq n - 1).$$

where

$$\begin{aligned} I_s &= u^{4s+1} \wedge u^{4s+2} - u^{4s+3} \wedge u^{4s+4}, \\ J_s &= u^{4s+1} \wedge u^{4s+3} - u^{4s+4} \wedge u^{4s+2}, \\ K_s &= u^{4s+1} \wedge u^{4s+4} - u^{4s+2} \wedge u^{4s+3}, \\ D_{pq} &= u^{4p+1} \wedge u^{4q+1} + u^{4p+2} \wedge u^{4q+2} + u^{4p+3} \wedge u^{4q+3} + u^{4p+4} \wedge u^{4q+4}, \\ E_{pq} &= u^{4p+1} \wedge u^{4q+2} - u^{4p+2} \wedge u^{4q+1} - u^{4p+3} \wedge u^{4q+4} + u^{4p+4} \wedge u^{4q+3}, \\ F_{pq} &= u^{4p+1} \wedge u^{4q+3} + u^{4p+2} \wedge u^{4q+4} - u^{4p+3} \wedge u^{4q+1} - u^{4p+4} \wedge u^{4q+2}, \\ G_{pq} &= u^{4p+1} \wedge u^{4q+4} - u^{4p+2} \wedge u^{4q+3} + u^{4p+3} \wedge u^{4q+2} - u^{4p+4} \wedge u^{4q+1}. \end{aligned}$$

Let ∇ be a B_2 -connection on E . Then over U , the curvature form R^∇ is written in the form

$$\begin{aligned} R^\nabla &= \sum_{0 \leq s \leq n-1} (i_s \otimes I_s + j_s \otimes J_s + k_s \otimes K_s) \\ &+ \sum_{0 \leq p < q \leq n-1} (d_{pq} \otimes D_{pq} + e_{pq} \otimes E_{pq} + f_{pq} \otimes F_{pq} + g_{pq} \otimes G_{pq}), \end{aligned}$$

where $i_s, j_s, k_s, d_{pq}, e_{pq}, f_{pq}$ and g_{pq} are smooth sections to $\text{End}(E)$ over U . In view of the first Bianchi identity $d^\nabla R^\nabla = 0$, we have

$$\begin{aligned} -\nabla_{4s+3} i_s + \nabla_{4s+2} j_s + \nabla_{4s+1} k_s &= 0, \\ \nabla_{4s+1} i_s - \nabla_{4s+4} j_s + \nabla_{4s+3} k_s &= 0, \\ \nabla_{4s+4} i_s + \nabla_{4s+1} j_s - \nabla_{4s+2} k_s &= 0, \\ \nabla_{4s+2} i_s + \nabla_{4s+3} j_s + \nabla_{4s+4} k_s &= 0, \end{aligned}$$

for s with $0 \leq s \leq n - 1$. Furthermore, if l is either p or q , the identity $d^\nabla R^\nabla = 0$ implies

$$\begin{aligned} (-1)^{\varepsilon(l)} \nabla_{4l+1} d_{pq} - \nabla_{4l+2} e_{pq} - \nabla_{4l+3} f_{pq} - \nabla_{4l+4} g_{pq} &= 0, \\ (-1)^{\varepsilon(l)} \nabla_{4l+1} d_{pq} - \nabla_{4l+3} e_{pq} + \nabla_{4l+2} f_{pq} + \nabla_{4l+1} g_{pq} &= 0, \\ (-1)^{\varepsilon(l)} \nabla_{4l+2} d_{pq} + \nabla_{4l+1} e_{pq} - \nabla_{4l+4} f_{pq} + \nabla_{4l+3} g_{pq} &= 0, \\ (-1)^{\varepsilon(l)} \nabla_{4l+3} d_{pq} + \nabla_{4l+4} e_{pq} + \nabla_{4l+1} f_{pq} - \nabla_{4l+2} g_{pq} &= 0, \end{aligned}$$

for all p, q with $0 \leq p < q \leq n - 1$, where $\varepsilon(p) := 0$ and $\varepsilon(q) := 1$.

Then a straightforward computation shows that $(d^\nabla * R^\nabla)(x_0) = 0$, as required.

3. Deformations of B_2 -connections. In this section, we shall give an elliptic complex whose first cohomology group canonically contains the space of infinitesimal deformations of B_2 -connections on M (see Salamon [S2] for a similar complex).

