Vector mesons on the wall

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ABSTRACT: A domain-wall configuration of the η' meson bounded by a string (called a pancake or a Hall droplet) is recently proposed to describe the baryons with spin $N_c/2$. In order to understand its baryon number as well as the flavor quantum number, we argue that the vector mesons (the ρ and ω mesons) should play an essential role for the consistency of the whole picture. We determine the effective theory of large-N_c QCD with N_f massless fermions by taking into account a mixed anomaly involving the θ-periodicity and the global symmetry. The anomaly matching requires the presence of a dynamical domain wall on which a $\mathrm{U}(N_f)_{-N_c}$ Chern-Simons theory is supported. We consider the boundary conditions that should be imposed on the edge of the domain wall, and conclude that there should be a boundary term that couples the $\mathrm{U}(N_f)_{-N_c}$ gauge field to the vector mesons. We discuss the impact on physics of the chiral phase transition and the relation to the "duality" of QCD.

Contents

1 Introduction

The low energy effective theory of QCD in the limit of the massless N_f-flavor quarks is a non-linear sigma model corresponding to the breaking $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$. The 't Hooft anomaly [1] for $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V$ is matched by the Wess-Zumino-Witten (WZW) term [2, 3]. In the large N_c limit [4] of massless QCD, the axial symmetry $\mathrm{U}(1)_A$ is restored and the η' meson is regarded as the Nambu-Goldstone (NG) boson associated with the spontaneous $\mathrm{U}(1)_A$ breaking [5, 6]. The η' field is, therefore, described as a $\mathrm{U}(1)$-valued pseudo-scalar field. The WZW term can be extended to include η'. In the large N_c limit, the baryons should be described as solitons since the mass and the couplings of order N_c and $1/N_c$, respectively, are naturally explained [7, 8]. Indeed, the Skyrmion configurations [9] of the pions are identified as the baryons by the coupling between the external baryon gauge field and the topological charge in the WZW term.

Recently, the notion of the 't Hooft anomalies and their matching has been extended to the discrete symmetries [10, 11] and higher form symmetries [12]. One of the important results is given in ref. [13], where a mixed anomaly involving time reversal and the 1-form center symmetry is discussed in the $\mathrm{SU}(N_c)$ Yang-Mills theory at $\theta = \pi$. The anomaly implies that if the confinement persists at $\theta = \pi$, there are two degenerate vacua corresponding to the spontaneous breaking of time reversal (or CP), and the effective theory

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on a domain wall separating two vacua is an $\mathrm{SU}(N_c)_1$ Chern-Simons (CS) theory.[1] One can also consider the 't Hooft anomaly for symmetries in the parameter space of the theory [16, 17]. In QCD, there is a mixed anomaly involving the θ-periodicity and the global symmetry [17]. The matching of such anomalies provides new restrictions on the effective theory especially for topologically non-trivial objects.

In the effective theory of QCD including η', there are topologically non-trivial configurations of η' in addition to the Skyrmions. One of them is the domain-wall configuration which connects two minima of the periodic potential of η', where the potential is generated at the next-to-leading order of the $1/N_c$ expansion. Although this object is stable within the effective theory, it should not be so in full QCD since there is no corresponding conserved charges. The object to destabilize the domain wall can be identified as a string around which η' winds. Komargodski recently pointed out that the domain wall bounded by a string, which is referred to as a pancake or a Hall droplet, can have stable excited states that can be regarded as a spin-$N_c/2$ baryon [18]. This proposal is based on an expectation that the effective theory on the domain wall is a CS theory, which implies that there must be chiral edge modes on the boundary of the domain wall. Excited states on the edge with a unit U(1) charge can be regarded as baryons if we identify the U(1) symmetry of the chiral edge modes as the baryon symmetry. It has been proposed that this baryon can be understood as a chiral bag in a $1 + 2$ dimensional strip using the Cheshire Cat principle [19].

The existence of a CS theory on the domain wall is an expectation from the spontaneous CP breaking in the Yang-Mills theory at $\theta = \pi$. If we insert a interface of θ changing from 0 to 2π in the Yang-Mills theory, there appears a domain wall with a CS theory on it at the location of the interface. When massless quarks are added, the θ dependence is eliminated by a shift of η', and thus a domain wall where η' shifts from a minimum of the potential to another corresponds to the θ interface in the Yang-Mills theory. This means that it is expected that the domain wall of η' supports a CS theory on it. However, this explanation is rather heuristic and there are also proposals that such a CS theory is replaced by a topologically trivial gapped theory when light quarks are added to the Yang-Mills theory [20, 21].

The pancake story provides a new insight into the role of the vector mesons (ρ and ω) as the dual gluons [20–23] in the sense of the Seiberg duality in $\mathcal{N} = 1$ supersymmetric QCD [24]. The interpretation of the vector mesons as the Seiberg dual gauge boson has been discussed in literature [25–31] where the vector mesons are identified as the Higgsed magnetic gauge bosons. Indeed, the effective theory of the vector mesons can be formulated consistently as the gauge theory of a hidden local symmetry in the non-linear sigma model [32]. In this formulation, the WZW term can be extended to include the vector mesons by adding a set of gauge invariant operators [33]. The Lagrangian of the vector mesons on the domain wall is determined up to four parameters. It is certainly interesting to note that the postulated CS theory on the wall is $\mathrm{U}(N_f)_{-N_c}$ whereas the vector mesons

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[1]There are other logical options but an $\mathrm{SU}(N_c)_1$ is the most plausible one [14]. See also [15] for an alternative argument for the existence of the phase transition.

can also form a U(N_f) CS Higgs theory on the wall. It is natural to suspect that there is some relation between these two theories. Since one can formally interpret the U(N_f)$_{-N_c}$ CS theory on the wall as that of the (level-rank) dual gluon, there seems to be a deep connection between the hidden local symmetry and the Seiberg duality through the theory on the domain wall. The topics of the baryon number of the pancake objects, the topological quantum field theory on the wall, new 't Hooft anomalies, and the Seiberg duality seem to be all related, and it is quite important now to look for the consistent picture and determine the effective (or dual) theory.

In this paper, we determine the effective theory of pions and η' including the background θ term as well as background gauge fields of global symmetries, SU(N_f)$_L$ × SU(N_f)$_R$ × U(1)$_V$. We include the domain wall as the dynamical degree of freedom, that is necessary to reproduce the cusp in the η' potential. We show that some topological quantum field theory should live on the domain wall by using the anomaly matching involving the periodicity of θ. This confirms the heuristic explanation of the presence of the U(N_f)$_{-N_c}$ theory on the domain wall. We find that the anomalous coupling between η' and the background fields implies that the global U(1) symmetry of the CS theory is actually identified with the baryon symmetry, and the exited states with a unit U(1) charge belong to the correct representation of the flavor symmetry. In addition, we propose a specific form of the coupling between the vector meson fields and the gauge field of the CS theory on the edge of the pancake in order to recover the gauge invariance of the SU(N_f)$_L$ × SU(N_f)$_R$ gauge field, which must be maintained for consistency. From this effective theory, one can understand the real meaning of the duality between the gluon and the vector mesons. We discuss the implication of the behavior of the vector mesons near the chiral phase transition. The effective theory we derived is quite different from the proposals in refs. [20, 21], where the theory on the pancake is claimed to be a CS Higgs theory of the vector mesons. We argue that the CS theory, not CS Higgs theory, is required by the matching of the anomaly while the vector mesons mix with the gauge field of the CS theory on the edge of the pancake. There are important differences in the formula of the baryon and flavor currents. The existence of the CS theory on the wall will also be quite important in the discussion of the physics of the vector mesons such as the anomalous coupling among hadrons as well as how they approach to the chiral symmetric phase.

This paper is organized as follows: In section 3, we determine the effective theory for η' including the anomalous coupling to the background fields. With an appropriate counter term, the constant θ dependence is compensated by a shift of η'. However, we find that naive coupling between η' and the background fields satisfying this feature is not adequate when we consider a violation of the θ-periodicity, which is originated from the division part of the global symmetry [SU(N_f)$_L$ × SU(N_f)$_R$ × U(1)$_V$]/[$\mathbb{Z}_{N_c}$ × ($\mathbb{Z}_{N_f}$)$_V$]. This violation is actually an anomaly when $\gcd(N_c, N_F) \neq 1$ [17]. As a result, at least when $\gcd(N_c, N_f) \neq 1$, we find that there has to be some topological quantum field theory on the domain wall of η'. We further restrict the effective theory by considering the large-N_c argument by Witten [5], which states that the part of the effective theory only including glueballs are the same as that of the Yang-Mills theory. The domain wall we consider corresponds to the change of η' over one period. This corresponds to the shift of θ from

0 to $2\pi N_f$. As we mentioned above, a topological quantum field theory on the interface of θ from 0 to 2π is $\mathrm{SU}(N_c)_1$. When θ changes from 0 to $2\pi N_f$, the theory $(\mathrm{SU}(N_c)_1)^{N_f}$ can undergo a transition to another topological quantum field theory [14]. It has been suggested that the theory is $\mathrm{SU}(N_c)_{N_f}$ [13]. Due to the level-rank duality, an $\mathrm{SU}(N_c)_{N_f}$ CS theory is identified with a $\mathrm{U}(N_f)_{-N_c}$ CS theory. Thus we can expect that there is a $\mathrm{U}(N_f)_{-N_c}$ CS theory on the domain wall. In section 4, we determine the effective theory on the pancake when there is the background $\mathrm{U}(N_f)_V/\mathbb{Z}_{N_c}$ gauge field. We find that if we couple the gauge field of the CS theory to the background gauge field properly, then we can maintain the gauge invariance of the background gauge field. As a result, we confirm that an excited state with a unit $\mathrm{U}(1)$ charge belongs to the correct representation of the global symmetry as a spin-$N_c/2$ baryon. In section 5, we consider the coupling between the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ background gauge field and the domain wall. Unlike the case in section 4, the gauge field of the CS theory cannot be used to recover the gauge invariance of the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ gauge field. We propose that the vector mesons as the gauge field of the hidden local symmetry should couple to the domain wall instead of the background fields. Due to the mass mixing of the vector mesons and the background fields, the representation for a spin-$N_c/2$ baryon is correctly reproduced. Also, it is found that the vector meson cannot have nonzero instanton charge for a consistency. In section 6, physical implications of the coupling between the vector mesons and the domain wall are explained. The effective theory we derived provides us with new understanding of the "duality" between the gluon and the vector mesons.

