



A vector-metric theory of gravity
by Ronald Ward Hellings

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Physics
Montana State University
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Abstract:

A new theory of gravity is presented in which gravity is produced by a massless vector field in addition to the usual metric field. It is found that the theory is compatible with present solar system experiments and cosmological observations. The theory also predicts the existence of black holes and of the most general types of weak plane waves.

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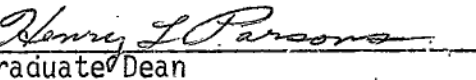
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ABSTRACT

A new theory of gravity is presented in which gravity is produced by a massless vector field in addition to the usual metric field. It is found that the theory is compatible with present solar system experiments and cosmological observations. The theory also predicts the existence of black holes and of the most general types of weak plane waves.

CHAPTER I

INTRODUCTION

Whenever a new theory is proposed, it should be made clear why the theory is needed. Usually this involves showing how existing theories fail to adequately explain certain experimental or observational results. Yet this need not be the only justification for a new theory, and it is not the justification for the new theory of gravity to be presented here.

As a matter of fact, there is presently nothing observationally wrong with Einstein's general relativity,¹ the most widely accepted theory of gravity today. As time goes on and the evidence mounts in favor of general relativity, it is strange to remember that a few years ago, as recently as 1960, only two of general relativity's predictions which differed from Newton's theory had actually been observed,² and based on only these two the theory was almost universally accepted. One thing which probably gave impetus to the growth of experimental gravity during the sixties and certainly was the motivation behind the recent light deflection and time-delay experiments, was the appearance in 1961 of a new respectable, covariant theory of gravity--the Brans-Dicke theory.

¹See a summary of experimental results in C.M. Will, Lectures in B. Bertotti, ed., Proceedings of Course 56 of the International School of Physics "Enrico Fermi", Academic Press (1973).

²These were the deflection of light and the perihelion precession of mercury. The third "classical test" of relativity, the gravitational red shift, was originally confirmed, but subsequent observations showed the results to be inconclusive. It was not until the Pound-Rebka experiment in 1960 that the red shift was verified. See E. Finlay-Freundlich, *Philos. Mag.* 45, 303 (1954); and R.V. Pound and G.A. Rebka, *Phys. Rev. Letters* 4, 337 (1960).

In fact, what had been largely missing during the period 1916-1960 was the usual interplay between theory and experiment. Most theorists were satisfied with general relativity and were reluctant to consider any other possibility. Most experimenters were reluctant to tackle any of the great difficulties involved in measuring the extremely weak post-Newtonian effects of gravity, especially when the expected outcome of the experiment would merely be a further confirmation of what everyone already "knew".

There is an additional difficulty for the experimenter that arises when there is only one theory present, and that is that it is not clear which experiments are significant. This is especially true for general relativity which predicts a value of zero for the result of many experiments. A null experimental value is really only meaningful against a background of other theories predicting non-null results for the same experiment. For example, general relativity predicts that all neutral test bodies fall at the same rate in a gravitational field (weak equivalence principle), and that the result of any local experiment is independent of the velocity of the apparatus (part of the strong equivalence principle). If the second prediction must follow from the first in any reasonable theory of gravity, then an experimental result verifying the second prediction does not increase our confidence in general relativity. However, if a theory appears which predicts the first but not the second (as the Vector-Metric Theory does), then a

verification of the second prediction serves to eliminate the interloper and increase our confidence in general relativity. The reason that the present theory is needed, as are other theories, is to point the way to future significant experiments, and to make "non-null" some of the latent null predictions of general relativity.

A. Metric Theories. An analysis by R. H. Dicke³ has shown that the high precision null experiments--Eötvös experiments, Hughes-Drever experiments, and ether drift experiments--rule out the existence of vector or additional tensor fields (besides the metric field) which couple directly to matter. As pointed out by Will and Nordtvedt,⁴ however, vector and tensor fields may exist along with the metric field as long as the equation of motion for matter does not include them explicitly. These fields may couple to the metric field and be involved in determining the functional form of the metric, but once the metric functions are found the matter equations of motion will depend only on them and on the matter variables (position, velocity, etc.). Theories which have this property are called "metric theories".⁵

³R. H. Dicke, The Theoretical Significance of Experimental Relativity, Gordon and Breach (1965).

⁴C. Will and K. Nordtvedt, Jr., Ap. J. 177, 757 (1972).

⁵K. Thorne and C. Will, Ap. J. 163, 595 (1970).

Gravitational experimental results depend on the metric fields and can be expected to indirectly detect the existence of the vector or tensor fields which go into determining the metric. In non-gravitational experiments, however, the metric only serves as a background. Coordinates are always chosen so that the metric is locally Lorentz, and the effect of whatever went into determining the metric is washed out. The Eötvös, Hughes-Drever, and ether experiments are all non-gravitational in this sense. They are based on the fundamental idea of a metric and measure non-metric perturbations to geodesic behavior.

B. Machian Effects. Will and Nordtvedt⁶ have studied extensively the effects of gravity which have been variously called "Machian",⁷ "preferred-frame", and "ether". Of these labels the last two are far inferior, implying the existence of absolute space in logical contradiction to the spirit of all relativity. In fact, one need not postulate absolute space in order to find a frame which is in some sense preferred. Since gravity is a long-range universal interaction, one might expect the global distribution of matter to affect local gravitational physics in a Machian way and to establish a preferred frame as the mean rest frame of the universe.

⁶Will and Nordtvedt, op cit.

⁷The term "Machian" has been loosely applied to refer to the determination of the properties of space by the global distribution of matter. See Ernst Mach, The Science of Mechanics, Open Court Publishing (1902), Chapter II.

As Will and Nordvedt point out the mystery is not, "How can we have a preferred rest frame in space?", but "How can we ever avoid having one, related to the universe rest-frame". Theories which have only a metric field, or only a metric and a scalar field, avoid these effects in the following way.

1. A theory which contains only a metric field yields local gravitational physics which is identical in all frames which are asymptotically Minkowskian. This follows from the invariance under boosts of $\eta_{\mu\nu}$ (the asymptotic form of $g_{\mu\nu}$), the only field coupling the local system asymptotically to the universe; and from general covariance, which allows us to find a coordinate system in which the metric takes this Minkowski form at the boundary between the universe and the local system.

2. A theory which contains a metric field and a scalar field yields physics which is identical in all asymptotic Minkowski frames. This follows from invariance of $\eta_{\mu\nu}$ and the scalar field under boosts (the scalar field, of course, is generally invariant).

In the present theory, we include with the metric a vector field, K_μ , whose components depend on the choice of Lorentz frame. In some Lorentz frame, the vector field at a point will have a zeroth component only. This is the only frame in which space is locally isotropic⁸ and

⁸Any non-zero space part of the background vector field must point in some preferred direction, destroying the local isotropy of space.

we define this frame to be the preferred frame in which, for simplicity, we will do much of the work to follow. The apparent isotropy of the matter distribution seen through the telescope leads us to believe that this preferred frame is also the mean rest frame of the universe, and that the preferred frame is determined in a "Machian" way.

C. Summary of Results. In the chapters to follow, we compute the predictions of the vector-metric theory for: 1) solar system experiments, 2) the existence of black holes or event horizons, 3) gravitational radiation, and 4) cosmology. The results are as follows:

1. Solar system experiments. It is found that with restrictions on the strength of some coupling constants the theory is compatible with all existing solar system experiments, and that for a particular choice of the coupling constants the theory makes the same predictions as does general relativity. This last point is of particular importance since it means that no solar system experiment presently envisioned can differentiate between this theory and general relativity. It is evident that experiments which are to choose between these theories must involve higher order effects such as occur in a) gravitational radiation, b) cosmology, or c) extremely precise solar system experiments. It is not clear which type of experiment offers the best possibility, but theories such as these should stand as a challenge to the gravitational experimenter to devise new and better ways to measure the extremely small effects of post-Newtonian and post-post-Newtonian gravity.

2. Black holes. It is found that an event horizon exists in a static spherically symmetric configuration. Moreover, this solution has the same metric behavior close to the horizon as the Schwarzschild solution in general relativity.

3. Gravitational radiation. We have found that there are two classes of radiation in the theory, depending on the type of coupling between the vector and metric fields. If the coupling is "scalar-type" (that is, K_μ appears only as $K_\mu K^\mu$ in the interaction Lagrangian), then we find waves of class N_3 ,⁹ the class typical of scalar-metric theories. In the most general coupling, the class is II_6 , reflecting the vector nature of the interaction. Also, these last waves are seen to propagate at speeds either greater or less than the speed of light, depending on the values of the coupling constants. This is discussed in Chapter V.

4. Cosmology. The vector-metric theory is found to possess an acceptable closed cosmology, the precise behavior depending on the coupling constants as in Brans-Dicke theory. It is found that the local constant of gravity is dependent on the cosmological strength of the vector field. The observed gravitational constant is obtained if one postulates an energy density in the universe which is 1,000 times the observed stellar matter density. This requirement sounds less outrageous

⁹For a description of this classification, see D. Eardley, et al., Phys. Rev. Letters 30, 884 (1973). There is also a summary of the scheme in Chapter V of this work.

When we realize that general relativity requires a factor of 100 more energy in the universe than the stellar matter presently observed in order to produce a closed universe.

CHAPTER II

THE VECTOR-METRIC THEORY OF GRAVITY

A. Notation. The notation used in the remainder of this work follows that of Adler, Bazin, and Schiffer's Introduction to General Relativity.¹

1. Greek indices take on values from 0 to 3. Latin indices run from 1 to 3. Repeated indices are summed over their range of values.
2. The metric tensor is denoted $g_{\mu\nu}$ and has signature (+ - - -). $\eta_{\mu\nu}$ is the Minkowski metric tensor with the same signature. Occasionally g_{ss} will be used to mean g_{11} , g_{22} , or g_{33} , and there will be no sum over s .
3. A comma (,) indicates ordinary partial differentiation. A vertical bar (|) indicates covariant differentiation:

$$A_{|\mu\nu} = A_{\mu,\nu} - \Gamma_{\mu\nu}^{\alpha} A_{\alpha}$$

4. Sign conventions in the differential geometry are as follows:

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2} g^{\alpha\beta} (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}),$$

$$R^{\alpha}_{\mu\beta\nu} = \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\lambda\nu} \Gamma^{\lambda}_{\beta\mu} - \Gamma^{\alpha}_{\lambda\beta} \Gamma^{\lambda}_{\mu\nu},$$

$$R_{\mu\nu} = R^{\lambda}_{\mu\alpha\nu}$$

¹R. Adler, M. Bazin, and M. Schiffer, Introduction to General Relativity, McGraw-Hill (1965).