(3.1) Let r be an integer with $r \geq 2$. By setting $N_r^c := \wedge^r(H^{(n)})^* \otimes_c S^r(H^{(1)})^*$ (cf. (2.1)), we can express the $Sp(n) \cdot Sp(1)$ -module $\wedge^r(H^{(n)} \otimes_c H^{(1)})^*$ as a direct sum $N_r^c \oplus L_r^c$, where L_r^c is the orthogonal complement of N_r^c in $\wedge^r(H^{(n)} \otimes_c H^{(1)})^*$. As in (2.1), the $Sp(n) \cdot Sp(1)$ -modules N_r^c and L_r^c respectively admit real forms N_r and L_r fixed by the natural real structure (cf. (1.1.2)). Since T^*M is associated to the $Sp(n) \cdot Sp(1)$ -module $(H^{(n)} \otimes_c H^{(1)})^*_R$ (see (1.2.1)), the vector bundle $\wedge^r T^*M$ is a direct sum $A_r \oplus B_r$ of its subbundles A_r, B_r corresponding respectively to N_r, L_r . Let $\pi^r: \wedge^r T^*M (= A_r \oplus B_r) \rightarrow A_r$ be the projection to the first factor. Then we have:

THEOREM (3.2). *For a B_2 -connection ∇ on E , the following is an elliptic complex:*

$$(3.2.1) \quad 0 \rightarrow \mathcal{E}(E) \xrightarrow{\nabla} \mathcal{E}(E \otimes T^*M) \xrightarrow{d_1} \mathcal{E}(E \otimes A_2) \xrightarrow{d_2} \mathcal{E}(E \otimes A_3) \xrightarrow{d_3} \dots \xrightarrow{d_{2n-1}} \mathcal{E}(E \otimes A_{2n}) \rightarrow 0,$$

where $d_i := (\text{id} \otimes \pi^{i+1}) \circ d^\nabla$ and for every vector bundle E' on M , we denote by $\mathcal{E}(E')$ the sheaf of germs of C^∞ -sections of E' .

PROOF. (i) Fix a section $s \in \Gamma(M, E \otimes A_i)$ ($i \geq 1$) and define a section $t \in \Gamma(M, E \otimes B_{i+1})$ by

$$d^\nabla s = d_i s + t.$$

Then from $(d^\nabla \circ d^\nabla)s = (d^\nabla \circ d_i)s + d^\nabla t$, we obtain

$$((\text{id} \otimes \pi_{i+2}) \circ d^\nabla \circ d^\nabla)s = (d_{i+1} \circ d_i)s + ((\text{id} \otimes \pi_{i+2}) \circ d^\nabla)t.$$

Since ∇ is a B_2 -connection, the A_{i+2} -component of $(d^\nabla \circ d^\nabla)s$ is zero, i.e.,

$$0 = (d_{i+1} \circ d_i)s + ((\text{id} \otimes \pi_{i+2}) \circ d^\nabla)t.$$

Write t as $t = \sum_k v_k \otimes b_k$ locally, where v_k, b_k is a local section of E, B_{i+1} , respectively. The $S^{i+1}(V^*)$ -component of b_k is zero, and hence the $S^{i+2}(V^*)$ -component of $\nabla(v_k) \wedge b_k$ is zero. Therefore,

$$((\text{id} \otimes \pi_{i+2}) \circ d^\nabla)t = \sum_k v_k \otimes db_k.$$

Since d is the composite of the Riemannian connection and the alternation operator, the $S^{i+2}(V^*)$ -component of db_k is zero. Thus, $(d_{i+1} \circ d_i)s = 0$, as required.

(ii) Secondly, we shall show that (3.1.1) is an elliptic complex. Then we need to calculate the symbol $\sigma(d_{i_x}, u)$ ($u \in T_x^*M - \{0\}$). Fix a point of M and an element s of $E_x \otimes A_{i_x}$. All computations below are taken at the point x .

$$\sigma(d_{i_x}, u)s := (d/dt)(e^{-tq}d_i(e^{tq}s))|_{t=0} = (\text{id} \otimes \pi_{i+1})(u \wedge s),$$

where q is a locally defined function such that $dq_x = u$. We next show that the following sequence is exact for every u :

$$(3.2.2) \quad E \otimes A_{i-1} \xrightarrow{\sigma(d_{i-1}, u)} E \otimes A_i \xrightarrow{\sigma(d_i, u)} E \otimes A_{i+1}.$$