2 Practice: a domain wall and a monopole

Before going into the technical detail, we start with a physical question in QCD where the baryon number symmetry, $\mathrm{U}(1)_B$, is weakly gauged. Let us consider a configuration where a magnetic monopole, whose magnetic charge under $\mathrm{U}(1)_B$ is unity, is surrounded by the domain wall that connects $\eta' = 0$ and $\eta' = 2\pi \sim 0$. The WZW term of QCD contains the following term:

$$\frac{i}{8\pi^2} \frac{N_f}{N_c} \eta' dA_B dA_B, \tag{2.1}$$

where A_B is the baryon gauge field where it is normalized such that the baryon has the charge unity. This term induces the Witten effect[2] [36] when η' changes from 0 to 2π. The electric charge, i.e., the baryon number, of this configuration is N_f/N_c. In the effective theory, such an object is not allowed by the Dirac quantization condition for general values of N_f and N_c.

This already provides an interesting puzzle. When we go back to QCD, the monopole with the unit magnetic charge is allowed by the Dirac quantization condition. For the first look it has a problem with the quarks which has baryon number $1/N_c$, but it is actually

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[2]The modern understanding of the Witten effect associated with the axion domain wall has been discussed in refs. [34, 35], where a higher-group structure is found among the higher form symmetries in the axion electrodynamics.

consistent by attaching the color magnetic flux to the monopole [37]. For consistency, the effective theory should also allow such a monopole to exist. But once it is allowed, since there is no gluon degrees of freedom in the effective theory, one cannot attach the color magnetic flux anymore, and the Dirac quantization condition seems to be just violated.

In order to fix this problem, we need something topologically nontrivial on the domain wall. The "something" should be either a term to cancel the Witten effect so that the configuration is allowed or some term to make the configuration impossible. We will see that QCD choose the latter by having a CS theory on the wall.

3 The effective theory for η'

We consider large-N_c QCD with N_f massless Dirac fermions. The (Euclidean) Lagrangian is

$$\mathcal{L} = \frac{N_c}{4\lambda^2} \operatorname{tr}(f \star f) + iN_c \bar{\psi} \slashed{D} \psi + i\theta \frac{1}{8\pi^2} \operatorname{tr} f^2. \tag{3.1}$$

At the leading order of the $1/N_c$ expansion, the $U(1)_A$ symmetry is recovered and we can write the effective action by using a $U(N_f)$-valued field U. We decompose U into $U(1)$ part and $SU(N_f)$ part as $U = e^{i\eta'} \widehat{U}$, $\widehat{U} \in SU(N_f)$, and focus on η'. This decomposition is not determined uniquely, but if we fix the pion field $\widehat{U}$, this ambiguity is not relevant.

When we add small masses to quarks, the effective Lagrangian should include a term:

$$\frac{f_\pi^2 \Lambda}{2} \operatorname{tr}(MU + M^\dagger U^\dagger), \tag{3.2}$$

where M is the $N_f \times N_f$ mass matrix and Λ is the scale of the strong interaction. In the following, we discuss the effective theory of QCD in the massless limit. The correction by the quark masses can always be taken into account by adding eq. (3) as a small perturbation.[3]

It is known that, to the next to leading order of $1/N_c$ expansion, the effective Lagrangian for η' is

$$\mathcal{L}_{\eta'}^{\text{eff}} = \frac{N_f f_\pi^2}{8} d\eta' \star d\eta' + \frac{f_\pi^2}{8N_f} m_{\eta'}^2 \min_{n \in \mathbb{Z}} (N_f \eta' + \theta - 2\pi n)^2, \tag{3.3}$$

where the N_c dependence of the parameters is given as

$$f_\pi^2 = \mathcal{O}(N_c), \quad m_{\eta'}^2 = \mathcal{O}(1/N_c). \tag{3.4}$$

Here the normalization of η' is chosen so that, under $U(1)_A$ transformation $\psi \to \exp(i\alpha\gamma^5)\psi$, it transforms as $\eta' \to \eta' + 2\alpha$. For this normalization, η' is 2π-periodic. The coefficients are determined so as to reproduce the Witten-Veneziano formula [5, 6]

$$m_{\eta'}^2 = \frac{4N_f}{f_\pi^2} \chi, \tag{3.5}$$

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[3]At $\theta = \pi$, with finite quark masses, m, the vacuum structure in the large N_c limit actually depends on the order of two limits, $m \to 0$ and $1/N_c \to 0$ [38]. Our discussion of the massless quarks corresponds to $m \to 0$ first, where the η' mass is larger than $m\Lambda$. However, the domain wall of our interest which connects $\eta' = \eta'_0$ and $\eta'_0 + 2\pi$ always exists independent of the vacuum structure, and the discussion of the theories on the domain wall and on its boundary are unchanged even in the other limit since the effective theory is constructed only based on anomaly matching and the gauge invariance.

where χ is the topological susceptibility in the $\mathrm{SU}(N_c)$ Yang-Mills theory. The periodicity and $1/N_c$ expansion imply that the η' potential has a cusp at $N_f \eta' + \theta = n\pi$ with n odd integers.

We consider how to couple the background gauge fields to the effective theory. The global symmetry of the theory is $[\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V]/[\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_V]$ [39]. This is because the group with the faithful action on the fermion field is $[\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V]/[\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_V]$ rather than the simple direct product $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V$. In this section, we focus on the subgroup

$$G_{\mathrm{sub}} := [\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_V]/[\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_V], \tag{3.6}$$

and couple the corresponding background fields to QCD.

3.1 Anomaly constraint

Recently, it is shown that QCD has an anomaly involving the θ-periodicity and the global symmetry [17]. We can use this anomaly to restrict the coupling between the background fields and η' at large N_c.

In the following discussion, the division part of G_{sub} plays the main role. In order to express the division part we should introduce the 2-form background gauge fields for the $\mathbb{Z}_{N_c}$ and $\mathbb{Z}_{N_f}$ groups (see refs. [13, 39]). A 2-form $\mathbb{Z}_{N_c}$ gauge field is realized as a pair of a 2-form $\mathrm{U}(1)$ gauge field and a 1-form $\mathrm{U}(1)$ gauge field, $(B_c^{(2)}, \widehat{A}_c)$, that satisfies a constraint $N_c B_c^{(2)} = d\widehat{A}_c$, where the normalization of $\widehat{A}_c$ is given as $\int_\Sigma d\widehat{A}_c \in 2\pi\mathbb{Z}$ for a closed 2-surface Σ. They transform under a 1-form $\mathrm{U}(1)$ gauge transformation as

$$B_c^{(2)} \to B_c^{(2)} + d\lambda_c^{(1)}, \quad \widehat{A}_c \to \widehat{A}_c + N_c \lambda_c^{(1)}, \tag{3.7}$$

where $\lambda_c^{(1)}$ is an arbitrary 1-form $\mathrm{U}(1)$ gauge field whose normalization is given as $\int_\Sigma d\lambda_c^{(1)} \in 2\pi\mathbb{Z}$. As the gluon degrees of freedom, we introduce a dynamical $\mathrm{U}(N_c)$ gauge field $\widetilde{a}$ instead of the $\mathrm{SU}(N_c)$ gauge field. To eliminate the extra degrees of freedom, we impose a constraint on the field strength, $\widetilde{f}$, as

$$\operatorname{tr} \widetilde{f} = d\widehat{A}_c. \tag{3.8}$$

Under the 1-form $\mathrm{U}(1)$ gauge transformation in eq. (3.7), $\widetilde{a}$ transforms as

$$\widetilde{a} \to \widetilde{a} + \lambda_c^{(1)} \mathbf{1} \tag{3.9}$$

to maintain the constraint. The field strength f for the $\mathrm{SU}(N_c)$ gauge field in the action is replaced by

$$\widetilde{f} - B_c^{(2)} \mathbf{1}. \tag{3.10}$$

Note that it is a 1-form gauge invariant $\mathfrak{su}(N_c)$-valued 2-form locally.