5. The box operator is defined $\square^2(\) \equiv \eta^{\alpha\beta}(\)_{,\alpha\beta}$ and we define a covariant box operator $\square^2(\) \equiv g^{\alpha\beta}(\)_{;\alpha\beta}$.
6. We work in units where $c = 1$.

B. The Lagrangian. Einstein's theory of gravity can be derived from variation of the action integral:

$$A = \int \sqrt{-g} [16\pi G L_m + R] d^4x$$

where g is the determinant of $g_{\mu\nu}$,

G is Newton's gravitational constant,

$L_m = L_m(g_{\mu\nu}, \text{matter variables})$ is the matter Lagrangian which is a function of $g_{\mu\nu}$ and its derivatives and of whatever variables are used to describe the state of matter in the system (position, velocity, etc.).

The requirement that the action be invariant under an infinitesimal variation of $g_{\mu\nu}$ produces the field equations of general relativity. Variation of the matter variables produces the equations of motion of matter.

In 1961, Brans and Dicke² proposed a theory in which gravitation was produced by two fields--a metric tensor field and an auxiliary scalar field. Their action integral is written

²C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

$$A = \int \sqrt{-g} [16\pi L_m + \phi R + \frac{\omega}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}] d^4x,$$

where ϕ is the scalar field and ω is a dimensionless parameter. Once again, L_m is a function of $g_{\mu\nu}$ and matter variables only; ϕ does not enter. The Brans-Dicke field equations come from independent variation of $g_{\mu\nu}$ and ϕ . The equations of motion of matter come from variation of the matter variables in L_m . Since ϕ does not appear in L_m , the equation of motion of matter will involve only the metric and will produce a metric theory of gravity.

Our reason for reviewing these theories is that the new theory to be presented here adds onto general relativity in a way similar to that of the Brans-Dicke theory. We propose a theory of gravity in which a massless vector field appears in addition to the metric field. Committed to the spirit as well as to the law of general covariance in physics, we introduce no a priori fields or reference frames into the theory. We require a Lagrangian subject to the following conditions:

1. The Lagrangian density is a four-scalar density.
2. It generates positive definite free field energies for both the metric and the vector fields.
3. It produces a "metric theory".
4. It generates field equations containing no higher than second derivatives of the fields.

Such a Lagrangian is

$$A = \int \sqrt{-g} [16\pi G_0 L_m + R - F_{\mu\nu} F^{\mu\nu} + \omega K_\mu K^\mu R + \eta K^\mu K^\nu R_{\mu\nu}] d^4x \quad (\text{II.1})$$

where $L_m = L_m(g_{\mu\nu})$, matter variables) as before,

$F_{\mu\nu} = K_{\mu\nu} - K_{\nu\mu}$ in analogy with electrodynamics,

K_μ is the vector field,

ω and η are dimensionless parameters, and

G_0 is an a priori or "bare" gravitational constant.

C. The Field Equations. The field equations are calculated by requiring that the action, equation II.1, be stationary under independent variation of the fields. Details of the derivation are given in Appendix E. Variation of $g_{\mu\nu}$ gives the equation

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R + \omega(K_\mu K_\nu R + KR_{\mu\nu} - \frac{1}{2}g_{\mu\nu} KR + K_{\mu\nu} - g_{\mu\nu} \square^2 K) \\ + \eta K^\alpha K^\beta (g_{\mu\alpha} R_{\nu\beta} + g_{\nu\beta} R_{\mu\alpha} - \frac{1}{2}g_{\mu\nu} R_{\alpha\beta}) - \frac{1}{2}\eta [g_{\mu\nu} (K^\alpha K^\beta)_{\alpha\beta} \\ + \square^2 (K_\mu K_\nu) - (K_\mu K^\alpha)_{\nu\alpha} - (K_\nu K^\alpha)_{\mu\alpha}] \\ + 2F_{\mu\alpha} F^\alpha_\nu + \frac{1}{2}g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = -8\pi G_0 T_{\mu\nu} \end{aligned} \quad (\text{II.2})$$

where $K \equiv K_\alpha K^\alpha$ and $T_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} 16\pi G_0 L_m)$. The contraction of equation II.2 is

$$R + (3\omega + \frac{1}{2}\eta) \square^2 K + \eta (K^\alpha K^\beta)_{\alpha\beta} = 8\pi G_0 g^{\mu\nu} T_{\mu\nu} \equiv 8\pi G_0 T. \quad (\text{II.3})$$

Variation of K_μ in equation II.1 gives

$$\omega R K^\mu + \eta R^{\mu\nu} K_\nu + 2F^{\mu\nu}{}_{\nu} = 0. \quad (\text{II.4})$$

D. The Solution. In the chapters to follow, the above equations will be solved in four different contexts. To each of the cases, there corresponds some approximation or symmetrization of the metric which can be used to simplify the field equations. The four cases are as follows:

1. Weak Gravity. It is assumed that there exists a region of space where gravity is weak and the motion of sources is slow. By weak gravity it is meant that, for each massive source of the metric, $\frac{GM}{r} \ll 1$ (where r is the distance from the source of mass M). Slow motion of the sources naturally means $v^2 \ll 1$. These conditions are satisfied in the solar system and in most local regions of the universe. When the conditions are satisfied, one can make a general expansion of the metric in powers of both $\frac{GM}{r}$ and v^2 . Keeping terms second order in the combination (that is, terms like G^2M^2/r^2 and GMv^2/r) produces the Parameterized Post-Newtonian (PPN) metric of Nordtvedt and Will.³ In Chapter III, we obtain the PPN metric for the vector-metric theory.

2. Event Horizon. Vishveshwara⁴ has shown that, for a static metric, a surface of infinite redshift (i.e., a surface on which $g_{00} = 0$)

³C. M. Will and K. Nordtvedt, Jr., Ap. J. 177, 757 (1972).

⁴C. V. Vishveshwara, J., Math. Phys. 9, 1319 (1968).

is also an event horizon or a one-way membrane. Therefore, the question of whether or not the vector-metric theory predicts the existence of black holes as does general relativity can be answered by a search for a surface of infinite redshift in a static configuration, which for simplicity we also take to be spherically symmetric. Ideally one would like to have an exact solution for this configuration, but this turns out to be an extremely difficult problem whose solution has not yet been found. Rather, we have expanded the fields in a power series about the event horizon, keeping the leading terms only.

3. Radiation. Weak plane gravitational waves are described by writing the metric and vector fields as the sum of a constant background part and a wave disturbance perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$K^\mu = \phi^\mu + A^\mu$$

and requiring that $h_{\mu\nu} \ll \eta_{\mu\nu}$ and that $A^\mu \ll \phi^\mu$. Here

$\eta_{\mu\nu}$ is the Minkowski metric tensor,

ϕ^μ is the cosmological vector background,

$h_{\mu\nu}$ is the metric wave,

A^μ is the vector wave;

$h_{\mu\nu}$ and A^μ are assumed proportional to $\exp(ik_\mu x^\mu)$, with k_μ as some constant propagation vector. Solutions giving relationships between

ϕ_μ , $h_{\mu\nu}$, A_μ and k_μ for weak radiation are found in Chapter V.

4. Cosmology. The Robertson-Walker cosmological metric with a neutral dust model for matter is used to establish a cosmology for the vector-metric theory. The metric is assumed to be of the form

$$g_{00} = 1$$
$$g_{ij} = - \frac{S(t)^2}{(1 + \frac{\kappa}{4} r^2)^2} \delta_{ij}.$$

In Chapter VI, the field equations are solved giving S , the "size" of the universe, as a function of time.

CHAPTER III

THE PPN METRIC

Will and Nordvedt¹ have arrived at a general form for the first post-Newtonian metric valid in any inertial coordinate frame.

$$\begin{aligned}
 g_{00} = & 1 - 2\sum_i \frac{GM_i}{r_i} + 2\beta \left(\sum_i \frac{GM_i}{r_i}\right)^2 - (2\gamma + 1 + \alpha_3 + \zeta_1) \sum_i \frac{GM_i v_i^2}{r_i} \\
 & - 2(1 - 2\beta + \zeta_2) \sum_i \frac{GM_i}{r_i} \sum_{j \neq i} \frac{GM_j}{r_{ij}} + \zeta_1 \sum_i \frac{GM_i}{r_i} (\vec{v}_i \cdot \vec{r}_i)^2 \\
 & + \alpha_2 \sum_i \frac{GM_i}{r_i} (\vec{w} \cdot \vec{r}_i)^2 + (\alpha_1 - \alpha_2 - \alpha_3) \sum_i \frac{GM_i w^2}{r_i} \\
 & + (\alpha_1 - 2\alpha_3) \sum_i \frac{GM_i}{r_i} (\vec{w} \cdot \vec{v}_i)
 \end{aligned}$$

$$\begin{aligned}
 g_{0k} = & \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1) \sum_i \frac{GM_i}{r_i} v_i^k + \frac{1}{2}(1 + \alpha_2 - \zeta_1) \sum_i \frac{GM_i}{r_i} \\
 & X(\vec{v}_i \cdot \vec{r}_i) x_i^k + \left(\frac{1}{2}\alpha_1 - \alpha_2\right) \sum_i \frac{GM_i}{r_i} w^k + \alpha_2 \sum_i \frac{GM_i}{r_i} (\vec{w} \cdot \vec{r}_i) x_i^k
 \end{aligned}$$

$$g_{\ell m} = -\delta_{\ell m} \left(1 + 2\gamma \sum_i \frac{GM_i}{r_i}\right)$$

where x_i^k are cartesian components of the i^{th} source-to-field-point vector,

v_i^k are cartesian velocity components of the i^{th} source

$$\left(v^k = \frac{dx^k}{dt}\right),$$

w^k are cartesian components of the velocity of the inertial coordinate system relative to the universe rest frame.

¹C. M. Will and K. Nordtvedt, Jr., Ap. J. 177, 757 (1972).

$$r_i = \left[\sum_{k=1}^3 (x_i^k)^2 \right]^{1/2},$$

$$r_{ij} = \left[\sum_{k=1}^3 (x_i^k - x_j^k)^2 \right]^{1/2},$$

M_i is the gravitational mass of the i^{th} body, and

G is the effective gravitational constant.