Without loss of generality, we may assume

$$u = e_1 \otimes h_1 + (e_1 \otimes h_1)^- (= e_1 \otimes h_1 + e_2 \otimes h_2),$$

where $\langle e_1, \dots, e_{2n} \rangle$ (resp. $\langle h_1, h_2 \rangle$) is a symplectic basis of $W^* \cong W$ (resp. $V^* \cong V$), i.e., an orthonormal basis and $j^{(n)}e_{2j+1} = e_{2j+2}$ (resp. $j^{(1)}h_1 = h_2$). Let $s \in E \otimes A_i$ be such that $\sigma(d_{i+1}, u)s = 0$. Note that $S^i V^* = \text{Span}(h_1^k \cdot h_2^{i-k}; 0 \leq k \leq i)$, where $h_1^k \cdot h_2^{i-k}$ denotes the symmetric component of $h_1^k \otimes h_2^{i-k}$. Hence, there are local sections s_0, \dots, s_i of $E \otimes \wedge^i W^*$ such that

$$s = \sum_{k=0}^i s_k \otimes h_1^k \cdot h_2^{i-k}.$$

We can now write $\sigma(d_{i+1}, s) = 0$ as follows:

$$\begin{aligned} 0 &= (\text{id} \otimes \pi_{i+1})(u \wedge s) = (\text{id} \otimes \pi_{i+1})((e_1 \otimes h_1 + e_2 \otimes h_2) \wedge \sum s_k \otimes h_1^k \cdot h_2^{i-k}) \\ &= \sum_{k=0}^i ((e_1 \wedge s_k) \otimes h_1^{k+1} \cdot h_2^{i-k} + (e_2 \wedge s_k) \otimes h_1^k \cdot h_2^{i+1-k}). \end{aligned}$$

Since the coefficient of the right-hand side in $h_1^k \cdot h_2^{i+1-k}$ is zero, we have:

$$\begin{aligned} (0) \quad & e_2 \wedge s_0 = 0, \\ (1) \quad & e_1 \wedge s_0 + e_2 \wedge s_1 = 0, \\ & \vdots \\ (i) \quad & e_1 \wedge s_{i-1} + e_2 \wedge s_i = 0, \\ (i+1) \quad & e_1 \wedge s_i = 0. \end{aligned}$$

By (0), there exists $r_0 \in \wedge^{i-1} W^*$ such that $s_0 = e_2 \wedge r_0$. Plugging this into (1), we obtain $e_2 \wedge (-e_1 \wedge r_0 + s_1) = 0$. Hence there exists $r_1 \in \wedge^{i-1} W^*$ such that $s_1 = e_1 \wedge r_0 + e_2 \wedge r_1$. Repeating this process inductively, we obtain $r_k \in \wedge^{i-1} W^*$ such that $s_k = e_1 \wedge r_{k-1} + e_2 \wedge r_k, 1 \leq k \leq i$. Now by

(i + 1), the identity $e_1 \wedge e_2 \wedge r_i = 0$ holds. It then follows that there exists $r'_i \in \wedge^{i-2} W^*$ such that $e_2 \wedge r_i = e_1 \wedge e_2 \wedge r'_i$. Since $e_2 \wedge (r_{i-1} + e_2 \wedge r'_i) = e_2 \wedge r_{i-1}$, we may replace r_{i-1} by $r_{i-1} + e_2 \wedge r'_i$. Therefore,

$$\begin{aligned} s_0 &= e_2 \wedge r_0, \\ s_1 &= e_1 \wedge r_0 + e_2 \wedge r_1, \\ &\vdots \\ s_i &= e_1 \wedge r_{i-1} \end{aligned}$$

Thus,

$$s = \sum_{k=0}^i s_k \otimes h_1^k \cdot h_2^{i-k} = \sigma(d_{i-1}, u) \left(\sum_{k=0}^{i-1} r_k \otimes h_1^k \cdot h_2^{i-1-k} \right),$$

i.e., the sequence (3.2.2) is exact, as required.

DEFINITION (3.3). Let \mathcal{E} be the set of all B_2 -connections on E with holonomy groups contained in a compact semisimple Lie group G . Assume that $\mathcal{E} \neq \emptyset$ and let $\nabla \in \mathcal{E}$. Then the frame bundle Q of E can be regarded as a principal G -bundle. Put $G_\theta := Q \times_\theta G$ and $\mathfrak{g}_\theta := Q \times_{\text{Ad}} \mathfrak{g}$, where θ is the group conjugation and $\text{Ad}: G \rightarrow GL(\mathfrak{g})$ is the adjoint representation of G . Now, a C^∞ -section to G_θ over M is called a *gauge transformation* of Q . Let \mathcal{G} be the set of all gauge transformations of Q . Then \mathcal{G} naturally acts on \mathcal{E} (see Atiyah-Hitchin-Singer [A-H-S]). We call $\mathcal{M} (:= \mathcal{E}/\mathcal{G})$ the *moduli space* of the B_2 -connections on E with holonomy groups in G .