In the same way, we introduce a 2-form $\mathbb{Z}_{N_f}$ gauge field $(B_f^{(2)}, \widehat{A}_f)$, and a 1-form $\mathrm{U}(N_f)$ gauge field $\widetilde{A}_f$ that satisfy a constraint

$$\operatorname{tr} \widetilde{F}_f = d\widehat{A}_f. \tag{3.11}$$

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They transform under a 1-form gauge transformation as

$$\widetilde{A}_f \to \widetilde{A}_f + \lambda_f^{(1)}\mathbf{1}, \quad B_f^{(2)} \to B_f^{(2)} + d\lambda_f^{(1)}, \quad \widehat{A}_f \to \widehat{A}_f + N_f\lambda_f^{(1)}. \tag{3.12}$$

Let $\widetilde{A}_V$ be the $\mathrm{U}(1)_V$ gauge field, which transforms under the 1-form gauge transformations as

$$\widetilde{A}_V \to \widetilde{A}_V - \lambda_c^{(1)} - \lambda_f^{(1)}. \tag{3.13}$$

Using these fields, the covariant derivative is defined as

$$D\psi = (d - i\widetilde{a} - i\widetilde{A}_f - i\widetilde{A}_V)\psi, \tag{3.14}$$

which is invariant under the 1-form gauge transformations. We define "the instanton charge densities" as 1-form gauge invariant quantities,

$$q_c := \frac{1}{8\pi^2}\left(\mathrm{tr}(\widetilde{f}^2) - \frac{1}{N_c}(d\widehat{A}_c)^2\right), \qquad q_f := \frac{1}{8\pi^2}\left(\mathrm{tr}(\widetilde{F}_f^2) - \frac{1}{N_f}(d\widehat{A}_f)^2\right),$$

$$q_V := \frac{1}{8\pi^2}\left(d\widetilde{A}_V + \frac{1}{N_c}d\widehat{A}_c + \frac{1}{N_f}d\widehat{A}_f\right)^2. \tag{3.15}$$

In order for a $\mathrm{U}(1)_A$ transformation $\psi \to \exp(i\alpha\gamma^5)\psi$ to compensate a constant shift of θ, we introduce the counter term

$$\mathcal{L}_c = i\theta\frac{1}{N_f}(N_c q_f + N_c N_f q_V). \tag{3.16}$$

Because a constant shift of θ is compensated by a shift of η', we should add the following term to the effective Lagrangian:

$$\mathcal{L}_{\mathrm{topo}}^{\mathrm{eff}} = i(N_f\eta' + \theta)\frac{1}{N_f}(N_c q_f + N_c N_f q_V). \tag{3.17}$$

Note that under the $\mathrm{U}(1)_A$ transformation $\psi \to \exp(i\alpha\gamma^5)\psi$, η' changes as $\eta' \to \eta' + 2\alpha$. However, as we will see in the following, this term alone added by eq. (3.3) cannot be the correct effective theory. We need more terms in the effective Lagrangian.

Because of the division part of the symmetry, we can find that full QCD has the following properties that should be maintained in the effective theory:

- *Fractional instanton charges.* The instanton charges can have fractional values. The values of the instanton charges are restricted as [40]

$$Q_c := \int q_c = -\frac{m_c m_c'}{N_c} + l_c, \qquad Q_f := \int q_f = -\frac{m_f m_f'}{N_f} + l_f,$$

$$Q_V := \int q_V = \left(\frac{m_c}{N_c} + \frac{m_f}{N_f} + l_V\right)\left(\frac{m_c'}{N_c} + \frac{m_f'}{N_f} + l_V'\right),$$

$$m_{c,f}, m_{c,f}' = 0, \dots, N_{c,f} - 1, \quad l_c, l_f, l_V, l_V' \in \mathbb{Z}. \tag{3.18}$$

Note that $m_c, m_c', m_f, m_f', l_f, l_V, l_V'$ are determined by the background fields. Only l_c depends on the dynamical field.

- $Z(\theta) = 0$ *for some background fields.* For some background fields, the partition function $Z(\theta)$ becomes zero. This is because the index of the Dirac operator is

$$\begin{aligned}
\operatorname{ind} i\slashed{D} &= N_f Q_c + N_c Q_f + N_c N_f Q_V \\
&= m_c m'_f + m'_c m_f + (N_f m_c + N_c m_f) l'_V + (N_f m'_c + N_c m'_f) l_V \\
&\quad + N_c l_f + N_f l_c + N_c N_f l_V l'_V \\
&= l_I + N_f (m_c l'_V + m'_c l_V + N_c l_V l'_V) + N_f l_c,
\end{aligned}$$

$$l_I := m_c m'_f + m'_c m_f + N_c(m_f l'_V + m'_f l_V + l_f). \tag{3.19}$$

Again note that only l_c depends on the dynamical fields and other integers are determined by the external fields. If l_I is not an integer multiple of N_f, the index is nonzero for all values of l_c, and thus in this case the partition function is zero. Conversely, when l_I is an integer multiple of N_f (for example $\widetilde{A}_f = \widehat{A}_f = 0$), the partition function is in general non-vanishing.

- *The anomaly between the θ-periodicity and G_{sub}.* Under the 2π shift of θ, the unnormalized expectation value $\langle X[\psi] \rangle_\theta$ of some operator $X[\psi]$ transforms as

$$\langle X[\psi] \rangle_{\theta + 2\pi} = \exp\left(2\pi i \frac{l_I}{N_f}\right) \langle X[\psi] \rangle_\theta, \tag{3.20}$$

for the choice of the counter term eq. (3.16). This violation of the 2π-periodicity of θ is actually anomaly if the greatest common divisor $L := \gcd(N_c, N_f)$ is not equal to 1, because in this case this phase shift cannot be eliminated by adding a counter term to the Lagrangian as shown in ref. [17].

Because we choose the counter term so that a constant shift of θ is compensated by a $\mathrm{U}(1)_A$ transformation, the $(\mathbb{Z}_{2N_f})_A$ transformation gives the same phase shift as

$$\left\langle X\left[\exp\left(-i\frac{2\pi}{2N_f}\gamma^5\right)\psi\right]\right\rangle_\theta = \exp\left(2\pi i \frac{l_I}{N_f}\right) \langle X[\psi] \rangle_\theta. \tag{3.21}$$

This means, when l_I is not an integer multiple of N_f, a non-vanishing path integral requires an appropriate insertion of fermion operators such as $(\bar{\psi}\psi)^{l_I}$. This is actually the anomaly between $(\mathbb{Z}_{N_f})_L$ and G_{sub} [39]. Note that for a different choice of the counter term, the 2π-shift of θ gives a different phase shift, and a constant θ shift cannot be compensated by a $\mathrm{U}(1)_A$ transformation.

We can see that the effective Lagrangian in eq. (3.17) added by eq. (3.3) is not satisfactory. By using the background with fractional instanton charges, that should be allowed, the partition functions and the θ dependence above do not quite match. Let us take l_I as an integer multiple of N_f. As we have seen, the partition function, $Z(\theta)$, should not vanish in this background. In the effective theory, $Z(\theta)$ is obtained as the path integral of η'. By changing the variable as $\eta' \to \widetilde{\eta}' = \eta' + 2\pi/N_f$, which leaves the kinetic and the mass terms invariant, we obtain

$$Z(\theta) = e^{-2\pi i \frac{m_c m'_c}{N_c}} Z(\theta), \tag{3.22}$$

from eq. (3.17). When $m_c m'_c \neq 0$, the phase is not unity, which implies $Z(\theta) = 0$.

The property of the 2π shift of θ is also not maintained. Because of the term (3.16), the 2π-shift of θ gives the phase $\exp(2\pi i(l_I/N_f + m_c m_c'/N_c))$, which is different from eq. (3.20).

In order to avoid these pathologies, we should add some term to the effective Lagrangian. It is expected that the term contains a 4-form $C^{(4)}$ that depends only on the background fields and satisfies

$$\int C^{(4)} = \frac{m_c m_c'}{N_c} \quad \mathrm{mod}\ 1. \tag{3.23}$$

For example, the term

$$-(N_f \eta' + \theta)C^{(4)} \tag{3.24}$$

seems to fix the pathologies. However, we cannot write this type of terms at least if $L := \gcd(N_c, N_f) \neq 1$. This is because, $C^{(4)}$ that satisfies the condition (3.23) cannot be written as a field that only contains the background fields and is invariant under the 1-form gauge transformations (3.7), (3.12) and (3.13) if $L \neq 1$ [17]. Since the original QCD Lagrangian is invariant under the 1-form gauge transformation, the effective theory must be invariant.

We show that $C^{(4)}$ cannot be written as a 1-form gauge invariant field by following ref. [17]. It is enough to show that, if $L \neq 1$, there are no solutions of the equation for variables s and t,

$$Q_f s + Q_V t = \frac{m_c m_c'}{N_c} \quad \mathrm{mod}\ 1. \tag{3.25}$$

Note that s and t do not depend on m_c, m_c', m_f, m_f', l_c, l_f or l_V, because these quantities only defined as the integrals over the whole space, but $C^{(4)}$ is a local quantity. In order to eliminate l_f, l_V and l_V', the variable s has to be an integer and t has to be an integer multiple of $N_c N_f / L$. Then the equation becomes

$$\frac{m_c m_c'}{N_c}\left(-1 + \frac{t}{N_c}\right) + \frac{m_f m_f'}{N_f}\left(-s + \frac{t}{N_f}\right) + \frac{m_c m_f' + m_c' m_f}{N_c N_f}t = 0 \quad \mathrm{mod}\ 1. \tag{3.26}$$

It follows from this that

$$-\frac{1}{N_c} + \frac{t}{N_c^2} = R \in \mathbb{Z}, \quad -\frac{s}{N_f} + \frac{t}{N_f^2} = P \in \mathbb{Z}, \quad \frac{t}{N_c N_f} = J \in \mathbb{Z}. \tag{3.27}$$

The first and last equation imply

$$1 = N_f J - N_c R. \tag{3.28}$$

Because the right-hand side is an integer multiple of L, there is a solution only if $L = 1$. Therefore, if $L \neq 1$ we cannot write $C^{(4)}$ as a 1-form gauge invariant field.

Since the coupling between the background fields and η' cannot solve the problems, we need to extend the effective theory to include more dynamical degrees of freedom. In the following we introduce a dynamical domain wall and a CS theory on it. In the potential

for η' in eq. (3.3), there are jumps between different branches. In each branch labeled by an integer n, the potential for η' is given as

$$\frac{f_\pi^2}{2} m_{\eta'}^2 (N_f \eta' + \theta - 2\pi n)^2. \tag{3.29}$$

The jump between branches indicates that a dynamical domain wall (made of some heavy degrees of freedom) attaches at the location where n changes. The location is dynamically chosen such that the energy is minimized. We propose that in the effective action, there is the term

$$-2\pi i \int n C^{(4)}, \tag{3.30}$$

where n takes different values on the each side separated by the domain wall. The action is accompanied by the term on the world volume,

$$i \int_{\partial X_4} c^{(3)}, \tag{3.31}$$

where X_4 is the interior of the domain wall, and $c^{(3)}$ should be some dynamical degree of freedom on the wall which is necessary to maintain the gauge invariance. The value of n inside X_4 is one greater than the value outside X_4.