It should be appreciated that this form is based on very few assumptions (see Nordtvedt²). The parameters $\beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2$ in the metric are theory-dependent and may depend on cosmological factors through the influence of cosmological fields. In general relativity the PPN parameters have the value

$$\gamma = \beta = 1$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = 0,$$

and in the Brans-Dicke scalar-metric theory

$$\gamma = \frac{1 + \omega}{2 + \omega}$$

$$\beta = 1$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = 0.$$

²K. Nordtvedt, Jr., Phys. Rev. 169, 1017 (1968).

Inspection of the PPN metric shows that one can obtain all the PPN parameters by calculating in the universe rest frame, and that one can obtain all but ζ_2 by considering a single source. For most of our work we therefore use

$$\begin{aligned}
 g_{00} &= 1 - 2\frac{GM}{r} + 2\beta\frac{G^2M^2}{r^2} - (2\gamma + 1 + \alpha_3 + \zeta_1)\frac{GM}{r}v^2 + \zeta_1\frac{GM}{r^3}(\vec{r} \cdot \vec{v})^2 \\
 g_{0k} &= \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)\frac{GM}{r}v^k + \frac{1}{2}(1 + \alpha_2 - \zeta_1)\frac{GM}{r^3}(\vec{r} \cdot \vec{v})x^k \\
 g_{\ell m} &= -\delta_{\ell m}\left(1 + 2\gamma\frac{GM}{r} + 2\delta\frac{G^2M^2}{r^2}\right)
 \end{aligned}
 \tag{III.1}$$

where it is noted that a post-post-Newtonian term is added to $g_{\ell m}$.

In addition to the PPN metric expansion, we will need a similar expansion for K_μ ,

$$\begin{aligned}
 K_0 &= \sqrt{\phi}\left[1 + a_1\frac{GM}{r} + b\frac{G^2M^2}{r^2} + a_2\frac{GM}{r}v^2 + a_3\frac{GM}{r^3}(\vec{r} \cdot \vec{v})^2 + f\frac{GM}{r}(\vec{r} \cdot \vec{a})\right] \\
 K_\ell &= \sqrt{\phi}\left[d\frac{GM}{r}v^\ell + d'\frac{GM}{r^3}(\vec{r} \cdot \vec{v})x^\ell\right]
 \end{aligned}
 \tag{III.2}$$

where ϕ is a constant, as are the parameters a_i , b , f , d , d' . These expansions are substituted into the field equations (II.2, II.3, and II.4), the details of the solution being given in Appendix A. The linearized equations are first solved for a static source, giving for γ

$$\gamma = 1 + \omega\phi \frac{2\omega - n - 2}{1 - \omega\phi(4\omega - 1)}$$

Light deflection and retardation experiments³ show that $\gamma \approx 1$, so we specialize to that case. There are three ways this condition can be realized. The first is to require that

$$\phi \ll 1.$$

This weak cosmological field condition reduces all results arbitrarily close to general relativity. For that reason, it is the least interesting. There are two other conditions which give $\gamma \approx 1$. These are

$$\omega = \frac{1}{2}n + 1 \quad (\text{Case I})$$

and

(III.3)

$$\omega = 0 \quad (\text{Case II})$$

Proceeding with the solutions for each case, it is found that

Case I

$$\gamma = 1$$

$$a_1 = -1$$

Case II

$$\gamma = 1$$

$$a_1 = \frac{1}{2}n$$

and the gravitational constant in each case is renormalized

$$G = \frac{G_0}{1 + \frac{1}{2}n\phi} \quad \text{Case I}$$

$$G = \frac{G_0}{1 + n\phi + \frac{1}{4}n^2\phi} \quad \text{Case II} \quad (\text{III.4})$$

³John D. Anderson, et al., Proceedings of the Conference on Experimental Tests of Gravitational Theories, (NASA-JPL Technical Memorandum 33-499; 1970).

This means that G , the effective gravitational constant which enters the metric and determines the strength of gravity in the solar system, depends on background strength of K_{μ} far from the solar system. In Chapter VI it is seen from cosmological considerations that this produces a time development of G coupled to the evolution of the universe; and that the extreme weakness of G in the solar system can be viewed as being due to a renormalization of G_0 (~ 1) by a large cosmological $n\phi$.

The parameters β and δ come from the static solution of the field equations to second order. For the two cases we get

(Case I)

$$\beta = 1$$

which agrees with the value in general relativity and

$$\delta = 1 + \frac{1 - 4\omega}{2\phi^{-1} + 2\omega(1 - 4\omega)} \quad (\text{III.5})$$

which does not, general relativity having $\delta = 1$. Unfortunately, this parameter does not affect existing experiments to a measurable degree.

In the other case, β and δ are found to be

(Case II)

$$\beta = 1 + \frac{\frac{1}{4}n(n+2)(n+4)\phi}{4 + n\phi(n+4)} \quad (\text{III.6})$$

$$\delta = 1 - \frac{1}{2n\phi(n+4)} \frac{(n+3) - \frac{1}{4}n\phi(n+4)}{4 + n\phi(n+4)}$$

The experimental result $\beta \approx 1$ requires $\eta \approx 0, -2, \text{ or } -4$ (none of which provides the small α_1 required below). The unobservability of δ effects in present solar system experiments limits $\phi \sim 1$ or less.

Solving the total dynamic linearized field equations gives the additional PPN parameters:

$$\begin{aligned} \text{(Case I)} \quad \zeta_1 = \alpha_3 = 0 \\ \frac{1}{2^{\alpha_1}} = \alpha_2 = \frac{4\eta}{4\phi^{-1} + 4 + 6\eta + \eta^2} \end{aligned} \quad \text{(III.7)}$$

$$\begin{aligned} \text{(Case II)} \quad \zeta_1 = \alpha_3 = 0 \\ \alpha_1 = \frac{2\eta\phi(3 + \eta)}{1 + \eta\phi + \frac{1}{4}\eta^2\phi} \\ \alpha_2 = \frac{1}{2^{\alpha_1}} - \frac{1}{2} \frac{3\eta\phi(\eta + 2)}{2 + 4\eta\phi + \eta^2\phi} \end{aligned} \quad \text{(III.8)}$$

An examination of the configuration of a point source inside a massive spherical shell yields the second order two-mass PPN parameter;

$$\zeta_2 = 0$$

for both cases.

The ζ parameters measure 4-momentum non-conservation and are expected to be zero in theories derived from Lagrangian action principles.⁴ The α parameters measure the existence of "Machian" effects

⁴Will and Nordtvedt, op cit.

in gravitation. Except for the case $\eta = 0$, both cases predict such preferred frame effects. Will and Nordtvedt⁵ have analyzed various geophysical and planetary orbital effects to arrive at upper limits on the α parameters. The restrictions are⁶

$$\alpha_1 < .1$$

$$\alpha_2 < .1$$

In Case I, there are two ways these restrictions can be met:

$$\eta \geq 34$$

(III.9)

$$\eta \leq .1$$

Of special interest is the case where $\eta = 0$ ($\omega = 1$) in equation III.7; then α_1 and α_2 are strictly zero. In this case, renormalization of G_0 is lost and the total set of PPN parameters is identical to those of general relativity. δ still differs, however, as can be seen by setting $\omega = 1$ in equation III.5.

$$\delta = 1 + \frac{1}{2} \frac{3\phi}{3\phi - 1}$$

In Case II, the α restrictions require

⁵C. M. Will and K. Nordtvedt, Jr., *Ap. J.* 177, 775 (1972).

⁶The original value of $\alpha_2 < .03$ given by Nordtvedt and Will probably represents an overly optimistic evaluation of gravimeter results. See K. Nordtvedt, Jr., *Science* 178, 1157 (1972).

$$\eta \leq \frac{1}{60}$$

Case II already had $\omega \approx 0$, so this additional restriction on the magnitude of η shows this case to represent weak coupling to the metric.

To sum up, experimental results limit the theory to a form fulfilling one of several conditions. The first condition is that of weak cosmological vector field,

$$\phi \ll 1,$$

which is not interesting because one can always postulate any kind of field one wants as long as it is too weak to affect experimental results.

A second condition is Case II where

$$\omega \ll 1$$

$$\eta \ll 1$$

which is also not interesting, involving as it does an extremely weak coupling of the vector field to the metric. The last possible way of satisfying experimental restrictions is that of Case I, which is

$$\omega = \frac{1}{2\eta} + 1$$

$$\eta \geq 34 \quad \text{or} \quad \eta \leq .1.$$

In the interest of generality, subsequent sections will not require these conditions a priori but will refer to them from time-to-time.

CHAPTER IV

THE EVENT HORIZON

We first produce the exact field equations for a source which is spherically symmetric, static, and at rest in the universe. Then an approximate solution is found which is valid near the event horizon. Choosing Schwarzschild-type variables, the fields are written

$$g_{00} = e^{\nu}$$

$$g_{rr} = -e^{\lambda}$$

$$g_{\theta\theta} = -r^2$$

$$g_{\phi\phi} = -r^2 \sin^2 \theta$$

$$K_0 = \sqrt{\phi} e^{\mu}$$

$$K_{\ell} = 0$$

where μ , ν , and λ are functions of r only. The spherical symmetry and time independence are manifest in the form of the metric. The absence of motion through the universe is reflected in the vanishing of K_{ℓ} . If a spherically symmetric black hole were moving with respect to the rest frame of the universe, an asymmetry could be communicated to the metric via a non-zero space part of the vector field, producing an asymmetric event horizon. This possibility is worth investigating for its astrophysical implications, though that will not be done here.