(3.4) Let $\nabla \in \mathcal{E}$ be irreducible in the sense that \mathfrak{g}_θ admits no non-zero parallel section over M . Fix a smooth one-parameter family ∇^t ($|t| < \varepsilon$) of connections in \mathcal{E} such that $\nabla^0 = \nabla$. Put $S = (d/dt)\nabla^t|_{t=0}$. We write the curvature form R^{∇^t} of ∇^t as

$$R^{\nabla^t} = R^\nabla + td^{\nabla'}S + \text{higher order terms in } t,$$

where ∇' is the connection on \mathfrak{g}_θ naturally induced by ∇ . Since R^{∇^t} is a \mathfrak{g}_θ -valued B_2 -form, the corresponding derivative $d^{\nabla'}S$ at $t = 0$ also satisfies

$$((\text{id} \otimes \pi^2) \circ d^{\nabla'})S = 0.$$

Let f^t ($|t| < \varepsilon$) be a one-parameter family of gauge transformations such that $f^0 = \text{id}$. Then,

$$\frac{d}{dt}(f^t(\nabla))|_{t=0} = \nabla'(f^\dot{}),$$

where $f^\dot{} := (d/dt)(f^t)|_{t=0}$. Since $f^t(\nabla) \in \mathcal{E}$ for all t , the same argument as above shows that the \mathfrak{g}_θ -valued 1-form $\nabla'(f^\dot{})$ satisfies

$$((\text{id} \otimes \pi^2) \circ d^{\nabla'}) (\nabla'(f)) = 0 .$$

For each $A \in \Gamma(\mathfrak{g}_Q)$, there exists a one-parameter family $f^t = \exp(tA)$ such that $(d/dt)f^t|_{t=0} = A$. Then together with (3.2), we immediately obtain the following:

THEOREM (3.5). *Assume that $\mathcal{E} \neq \emptyset$ and let $\nabla \in \mathcal{E}$ be irreducible. Then the space of infinitesimal (essential) deformations at ∇ of connections in \mathcal{E} , that is, the tangent space of \mathcal{M} at ∇ is a linear subspace of the first cohomology group of the elliptic complex*

$$\begin{aligned} 0 \rightarrow \mathcal{E}(\mathfrak{g}_Q) \xrightarrow{\nabla'} \mathcal{E}(\mathfrak{g}_Q \otimes T^*M) \xrightarrow{d'_1} \mathcal{E}(\mathfrak{g}_Q \otimes A_2) \\ \xrightarrow{d'_2} \mathcal{E}(\mathfrak{g}_Q \otimes A_3) \xrightarrow{d'_3} \dots \xrightarrow{d'_{2n-1}} \mathcal{E}(\mathfrak{g}_Q \otimes A_{2n}) \rightarrow 0 , \end{aligned}$$

where $d'_i := (\text{id} \otimes \pi^{i+1}) \circ d^{\nabla'}$.

4. Einstein-Hermitian connections associated with B_2 -connections.

In this section we shall prove Theorem (0.2) (see the Introduction) which clarifies the relationship between B_2 -connections and the corresponding Einstein-Hermitian connections.

PROOF OF (0.2). (i) Let (E, D_E) be a Hermitian pair. Then by the definition of B_2 -connections, the curvature form corresponding to the connection D_E is an $\text{End}(E)$ -valued B_2 -form, and by Lemma (2.3) the curvature form corresponding to the connection p^*D_E on p^*E is an $\text{End}(p^*E)$ -valued $(1, 1)$ -form. Hence the connection p^*D_E induces naturally an integrable complex structure on p^*E as follows: Put $l := \text{rank}(E)$ and denote by $q: p^*E \rightarrow Z$ the natural projection. Let (s_1, \dots, s_l) (resp. (y^1, \dots, y^l)) be a local unitary frame for p^*E (resp. the dual frame corresponding to (s_1, \dots, s_l)). Then the vector subbundle $\wedge^{1,0}T^*(p^*E)$ of type $(1, 0)$ in the complexification $T^*(p^*E)^c$ of the cotangent bundle $T^*(p^*E)$ is defined as the direct sum of the pull-back $q^*(\wedge^{1,0}T^*Z)$ and the space spanned by $\{dy^j + \sum_{i=1}^l y^i q^* \theta_{ij}, 1 \leq j \leq l\}$, where (θ_{ij}) is the connection matrix for p^*D_E with respect to the frame (s_1, \dots, s_l) (i.e., $(p^*D_E)s_j = \sum_{i=1}^l s_i \theta_{ij}$). Now, we may take the frame (s_1, \dots, s_l) as the pull-back (p^*t_1, \dots, p^*t_l) of a local unitary frame (t_1, \dots, t_l) on E . Then the 1-forms $\theta_{ij}, 1 \leq i, j \leq l$, are written as $p^*\psi_{ij}$, where (ψ_{ij}) denotes the connection matrix for D_E with respect to the frame (t_1, \dots, t_l) . Let $q': (p^*E)^* \rightarrow Z$ be the projection naturally induced from $q: p^*E \rightarrow Z$. Since the real structure $\tau: Z \rightarrow Z$ is antiholomorphic (cf. Nitta and Takeuchi [N-T]), and since the mapping $q' \circ \sigma: p^*E \rightarrow Z$ is equal to $\tau \circ q$, the mapping $\sigma: p^*E \rightarrow (p^*E)^*$ is clearly an antiholomorphic bundle automorphism by the definition of the complex structures on p^*E and $(p^*E)^*$.

