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VECTOR FIELDS AND DIFFERENTIAL FORMS ON GENERALIZED RAYNAUD SURFACES

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Abstract. We consider the tangent and cotangent sheaves of generalized Raynaud surfaces, which have cuspidal fibrations and which are mostly of general type. In particular, we compute the dimensions of the space of vector fields and of non-closed differential 1-forms.

Let $f: V \to B$ be a fibration from a smooth projective surface to a smooth projective curve over an algebraically closed field k. In the case of characteristic zero, almost all fibres of f are nonsingular. In the case of positive characteristic, however, there exists fibrations whose general fibres have singularities. Generalized Raynaud surfaces are typical examples of surfaces which have such fibrations. Moreover, these surfaces have interesting geometry in positive characteristic. In the present article, we compute the dimension of the space of vector fields and give an estimate for the difference of the dimension of the space of all global differential 1-forms from that of the space of closed differential 1-forms on generalized Raynaud surfaces.

1. Generalized Raynaud surfaces. Throughout this article, we assume that k is an algebraically closed field of characteristic $p \geqq 3$. To begin with, we define generalized Tango curves. Let C be a smooth projective curve over k and let $\mathcal{N}$ be an invertible sheaf on C with positive degree. Suppose that there exist local sections $\{\xi_i \in \Gamma(U_i, \mathcal{O}_C)\}_{i \in I}$ whose differentials $\{d\xi_i\}$ are local generators of the sheaf of differentials Ω_C^1 satisfying $d\xi_i = a_{ij}^{pn} d\xi_j$, where $\{a_{ij}\}_{i,j \in I}$ are transition functions of $\mathcal{N}$ for an affine open covering $\{U_i\}_{i \in I}$ and where n is a positive integer with $n \not\equiv 0 \pmod p$ and $n > 1$. Then we call the triple $(C, \mathcal{N}, \{d\xi_i\})$ a *generalized Tango curve of index n*. Note that $\mathcal{N}^{np} \cong \Omega_C^1$. The following lemma is useful for constructing generalized Tango curves. For the proof, we refer to Takeda [7].

LEMMA 1.1 (Kurke [1]). *Let ω be an exact differential on a smooth projective curve C. Suppose that the divisor of ω has the form pnD, where D is a nonzero effective divisor and n is a positive integer with $n \not\equiv 0 \pmod p$ and $n > 1$. Then there exist local sections*

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$\{\xi_i \in \Gamma(U_i, \mathcal{O}_C)\}_{i\in I}$ such that $(C, \mathcal{O}_C(D), \{d\xi_i\})$ is a generalized Tango curve of index n.

We shall give an example of a generalized Tango curve.

EXAMPLE 1.2 (Raynaud [5]). Let C be an affine plane curve defined by the equation $y^{np}-y=x^{np-1}$, where n is a positive integer with $n\not\equiv 0 \pmod p$ and $n>1$. Since the genus of C is $(np-1)(np-2)/2$, the divisor of the exact differential dx is $np(np-3)P_\infty$, where P_∞ is the point at infinity. By Lemma 1.1, we know that $(C, \mathcal{O}_C((np-3)P_\infty), dx)$ is a generalized Tango curve of index n. We note that $H^0(C, \mathcal{O}_C((np-3)P_\infty))\neq 0$ and $\dim H^0(C, \mathcal{O}_C(p(np-3)P_\infty))>1$ for §2 and §3.

Now, we shall construct a generalized Raynaud surface. Let $(C, \mathcal{N}, d\xi)$ be a generalized Tango curve of index n. Then there exist an affine open covering $\{U_i\}_{i\in I}$ of C and local sections $\xi_i \in \Gamma(U_i, \mathcal{O}_C)$ satisfying $d\xi_i = a_{ij}^{np} d\xi_j$, where a_{ij} is the transition functions for $\mathcal{N}$ on $U_i \cap U_j$. Moreover, there exist sections $b_{ij}\in \Gamma(U_i\cap U_j, \mathcal{O}_C)$ such that $\xi_i = a_{ij}^{np}\xi_j + b_{ij}^p$. We define a Gorenstein scheme Y and a morphism $\phi\colon Y\to C$ by giving local data as

$$\phi^{-1}(U_i)=\operatorname{Spec}\Gamma(U_i, \mathcal{O}_C)[x_i, y_i]/(x_i^p-y_i^n-\xi_i)\cup \operatorname{Spec}\Gamma(U_i, \mathcal{O}_C)[s_i, t_i]/(s_i^p-t_i^q-t_i^{hp}\xi_i)\,,$$