Continuing with the solution, we write the action integral in terms of the Swarzschild variables. First,

$$R = \frac{e^{-\lambda}}{r^2} (2e^\lambda - 2 + 2r\lambda' + \frac{1}{2}r^2\lambda'v' - r^2v'' - 2rv' - \frac{1}{2}r^2v'^2)$$

$$R_{00} = e^{v-\lambda} (\frac{1}{4}\lambda'v' - \frac{1}{2}v'' - \frac{1}{4}v' - \frac{1}{4}v'^2)$$

$$K_{01r} = (\mu' - \frac{1}{2}v')\sqrt{\phi}e^\mu$$

$$K_{r10} = -\frac{1}{2}v'\sqrt{\phi}e^\mu$$

with all other covariant derivatives of K_μ being zero. (Prime denotes differentiation with respect to r .) Equation II.1 can then be written

$$A = \int d^4x r^2 e^{\frac{v+\lambda}{2}} [16\pi G_0 L_m + R + \omega\phi R e^{2\mu-v} + \eta\phi R_{00} e^{2\mu-2v} + 2\phi\mu'^2 e^{2\mu-v-\lambda}]$$

Invariance of the action under variation of μ gives the equation

$$4\frac{dP}{dr} + (2\omega + \eta)\frac{dQ}{dr} - 4F - 4\omega\phi e^\zeta (e^\lambda - 1 + r\lambda') = 0. \quad (IV.1)$$

Variation of v gives

$$(2\omega + \eta)\frac{dP}{dr} - (2\omega + \eta)\frac{dQ}{dr} - e^{\frac{v-\lambda}{2}} (e^\lambda - 1 + r\lambda') + \omega\phi e^\zeta (e^\lambda - 1 + r\lambda') + F = -8\pi G_0 T_0^0. \quad (IV.2)$$

Variation of λ gives

$$e^{\frac{v-\lambda}{2}} (e^\lambda - 1 - rv') + \omega\phi e^\zeta (e^\lambda - 1 + rv' - 4r\mu') - F = 0. \quad (IV.3)$$

In all of the above

$$\zeta = 2\mu - \frac{1}{2}\nu - \frac{1}{2}\lambda$$

$$P = \phi r^2 e^{\zeta} \mu'$$

$$Q = \phi r^2 e^{\zeta} \nu'$$

$$F = (2\omega + \eta) \phi r^2 e^{\zeta} \frac{1}{2} \nu' (\mu' - \frac{1}{2} \nu') + \phi r^2 e^{\zeta} \mu'^2$$

$$T_0^0 \equiv - \frac{\delta L_m}{\delta \nu}$$

These three field equations can be made simpler both for exact solution and for the present approximate solution, by some recombination. Adding together IV.1 and IV.2 gives

$$(2\omega + \eta) \frac{d}{dr} (P - Q) - e^{\frac{\nu-\lambda}{2}} (r\nu' + r\lambda') + 2\omega\phi e^{\zeta} (e^{\lambda} - 1 - r\zeta') = 0 \quad (\text{IV.4})$$

where the solution is to be taken external to the source ($T_0^0 = 0$).

One-fourth of equation IV.1 subtracted from equation IV.3 gives

$$\frac{dP}{dr} + \frac{1}{4}(2\omega + \eta) \frac{dQ}{dr} - 2\omega\phi e^{\zeta} (e^{\lambda} - 1 - r\nu') - e^{\frac{\nu-\lambda}{2}} (e^{\lambda} - 1 - r\zeta') = 0. \quad (\text{IV.5})$$

The field equations for a static source (IV.1 to IV.5) have been written in a form which, it is hoped, should be amenable to exact solution, though we have not made progress in this direction. Here we will describe an approximate solution, valid near an event horizon,

obtained by a method which should be useful in other complicated theories of gravity as well.

The event horizon around a static spherically symmetric source is a surface $r = r_0 = \text{const}$ on which $g_{00} = 0$. Close to the horizon $\frac{r - r_0}{r_0} < 1$ and we can expand the fields in a power series.

$$\begin{aligned} g_{00} &= e^v = \left(\frac{r - r_0}{r_0}\right)^s \left[a + \sum_{n=1}^{\infty} a_n \left(\frac{r - r_0}{r_0}\right)^n \right] \\ g_{rr} &= -e^\lambda = -\left(\frac{r - r_0}{r_0}\right)^t \left[b + \sum_{n=1}^{\infty} b_n \left(\frac{r - r_0}{r_0}\right)^n \right] \\ K_0 &= \sqrt{\phi} e^\mu = \left(\frac{r - r_0}{r_0}\right)^u \left[c + \sum_{n=1}^{\infty} c_n \left(\frac{r - r_0}{r_0}\right)^n \right]. \end{aligned} \quad (\text{IV.6})$$

Requiring that $s > 0$ guarantees that $r = r_0$ is indeed an event horizon. Sufficiently close to the horizon, the metric and vector fields may be approximated by their leading terms alone. When this approximation is substituted into equations IV.5, IV.4, and IV.3, they become, respectively

$$\begin{aligned} \lambda(u - s) \left[(w - 1)x^{w-2} + 2x^{w-1} \right] - \frac{a}{c} x^{p-1} (s + t) + 2\omega x^w \left(bx^t - 1 - \frac{w}{x} \right) \\ = 0 \end{aligned} \quad (\text{IV.7})$$

$$\begin{aligned} \left(u + \frac{1}{4} \lambda s \right) \left[(w - 1)x^{w-2} + 2x^{w-1} \right] - \frac{a}{c} x^p \left(bx^t - 1 - \frac{s}{x} \right) \\ - 2\omega x^w \left(bx^t - 1 - \frac{w}{x} \right) = 0 \end{aligned} \quad (\text{IV.8})$$

$$\lambda x^{w-2} \frac{1}{2s} (u - \frac{1}{2}s) + x^{w-2} u^2 - \frac{a}{2x^p} (bx^t - 1 - \frac{s}{x}) + \omega x^w (bx^t - 1 - \frac{s}{x} - \frac{4u}{x}) = 0 \quad (V.9)$$

where we have defined

$$x = \frac{r - r_0}{r_0},$$

and

$$w = 2u - \frac{1}{2}s - \frac{1}{2}t$$

$$p = \frac{1}{2}s - \frac{1}{2}t$$

$$\lambda = 2\omega + \eta.$$

The details of the solution for s , t , and u are given in Appendix

B. The result is

$$g_{00} = a \frac{r - r_0}{r_0}$$

$$g_{rr} = -[1 + \frac{1}{4} \frac{c^2}{a} (2\omega + \eta + 4)] \frac{r_0}{r - r_0}$$

$$K_0 = c \frac{r - r_0}{r_0}$$

with a , c , and r_0 still arbitrary. The arbitrariness of a stems from the freedom to redefine time anyway we choose. c just represents the freedom in the cosmological value of the vector field.

Since the metric has the same dependence on $r - r_0$ as does the Schwarzschild metric, the behavior near the horizon is similar to that of general relativity. In particular, the singularity in g_{rr} at $r = r_0$ is only a coordinate singularity, the physical components of the Riemann tensor remaining regular across the boundary.

CHAPTER V

RADIATION

A. Linearized Gravitational Waves. We now consider the propagation of weak gravitational waves through a region of empty space. The fields are written

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$K^\mu = \phi^\mu + A^\mu$$

where $\eta_{\mu\nu}$ and ϕ^μ are the constant cosmological background values of the fields and $h_{\mu\nu}$ and A^μ are the local wave perturbations. By "weak" waves, it is meant that, over the region of space to be considered, $h_{\mu\nu} \ll \eta_{\mu\nu}$ and $A^\mu \ll \phi^\mu$. Coordinates are chosen so that $\eta_{\mu\nu}$ is the Minkowski metric. In all that follows we consider plane waves, written

$$h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik_\mu x^\mu}$$

$$A^\mu = a^\mu e^{ik_\mu x^\mu}.$$

k_μ is a constant propagation four-vector which will be taken to represent waves moving in the z-direction, i.e., $k_\mu = (\omega, 0, 0, -k)$. The speed of propagation of the radiation is

$$c = \frac{\omega}{k}$$

and may or may not be equal to the speed of light (1 in the units we have chosen).

Substituting these expressions for the fields into the field equations and discarding terms second order or higher (i.e., terms like A^2 , Ah , h^2) produces the following linearized equations (details of the derivation are given in Appendix C).

The g (contracted) equation (equation II.3) is

$$R + \eta \phi^\alpha \phi^\beta R_{\alpha\beta} + 3\Box^2 P - 2\eta \phi^\alpha F_{\alpha,\beta}^\beta = 0$$

where $P \equiv (2\omega + \eta)(\phi^\alpha A_\alpha - \frac{1}{2}\phi^\alpha \phi^\beta h_{\alpha\beta})$ has been defined for notational ease. The K_μ equation (equation II.4) is

$$\omega \phi_\mu R + \eta \phi^\alpha R_{\mu\alpha} + 2F_{\mu,\alpha}^\alpha = 0$$

The $g_{\mu\nu}$ equation (II.2) is simplified by adding combinations of other equations to it;

$$\begin{aligned} & R_{\mu\nu} (1 + \omega \phi^2) + \eta \phi^\alpha \phi^\beta R_{\mu\alpha\nu\beta} - \omega \phi_\mu \phi_\nu R + \frac{1}{2} g_{\mu\nu} \Box^2 P + P_{,\mu\nu} \\ & + g_{\mu\nu} \phi^\alpha F_{\alpha,\beta}^\beta - \frac{1}{2} \eta \phi^\alpha (F_{\alpha\mu,\nu} + F_{\alpha\nu,\mu}) - \left(\frac{1}{2}\eta + 2\right) (\phi_\mu F_{\nu,\alpha}^\alpha - \phi_\nu F_{\mu,\alpha}^\alpha) \\ & = 0. \end{aligned}$$

There are a total of fourteen functions which appear in these equations, ten components of $h_{\mu\nu}$ and four of K_μ . However, except for the scalar P , $h_{\mu\nu}$ appears only in the components of the Riemann tensor of which only six are independent, and A_μ appears only in $F_{\mu\nu,\alpha}$ which only

has three independent components. Therefore, a careful check of any solution is necessary to be sure that it is consistent with all fourteen independent equations.

As was mentioned, coordinates were chosen so that the background metric is locally Minkowskian and the z-axis is along the direction of propagation of the wave. There still remains the freedom of inertial frame which we choose to be the rest frame of the universe (i.e., that frame in which $\phi_k = 0$). The vanishing or non-vanishing of the various components of $R_{\mu\alpha\nu\beta}$ and A_μ will of course depend on this choice of Lorentz frame, but, once the linearized solutions have been worked out in this frame, they may be found in any frame by Lorentz transformations since we are working in a flat background metric.

B. Results. The solutions are worked out in detail in Appendix C. Here we simply present the results within a classification scheme worked out by Douglas Eardley, et al.¹ In this scheme, the six independent components of the Riemann tensor are combined into four Newman-Penrose functions, two of which are complex. The reason for this recombination is that each of these functions affects a gravitational wave antenna in a characteristic way. The reader is referred to the work of

¹D. Eardley, et al., Phys. Rev. Letters 30, 884 (1973).