(ii) We next fix an arbitrary excellent pair (F, D_F) on Z . Then by the condition (a) in the definition of excellent pair (see the Introduction), we can choose an open cover $\{U_\lambda\}$ of M , and a local unitary frame $(f_1^\lambda, \dots, f_r^\lambda)$ ($r = \text{rank of } F$) of $F|_{p^{-1}(U_\lambda)}$ such that each restriction $(f_{1|p^{-1}(x)}^\lambda, \dots, f_{r|p^{-1}(x)}^\lambda)$ over $p^{-1}(x)$ ($x \in U_\lambda$) forms a holomorphic frame for $F|_{p^{-1}(x)}$. When $U_\lambda \cap U_\mu \neq \emptyset$, the transition matrix for F in terms of the frames $(f_1^\lambda, \dots, f_r^\lambda), (f_1^\mu, \dots, f_r^\mu)$ is holomorphic (and hence constant) along each fibre $p^{-1}(x)$ ($x \in U_\lambda \cap U_\mu$). Hence there exists a Hermitian vector bundle E on M such that, including metrics, we have $p^*E = F$. In particular, we obtain a local unitary frame $(f_1^{\prime\lambda}, \dots, f_r^{\prime\lambda})$ for $E|_{U_\lambda}$ such that $(p^*f_1^{\prime\lambda}, \dots, p^*f_r^{\prime\lambda})$ coincides with the previous $(f_1^\lambda, \dots, f_r^\lambda)$ over $p^{-1}(U_\lambda)$. Fix an arbitrary λ . If there is no fear of confusion, we shall omit the suffix λ and denote $U_\lambda, (f_1^\lambda, \dots, f_r^\lambda), \dots$ simply by $U, (f_1, \dots, f_r), \dots$, respectively. Let (ω_{ij}) be the connection matrix of D_F with respect to the frame (f_1, \dots, f_r) , i.e., $D_F f_j = \sum_{i=1}^r f_i \omega_{ij}$. Furthermore, we choose a local symplectic basis (e_1, \dots, e_{2n}) (resp. (h_1, h_2)) for $W^*|_U$ (resp. $V^*|_U$) (see Section 3). Now, since D_F is a Hermitian connection, we have:

$$(1) \quad \omega_{ij} + \overline{\omega_{ji}} = 0, \quad \text{for } 1 \leq i, j \leq r.$$

Then the construction of D_E is reduced to showing that there exist 1-forms ω'_{ij} ($1 \leq i, j \leq r$) on U satisfying $\omega_{ij} = p^* \omega'_{ij}$. In fact, once we can find such 1-forms ω'_{ij} , they define a Hermitian connection on E , such that the corresponding curvature form is pulled back by p to an $\text{End}(F)$ -valued $(1, 1)$ -form on Z , which together with Lemma (2.3) implies that our connection on E is a B_2 -connection. Recall that, for each $x \in U$, the frame $(f_{1|p^{-1}(x)}, \dots, f_{r|p^{-1}(x)})$ for $F|_{p^{-1}(x)}$ is trivial. Hence,

$$(2) \quad \omega_{ij}(v) = 0, \quad 1 \leq i, j \leq r,$$

for every vector v tangent to $p^{-1}(x)$ ($\cong P^1C$). Since $(e_1 \otimes h_1, e_1 \otimes h_2, \dots, e_{2n} \otimes h_1, e_{2n} \otimes h_2)$ is a frame for $T^*M^c|_U = W^*|_U \otimes V^*|_U$, there exist by (2) C^∞ -functions a_{ij}^k, b_{ij}^k ($1 \leq i, j \leq r, 1 \leq k \leq 2n$) on $p^{-1}(U)$ such that

$$(3) \quad \omega_{ij} = \sum_{k=1}^{2n} (a_{ij}^k p^*(e_k \otimes h_1) + b_{ij}^k p^*(e_k \otimes h_2)), \quad 1 \leq i, j \leq r.$$