One can find that the problems are solved as follows. If we change the variable in the path integral as $\eta' \to \widetilde{\eta}' = \eta' + 2\pi/N_f$, we should also change n as $n \to \widetilde{n} = n+1$ in order for the potential in eq. (3.29) to be unchanged. Due to the contribution from eq. (3.30), the unwanted phase in eq. (3.22) is cancelled. The phase shift eq. (3.20) is also reproduced in the same way.

The condition (3.23) does not determine the form of $C^{(4)}$ and $c^{(3)}$ uniquely, but the simplest possibility is to set

$$C^{(4)} = \frac{N_c}{8\pi^2} (B_c^{(2)})^2, \tag{3.32}$$

and introduce $c^{(3)}$ as the $\mathrm{U}(1)_{-N_c}$ CS theory,

$$i \int_{\partial X_4} c^{(3)} = -i \frac{1}{4\pi} \int_{\partial X_4} (N_c c dc + 2 c d\widehat{A}_c), \tag{3.33}$$

where c transforms as $c \to c - \lambda_c^{(1)}$ and is normalized as $\int_\Sigma dc \in 2\pi\mathbb{Z}$. The normalization condition is necessary for the CS action to be well-defined. With this choice, the whole action is invariant under the 1-form transformation. This provides a consistent effective theory.

Generally, if $C^{(4)}$ is given as eq. (3.32), a theory on the domain wall has to be able to couple to the 2-form $\mathbb{Z}_{N_c}$ gauge field $(B_c^{(2)}, \widehat{A}_c)$, but it cannot be 1-form gauge invariant. This means that the theory has to have an anomaly for a $\mathbb{Z}_{N_c}$ 1-form symmetry. In other words, the path integral of the theory on the wall is defined by the combination with the four-dimensional symmetry protected topological (SPT) phase (3.32), and it depends on how to take the four-dimensional space X_4. If the domain wall separates the spacetime

into two regions X_4 and X_4', the difference between the two choices is the integral over the whole space,

$$- i \frac{N_c}{4\pi} \left(\int_{X_4} (B_c^{(2)})^2 - \int_{X_4'} (B_c^{(2)})^2 \right)$$

$$= -i \frac{N_c}{4\pi} \int_{\text{whole space}} (B_c^{(2)})^2 = -2\pi i \frac{m_c m_c'}{N_c} \mod 2\pi i. \qquad (3.34)$$

Thus, the dependence on the choice of X_4 is characterized by an element of $\mathbb{Z}_{N_c}$ in this case.

In the next subsection, we will see that eq. (3.32) is the most plausible form of $C^{(4)}$, but logically, there are other possibilities of $C^{(4)}$, e.g.,

$$C^{(4)} = -\frac{R}{L} \frac{N_c^2}{8\pi^2} (B_c^{(2)})^2 - \frac{J}{L} \frac{N_c N_f}{4\pi^2} B_c^{(2)} (d\widetilde{A}_V + B_f^{(2)}) + \frac{J N_c N_f}{L} q_V + \frac{J N_c}{L} q_f, \qquad (3.35)$$

where J and R are integers that satisfy

$$J N_f - R N_c = L. \qquad (3.36)$$

For this $C^{(4)}$, a theory on the domain wall has to have an anomaly that is controlled by the first two terms in eq. (3.35). Note that the last two terms in eq. (3.35) is 1-form gauge invariant. The difference between two choices of the interior is calculated as

$$-2\pi i \int \left[-\frac{R}{L} \frac{N_c^2}{8\pi^2} (B_c^{(2)})^2 - \frac{J}{L} \frac{N_c N_f}{4\pi^2} B_c^{(2)} (d\widetilde{A}_V + B_f^{(2)}) \right] = -2\pi i \frac{m}{L} \mod 2\pi i,$$

$$m := -R m_c m_c' - J(m_c m_f' + m_c' m_f). \qquad (3.37)$$

Here m can be any integer because R and J are relatively prime. Thus we find that the theory on the wall has a $\mathbb{Z}_L$ anomaly for this $C^{(4)}$, which is different from the case (3.32) where a theory on the wall has a $\mathbb{Z}_{N_c}$ anomaly. It is hard to believe that the theory on the domain wall depends on L, but we cannot deny this possibility only from the anomaly argument. Especially, if $L = 1$, we cannot show even the necessity of dynamical degrees of freedom on the wall. In this case, $C^{(4)}$ can be written only using 1-form gauge invariant background fields as

$$C^{(4)} = J N_c N_f q_V + J N_c q_f. \qquad (3.38)$$

If $C^{(4)}$ is actually this form, there does not need to be dynamical degrees of freedom on the wall.

When we consider the situation that the $N_f \eta' + \theta$ changes rapidly from 0 to $2\pi k$, there are additional possibilities of the theory on the wall other than the theory obtained by simply adding k copies of the same theory. In this case, the theory has to have an anomaly controlled by

$$-2\pi i k \int_{X_4} C^{(4)}. \qquad (3.39)$$

For example, if $C^{(4)}$ is eq. (3.32), the theory on the wall can be an $SU(N_c)_k$ CS theory rather than $(U(1)_{-N_c})^k$.

In summary, the effective action for the domain wall corresponding to $2\pi k$ shift of $N_f \eta' + \theta$ is

$$i\frac{2\pi k}{N_f} \int_{X_4} (N_c q_f + N_c N_f q_v - N_f C^{(4)}) + i \int_{\partial X_4} c_k^{(3)}, \tag{3.40}$$

where X_4 is an interior of the domain wall, $C^{(4)}$ is a background 4-form field that satisfies the condition (3.23), and $c_k^{(3)}$ is a dynamical 3-form field for which the theory $i \int c_k^{(3)}$ has an anomaly controlled by the bulk action $-2\pi i k \int C^{(4)}$.

3.2 Additional large-N_c argument

Using a large-N_c argument, we further restrict the effective action. As discussed in ref. [5], at the leading order of the $1/N_c$ expansion, the terms in the effective action for QCD including only glueballs are the same as that of the Yang-Mills theory except for the replacement of θ by $N_f \eta' + \theta$. Now we consider the effect of the background fields. The only background fields that affect the gluon field f is $B_c^{(2)}$. If we focus on the gluon part, $B_c^{(2)}$ can be regarded as the background 2-form gauge field for the 1-form $\mathbb{Z}_{N_c}$ symmetry in the Yang-Mills theory. It is known that, in the Yang-Mills theory with θ parameter, there is a domain wall where θ crosses π, and the effective theory on the domain wall has an anomaly controlled by eq. (3.23) (if the spacetime is a spin manifold) [13]. Therefore we can conclude that eq. (3.23) is the correct form of $C^{(4)}$ irrespective of $\gcd(N_c, N_f)$. In addition, it is suggested in ref. [13] that, in the Yang-Mills theory, if θ changes rapidly from 0 to $2\pi k$, the theory on the wall is $SU(N_c)_k$. This expectation comes from an analogy with the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. Alternatively, by considering that the theory on the wall is $SU(N_c)_k$ in the UV, we can expect the same theory at low energy if we assume that the confinement in the bulk does not affect the theory on the wall.

According to this argument, we conclude that a theory on the domain wall corresponding to $2\pi k$ shift of $N_f \eta' + \theta$ is an $SU(N_c)_k$ CS theory. Because of the level-rank duality, an $SU(N_c)_k$ CS theory is identified with a $U(k)_{-N_c}$ CS theory. Therefore, in the rest of this paper, we use

$$C^{(4)} = \frac{N_c}{8\pi^2}(B_c^{(2)})^2, \quad c_k^{(3)} = -\frac{1}{4\pi}\left[N_c \operatorname{tr}\left(cdc - i\frac{2}{3}c^3\right) + 2\operatorname{tr}(c)d\widehat{A}_c\right], \tag{3.41}$$

where c is a 1-form $U(k)$ gauge field and transforms under the 1-form gauge transformation as

$$c \to c - \lambda_c^{(1)}\mathbf{1}. \tag{3.42}$$

3.3 Resolution of the puzzle

We can see the resolution of the puzzle in section 2. A monopole surrounded by the domain wall gives the integration of the magnetic flux over a time slice X_2 of the wall as

$$\frac{1}{2\pi} \int_{X_2} \left(d\widetilde{A}_V + \frac{1}{N_c}d\widehat{A}_c + \frac{1}{N_f}\widehat{A}_f\right) = \frac{1}{N_c}, \tag{3.43}$$
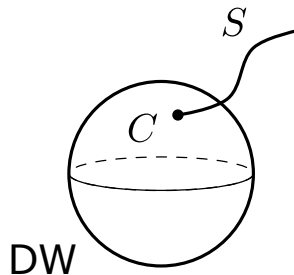
Figure 1. A time slice of the domain wall with the operator (3.45). The surface S is infinitely long or ended at another quark representing the color-electric flux tube.

which means $\widehat{A}_c$ is non-vanishing on the wall. By the Gauss law constraint on the wall from eq. (3.41), $dc - ic^2 = -d\widehat{A}_c \mathbf{1}/N_c$, c is forced to have the magnetic flux,

$$\frac{1}{2\pi} \int_{X_2} d\operatorname{tr}(c) = -\frac{N_f}{2\pi N_c} \int_{X_2} d\widehat{A}_c = -\frac{N_f}{N_c} \mod 1, \tag{3.44}$$

which does not satisfy the quantization condition $\int_{\Sigma} d\operatorname{tr}(c) \in 2\pi\mathbb{Z}$. Such a configuration is not allowed in accordance with the discussion of the Witten effect and the Dirac quantization.