Eardley, et al., for details. The four functions are (where the 3-axis is taken along the direction of propagation)

$$\psi_2 = -\frac{1}{6}R_{0303}$$

$$\psi_3 = \frac{1}{2}(-R_{0103} + iR_{0203})$$

$$\psi_4 = R_{0101} - R_{0202} + 2iR_{0102}$$

$$\phi_{22} = -(R_{0101} + R_{0202})$$

Besides affecting an antenna, the passage of radiation could also affect the gravitational constant in PPN-type experiments (see Chapter III). The effect is proportional to

$$K_\mu K_\nu g^{\mu\nu} = (\phi_\mu + A_\mu)(\phi_\nu + A_\nu)(\eta^{\mu\nu} + h^{\mu\nu})$$

In the universe rest frame this becomes

$$K_\mu K_\nu g^{\mu\nu} = \phi_0^2 + 2\phi_0 A_0 - \phi_0^2 h_{00}$$

where the fact that $h^{00} = -h_{00}$ has been used.

We describe the separate modes in Tables I and II.

TABLE I. $n = 0$; five independent modes.

Independent Mode	Speed ($c^2 =$)	Other Fields	Correction to $K_{\mu} K^{\mu}$
Φ_{22}	1	None	$\frac{1 + \omega \phi^2}{\omega} (-k^2) \Phi_{22}$
$\text{Re}\psi_4$	1	None	None
$\text{Im}\psi_4$	1	None	None
A_1	1	None	None
A_2	1	None	None

$$\text{Re}\psi_3 = \text{Im}\psi_3 = \psi_2 = 0$$

At this point it is recalled that the solution has been found only in a particular Lorentz frame. Eardley, *et al.*, have shown, however, that if ψ_3 and ψ_2 are null waves ($c = 1$) and if $\psi_3 = \psi_2 = 0$ in any frame, then they are zero in all frames.

In the most general case ($n \neq 0$) it is found that all four of the Newman-Penrose functions are present, though they travel at different speeds.

TABLE II. $n \neq 0$; five independent modes.

Independent Mode	Speed ($c^2 =$)	Other Fields	Correction to $K_\mu K^\mu$
$\text{Re}\psi_4$	$1 - \frac{\eta\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	None	None
$\text{Im}\psi_4$	$1 - \frac{\eta\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	None	None
$\text{Re}\psi_3$	$1 + \frac{1}{4} \frac{\eta^2 \phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	$\square^2 A_1 = \frac{\eta\phi}{2c} \text{Re}\psi_3$	None
$\text{Im}\psi_3$	$1 + \frac{1}{4} \frac{\eta^2 \phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	$\square^2 A_2 = \frac{\eta\phi}{2c} \text{Im}\psi_3$	None
ϕ_{22}	$1 + \frac{1}{3} \frac{\eta\phi^2 + \eta^2 \phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	$\psi_2 = \frac{c^2}{1 - c^2} \phi_{22}$ $F_0^\alpha{}_{,\alpha} = \frac{\eta\phi}{2c} \phi_{22}^*$	$2 \frac{1 + \omega\phi^2 + \eta\phi^2}{2\omega + \eta}$ $\chi(-c^2 k^2) \phi_{22}$

* $F_0^\alpha{}_{,\alpha}$ is used instead of A_0 or A_3 to avoid possible non-physical "coordinate ripple" waves. See Appendix D.

Eardley, et al., have classified theories of gravity according to which of the metric-wave functions vanish, as follows:

Class II₆. $\psi_2 \neq 0$. All observers in such Lorentz frames measure a non-zero amplitude in the ψ_2 mode and agree on the value of this amplitude. (But they will generally disagree about the presence or absence and amplitude of all other modes.)

Class III₅. $\psi_2 \equiv 0 \neq \psi_3$. All observers agree on the absence of ψ_2 and the presence of ψ_3 . (But they generally disagree about the presence or absence of ψ_4 and ϕ_{22} .)

Class N₃. $\psi_2 \equiv 0 \equiv \psi_3$; $\psi_4 \neq 0 \neq \phi_{22}$. All observers agree about the presence or absence of all modes.

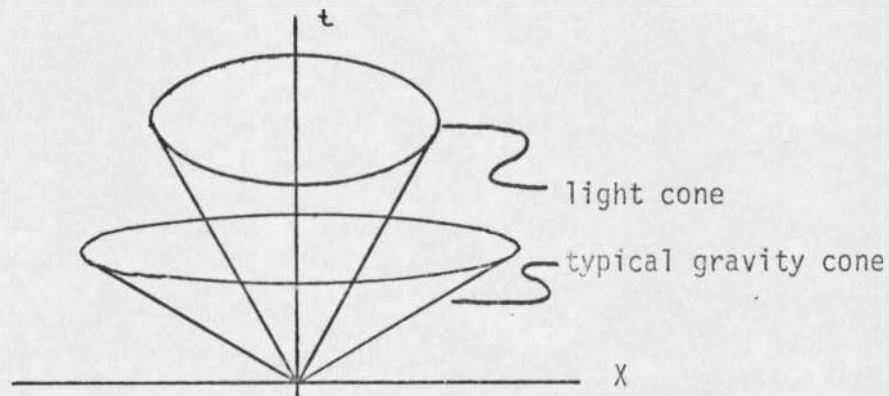
Class N₂. $\psi_2 \equiv 0 \equiv \psi_3$; $\psi_4 \neq 0 \equiv \phi_{22}$. All observers agree.

Class O₁. $\psi_2 \equiv 0 \equiv \psi_3$; $\psi_4 \equiv 0 \neq \phi_{22}$. All observers agree.

The version of the theory with $\eta = 0$ is therefore of class N₃, while the general $\eta \neq 0$ version is of class II₆. In addition the $\eta \neq 0$ case has gravitational waves traveling at speeds greater or less than the speed of light, depending on the values of ω and η . The slower speed is interesting but not shocking. The faster speed requires some additional discussion.

C. Causality. The usual argument against the existence of real faster-than-light signals is that such signals would violate causality. If by causality it is meant that causes must precede their effects, as seen by all observers, then causality is certainly violated, but that is an arbitrary and unnecessary concept of causality. If by causality it is simply meant that logical paradoxes (i.e., A both sends and does not send a signal to B) may not occur, then causality in this sense is not violated by the faster-than-light signals of this theory.

In the rest frame of the universe, there are three "gravity cones" in addition to the light cone, corresponding to the three different speeds of propagation of the II_6 waves.



As long as there exists a maximum speed of propagation, it is physically impossible to produce closed causal loops in space-time, so it is impossible for any experimenter to influence his past and produce causal paradoxes.

CHAPTER VI

COSMOLOGY

To investigate cosmological solutions of the theory, we use a homogeneous dust model for matter and a Robertson-Walker cosmological metric,

$$g_{00} = 1$$

$$g_{ij} = - \frac{S(t)^2}{(1 + \frac{\kappa r^2}{4})^2} \delta_{ij},$$

from which we can calculate

$$R_{00} = 3\frac{\dot{S}}{S}$$

$$R = \frac{6}{S^2}(S\ddot{S} + \dot{S}^2 + \kappa),$$

the dot denoting time differentiation. In the universe rest frame

$$K_0 \equiv \sqrt{\phi} \neq 0$$

$$K_\ell = 0,$$

and we can write down the K_μ derivatives.

$$(K^{\alpha\beta})_{|\alpha\beta} = \ddot{\phi} + 3\phi\frac{\ddot{S}}{S} + 6\phi\frac{\dot{S}^2}{S^2} + 6\dot{\phi}\frac{\dot{S}}{S}$$

$$\square^2\phi = \ddot{\phi} + 3\dot{\phi}\frac{\dot{S}}{S}$$

$$\phi_{|00} = \ddot{\phi}$$

$$\square^2(K_0 K_0) = \ddot{\phi} + 3\dot{\phi} \frac{\dot{S}}{S} - 6\phi \frac{\dot{S}^2}{S^2}$$

$$(K^\alpha K_0)_{;0} = \ddot{\phi} + 3\dot{\phi} \frac{\dot{S}}{S} - 3\phi \frac{\dot{S}^2}{S^2}$$

Substituting these into equation II.4, gives

$$(2\omega + \eta) \frac{\ddot{S}}{S} + 2\omega \frac{\dot{S}^2 + \kappa}{S^2} = 0. \quad (\text{VI.1})$$

And in equation II.2, there results

$$(1 - \omega\phi) \frac{\dot{S}^2 + \kappa}{S^2} + \eta\phi \frac{\dot{S}^2}{S^2} - (2\omega + \eta)\phi \frac{\ddot{S}}{S} + (2\omega + \eta) \frac{1}{2} \frac{\dot{S}}{S} = \frac{8\pi G_0 T_{00}}{3}$$

Using VI.1 to eliminate second derivatives of S reduces this to

$$(1 + \omega\phi) \frac{\dot{S}^2 + \kappa}{S^2} + \eta\phi \frac{\dot{S}^2}{S^2} + (2\omega + \eta) \frac{1}{2} \frac{\dot{S}}{S} = \frac{8\pi G_0 T_{00}}{3} \quad (\text{VI.2})$$

The K_ℓ and $g_{0\ell}$ equations vanish identically, and the g_{ij} equation is consistent with VI.1 and VI.2.

Equation VI.1 integrates to give

$$\dot{S}^2 + \kappa = \left(\frac{S_0}{S}\right)^p \quad (\text{VI.3})$$

where S_0^p is a constant of integration,

$$p \equiv \frac{2\omega}{\omega + \frac{1}{2}\eta}, \text{ and we define}$$

$$q \equiv \frac{\eta}{\omega + \frac{1}{2}\eta} \text{ for future use.}$$

Equation VI.3 integrates again, yielding

$$t = \int \frac{S^{\frac{1}{2}p} dS}{\sqrt{S_0^p - S^p}} \quad (\text{VI.4})$$

The time development of S depends only on the scale constant S_0 and on the parameters ω and η . It is independent of $T_{\mu\nu}$.