For every form η on $Z|_U$, we denote by $\hat{\eta}$ the pull-back of η to $(V - \{\text{zero section}\})|_U$. Then by (3), we have:

$$\begin{aligned} \hat{R}_{ij} &= d\hat{\omega}_{ij} + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{tj} \\ &= \sum_{k=1}^{2n} d(\hat{a}_{ij}^k \hat{p}^*(e_k \otimes h_1)) + d(\hat{b}_{ij}^k \hat{p}^*(e_k \otimes h_2)) + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{tj}. \end{aligned}$$

Fix an arbitrary point x on U . Choosing an appropriate (e_1, \dots, e_{2n}) (resp.

(h_1, h_2) , we may assume that $(\nabla^{V^*} e_k)(x) = 0, k = 1, 2, \dots, 2n$ (resp. $(\nabla^{W^*} h_i)(x) = 0, i = 1, 2$), where ∇^{V^*} (resp. ∇^{W^*}) denotes the connection of V^* (resp. W^*) canonically induced by that of P (cf. Example (2.4)). Then, on $\hat{p}^{-1}(x)$,

$$\hat{R}_{ij} = \sum_{k=1}^{2n} \{d(\hat{a}_{ij}^k) \wedge \hat{p}^*(e_k \otimes h_1) + d(\hat{b}_{ij}^k) \wedge \hat{p}^*(e_k \otimes h_2)\} + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{ij}.$$

Recall that the complex structure on the twistor space $Z (= (V - \{\text{zero section}\})/\mathbf{C}^*)$ is induced by the complex structure on $V - \{\text{zero section}\}$ (see Section 1). Since \hat{R}_{ij} is of type $(1, 1)$, we have:

$$(4) \quad \sum_{k=1}^{2n} \{\partial(\hat{a}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_1))^{(1,0)} + \partial(\hat{b}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_2))^{(1,0)}\} + \sum_{t=1}^r \hat{\omega}_{it}^{(1,0)} \wedge \hat{\omega}_{ij}^{(1,0)} = 0 \quad \text{on } p^{-1}(x);$$

$$(5) \quad \sum_{k=1}^{2n} \{\bar{\partial}(\hat{a}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_1))^{(0,1)} + \bar{\partial}(\hat{b}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_2))^{(0,1)}\} + \sum_{t=1}^r \hat{\omega}_{it}^{(0,1)} \wedge \hat{\omega}_{ij}^{(0,1)} = 0 \quad \text{on } p^{-1}(x),$$

where for every 1-forms ζ on $(V - \{\text{zero section}\})_{|U}$, $\zeta^{(1,0)}$ (resp. $\zeta^{(0,1)}$) always denotes the $(1, 0)$ -component (resp. $(0, 1)$ -component) of ζ . Let (z^1, z^2) be the local triviality for $V_{|U}$ corresponding to (h_1, h_2) . Then, by the definition of the complex structure of $(V - \{\text{zero section}\})$, we obtain from (4) and (5) the following:

$$(4') \quad \sum_{k=1}^{2n} \left\{ \left(\frac{\partial}{\partial z^1} \hat{a}_{ij}^k dz^1 + \frac{\partial}{\partial z^2} \hat{a}_{ij}^k dz^2 \right) \wedge \bar{z}^1 (z^1 \hat{p}^*(e_k \otimes h_1) + z^2 \hat{p}^*(e_k \otimes h_2)) + \left(\frac{\partial}{\partial z^1} \hat{b}_{ij}^k dz^1 + \frac{\partial}{\partial z^2} \hat{b}_{ij}^k dz^2 \right) \wedge \bar{z}^2 (z^1 \hat{p}^*(e_k \otimes h_1) + z^2 \hat{p}^*(e_k \otimes h_2)) \right\} = 0 \quad \text{on } \hat{p}^{-1}(x);$$

$$(5') \quad \sum_{k=1}^{2n} \left\{ \left(\frac{\partial}{\partial \bar{z}^1} \hat{a}_{ij}^k d\bar{z}^1 + \frac{\partial}{\partial \bar{z}^2} \hat{a}_{ij}^k d\bar{z}^2 \right) \wedge (-z^2)(\bar{z}^1 \hat{p}^*(e_k \otimes h_2) - \bar{z}^2 \hat{p}^*(e_k \otimes h_1)) + \left(\frac{\partial}{\partial \bar{z}^1} \hat{b}_{ij}^k d\bar{z}^1 + \frac{\partial}{\partial \bar{z}^2} \hat{b}_{ij}^k d\bar{z}^2 \right) \wedge z^1(\bar{z}^1 \hat{p}^*(e_k \otimes h_2) - \bar{z}^2 \hat{p}^*(e_k \otimes h_1)) \right\} = 0 \quad \text{on } \hat{p}^{-1}(x).$$