where $x_i=a_{ij}^n x_j+b_{ij}$, $y_i=a_{ij}^p y_j$, $s_i=a_{ij}^{-q}s_j+a_{ij}^{-hp}t_j^h b_{ij}$, $t_i=a_{ij}^{-p}t_j$, $x_i=t_i^{-h}s_i$, $y_i=t_i^{-1}$, and $n+q=hp$ ($q, h\in \boldsymbol{Z}$, $0<q<p$). Unless $n+1\equiv 0 \pmod p$, Y is not normal. So, we take the normalization $\mu\colon X\to Y$ and denote $\varphi=\phi\circ\mu$. We call this surface X a *generalized Raynaud surface* over a generalized Tango curve $(C, \mathcal{N}, d\xi)$. The generalized Raynaud surface X has the following properties:

(1) X is a nonsingular relatively minimal surface and X is of general type provided $(p, n)\neq(3, 2)$.

(2) φ is a fibration and all fibres of φ are rational curves with one cusp of type $x^p=y^n$. The singular locus Σ of the fibres, i.e., the locus of the moving singularities, is locally defined by $y_i=0$ in the same notation as above.

(3) There exists a section E of φ lying over $\bigcup_{i\in I}\{t_i=0\}$ such that the normal sheaf of E is isomorphic to $\mathcal{N}$ and $\mathcal{O}_X(\Sigma)=\mathcal{O}_X(pE)\otimes\varphi^*\mathcal{N}^{-p}$.

(4) X has the following numerical invariants:

(i) $\omega_X=\mathcal{O}_X((np-p-n-1)E)\otimes\varphi^*\mathcal{N}^{n+p}$.

(ii) $(K_X^2)=(n^2p^2-p^2-n^2-4np+1)d$, where $d=\deg\mathcal{N}$.

(iii) The Euler number of X is $e(X)=-2npd$.

(iv) $\chi(\mathcal{O}_X)=(n^2p^2-p^2-n^2-6np+1)d/12$.

(5) $\mathcal{O}_X(E)\otimes\varphi^*\mathcal{N}$ is an ample invertible sheaf on X and

$$H^1(X, \mathcal{O}_X(-E)\otimes\varphi^*\mathcal{N}^{-1})\neq 0\,.$$

For the proof of these facts and more detailed discussions on generalized Raynaud surfaces, see [7].

Throughout this article, we keep the notation in this section.

2. Vector fields on generalized Raynaud surfaces. In this section, we prove the following theorem:

THEOREM 2.1. *Let Θ_X be the tangent sheaf of a generalized Raynaud surface X. Then we have*

$$H^0(X, \Theta_X) = \begin{cases} H^0(C, \mathcal{N}) & \text{if } \ n+1 \equiv 0 \pmod p, \\ 0 & \text{otherwise}. \end{cases}$$

We note that X is of general type provided $(p, n) \neq (3, 2)$, and that $H^0(C, \mathcal{N}) \neq 0$. This theorem is a refined version of Lang [3, Theorem 1]. Our proof of this theorem begins with the following:

LEMMA 2.2 (Lang [2]). *There is an exact sequence*

$$0 \to \varphi^*\Omega_C^1 \otimes \mathcal{O}_X((n-1)\Sigma) \to \Omega_X^1 \to \omega_{X/C} \otimes \mathcal{O}_X(-(n-1)\Sigma) \to 0 .$$

PROOF. Since φ is smooth on $X-\Sigma$, we obtain the required exact sequence on $X-\Sigma$. Near Σ, X is locally defined by the equation $x_i^p - y_i^n - \xi_i = 0$ (see §1). By exterior differentiation, we have $d\xi_i = -ny_i^{n-1}dy_i$ or $d\xi_i/y_i^{n-1} = -ndy_i$. We know that (x_i, y_i) is a local coordinate system and that Σ is locally defined by $y_i = 0$. Hence there is an injection $\varphi^*\Omega_C^1 \otimes \mathcal{O}_X((n-1)\Sigma) \to \Omega_X^1$ whose cokernel is locally free. Now, the assertion is clear. q.e.d.