Equation VI.2 is made first order in $\frac{d\phi}{dS}$ as follows. Using equation VI.3 to eliminate \dot{S} and dividing by $\omega + \frac{1}{2\eta}$, equation VI.2 becomes

$$\frac{d\phi}{dS} \frac{1 - \kappa x^p}{S x^p} + \left(\frac{1}{2}p + q\right) \phi \frac{1 - \kappa x^p}{S^2 x^p} + \frac{1}{2}p \phi \frac{\kappa}{S^2} = \frac{1}{\omega + \frac{1}{2\eta}} \left(\frac{8\pi G_0 T_{00}}{3} - \frac{1}{S^2 x^p} \right)$$

where we have defined $x \equiv \frac{S}{S_0}$. Multiplying by S_0^2 and defining

$$Y = \phi \frac{1 - \kappa x^p}{x^{p+1}}$$

$$\mu = \frac{2G_0 M}{S_0} = \frac{8\pi G_0 T_{00} S_0^3}{3S_0}$$

produces the first order equation

$$\frac{dY}{dx} + \frac{Y}{x} \left(1 + q + \frac{\frac{3}{2}p}{1 - \kappa x^p}\right) = \frac{1}{\omega + \frac{1}{2\eta}} \left(\frac{\mu}{x^3} - \frac{1}{x^{p+2}} \right),$$

with solution

$$\phi = \frac{1 - \kappa x^p}{(\omega + \frac{1}{2\eta}) x^{\frac{1}{2}q+1}} \int \frac{\mu x^{\frac{1}{2}p} - x^{\frac{1}{2}q}}{(1 - \kappa x^p)^{\frac{3}{2}}} dx + \lambda \frac{\sqrt{1 - \kappa x^p}}{x^{\frac{1}{2}q+1}} \quad (\text{VI.5})$$

where λ is a constant of integration.

The special case $\eta = 0$, $\omega = 1$ is of particular interest. It is the case which exactly reproduces the PPN parameters of general relativity. It will here be shown to have a cosmological solution different from the cosmology of general relativity. Also this case represents an approximate solution for one of the cases allowed by equation III.9, namely $\eta < .1$, $\omega \approx 1$. Choosing the $\kappa = 1$ of a closed cosmology and noting $p = 2$, $q = 0$, equation VI.4 integrates to give

$$S = \sqrt{t(2S_0 - t)},$$

and equation VI.5 yields

$$1 + \phi = \frac{\mu}{x} - \lambda \frac{\sqrt{1 - x^2}}{x}.$$

If we demand $\phi \rightarrow 0$ as $t \rightarrow 0$ (the initial cosmological singularity, see Dicke¹), then for $\phi \gg 1$ (equivalent to $G_0 \gg G$),

$$\phi = \frac{2G_0 M}{S} (1 - \sqrt{1 - x^2}).$$

Assuming $S_0 \gg S$, these become

$$\begin{aligned} S &\approx \sqrt{2S_0 t} \\ \phi &\approx \frac{G_0 M S}{S_0^2} \end{aligned} \tag{VI.6}$$

¹C. Brans and R. H. Dicke, Phys. Rev. 124, 425 (1961).

The relationship between Hubble time (T_H) and the present age of the universe t_u is

$$\frac{1}{H} \equiv T_H \equiv \frac{S}{\dot{S}} \approx 2t_u$$

Writing

$$M \equiv \frac{4\pi}{3}\rho S^3 = \frac{4\pi}{3}\rho (S_0 T_H)^{3/2},$$

some substitutions give $\frac{\phi}{G_0}$ in terms of observables,

$$\frac{\phi}{G_0} = \frac{4\pi}{3}\rho T_H^2.$$

Using this in equation III.4 gives

$$\frac{1}{G} = \frac{2\pi}{3}\rho n T_H^2 \sim 10^4 \quad (\text{c.g.s units}) \quad (\text{VI.7})$$

based on $\rho \approx 10^{-31}$ g/cm³ and $T_H \approx 2 \times 10^{10}$ years. The actual value of $\frac{1}{G}$ is about 10^7 , but there is substantial uncertainty in the total energy density (ρ) of the universe. From equation VI.6 the early-time behavior of the universe is

$$S \sim t^{1/2} \quad (\text{VI.8})$$

as compared with general relativity's

$$S \sim t^{2/3}$$

A second limiting cosmology, compatible with the second case allowed by equation III.9, is $\omega \approx \frac{1}{2n} \gg 1$. Then $p = q = 1$, and for $\kappa = 1$ we find

$$\frac{t}{S_0} = \sin^{-1} \sqrt{x} - \sqrt{x - x^2}$$

$$\phi = 2 \frac{(\mu - 1) \sqrt{1 - x}}{n x} \left(\frac{1}{\sqrt{1 - x}} - \frac{\sin^{-1} \sqrt{x}}{\sqrt{x}} \right)$$

where we have again assume $S \rightarrow 0$, $\phi \rightarrow 0$ as $t \rightarrow 0$. The early time behavior is

$$t = \frac{2}{3} S_0 x^{3/2}, \quad (VI.9)$$

or

$$S \sim t^{2/3}, \quad (VI.10)$$

and

$$\frac{\phi}{G_0} = \frac{8MS}{15nS_0^2}$$

From equation VI.9, the age of the universe is related to the Hubble age by

$$T_H \equiv \frac{S}{\dot{S}} = \frac{3}{2} t_u$$

Then M can be written

$$M = \frac{4\pi}{3} \rho S_0 T_H^2,$$

and ϕ/G_0 becomes

$$\frac{\phi}{G_0} = \frac{32\pi}{45\eta} \rho T_H^2 \left(\frac{T_H}{S_0}\right)^{2/3}.$$

Finally, using this in equation III.4 gives

$$\frac{1}{G} = \frac{32\pi}{90} \rho T_H^2 \left(\frac{T_H}{S_0}\right)^{2/3} \sim 10^4 \quad (\text{VI.11})$$

if $T_H/S_0 \sim .1$.

A final choice of parameters compatible with Chapter III's case II is $\omega = 0$. In this case, equation VI.1 reduces to

$$\ddot{\frac{S}{\eta}} = 0,$$

having solution

$$S = vt$$

where v is a constant. Equation VI.2 becomes

$$\frac{\dot{S}^2 + \kappa}{S^2} + \eta \phi \frac{\dot{S}^2}{S^2} + \frac{1}{2} \eta \phi \frac{\dot{S}}{S} = \frac{8\pi G_0 T_{00}}{3}$$

which has solution

$$\phi = \frac{2\lambda}{\eta v^3} t^{-2} + \frac{4G_0 M}{\eta v^3} t^{-1} - \frac{v^2 + \kappa}{\eta v^2}$$

and is singular at the origin.

Therefore here, as in Chapter III, case II is seen to be much less interesting than case I. The most interesting predictions in the case I solutions are the early time behavior (equations VI.8 and VI.10) and the functional form of the variation of G with the evolution of the universe (equations VI.7 and VI.11). The former is important in formation of the elements during the big bang;² the latter determines the history of the gravitational constant in the universe.

²See for example G. Gamow, *Revs. Modern Phys.* 21, 367 (1949).

CHAPTER VII

CONCLUSIONS

A. Comparison of Theory and Experiment. In this work we have made several predictions of observable results of the vector-metric theory. We would like to review these here and make some comments on the empirical status of each result. A good summary of experimental gravity is found in Kip Thorne's Lecture to the International Union of Pure and Applied Physics in 1972.¹

1. PPN Parameter γ . The most precise test of γ to date has been the time delay experiment using active radar to Mariners VI and VII. Anderson, et al.,² find a result of

$$\frac{1}{2}(1 + \gamma) = 1.00 \pm .04,$$

or

$$\gamma = 1.00 \pm .08.$$

We chose to consider $\omega = \frac{1}{2} + 1$ so as to predict a value of

$$\gamma = 1,$$

¹K. S. Thorne, Available as Cal Tech Preprint OAP-321 (1972).

²John D. Anderson, et al., Proceedings of the Conference on Experimental Tests of Gravitational Theories (NASA-JPL Technical Memo 33-499, 1970).

in agreement with experiment.

2. PPN Parameter β . The vector-metric theory predicts a value

$$\beta = 1..$$

The best experimental determination of β comes from the relativistic perihelion shift of Mercury, which is known to be³

$$\delta\Omega = 43 \pm 1 \text{ secs./century} = 43(1.00 \pm .02)$$

When $\delta\Omega$ is determined theoretically it is found that it is actually proportional to β , γ , and all the α -parameters, but it can be shown, by comparison with the earth's perihelion shift, that the α dependence is small and the deflection is basically just

$$\delta\Omega = \frac{43}{3}(2\gamma + 2 - \beta).$$

The limiting factor on the accuracy is seen to be the accuracy of γ . Therefore β can be no more accurate than γ :

$$\beta = 1.00 \pm .08.$$

The vector-metric theory is in agreement with the experimental β .

³C. M. Will, Lectures in B. Bertotti, ed., Proceedings of Course 56 of the International School of Physics "Enrico Fermi", Academic Press (1973), p. 129.

3. PPN Parameters α_j . The vector-metric theory predicts values for α_j of

$$\alpha_1 = \frac{8\eta}{4\phi^{-1} + 4 + 6\eta + \eta^2}$$

$$\alpha_2 = \frac{4\eta}{4\phi^{-1} + 4 + 6\eta + \eta^2}$$

$$\alpha_3 = 0.$$

A combination of earth tide, earth rotation rate, and perihelion shift data⁴ give experimental limits of

$$\alpha_1 < .1$$

$$\alpha_2 < .1$$

$$\alpha_3 < 10^{-6}.$$

The predicted α_3 parameter is well within experimental limits. The required smallness of the α_1 and α_2 parameters can only be met if

$$\eta \leq .1$$

or

⁴Ibid., p. 146 and K. Nordtvedt, Jr., Science 178, 1157 (1972).

$$\eta \geq 34$$

Therefore, a version of the theory in which η satisfies these restrictions will predict α -parameters in agreement with experiment.

4. PPN Parameters ζ_1 . The vector-metric theory predicts values of zero for both ζ_1 and ζ_2 , as do all theories in which energy and momentum are conserved. The best experimental test of these parameters is the Lunar Laser Ranging experiment, whose results are not yet in.⁵ The predicted change in lunar range is given by

$$\delta S = 20(4\beta - 3 - \gamma - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2) \text{ meters,}$$

and the expected accuracy of the experiment should be about 0.1 meters. A non-zero value of $\zeta_1 + \frac{1}{2}\zeta_2$ returned by this experiment would certainly disprove the vector-metric theory, but it would also disprove general relativity, Brans-Dicke theory, and any other conservative theory.