Since both $z_{|p^{-1}(x)}^1$ and $z_{|p^{-1}(x)}^2$ are holomorphic on $\hat{p}^{-1}(x) \cong \mathbf{C}^2 - \{0\}$, we have

$$\frac{\partial}{\partial \bar{z}^i} (z^1 \bar{\hat{a}}_{ij}^k + z^2 \bar{\hat{b}}_{ij}^k) = \frac{\partial}{\partial \bar{z}^i} (-z^2 \hat{a}_{ij}^k + z^1 \hat{b}_{ij}^k) = 0 \quad (i = 1, 2),$$

on $p^{-1}(x)$, i.e., both $f_1(z^1, z^2) := z^1 \bar{\hat{a}}_{ij}^k + z^2 \bar{\hat{b}}_{ij}^k$ and $f_2(z^1, z^2) := -z^2 \hat{a}_{ij}^k + z^1 \hat{b}_{ij}^k$ are holomorphic on $C^2 - \{0\}$. By Hartogs' theorem, both f_1 and f_2 extend further to holomorphic functions on C^2 . Since $f_i(cz^1, cz^2) = cf_i(z^1, z^2)$ for all $z = (z^1, z^2) \in C^2$ and $c \in C^*$ ($i = 1, 2$), there exist constants $\alpha_{ij}^k, \beta_{ij}^k, \gamma_{ij}^k, \delta_{ij}^k \in C$ independent of z such that

$$(6) \quad z^1 \bar{\hat{a}}_{ij}^k + z^2 \bar{\hat{b}}_{ij}^k = z^1 \bar{\alpha}_{ij}^k + z^2 \bar{\beta}_{ij}^k,$$

$$(7) \quad -z^2 \hat{a}_{ij}^k + z^1 \hat{b}_{ij}^k = -z^2 \gamma_{ij}^k + z^1 \delta_{ij}^k, \quad (1 \leq k \leq 2n).$$

Let $\Gamma(Z, F^*)$ (resp. $\Gamma(Z, F^* \otimes T^*Z^c)$) be the space of global C^∞ -sections over Z to F^* (resp. $F^* \otimes T^*Z^c$). Let $\psi: \Gamma(Z, F^*) \rightarrow \Gamma(Z, F^* \otimes T^*Z^c)$ be the C -linear map sending each $s \in \Gamma(Z, F^*)$ to an element $\psi(s)$ of $\Gamma(Z, F^* \otimes T^*Z^c)$ defined by

$$\psi(s)(X) := \sigma((D_F)_{\tau^*(X)}(\sigma^{-1}s)) \in F_z^*,$$

for $X \in T_z Z^c$ ($z \in Z$).

Then by the condition (b) in the Introduction, this ψ defines a Hermitian $(1, 0)$ -connection on the holomorphic vector bundle F^* . The corresponding connection matrix with respect to the frame $(\sigma f_1, \dots, \sigma f_r)$ for $F_{(p^{-1}(U))}^*$ is written as $(\tau^* \omega_{ij})$. By the definition of σ , it is easy to check that the frame $(\sigma f_1, \dots, \sigma f_r)$ is dual to our previous (f_1, \dots, f_r) . Hence the uniqueness of the $(1, 0)$ -connection on the Hermitian vector bundle F^* implies the equality $(\tau^* \omega_{ij})^- = \omega_{ij}^*$, where $\omega_{ij}^* := -\omega_{ji}$. In view of (1), we have $\tau^* \omega_{ij} = \omega_{ij}$ and $\hat{\tau}^* \hat{\omega}_{ij} = \hat{\omega}_{ij}$. By (3) and $\hat{p} \circ \hat{\tau} = \hat{p}$, we obtain:

$$(8) \quad \hat{\tau}^* \hat{a}_{ij}^k = \hat{a}_{ij}^k \quad \text{and} \quad \hat{\tau}^* \hat{b}_{ij}^k = \hat{b}_{ij}^k \quad (1 \leq k \leq 2n).$$