The monopole can only be surrounded by the domain wall on which appropriate "quark" operators are inserted. An example of such an operator is [41]

$$\exp\left(i\int_C \operatorname{tr} c\right) \exp\left(i\int_S B_c^{(2)}\right), \tag{3.45}$$

where C is a contour on the domain wall, and S is a two-dimensional surface whose boundary contains C. The first part of the operator is the Wilson loop in a charge-N_f representation of the U(1) subgroup. Let C intersect with each time slice of the domain wall only once. It is known that in the CS theory, the insertion of the Wilson loop causes the same effect as the singular gauge transformation [42]. In this case, the operator corresponds to the gauge transformation by $g(\varphi) = \exp(i\varphi\mathbf{1}/N_c)$ with an angular variable φ around C. Thus, when we insert the operator (3.45), the Gauss law constraint becomes

$$dc^g - i(c^g)^2 = -\frac{1}{N_c}d\widehat{A}_c\mathbf{1}, \qquad c^g := gcg^{-1} + igdg^{-1}, \tag{3.46}$$

where we only include the spatial components. Due to the singularity, the integral of $d\operatorname{tr}(c^g)$ should be evaluated over $X_2 \setminus D_\varepsilon$ with an infinitely small subregion D_ε around C as

$$\frac{1}{2\pi} \int_{X_2 \setminus D_\varepsilon} d\operatorname{tr}(c^g) = -\frac{1}{2\pi} \int_{\partial D_\varepsilon} d\operatorname{tr}(igdg^{-1}) \mod 1 = -\frac{N_f}{N_c} \mod 1. \tag{3.47}$$

Thus the Gauss law constraint can be satisfied.

The operator (3.45) itself does not have the baryon charge. However, the whole object has the baryon charge N_f/N_c due to the Witten effect.

In full QCD, the field $B_c^{(2)}$ in the definition of the operator (3.45) represents the color-electric flux tube emerging from the quark. Because C intersects with each time slice of the

domain wall only once, the surface S has to be infinitely long or ended at another quark. See eq. (1). The color electric flux can also be understood as the one generated by the Witten effect since the monopole sources the color magnetic flux.

4 The effective theory on the pancake

The shift of η' by $2\pi k/N_f$ for an integer k can be reversed by shifting U while keeping $\det U$ fixed, where U is the $\mathrm{U}(N_f)$-valued field to describe the pions and η'. As a consequence, the domain wall corresponding to $2\pi k$ shift of $N_f\eta' + \theta$ could be bounded by a string. At the core of the string, the chiral symmetry has to be recovered because otherwise the energy diverges there due to the kinetic term for U.

In this paper, we mainly consider domain walls corresponding to $2\pi N_f$ shift of $N_f\eta' + \theta$ bounded by a string around which η' has the monodromy $\eta' \to \eta' + 2\pi$. We call such an object a pancake. For this type of domain walls, the pion field does not need to change because $\eta' = \eta_0'$ and $\eta' = \eta_0' + 2\pi$ is the same point in the target space of U.

In ref. [18], it is proposed that a pancake can be regarded as a baryon. In this paper, we clarify how to couple the background fields to pancakes and confirm that the objects belong to an appropriate representation.

In the previous section, we have found that, in general, the effective theory on the domain wall depends on the choice of the interior as eq. (3.40). We cannot consider that such a domain wall breaks and has a boundary. However, when $k = N_f$, the wall theory does not depend on the choice of the interior because

$$2\pi i \int (N_c q_f + N_c N_f q_V - N_f C^{(4)}) \in 2\pi i\mathbb{Z}, \tag{4.1}$$

where the integral is performed over the whole spacetime. This consists with the fact that $\eta' = 0$ and $\eta' = 2\pi$ represents the same vacuum when $\theta = 0$. Then the action of the theory on the domain wall for $k = N_f$ reduces to

$$i\frac{1}{4\pi}\int_{Y_3}\left[-N_c\operatorname{tr}\left(cdc - i\frac{2}{3}c^3\right) + N_c\operatorname{tr}\left(\mathcal{A}_f d\mathcal{A}_f - i\frac{2}{3}(\mathcal{A}_f)^3\right) + 2\left(\operatorname{tr}(\mathcal{A}_f) - \operatorname{tr}(c)\right)d\widehat{A}_c\right], \tag{4.2}$$

where we define a $\mathrm{U}(N_f)$ gauge field as

$$\mathcal{A}_f := \widetilde{A}_f + \widetilde{A}_V\mathbf{1}. \tag{4.3}$$

Actually, this is the gauge field for the $\mathrm{U}(N_f)_V$ global symmetry. This field transforms under the 1-form gauge transformation as

$$\mathcal{A}_f \to \mathcal{A}_f - \lambda_c^{(1)}\mathbf{1}. \tag{4.4}$$

We can check that this action is invariant under the zero-form and 1-form gauge transformations if Y_3 is a closed manifold.

For a domain wall bounded by strings, the action (4.2) is not gauge invariant. In order to recover the gauge invariance we need to modify the action on the boundary. Because

c is a dynamical field only on the domain wall, we can impose a boundary condition. On the other hand, since $\mathcal{A}_f$ and $\widehat{A}_c$ are background gauge fields living in the four dimensional spacetime, we cannot impose a boundary condition. A simple way to maintain the gauge invariance for the background fields is to add the boundary term

$$-i\frac{1}{4\pi}\int_{\partial Y_3} N_c \operatorname{tr}(\mathcal{A}_f c). \tag{4.5}$$

For this action, we can check the 1-form gauge invariance immediately. To check the zero-form gauge invariance, we should consider a boundary condition for c. Because a boundary condition should be imposed so that there are no boundary correction to the equation of motion, the condition is that one of the components of $c - \mathcal{A}_f$ is zero. Note that this condition is 1-form gauge invariant. To maintain the Lorentz invariance, the choice of a component that is imposed to be zero is restricted. We decompose the gauge fields such as $c = c_t dt + \widetilde{c}$, and let t denote such a component, i.e., we impose

$$c_t - \mathcal{A}_t^f = 0. \tag{4.6}$$

This condition implies that c transforms under the $\mathrm{U}(N_f)_V$ transformation in the same way as $\mathcal{A}_f$ on the boundary. Then we can check that the action is invariant under the zero-form gauge transformations.

Due to the boundary condition, the theory on the wall is equivalent to the one with only including dynamical edge degrees of freedom, that is the chiral version of the Wess-Zumino-Witten (WZW) model [43]. In the following, we set $\widehat{A}_c = 0$. By integrating out c_t we obtain the Gauss law constraint as

$$\widetilde{f}_c = 0. \tag{4.7}$$

The solution of this constraint is

$$\widetilde{c} = iW\widetilde{d}W^{-1}, \quad W \in \mathrm{U}(N_f). \tag{4.8}$$

Then the action reduces to

$$\begin{aligned}
S_p &= i\frac{N_c}{4\pi}\int_{\partial Y_3} \operatorname{tr}\left(W\widetilde{d}W^{-1}W\partial_t W^{-1}dt + 2iW\widetilde{d}W^{-1}\mathcal{A}_t^f dt - \widetilde{\mathcal{A}}_f \mathcal{A}_t^f dt\right) \\
&\quad + i\frac{N_c}{4\pi}\int_{Y_3} \operatorname{tr}\left(-\frac{1}{3}(WdW^{-1})^3 + \mathcal{A}_f d\mathcal{A}_f - i\frac{2}{3}\mathcal{A}_f^3\right).
\end{aligned} \tag{4.9}$$

Note that under the $\mathrm{U}(N_f)_V$ gauge transformation, the fields transform as

$$\mathcal{A}_f \to g_f \mathcal{A}_f g_f^{-1} + i g_f dg_f^{-1}, \quad W \to g_f W, \quad g_f \in \mathrm{U}(N_f)_V, \tag{4.10}$$

on the boundary.

We can read off the baryon charge B from the action (4.9),

$$B = \frac{1}{2\pi}\int_{\partial Y_2} \operatorname{tr}(\mathcal{A}_f - iWdW^{-1}) - \frac{1}{2\pi}\int_{Y_2} \operatorname{tr}(\mathcal{F}_f), \tag{4.11}$$

where Y_2 is the slice of Y_3 at a fixed t. There is no contribution from the external fields when we take a gauge such that no Dirac string penetrates the pancake due to the cancellation

between the bulk and boundary contributions. Therefore the baryon number is the winding number of W around the slice corresponding to $\pi_1(\mathrm{U}(N_f)) = \mathbb{Z}$.

It has been proposed that the operator $\rho(W)$ for the $\mathrm{Sym}^{N_c}(\Box)$ representation ρ describes a baryon with spin $N_c/2$ [18]. This operator actually belongs to the correct representation of spin $N_c/2$ baryons because under the $\mathrm{U}(N_f)_V$ transformation, it transforms as

$$\rho(W) \to \rho(g_f)\rho(W), \tag{4.12}$$

which means that the corresponding state has baryon number one, and belongs to the representation $\mathrm{Sym}^{N_c}(\Box)$ of $\mathrm{SU}(N_f)_V$.

5 The role of the vector mesons

Now we obtain the consistent effective theory and boundary conditions on the wall. We have seen that we obtained the consistent picture by coupling c to the external field $\mathcal{A}_f$ on the boundary. Here, we try to recover all the background fields A_L and A_R for the $\mathrm{SU}(N_f)_L$ and $\mathrm{SU}(N_f)_R$ symmetry. Once they are recovered, the vectorial part of the external field $\mathcal{A}_f$ does no longer transform properly under the full global symmetry, and thus the effective action in eqs. (4.2) and (4.5) is not invariant under the $\mathrm{SU}(N_f)_{L,R}$ gauge transformations. This is a problem since this breaks the anomaly matching condition for $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$. One may use the Nambu-Goldstone field U to compensate the gauge transformation, but it is not possible on the boundary, since the chiral symmetry is unbroken there. We propose to include the vector meson in the theory so that the appropriate boundary condition can be imposed.