5. Black Holes. The vector-metric theory predicts the existence of black holes as do general relativity and many other theories. The precise details of collapse and analysis of the implications of possible non-spherical event horizons (see page 24) might have observable

⁵Nordtvedt, op cit.

astrophysical properties but that work has not yet been done. As for the existence of black holes, observational evidence is still very tentative.

6. Speed of Gravity Waves. One form ($\eta = 0$) of the vector-metric theory predicts that gravity waves will travel at the speed of light. The other form ($\eta \neq 0$) predicts as many as three separate speeds which may be greater or less than the speed of light, depending on the values of ω and η . If $\omega = \frac{1}{2}\eta + 1$ and if the background strength of the vector field is taken to be large compared to one, the speeds are

$$c^2 = 1 - \frac{2\eta}{3\eta + 2}$$

$$c'^2 = 1 + \frac{1}{2} \frac{\eta^2}{3\eta + 2}$$

$$c''^2 = 1 + \frac{2}{3} \frac{\eta + \eta^2}{3\eta + 2}$$

No experiment has been done to measure the speed of propagation of gravity. What is needed is additional refinement of gravity wave detectors to give us more confidence in them and to enable us to detect the incoming direction with more accuracy. Then the time of arrival of gravity waves and electromagnetic waves created by a supernova or other discrete event can be compared to determine the speed. Eardley, et al.,⁶

⁶D. Eardley, et al., Phys. Rev. Letters 30, 884 (1973).

have considered this type of experiment and expect that it could have an accuracy of $-10^{-9} \times \frac{\text{time lag precision}}{\text{week}}$.

7. Modes of Gravity Waves. The two forms of the vector-metric theory make different predictions about the kinds of polarizations present. The $n = 0$ version has

ϕ_{22} , ψ_4 present;

ψ_3 , ψ_2 not present.

The $n \neq 0$ version has all four functions (both real and complex parts) present. The only detection of gravitational radiation to date has been that of Weber⁷ whose antenna is not sensitive to the polarization of the wave which is exciting the apparatus. However, he has used a special detector, a disk, which should be preferentially sensitive to ϕ_{22} type waves and he has seen no radiation in this mode. It should also be pointed out that just recently Tyson⁸ has published results describing his observations with a detector about 10^2 times more sensitive than Weber's original antenna, and he has seen no radiation in any mode. The whole question of detection of gravity waves is yet to

⁷J. Weber in B. Bertotti, ed., Proceedings of Course 56 of the International School of Physics "Enrico Fermi", Academic Press (1973).

⁸J. A. Tyson, Phys. Rev. Letters 31, 261 (1973).

be settled. When it is, it should prove to be a powerful tool for testing gravity theories.

8. Renormalization of G. When $n = 0$ there is no renormalization of the gravitational constant (see equation III.4), but $n \neq 0$ allows us to calculate G from some basic cosmological data. When n is small but not zero, the gravitational constant is given by

$$\frac{1}{G} = \frac{2\pi}{3} \rho T_H^2 n.$$

When n is large, it is

$$\frac{1}{G} = \frac{32\pi}{90} \rho T_H^2 \left(\frac{T_H}{S_0}\right)^{2/3}.$$

T_H is the Hubble time which is calculated from cosmological observations of redshift versus distance. It has changed over the years⁹ as astronomers have re-evaluated such things as the period-luminosity ratio for Cepheid variables, but it appears to have stabilized at about 2×10^{10} years. The density, ρ , is also tied up with the determination of the Hubble time, since we use Hubble's constant ($H = \frac{1}{T_H}$) to determine the distance to galaxies whose mass we are counting in order to arrive at a mean density. The present estimate of ρ is $\rho = 10^{-31} \text{g/cm}^3$, based

⁹See for example E. L. Schutzman, The Structure of the Universe, McGraw-Hill (1968), p. 12.

on the counting of galaxies. If these values are substituted into the expressions for $\frac{1}{G}$, there results

$$\frac{1}{G} \sim 10^5 \times \begin{cases} n & n \ll 1 \\ (T_H/S_0)^{2/3} & n \gg 1. \end{cases}$$

The actual value of $\frac{1}{G}$ is 10^7 in c.g.s units. At this point, however, it is not known how much energy density might be present in the universe in terms of gas, dust, or other forms of energy. Therefore, observational cosmology is not in a position at present to either prove or disprove the theory.

9. Time Variation of G. If ϕ is large, we have the relation

$$G = \frac{G_0}{1 - 2^n \phi}.$$

Therefore $\frac{\dot{G}}{G}$ is given by

$$\frac{\dot{G}}{G} = - \frac{\dot{\phi}}{\phi} = - \frac{\dot{S}}{S} = -H = -10^{-10} \text{ years}^{-1}.$$

since in each case $\phi \propto S$. Dicke has shown¹⁰ that such a variation is not inconsistent with geophysical, planetary, and astrophysical data.

¹⁰R. H. Dicke and P.J.E. Peebles, Space Science Reviews 4, 419 (1965).

B. Future Experiments. We have described nine predictions of the vector-metric theory, some of which are well within well established experimental results, some of which may well be proven wrong by future experiments. Here we will briefly describe what we feel is the most promising way to completely eliminate the vector-metric theory using future experiments and observations. We treat the $\eta \neq 0$ and the $\eta = 0$ cases separately.

1. $\eta \neq 0$. As was seen in number 3 above, present results for α_1 and α_2 effects already limit η to

$$\eta < .1$$

or

$$\eta > 34.$$

Greater accuracy in these kinds of Machian effect experiments or new types of more sensitive experiments could further restrict η . No matter how accurately these experiments are done, however, the η parameter can always slide away in either the very small or the very large direction to provide agreement with experiment. What is needed are two additional observations which nail the theory down--one for each direction.

The best candidate to stop the $n \ll 1$ slide is an observational estimate of the energy density in the universe. $\frac{1}{G}$ is proportional to n ,

$$\frac{1}{G} = \frac{2\pi}{3} T_H^2 n.$$

The Hubble time (T_H) is fairly well known. If astronomers can set a maximum order of magnitude value on the energy density of the universe (ρ) then the observed gravitational constant ($\frac{1}{G} \sim 10^7$) can provide a lowest acceptable value for n .

In the $n \neq 0$ radiation solutions (see number 6 above) we found that the speed of propagation of some of the waves was proportional to n ,

$$c^2 = 1 + \frac{1}{2} \frac{n^2}{3n + 2}.$$

When $n \gg 1$, this reduces to $c^2 = 1 + \frac{1}{6}n$. If current work in gravitational radiation detection continues, one might expect to observe the light and the gravity waves from a supernova or other large discrete event and compare their arrival times to set a limit on the amount by which the speed of propagation of the gravity wave differs from the speed of light. This can set a maximum value on n .

Therefore a combination of Machian effect experiments and energy density observations, and Machian effect experiments and gravity wave propagation speed experiments stands a good chance of showing the $n \neq 0$ version of the vector-metric theory to be incorrect.

2. $\eta = 0$. The experimental disproof of this form of the theory will be much harder. Its PPN metric is exactly the same as general relativity's. Its gravity waves travel at the speed of light. There is no G renormalization to be tested. The only real opportunity to negate this theory during the next decade or so is in the polarization of gravitational radiation. General relativity predicts that matter generates and that space propagates Ψ_4 -polarized waves. We have seen that the $\eta = 0$ vector-metric theory predicts the propagation of both Ψ_4 and Φ_{22} waves. It is crucial here to solve the problem of generation of gravitational radiation in the vector-metric theory to see if these scalar-type Φ_{22} waves are created by gravitational events or if they are in some way inhibited. If the creation problem is solved to show that Φ_{22} waves should indeed exist, then the demonstration of their non-existence (via disk-type antenna detectors) would eliminate this theory along with several others and would greatly increase our confidence in general relativity. Of course, the detection of Φ_{22} waves would favor the vector-metric theory over general relativity (though scalar-metric theories also predict Φ_{22} waves).

APPENDICES

APPENDIX A

The calculation of the PPN metric parameters consists of three parts. First, solution of the dynamic (moving sources) linearized equations is undertaken to obtain all potentials first order in $\frac{GM}{r}$. Second, the static field equations are solved to second order in $\frac{GM}{r}$ in order to find β and δ . Third, a method is devised to get the two-source metric potential with PPN coefficient involving ζ_2 :

Using the PPN metric for a single source (equation III.1), and the expansion of K_μ (equation III.2), one can calculate the following derivatives to linear order in $\frac{GM}{r}$:

$$K_0 K_{0100} = \phi \frac{GM}{r^3} (1 + a_1) \left[3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2 + \vec{r} \cdot \vec{a} \right]$$

$$K_0 K_{010k} = K_0 K_{01k0} = -\phi \frac{GM}{r^3} (1 + a_1) \left[3 \frac{\vec{r} \cdot \vec{v}}{r^2} x^k - v^k \right]$$

$$K_0 K_{01kk} = \phi \frac{GM}{r^3} (\zeta_1 - 2a_3) \left[3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2 \right] - 2\phi \frac{GM}{r^3} \vec{r} \cdot \vec{a}$$

$$-2\pi\phi GM\delta(\vec{r}) [2a_1 + 2a_2 v^2 + 2 + v^2(2\gamma + 1 + \alpha_3 + \zeta_1)]$$

$$K_0 K_{k100} = -\phi \frac{GM}{r^3} \left[3 \frac{\vec{r} \cdot \vec{v}}{r^2} x^k - v^k \right]$$

$$K_0 K_{k10k} = \phi \frac{GM}{r^3} (\zeta_1 + d' - d) \left[3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2 \right] + \phi \frac{GM}{r^3} (d' - d) \vec{r} \cdot \vec{a}$$

$$-2\pi\phi GM\delta(\vec{r}) [2 + v^2(2\gamma + 1 + \alpha_3 + \zeta_1)]$$

$$K_0^0 K_{k1k0} = \phi \frac{GM}{r^3} (d' - d + \frac{1}{2}\alpha_1 - \alpha_2 + 1 + \zeta_1 - \gamma) [3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2 + \vec{r} \cdot \vec{a}]$$

$$K_0^0 K_{k1\ell\ell} - K_0^0 K_{\ell 1k\ell} = -\phi \frac{GM}{r^3} (d' - d) [3 \frac{\vec{r} \cdot \vec{v}}{r^2} x^k - v^k] - 4\pi GM \phi \delta(\vec{r}) dv^k.$$

Components of $R_{\mu\nu}$ are

$$\begin{aligned} R_{00} &= \frac{GM}{r^3} (\gamma - 1 + \alpha_2 - \frac{1}{2}\alpha_1) [3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2] + \frac{GM}{r^3} (\gamma - 1 + \alpha_2 - \frac{1}{2}\alpha_1) \\ &\quad - \zeta_1 \vec{r} \cdot \vec{a} - 2\pi GM \delta(\vec{r}) [2 + v^2 (1 + 2\gamma + \alpha_3 + \zeta_1)] \\ R_{0k} &= \frac{GM}{r^3} (1 - \gamma + \frac{1}{4}\alpha_1) [3 \frac{\vec{r} \cdot \vec{v}}{r^2} x^k - v^k] - \pi GM v^k \delta(\vec{r}) (4\gamma + 3 + \alpha_1 \\ &\quad - \alpha_2 + \zeta_1). \end{aligned}$$

We will need the δ -function part of R to static order only

$$R = 2R_{00} + 16\pi GM \gamma \delta(r).$$

Next we write the first order approximations to the field equations.