Note that $\Theta_X \cong \Omega_X^1 \otimes \omega_X^{-1}$ and recall that $\mathcal{O}_X(\Sigma) = \mathcal{O}_X(pE) \otimes \varphi^*\mathcal{N}^{-p}$ and $\omega_X = \mathcal{O}_X((np-p-n-1)E) \otimes \varphi^*\mathcal{N}^{n+p}$ (see §1). So, we have:

COROLLARY 2.3. *There is an exact sequence*

$$0 \to \mathcal{O}_X((n+1)E) \otimes \varphi^*\mathcal{N}^{-n} \to \Theta_X \to \mathcal{O}_X(-(n-1)pE) \otimes \varphi^*\mathcal{N}^{-p} \to 0 .$$

Since $H^0(X, \mathcal{O}_X(-(n-1)pE) \otimes \varphi^*\mathcal{N}^{-p}) = 0$, we know that $H^0(X, \Theta_X) = H^0(X, \mathcal{O}_X((n+1)E) \otimes \varphi^*\mathcal{N}^{-n}) = H^0(C, \varphi_*\mathcal{O}_X((n+1)E) \otimes \mathcal{N}^{-n})$. Hence we consider the direct image $\varphi_*\mathcal{O}_X((n+1)E)$. Let $\boldsymbol{F}$: $C' \to C$ be the k-Frobenius morphism, i.e., C' is the normalization of C in $k(C)^{1/p}$. Let $X' = X \times_C C'$ and let φ': $X' \to C'$ and π: $X' \to X$ be the projections. Take the normalization ν: $Z \to X'$ and denote $\psi = \varphi' \circ \nu$. Then ψ: $Z \to C'$ is nothing but the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(\mathcal{O}_{C'} \oplus \boldsymbol{F}^*\mathcal{N}) \to C'$ (see [7, §3]).

$$\begin{array}{ccccc} Z & \xrightarrow{\nu} & X' & \xrightarrow{\pi} & X \\ \psi \downarrow & & \varphi' \downarrow & & \downarrow \varphi \\ C' & \xrightarrow[\mathrm{id}_{C'}]{} & C' & \xrightarrow[\boldsymbol{F}]{} & C \end{array}$$

Since the set of nonnormal points of X' is $\pi^{-1}(\Sigma)$, we see that X' is isomorphic to Z except over $\pi^{-1}(\Sigma)$. Let $U_i' = \boldsymbol{F}^{-1}(U_i)$ and let $\eta_i = \xi_i^{1/p}$, where U_i and ξ_i are the same as in §1. Then $\varphi'^{-1}(U_i')$ is defined by $x_i^p - y_i^n - \eta_i^p = 0$ near $\pi^{-1}(\Sigma)$. Since p and n are relatively prime, there exist positive integers α and β such that $\alpha p-\beta n=1$. Set $z_i=y_i^{\alpha}(x_i-\eta_i)^{-\beta}$. Then we see that $z_i^p=y_i$, $z_i^n=x_i-\eta_i$ and that (z_i, η_i) is a local coordinate system on Z. Let $\mathscr{R}$ be the cokernel of the natural injection $\mathcal{O}_{X'}\to v_*\mathcal{O}_Z$. Then $\operatorname{Supp}\mathscr{R}=\pi^{-1}(\Sigma)$. It is easy to verify that $\mathscr{R}|_{\varphi'^{-1}(U_i')}\cong\bigoplus_m\mathcal{O}_{U_i'}z_i^m$, where z_i is the same as above, and where m ranges over all positive integers except $Np+Nn$, p, and n. Denote $S=v^*(\pi^*E)=\pi^*E$. By tensoring $\pi^*\mathcal{O}_X((n+1)E)$, we have

$$0\to\pi^*\mathcal{O}_X((n+1)E)\to v_*\mathcal{O}_Z((n+1)S)\to\mathscr{R}\otimes\pi^*\mathcal{O}_X((n+1)E)\to 0\,.$$