The background gauge field for $\mathrm{SU}(N_f)_{L,R}$ is introduced via $\mathrm{U}(N_f)$ valued gauge fields $\widetilde{A}_{L,R}$ in the same way as $\mathrm{SU}(N_f)_V$ gauge field. We impose constraints

$$\mathrm{tr}\,\widetilde{F}_{L,R} = d\widehat{A}_f, \tag{5.1}$$

and the fields transform under the 1-form gauge transformation as

$$\widetilde{A}_{L,R} \to \widetilde{A}_{L,R} + \lambda_f^{(1)}\mathbf{1}. \tag{5.2}$$

Similarly to the $\mathrm{SU}(N_f)_V$ representation, they couple to the domain wall via $N_f\eta' + \theta$ as[4]

$$i(N_f\eta' + \theta)\frac{1}{N_f}\left(\frac{1}{2}N_c q_L + \frac{1}{2}N_c q_R + N_c N_f q_V\right) \tag{5.4}$$

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[4]The counter term is chosen so that a $\mathrm{U}(1)_A$ transformation compensates a constant shift of θ as

$$\mathcal{L}_c = i\theta\frac{N_c}{8\pi^2}(N_c q_L/2 + N_c q_R/2 + N_c N_f q_V).$$

This is because, under the $\mathrm{U}(1)_A$ transformation $\psi \to \exp(i\alpha\gamma^5)\psi$, the Lagrangian is shifted as

$$\mathcal{L} \to \mathcal{L} + i\alpha(2N_f q_c + N_c q_L + N_c q_R + 2N_c N_f q_V). \tag{5.3}$$

This shift is obtained as follows. If the massless Dirac fermions ψ couple to the gauge fields $\mathscr{L}$ and $\mathscr{R}$ as $(d - i\mathscr{L}P_L - i\mathscr{R}P_R)\psi$ with the projection operators $P_{L,R}$, the anomaly is characterized by the 6-dimensional action

$$S_6 = \frac{2\pi}{3!(2\pi^3)}\int \mathrm{tr}[(d\mathscr{L} - i\mathscr{L}^2)^3 - (d\mathscr{R} - i\mathscr{R}^2)^3].$$

$$= i(N_f \eta' + \theta) \frac{1}{4\pi N_f} \left[\frac{1}{2} N_c \operatorname{tr}(\mathcal{F}_L^2) + \frac{1}{2} N_c \operatorname{tr}(\mathcal{F}_R^2) + (d \operatorname{tr}(\mathcal{A}_L) + d \operatorname{tr}(\mathcal{A}_R)) d\widehat{A}_c \right],$$

where we define

$$q_{L,R} := \frac{1}{8\pi^2} \left(\operatorname{tr}(\widetilde{F}_{L,R}^2) - \frac{1}{N_f}(d\widehat{A}_f)^2 \right), \quad \mathcal{A}_{L,R} := \widetilde{A}_{L,R} + \widetilde{A}_V, \quad \mathcal{F}_{L,R} := d\mathcal{A}_{L,R} - i\mathcal{A}_{L,R}^2. \tag{5.5}$$

This gives the theory on a domain wall for $k = N_f$ as[5]

$$i\frac{1}{4\pi} \int_{Y_3} \left[-N_c \operatorname{tr}\left(cdc - i\frac{2}{3}c^3 \right) + 2\left(\operatorname{tr}(\mathcal{A}_L)/2 + \operatorname{tr}(\mathcal{A}_R)/2 - \operatorname{tr}(c) \right) d\widehat{A}_c \right.$$
$$\left. + \frac{1}{2} N_c \operatorname{tr}\left(\mathcal{A}_L d\mathcal{A}_L - i\frac{2}{3}(\mathcal{A}_L)^3 \right) + \frac{1}{2} N_c \operatorname{tr}\left(\mathcal{A}_R d\mathcal{A}_R - i\frac{2}{3}(\mathcal{A}_R)^3 \right) \right]. \tag{5.6}$$

However, for Y_3 with a boundary, we cannot recover the gauge invariance in the same way as in the previous section because there are two independent background fields $\mathcal{A}_L$ and $\mathcal{A}_R$. Although it seems possible that this problem is solved by using U and replacing $\mathcal{A}_R$ by $U\mathcal{A}_R U^{-1} + iU dU^{-1}$, we cannot use U on the boundary because the chiral symmetry is restored there. Our claim is that, to solve this problem, we should use the vector mesons as the gauge field for the hidden local symmetry.

5.1 Hidden local symmetry

In the formulation of the hidden local symmetry, the effective Lagrangian of the pions and the vector mesons is written in terms of a gauge theory as follows:

$$\mathcal{L}_h = \mathcal{L}_V + a\mathcal{L}_A,$$
$$\mathcal{L}_V = -\frac{f_\pi^2}{4} \operatorname{tr}(D_\mu \xi_L \xi_L^\dagger + D_\mu \xi_R \xi_R^\dagger)^2,$$
$$\mathcal{L}_A = -\frac{f_\pi^2}{4} \operatorname{tr}(D_\mu \xi_L \xi_L^\dagger - D_\mu \xi_R \xi_R^\dagger)^2,$$
$$D\xi_{L,R} = d\xi_{L,R} - iv\xi_{L,R} + i\xi_{L,R}\mathcal{A}_{L,R}, \tag{5.7}$$

where v is the gauge field for the hidden local symmetry $\mathrm{U}(N_f)_h$, and ξ_L and ξ_R relate to the $\mathrm{U}(N_f)$-valued field U that involves the pion field and η' as

$$\xi_L^\dagger \xi_R = U. \tag{5.8}$$

We set $\mathscr{L} = \widetilde{a} + \widetilde{A}_L + \widetilde{A}_V \mathbf{1} + A_A \mathbf{1}$ and $\mathscr{R} = \widetilde{a} + \widetilde{A}_R + \widetilde{A}_V \mathbf{1} - A_A \mathbf{1}$. The linear term in A_A is given as the surface integral

$$S_6 \supset \frac{1}{8\pi^2} \int [A_A(2N_f q_c + N_c q_L + N_c q_R + 2N_c N_f q_V].$$

This means that under the $\mathrm{U}(1)_A$ transformation, the Lagrangian changes as eq. (5.3) if $A_A = 0$.

[5]The domain wall action depends on the interior if $N_c \int (\operatorname{tr}(\mathcal{F}_L^2)/2 + \operatorname{tr}(\mathcal{F}_R^2)/2)/(8\pi^2)$ is not an integer. To avoid this, we consider $\mathcal{A}_{L,R}$ with the same instanton numbers. If we couple $\mathcal{A}_L$ and $\mathcal{A}_R$ with different instanton charges among them, U becomes singular at a point as we discuss later. We need to modify the effective theory to describe such a point.

Under the gauge transformations, the fields transform as

$$v \to hvh^{-1} + ihdh^{-1}, \qquad \mathcal{A}_{L,R} \to g_{L,R}\mathcal{A}_{L,R}g_{L,R}^{-1} + ig_{L,R}dg_{L,R}^{-1}, \qquad \xi_{L,R} \to h\xi_L g_{L,R}^{-1},$$
$$h \in \mathrm{U}(N_f)_h, \qquad\qquad\qquad g_{L,R} \in \mathrm{U}(N_f)_{L,R}. \tag{5.9}$$

To maintain the 1-form gauge invariance, v transforms as

$$v \to v - \lambda_c^{(1)}\mathbf{1}. \tag{5.10}$$

The term $\mathcal{L}_A$ does not depend on v and reduces to the kinetic term for U. The term $\mathcal{L}_V$ is regarded as the mass term for v. Due to this term, fields with the $\mathrm{U}(N_f)_h$ charge also couple to A_L and A_R. Actually, when we fix the gauge so that $\xi_L^\dagger = \xi_R =: \xi$, to maintain this gauge under the global transformation, fields with the $\mathrm{U}(N_f)_h$ charge are transformed by $h(\xi, g_L, g_R)$ that satisfies

$$g_L \xi h^{-1}(\xi, g_L, g_R) = h(\xi, g_L, g_R)\xi g_R^{-1}. \tag{5.11}$$

Therefore if $g_L = g_R =: g_f$, then $h(\xi, g_L, g_R) = g_f$, which means that fields with the $\mathrm{U}(N_f)_h$ charge has the corresponding $\mathrm{U}(N_f)_V$ charge. The effective theory including v should reduce to the effective theory for U if the mass term for v is minimized, i.e., if it is satisfied that

$$v = \frac{1}{2}(\mathcal{A}_L^{\xi_L} + \mathcal{A}_R^{\xi_R}),$$
$$\mathcal{A}_{L,R}^{\xi_{L,R}} := \xi_{L,R}\mathcal{A}_{L,R}\xi_{L,R}^{-1} + i\xi_{L,R}d\xi_{L,R}^{-1}. \tag{5.12}$$

5.2 Coupling between the pancake and the vector mesons

We propose that, the theory on the pancake should be written using the vector mesons v as

$$i\frac{1}{4\pi}\int_{Y_3}\left[-N_c\,\mathrm{tr}\left(cdc - i\frac{2}{3}c^3\right) + N_c\,\mathrm{tr}\left(vdv - i\frac{2}{3}v^3\right) + 2(\mathrm{tr}(v) - \mathrm{tr}(c))d\widehat{A}_c\right] \tag{5.13}$$
$$+ i\frac{N_c}{4\pi}\int_{\partial Y_3}\mathrm{tr}(vc),$$

up to terms we discuss in section 5.3.[6] For this action, the gauge invariance of the effective theory on the pancake is maintained in the same way as in section 4. In order for the CS term for v to be included, $\mathrm{tr}(f_v^2)$ must couple to η', and the coupling (5.4) has to be reproduced when we minimize the mass term for v, i.e., by substituting eq. (5.12). Therefore the bulk action contains the coupling of v to η' and θ as follows:

$$i\eta' N_c q_v + i\theta\left(bN_c q_v + (1-b)\frac{1}{N_f}(N_c q_L/2 + N_c q_R/2 + N_c N_f q_V)\right),$$
$$q_v := \frac{N_c}{8\pi^2}\,\mathrm{tr}\left[\left(f_v + \frac{1}{N_c}d\widehat{A}_c\mathbf{1}\right)^2\right], \tag{5.14}$$