Therefore,

$$-\bar{z}^2 \hat{\tau}^* \bar{\hat{a}}_{ij}^k + \bar{z}^1 \hat{\tau}^* \bar{\hat{b}}_{ij}^k = -\bar{z}^2 \bar{\alpha}_{ij}^k + \bar{z}^1 \bar{\beta}_{ij}^k \quad (1 \leq k \leq 2n).$$

Moreover by (6),

$$(9) \quad -z^2 \hat{a}_{ij}^k + z^1 \hat{b}_{ij}^k = -z^2 \alpha_{ij}^k + z^1 \beta_{ij}^k \quad (1 \leq k \leq 2n).$$

Hence by (7) and (9), we obtain:

$$(10) \quad \alpha_{ij}^k = \gamma_{ij}^k \quad \text{and} \quad \beta_{ij}^k = \delta_{ij}^k \quad (1 \leq k \leq 2n).$$

Now, in view of (6), (7) and (10), we see that

$$\begin{pmatrix} \bar{z}^1, & \bar{z}^2 \\ -z^2, & z^1 \end{pmatrix} \begin{pmatrix} \hat{a}_{ij}^k - \alpha_{ij}^k \\ \hat{b}_{ij}^k - \beta_{ij}^k \end{pmatrix} = 0 \quad (1 \leq k \leq 2n),$$

where $(z^1, z^2) \in C^2 - \{0\}$ ($= \hat{p}^{-1}(x)$). Thus, $\hat{a}_{ij}^k = \alpha_{ij}^k$ and $\hat{b}_{ij}^k = \beta_{ij}^k$ ($1 \leq k \leq 2n$), i.e., both α_{ij}^k and β_{ij}^k are constant along $p^{-1}(x)$, as required.

REMARK (4.1). In some sense, our Theorem (0.2) completely clarifies

the following result by Salamon [S2] (see Berard Bergery and Ochiai [B-O] for another generalization):

*For a Hermitian pair (E, D_E) on M , the pull-back (p^*E, p^*D_E) to Z is a Hermitian holomorphic vector bundle over Z .*

COROLLARY (4.2). *Let (F, D_F) be an excellent pair on Z . If the quaternionic Kähler manifold M has positive scalar curvature, then F with D_F is a Ricci-flat Einstein Hermitian vector bundle over Z .*

PROOF. Consider the twistor space $p: Z \rightarrow M$. Then the horizontal component of the Kähler form on Z is a p^*A_2' -form (cf. (1.2), (1.3)). Recall that the curvature of D_F is an $\text{End}(F)$ -valued p^*B_2 -form. Hence the Hermitian vector bundle F with D_F is Ricci-flat.

REMARK (4.3). We have the decomposition of $TZ = T^h \oplus T^v$, where T^h (resp. T^v) is the horizontal (resp. vertical) distribution in terms of the connection on Z induced by that of P . Since the complex structure on TZ is a direct sum of complex structures on T^h and T^v , the holomorphic part $TZ^{(1,0)}$ admits the corresponding decomposition $TZ^{(1,0)} = T^{h(1,0)} \oplus T^{v(1,0)}$, where $T^{h(1,0)}$ (resp. $T^{v(1,0)}$) denotes $T^{hC} \cap TZ^{(1,0)}$ (resp. $T^{vC} \cap TZ^{(1,0)}$). Recently, Zandi [Z] obtained the following:

The vector bundle $(T^{h(1,0)}, D^h)$ is an Einstein-Hermitian vector bundle, where D^h is the connection on $T^{h(1,0)}$ obtained as the restriction of the Riemannian connection on TZ to $T^{h(1,0)}$.

This result can be regarded as a straightforward consequence of our (4.2). We denote by L a locally defined (line) subbundle of p^*W (cf. (2.4)) such that, along each fibre $p^{-1}(x) = P^1C$ ($x \in M$), it restricts to a universal bundle over P^1C . Let ∇^V (resp. ∇^W) denote the connection of V (resp. W) canonically induced by that of P and ∇^L the restriction of $p^*\nabla^W$ to L . Then the vector bundle $(T^{h(1,0)}, D^h)$ is nothing but $(p^*W \otimes L^*, p^*\nabla^W \otimes (\nabla^L)^*)$, where $(L^*, (\nabla^L)^*)$ is dual to (L, ∇^L) (see Salamon [S1]). Since L^* is a locally defined line bundle and since ∇^W is a B_2 -connection on W , Corollary (4.2) clearly implies Zandi's result.

Added in proof. After the completion of this paper, the author received a preprint by M. M. Capria and S. M. Salamon entitled "Yang-Mills fields on quaternionic Kähler spaces", which gives (i) a result slightly stronger than (2.6) and (ii) a statement similar to (3.2).

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