Since $\pi^{-1}(E)\cap\operatorname{Supp}\mathscr{R}=\varnothing$, we have $\mathscr{R}\otimes\pi^*\mathcal{O}_X((n+1)E)=\mathscr{R}$. Restrict this sequence to $\varphi'^{-1}(U_i')$ and take the direct images. Then we have

$$0\to\varphi'_*\pi^*\mathcal{O}_X((n+1)E)|_{U_i'}\to\psi_*\mathcal{O}_Z((n+1)S)|_{U_i'}\xrightarrow{\tau}\bigoplus_m\mathcal{O}_{U_i'}z_i^m\,,$$

where m is the same as above. Since $\psi: Z\to C'$ is a $\boldsymbol{P}^1$-bundle, we obtain $\psi_*\mathcal{O}_Z((n+1)S)|_{U_i'}=\mathcal{O}_{U_i'}\oplus\mathcal{O}_{U_i'}z_i\oplus\cdots\oplus\mathcal{O}_{U_i'}z_i^{n+1}$. Considering the kernel of τ, we have

$$\begin{aligned}\varphi'_*\pi^*\mathcal{O}_X((n+1)E)|_{U_i'}&=\mathcal{O}_{U_i'}\oplus\mathcal{O}_{U_i'}z_i^n\oplus\mathcal{O}_{U_i'}z_i^p\oplus\cdots\oplus\mathcal{O}_{U_i'}z_i^{[(n+1)/p]p}\\&=\mathcal{O}_{U_i'}\oplus\mathcal{O}_{U_i'}(x_i-\eta_i)\oplus\mathcal{O}_{U_i'}y_i\oplus\cdots\oplus\mathcal{O}_{U_i'}y_i^{[(n+1)/p]}\\&=\mathcal{O}_{U_i'}\oplus\mathcal{O}_{U_i'}x_i\oplus\mathcal{O}_{U_i'}y_i\oplus\cdots\oplus\mathcal{O}_{U_i'}y_i^{[(n+1)/p]}\,,\end{aligned}$$

where $[x]$ denotes the greatest integer not exceeding x. Since $\varphi'_*\pi^*\mathcal{O}_X((n+1)E)|_{U_i'}=\boldsymbol{F}^*\varphi_*\mathcal{O}_X((n+1)E)|_{U_i'}$ and since x_i and y_i are sections of $\mathcal{O}_{U_i}$, we get

$$\varphi_*\mathcal{O}_X((n+1)E)|_{U_i}=\mathcal{O}_{U_i}\oplus\mathcal{O}_{U_i}x_i\oplus\mathcal{O}_{U_i}y_i\oplus\cdots\oplus\mathcal{O}_{U_i}y_i^{[(n+1)/p]}\,.$$

Recall that $x_i=a_{ij}^nx_j+b_{ij}$ and $y_i=a_{ij}^py_j$. So, we have:

LEMMA 2.4. *In the same notation and under the same assumptions as above, we have*

$$\varphi_*\mathcal{O}_X((n+1)E)=\begin{cases}V\oplus\mathcal{N}^p\oplus\cdots\oplus\mathcal{N}^{[(n+1)/p]p} & \text{if } n+1\geq p\,,\\ V & \text{if } n+1<p\,,\end{cases}$$

where V is the locally free sheaf on C which is locally generated by 1 *and* x_i.

Now, we are ready to prove Theorem 2.1. We already know that $H^0(X,\Theta_X)=H^0(C,\varphi_*\mathcal{O}_X((n+1)E)\otimes\mathcal{N}^{-n})$. So, by the previous lemma and $\deg\mathcal{N}>0$, we have

$$H^0(X,\Theta_X)=\begin{cases}H^0(C, V\otimes\mathcal{N}^{-n}\oplus\mathcal{N}) & \text{if } n+1\equiv 0\ (\mathrm{mod}\, p)\,,\\ H^0(C, V\otimes\mathcal{N}^{-n}) & \text{otherwise}\,.\end{cases}$$

By identifying x_i with η_i, we can regard V as a subsheaf of $\boldsymbol{F}_*\mathcal{O}_{C'}$. Meanwhile $\boldsymbol{F}_*\mathcal{O}_{C'}\otimes\mathcal{N}^{-n}$ is isomorphic to $\mathcal{N}^{-np}$ as a sheaf of abelian groups. Hence $H^0(C, V\otimes\mathcal{N}^{-n})\subset H^0(C,\boldsymbol{F}_*\mathcal{O}_{C'}\otimes\mathcal{N}^{-n})=0$. Therefore, the proof of Theorem 2.1 is complete.