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[6]When we consider the pancake configuration made of a subset of flavors, i.e., the $\mathrm{U}(n_f)$ subgroup of $\mathrm{U}(N_f)$, we can start from the hidden local symmetry only for the subgroup $\mathrm{U}(n_f)$ and ignore the rest of NG bosons and vector mesons. Then the gauge group of c and v are both reduced to $\mathrm{U}(n_f)$ while the CS levels $-N_c$ are unchanged.

where the parameter b is left arbitrary within our discussion. We can check that this reproduces eq. (5.4). When the mass term is minimized, it is satisfied that

$$\text{tr}(f_v^2) = \frac{1}{2}\text{tr}(\mathcal{F}_L^2) + \frac{1}{2}\text{tr}(\mathcal{F}_R^2) - \frac{1}{2}d\,\text{tr}[(\mathcal{A}_L^{\xi_L} - \mathcal{A}_R^{\xi_R})(\xi_L\mathcal{F}_L\xi_L^{-1} - \xi_R\mathcal{F}_R\xi_R^{-1})],$$

$$\text{tr}(f_v) = N_f d\widetilde{A}_V + d\widehat{A}_f. \tag{5.15}$$

By substituting this, we obtain eq. (5.4) added by the term

$$i\frac{N_c}{16\pi^2}(\eta' + b\theta)d\,\text{tr}[(\mathcal{A}_L^{\xi_L} - \mathcal{A}_R^{\xi_R})(\xi_L\mathcal{F}_L\xi_L^{-1} - \xi_R\mathcal{F}_R\xi_R^{-1})]. \tag{5.16}$$

This term is admissible because η' couples to the exterior derivative of a locally gauge invariant field, and thus the term is invariant under constant shifts of η'. Such a term does not change the property that constant shifts of θ are compensated by shifts of η', which we used to determine the coupling of η' to the background fields.

It is not enough that eq. (5.4) is reproduced when the mass term is minimized. Moreover, we need to check that the anomaly between $(\mathbb{Z}_{2N_f})_A$ and G_{sub}, eq. (3.21), is reproduced. Under the $(\mathbb{Z}_{2N_f})_A$ transformation, η' is shifted as $\eta' \to \eta' + 2\pi/N_f$. The phase shift is reproduced if and only if the instanton number for v satisfies

$$\frac{1}{8\pi^2}\int (N_c q_v - N_c q_L/2 - N_c q_R/2 - N_c N_f q_V) \in N_f\mathbb{Z} \tag{5.17}$$

This means that if there are no background fields, the instanton charge for v is restricted as an integer multiple of N_f. The simplest possibility is that v cannot have instantons on their own.

From the low-energy point of view, the left-hand side of eq. (5.17) is always zero for the following reason. For simplicity, we set $\widehat{A}_c = 0$. If the background fields $\mathcal{A}_{L,R}$ have nonzero instanton charge, there should be topologically nontrivial transition functions $g_{L,R}$ of the $U(N_f)_{L,R}$ transformation. Since the instanton charge is gauge invariant, the instanton charge for $v^{\xi_{L,R}} := \xi_{L,R}v\xi_{L,R}^{-1} + i\xi_{L,R}d\xi_{L,R}^{-1}$ is the same as that for v. Since $v^{\xi_{L,R}}$ gauge transform in the same way as $\mathcal{A}_{L,R}$, they have the same transition function $g_{L,R}$ and thus the same instanton charge as $\mathcal{A}_{L,R}$. This means the left-hand side of eq. (5.17) is zero.[7]

The condition (5.17) has to be maintained as long as the effective theory makes sense such as at finite temperatures. One thing we already have assumed is that the full theory admits a string around which η' winds by restoring the chiral symmetry at the core. Here we additionally need to assume that the effective theory does not admit instantons for v. This assumption seems to prohibit a simple UV completion of the hidden local symmetry by embedding ξ_L and ξ_R into scalar fields whose target space is linear. In such a UV theory, it is expected that the mass term for v can vanish, and thus the singular points of $\xi_{L,R}$ are allowed, which means there are configurations with non-zero instanton number for v even if there are no instanton solutions.

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[7]This also means that the instanton charge for $\mathcal{A}_L$ and $\mathcal{A}_R$ is the same as long as we assume that ξ_L and ξ_R are non-singular.

Another thing we should check is that whether the domain wall can be identified with a spin-$N_c/2$ baryon. From the point of view of the effective theory on the domain wall, v should be regarded as a background field although it is dynamical in the four-dimensional spacetime. The field $\rho(W)$ on the edge, which is defined similarly in the previous section, transforms under the $\mathrm{Sym}^{N_c}(\square)$ representation of $\mathrm{U}(N_f)_h$. In the four-dimensional theory, the pancake where the edge state corresponding to $\rho(W)$ excites has the correct charge of $\mathrm{U}(N_f)_V$ due to the mass term for v. Thus the object is regarded as a spin-$N_c/2$ baryon correctly. The baryon number is given by eq. (4.11) with the replacement of $\mathcal{A}_f$ by v,

$$B = \frac{1}{2\pi} \int_{\partial Y_2} \mathrm{tr}(v - iW\widetilde{d}W^{-1}) - \frac{1}{2\pi} \int_{Y_2} \mathrm{tr}(f_v), \tag{5.18}$$

The contribution from v is again cancelled as long as we take a gauge where a Dirac string does not penetrate the pancake.

The coupling of θ to the vector mesons and background fields in eq. (5.14) affects what happens if we insert an interface of θ. Even if b is not an integer, the effective theory on the interface from 0 to $2\pi N_f$ does not depend on the choice of the interior. This is because by using ξ_L and ξ_R, we can rewrite the term linear in b in eq. (5.14) as

$$ib\theta \left(N_c q_v - \frac{1}{N_f}(N_c q_L/2 + N_c q_R/2 + N_c N_f q_V) \right)$$
$$= i\frac{b\theta}{2} \sum_{H=\{L,R\}} \frac{1}{4\pi} d\left(N_c \mathrm{tr}\left[(v - \mathcal{A}_H^{\xi_H})(f_v + \xi_H \mathcal{F}_H \xi_H^{-1}) + i\frac{1}{3}(v - \mathcal{A}_H^{\xi_H})^3 \right] \right.$$
$$\left. + 2\,\mathrm{tr}(v - \mathcal{A}_H^{\xi_H})d\widehat{A}_c \right), \tag{5.19}$$

where we have used

$$\mathrm{tr}(f_v^2) - \mathrm{tr}(\mathcal{F}_{L,R}^2) = d\,\mathrm{tr}\left[(v - \mathcal{A}_{L,R}^{\xi_{L,R}})(f_v + \xi_{L,R}\mathcal{F}_{L,R}\xi_{L,R}^{-1}) + i\frac{1}{3}(v - \mathcal{A}_{L,R}^{\xi_{L,R}})^3 \right]. \tag{5.20}$$

Here since the exterior derivative acts on the locally gauge invariant quantity, we can safely use the Stokes theorem. Thus the effective action on the interface is the sum of eq. (5.6) and

$$i\frac{b}{2} \sum_{H=\{L,R\}} \frac{1}{4\pi} \int_{V_3} \left(N_c \mathrm{tr}\left[(v - \mathcal{A}_H^{\xi_H})(f_v + \xi_H \mathcal{F}_H \xi_H^{-1}) + i\frac{1}{3}(v - \mathcal{A}_H^{\xi_H})^3 \right] \right.$$
$$\left. + 2\,\mathrm{tr}(v - \mathcal{A}_H^{\xi_H})d\widehat{A}_c \right). \tag{5.21}$$

We can consider the configuration that the pancake attaches to the interface. This happens depending on how fast θ varies on the interface. On the edge of the pancake, we cannot use ξ_L and ξ_R because the chiral symmetry is restored there. This means that near the core of the string where the order parameter of the chiral symmetry breaking, f_π, approaches to zero, the value of b should approach to an integer, m,

$$b \to m. \tag{5.22}$$

The contribution from the integer part, m, of eq. (5.21) does not depend on ξ_L and ξ_R, and is given by

$$i\frac{m}{4\pi}\int_{V_3}\left[N_c\,\mathrm{tr}\left(vdv-i\frac{2}{3}v^3\right)+2(\mathrm{tr}(v)-\mathrm{tr}(\mathcal{A}_L)/2-\mathrm{tr}(\mathcal{A}_R)/2)d\widehat{A}_c\right.$$
$$\left.-\frac{1}{2}N_c\,\mathrm{tr}\left(\mathcal{A}_L d\mathcal{A}_L-i\frac{2}{3}\mathcal{A}_L^3\right)-\frac{1}{2}N_c\,\mathrm{tr}\left(\mathcal{A}_R d\mathcal{A}_R-i\frac{2}{3}\mathcal{A}_R^3\right)\right]. \quad (5.23)$$

In this sense, we can say that on the interface, there is a CS term for v with the level mN_c. The following two values of b are rather special:

- $b = 0$: the vector mesons v do not couple to the interface. When the pancake attaches to the interface, there is only the CS term for v on the pancake and not those for c and the background fields. On the other region of the interface, there are the CS terms for c and the background fields, and not that for v.