3. Differential 1-forms on generalized Raynaud surfaces. In this section, we consider the differential 1-forms and their closedness on generalized Raynaud surfaces. Lemma 2.2 implies that there is an exact sequence

$$0 \to \mathcal{O}_X((n-1)pE)\otimes\varphi^*\mathcal{N}^{(1-n)p}\otimes\varphi^*\Omega_C^1 \to \Omega_X^1 \to \mathcal{O}_X(-(n+1)E)\otimes\varphi^*\mathcal{N}^n \to 0\,.$$

Since $H^0(X, \mathcal{O}_X(-(n+1)E)\otimes\varphi^*\mathcal{N}^n)=0$, we have that $H^0(X, \Omega_X^1)=H^0(X, \mathcal{O}_X((n-1)pE)\otimes\varphi^*\mathcal{N}^{(1-n)p}\otimes\varphi^*\Omega_C^1)=H^0(C, \varphi_*\mathcal{O}_X((n-1)pE)\otimes\mathcal{N}^{(1-n)p}\otimes\Omega_C^1)$. On the other hand, by the same argument as in Lemma 2.4, we get the following Lemma:

LEMMA 3.1. *Retain the same notation and assumptions as above. Then we have*

$$\varphi_*\mathcal{O}_X((n-1)pE)=\bigoplus_{l=0}^{n-1}(W_l\otimes\mathcal{N}^{lp})\,,$$

where W_l is the locally free $\mathcal{O}_C$-module which is locally generated by $1, x_i, \dots, x_i^{[(n-l-1)p/n]}$.

From this lemma, it follows that

$$H^0(X, \Omega_X^1)=H^0\left(C, \bigoplus_{l=0}^{n-1}(W_l\otimes\mathcal{N}^{(l+1-n)p}\otimes\Omega_C^1)\right).$$

Recall that y_i is a local generator of $\mathcal{N}^p$. Hence we know that global 1-forms of X are locally of the form

$$\sum_{l=0}^{n-1}\sum_{m=0}^{[(n-l-1)p/n]} f_{lmi}x_i^m y_i^{-l}d\xi_i\,,$$

where f_{lmi}'s are local sections of $\mathcal{O}_C$. Since $y_i^{-(n-1)}d\xi_i=-ndy_i$ and since (x_i, y_i) is a local coordinate system of X, we see that a global 1-form is closed if and only if it is locally of the form $\sum_{l=0}^{n-1}f_{l0i}y_i^{-l}d\xi_i$ in the above expression. Therefore, we have:

THEOREM 3.2. *Let $\mathcal{Z}_X^1$ be the sheaf of closed 1-forms on X. Then*

$$\begin{aligned}H^0(X, \mathcal{Z}_X^1)&=H^0(C, \mathcal{N}^{-p(n-1)}\otimes\Omega_C^1\oplus\mathcal{N}^{-p(n-2)}\otimes\Omega_C^1\oplus\cdots\oplus\Omega_C^1)\\&\cong H^0(C, \mathcal{N}^p)\oplus H^0(C, \mathcal{N}^{2p})\oplus\cdots\oplus H^0(C, \mathcal{N}^{np})\,.\end{aligned}$$

COROLLARY 3.3. *In the same notation, we have*

$$\dim H^0(X, \Omega_X^1)-\dim H^0(X, \mathcal{Z}_X^1)\geqq\dim H^0(C, W_0\otimes\mathcal{N}^p)-\dim H^0(C, \mathcal{N}^p)\,.$$

When $n>p$, we have $W_0\cong F_*\mathcal{O}_{C'}$ by identifying x_i with η_i. Hence we have the following estimate:

COROLLARY 3.4. *When $n>p$, we have*

$$\dim H^0(X, \Omega_X^1)-\dim H^0(X, \mathcal{Z}_X^1)\geqq(p-1)(\dim H^0(C, \mathcal{N}^p)-1)\,.$$

PROOF. By assumption, we have $W_0\otimes\mathcal{N}^p\cong F_*\mathcal{O}_{C'}\otimes\mathcal{N}^p\cong\mathcal{N}^{p^2}$, where the last isomorphism is not as $\mathcal{O}_C$-modules, but as sheaves of abelian groups. Meanwhile, for invertible sheaves $\mathcal{L}$ and $\mathcal{M}$, we know that $\dim H^0(C, \mathcal{L}\otimes\mathcal{M})\geqq\dim H^0(C, \mathcal{L})+$

$\dim H^0(C, \mathscr{M})-1$. So, we have $\dim H^0(C, \mathscr{N}^{2p}) \geqq 2\dim H^0(C, \mathscr{N}^p)-1$ and so on.

q.e.d.

To close this article, we note that Example 1.2 is an example for which $\dim H^0(C, \mathscr{N}^p)-1$ is positive.

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