- $b = 1$: the vector mesons v couple to the interface via the CS term with the level N_c. When the pancake attaches to the interface, there are no fields on the pancake except for the edge. On the other region of the interface, there are the CS terms for c and v, and not those for the background fields.

To summarize, we consider the pancake and the interface for $N_f = 1$. For simplicity, we set $\widehat{A}_c = 0$, and only consider the ω meson field, the U(1) gauge field c of the CS theory, and the U(1)$_V$ background gauge field A_V. We set $\xi_{L,R} = \exp(i\phi_{L,R})$. The coupling of these fields to η' and θ is

$$N_c\eta'd\omega^2+N_c\theta(b\,d\omega^2+(1-b)dA_V^2). \quad (5.24)$$

The theory on the pancake is

$$i\frac{N_c}{4\pi}\int_{Y_3}(-cdc+\omega d\omega)+i\frac{N_c}{4\pi}\int_{\partial Y_3}\omega c. \quad (5.25)$$

The theory on the interface is

$$i\frac{N_c}{4\pi}\int_{X_3}(b(\omega-A_V+d\phi_L/2+d\phi_R/2)(d\omega+dA_V)-cdc+A_VdA_V)$$
$$=\begin{cases}i\dfrac{N_c}{4\pi}\displaystyle\int_{X_3}(-cdc+A_VdA_V) & \text{for }b=0,\\[3mm] i\dfrac{N_c}{4\pi}\displaystyle\int_{X_3}(-cdc+\omega d\omega) & \text{for }b=1.\end{cases} \quad (5.26)$$

5.3 Relation to the generalized WZW term with the vector mesons

In the context of the hidden local symmetry, the possible generalization of WZW term that includes the vector mesons has been discussed [33]. We consider the relation between our proposal of the coupling between η' and v in eq. (5.14), and the generalized WZW term. In this subsection, we only consider the case where $\widehat{A}_c = 0$, which means the 1-form symmetry corresponding to $\lambda_c^{(1)}$ is fixed. Especially, $\mathcal{A}_{L,R}$ are 1-form gauge invariant.

Note that we consider the WZW term corresponding to the anomaly for $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ because the axial U(1) symmetry is restored at the leading order of the $1/N_c$ expansion. The external gauge field for the axial symmetry is later taken to be vanishing, $A_A = 0$, since the symmetry is broken when we include the $1/N_c$ corrections in the effective Lagrangian.

We can only add the gauge invariant terms to the WZW term in order to match the 't Hooft anomaly, and thus possible additional terms consist of the gauge covariant building blocks f_v, $\widehat{\mathcal{F}}_{L,R} := \xi_{L,R}\mathcal{F}_{L,R}\xi_{L,R}^{-1}$ and $\widehat{\alpha}_{L,R} := \mathcal{A}_{L,R}^{\xi_{L,R}} - v$. The additional term to the WZW term is written as a linear combination of four terms as [44]

$$\Gamma[\xi_L, \xi_R, v, \mathcal{A}_L, \mathcal{A}_R] = \Gamma_{\mathrm{WZW}}[\xi_L^\dagger \xi_R, \mathcal{A}_L, \mathcal{A}_R] + \Gamma_v,$$

$$\Gamma_v := -i\frac{N_c}{16\pi^2}\int_{M^4}\sum_{i=1}^4 c_i\mathcal{L}_i,$$

$$\mathcal{L}_1 = \mathrm{tr}(\widehat{\alpha}_L^3\widehat{\alpha}_R - \widehat{\alpha}_R^3\widehat{\alpha}_L),$$

$$\mathcal{L}_2 = \mathrm{tr}(\widehat{\alpha}_L\widehat{\alpha}_R\widehat{\alpha}_L\widehat{\alpha}_R),$$

$$\mathcal{L}_3 = \mathrm{tr}(f_v(\widehat{\alpha}_L\widehat{\alpha}_R - \widehat{\alpha}_R\widehat{\alpha}_L)),$$

$$\mathcal{L}_4 = \frac{1}{2}\mathrm{tr}\left(\widehat{\mathcal{F}}_L(\widehat{\alpha}_L\widehat{\alpha}_R - \widehat{\alpha}_R\widehat{\alpha}_L) - \widehat{\mathcal{F}}_R(\widehat{\alpha}_R\widehat{\alpha}_L - \widehat{\alpha}_L\widehat{\alpha}_R)\right). \tag{5.27}$$

The four terms, $\mathcal{L}_i$, are the general gauge invariant 4-forms that conserve parity and charge conjugation but violate the intrinsic parity.

The term (5.14) is included if we choose $c_1 = 2/3, c_2 = -1/3, c_3 = 1, c_4 = 1$ as[8]

$$\Gamma \supset i\frac{N_c}{8\pi^2}\int \eta' \, \mathrm{tr}(f_v^2). \tag{5.28}$$

However, this does not mean that the values of $c_1, \ldots, c_4$ are fixed for the following reason. We can add freely terms in the form $d\eta'A^{(3)}$ for a locally gauge invariant 3-form field $A^{(3)}$ because they only add locally gauge invariant terms to the effective action on the wall. The problem is how to deal with them on the edge of the domain wall if $A^{(3)}$ includes $\xi_{L,R}$. On the edge, the chiral symmetry is restored and we cannot use $\xi_{L,R}$. A simple possibility is that they become zero at the point where the chiral symmetry is restored. By considering the finite thickness of the string, it is possible that the coefficients c_i gradually deform into a required combination as we approach to the core of the string. If the terms behaves like that, we can add them to the effective theory without affecting the argument in the previous section.

6 Summary and discussion

By the consideration of the large N_c limit of QCD, the η' potential should have cusps at $N_f\eta' + \theta = n\pi$ with n odd integers. We find that the consistent effective theory of the η' meson should be accompanied by the dynamical domain wall which appears when $N_f\eta' + \theta$

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[8]The same values of c_i are proposed in ref. [21] in the similar context.

goes across the cusp. On the domain wall, there should be a Chern-Simons (CS) theory in order for the anomaly matching to be satisfied.

The domain wall can have an edge where the chiral symmetry is restored. In this circumstance, in order to impose a consistent boundary condition, we find that the gauge boson of the hidden local symmetry, i.e., the vector mesons, should couple to that of the CS theory. The whole framework gets consistent by this construction, and the quantum numbers of the pancake excitations are correctly obtained as the spin-$N_c/2$ baryons.

It is interesting to find that the existence of the pancake configuration is made possible by the presence of the vector mesons, and thus it is essential to include them in the effective theory.

Another interesting finding is that the consistency of the periodicity of η' requires that the vector mesons should not have a non-vanishing instanton number. This indicates that a simple UV completion of the hidden local symmetry by a linear sigma model is not possible. This has some implication for the discussion of the finite temperature system or in general for the chiral phase transition. The vector mesons should not simply go to massless near the transition point.

By the analogy with the Seiberg duality [24] in supersymmetric QCD, the effective theory as a U(N_f) gauge theory based on the hidden local symmetry can be interpreted as some "dual" description of QCD [25–31]. Our discussion clarifies what is really the relation between the gluon and the vector mesons. In the UV theory, if we insert a sharp interface of θ from 0 to $2\pi N_f$, there appears an SU(N_c)$_{N_f}$ CS theory whose gauge field is the gluon field. If we assume that confinement in the bulk does not affect the theory on the wall, we find that even at low energy, the gauge field of the CS theory on the wall is actually the gluon field. The field c in our description of the wall theory, eq. (5.13), is the gauge field in the level-rank dual of the SU(N_c)$_{N_f}$ CS theory. In this sense, c can be regarded as the "dual gluon field." The CS theory of c and the hidden local symmetry of v have the same gauge group, U(N_f), which makes it possible for them to mix. Indeed, we find that such a mixing is necessary on the edge of the pancake as in eq. (5.13) in order to maintain the gauge invariance of the external gauge fields, i.e., for consistency.

Let us go back to the discussion of the phase transition. When we go to the finite temperature system, η' can cross the cusps of the potential more easily and frequently at high temperatures. In this situation, c can propagate in the four-dimensional space due to creation of the pancakes, and mixes with v and they are eventually indistinguishable. Although a simple linear sigma model is not possible as we discussed above, the vector mesons can gradually deform into the dual gluons in this picture. If this is the correct picture, the pancakes are essential for the description of the behavior of the vector resonances near the phase transition.

It is interesting to consider the S^1 compactified QCD where θ winds around the S^1 direction N_f times. The three-dimensional effective theory for a small radius is known to be an SU(N_c)$_{N_f}$ CS theory, and for a large radius, the effective theory is the one we discussed in this paper. In ref. [22], it has been claimed that the two limits of the theory are interpolated by the Higgs mechanism of a U(N_f)$_{-N_c}$ CS theory with $2N_f$ scalar fields. Our result is consistent with this. At a small radius where the change of θ is so rapid, η'

cannot follow the change of θ, which means there appears the domain wall with $k = N_f$. The $\mathrm{U}(N_f)_{-N_c}$ CS theory of c is realized on the wall, which is naturally identified as the level-rank dual of the theory in the small radius limit, $\mathrm{SU}(N_c)_{N_f}$. The gauge boson of $\mathrm{SU}(N_c)_{N_f}$ is nothing but the gluons. On the other hand, for a large radius, η' follows the change of θ, and then there is no domain wall. Instead of the CS theory for c, there is a CS Higgs theory for the vector mesons (at least if m in eq. (5.23) is not equal to 1). Again, the creation of the pancakes can make the transition smooth by the mixing between v and c on the edges. Although the simple Higgs mechanism does not work in four dimensions, the transition of the lowest modes, i.e., the three-dimensional effective theory, can be a smooth transition from the CS Higgs theory of vector mesons to the CS theory of c as the radius gets smaller.

Acknowledgments

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