First, the g_{00} -equation (II.2) is written

$$(R_{00} - \frac{1}{2}R)(1 - \omega\phi) + (2\omega + \eta)(\phi R_{00} + K_0^0 K_{1\ell\ell}) = -8\pi G_0 T_{00}; \quad (A.1)$$

next, the g -equation (II.3)

$$\begin{aligned} R + (6\omega + 3\eta)K_0^0 K_{100} - (6\omega + \eta)K_0^0 K_{1\ell\ell} - \eta K_0^0 (K_{\ell 1\ell 0} + K_{\ell 10\ell}) \\ = 8\pi G_0 T; \end{aligned} \quad (A.2)$$

and the K_0 -equation (II.4)

$$\omega K_0^R + \eta K_0^R{}_{00} = 2(K_{01\ell\ell} - K_{\ell 10\ell}). \quad (\text{A.3})$$

It is profitable to eliminate R in the above three equations by adding $\frac{1}{2}$ of A.2 and $-\frac{1}{2}K_0$ times A.3 to A.1, giving

$$\begin{aligned} R_{00}(1 + \omega\phi + \frac{1}{2}\eta\phi) + K_0 K_{01\ell\ell}(1 - \omega + \frac{1}{2}\eta) + K_0 K_{0100}(3\omega + \frac{3}{2}\eta) \\ - \frac{1}{2}\eta K_0 K_{\ell 1\ell 0} - (1 + \frac{1}{2}\eta)K_0 K_{\ell 10\ell} = -8\pi G_0 T_{00} + 4\pi G_0 T. \end{aligned} \quad (\text{A.4})$$

Keeping only the δ -function parts of each equation and setting v to zero, the first order static equations can be written. Thus A.1, A.3, and A.4 become, respectively,

$$(1 - \omega\phi)2GM\gamma\delta(r) + (2\omega + \eta)(a_1 + 2)\phi GM\delta(r) = 2G_0 T_{00} \quad (\text{A.5})$$

$$(2a_1 + 4\omega\gamma - 2\omega - \eta)\phi GM\delta(r) = 0 \quad (\text{A.6})$$

$$\begin{aligned} (1 + \omega\phi + \frac{1}{2}\eta\phi)GM\delta(r) + (1 - \omega + \frac{1}{2}\eta)(a_1 + 1)\phi GM\delta(r) \\ - (\frac{1}{2}\eta + 1)\phi GM\delta(r) = G_0(2T_{00} - T) \end{aligned} \quad (\text{A.7})$$

Using some relations valid in Minkowski space,

$$T_{00} d^3x = \rho \left(\frac{dt}{ds}\right)^2 d^3x = \rho \frac{dt}{ds} \frac{d^3x}{\sqrt{1-v^2}} = \frac{M}{\sqrt{1-v^2}} \quad (\text{A.8})$$

$$T d^3 x = \rho \frac{dt}{ds} \frac{ds}{dt} d^3 x = \rho \frac{dt}{ds} \sqrt{1 - v^2} d^3 x = M \sqrt{1 - v^2}, \quad (\text{A.9})$$

equations A.5 to A.7 integrate (with $v = 0$) to

$$2\gamma(1 - \omega\phi) + (2\omega + \eta)\phi a_1 - 2\frac{G_0}{G} = -2(2\omega + \eta)d \quad (\text{A.10})$$

$$2\gamma(4\omega) + 4a_1 = 2(2\omega + \eta) \quad (\text{A.11})$$

$$(2 - 2\omega + \eta)\phi a_1 - 2\frac{G_0}{G} = -2(1 + \frac{1}{2}\eta\phi). \quad (\text{A.12})$$

These can be solved simultaneously to give

$$\gamma = \frac{1 - \omega\phi(2\omega + \eta + 1)}{1 + \omega\phi(1 - 4\omega)}.$$

Experimental evidence indicates $\gamma \approx 1$, so we proceed with the special case

$$\omega = \frac{1}{2\eta} + 1. \quad (\text{A.13})$$

The procedure for the alternative possibility, $\omega = 0$, is similar and will not be given here. Using A.13 and A.11 gives

$$a_1 = -1.$$

Keeping the v^2 part of the delta-function terms and using A.13, A.4 integrates to

$$(1 + \frac{1}{2}\eta\phi)[2 + v^2(1 + 2\gamma + \alpha_3 + \zeta_1)] = 2\frac{G_0}{G} \left(\frac{2}{\sqrt{1 - v^2}} - \sqrt{1 - v^2} \right)$$

which gives

$$G = \frac{G_0}{1 + \frac{1}{2}\eta\phi} \quad (\text{A.14})$$

as a renormalization of G_0 , and the relationship

$$1 + 2\gamma + \alpha_3 + \zeta_1 = 3. \quad (\text{A.15})$$

Having found the δ -function source terms, we next find solutions to the total dynamic field equations outside the source. The g_{00} equation (A.1) becomes

$$\begin{aligned} (2\omega + \eta)\phi \frac{GM}{r^3} \left[3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2 \right] (2\zeta_1 + 3\gamma + \alpha_2 - \frac{1}{2}\alpha_1 - 1 - 2\gamma - \zeta_1 - 2a_3) \\ + \vec{r} \cdot \vec{a} (3\gamma + \alpha_2 - \frac{1}{2}\alpha_1 - 1 - 2\gamma - \zeta_1 - 2f) = 0. \end{aligned} \quad (\text{A.16})$$

This gives two requirements

$$\begin{aligned} \zeta_1 + \alpha_2 - \frac{1}{2}\alpha_1 - 2a_3 &= 0, \\ -\zeta_1 + \alpha_2 - \frac{1}{2}\alpha_1 - 2f &= 0, \end{aligned} \quad (\text{A.17})$$

where we have used $\gamma = 1$. Subtracting one from the other gives

$$\zeta_1 = a_3 - f. \quad (\text{A.18})$$

Equation A.3 becomes

$$\begin{aligned}
& (2\omega + \eta)\phi \frac{GM}{r^3} \left[\left(3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2 \right) \left(\alpha_2 - \frac{1}{2}\alpha_1 \right) + r \cdot a \left(\alpha_2 - \frac{1}{2}\alpha_1 - \zeta_1 \right) \right] \\
& + 2\phi \frac{GM}{r^3} \left[\left(3 \frac{(\vec{r} \cdot \vec{v})^2}{r^2} - v^2 \right) (2a_3 - d + d') + r \cdot a (2f - d + d') \right] = 0,
\end{aligned} \tag{A.19}$$

giving

$$(2\omega + \eta) \left(\alpha_2 - \frac{1}{2}\alpha_1 \right) = 2(d - d' - 2a_3) \tag{A.20}$$

$$(2\omega + \eta) \left(\alpha_2 - \frac{1}{2}\alpha_1 - \zeta_1 \right) = 2(d - d' - 2f).$$

Subtracting and using A.18 yields

$$(2\omega + \eta + 4)\zeta_1 = 0.$$

This requires that

$$\zeta_1 = 0, \tag{A.21}$$

unless $2\omega + \eta + 4 = 0$. However, if this last relation is combined with $\omega = \frac{1}{2}\eta + 1$, one finds unique values for ω and η which, when used in equation A.4, give $\zeta_1 = 0$ anyway. Equation A.18 yields the additional result

$$a_3 = f. \tag{A.22}$$

In equation A.16 it was assumed that $2\omega + \eta \neq 0$. In the other case, $2\omega + \eta = 0$, equation A.19 will give $a_3 = f$. Using this in A.4

results again in $\zeta_1 = 0$. Therefore the same PPN parameters are obtained, regardless of special relationships between ω and η .

$\zeta_1 = 0$ and $\gamma = 1$ in equation A.15 give the additional result

$$\alpha_3 = 0.$$

Using $\zeta_1 = 0$ and $a_3 = f$; A.17, A.20, and A.4 become

$$\alpha_2 - \frac{1}{2}\alpha_1 = 2f$$

$$(\eta + 1)\left(\alpha_2 - \frac{1}{2}\alpha_1\right) = d' - d - 2f$$

$$(1 + \phi + \frac{3}{2}\eta\phi)\left(\alpha_2 - \frac{1}{2}\alpha_1\right) = (1 + \eta)\phi(d' - d).$$

Solving these simultaneously gives

$$2f = \alpha_2 - \frac{1}{2}\alpha_1 = d' - d = 0. \quad (\text{A.23})$$

To get the individual values of α_1 and α_2 , the K_{ℓ} and $g_{0\ell}$ equations are needed. Equation II.4 gives

$$\eta K_{00\ell}^R = 2(K_{\ell 1m m} - K_{m 1\ell m} - K_{\ell 100}),$$

or

$$\frac{GM}{r^3} \left[3 \frac{\vec{r} \cdot \vec{v}}{r^2} x^\ell - v^\ell \right] \left[\frac{1}{4}\eta\alpha_1 + 2(d + d' - 1) \right] = 0.$$

Equation II.2 yields

$$(1 + \omega\phi + \eta\phi)R_{0\ell} + (2\omega + \eta)K_0^0 K_{10\ell}^0 + \frac{1}{2}\eta K_0(K_{\ell 1mm} - K_{m1\ell m}) = 0,$$

or

$$\frac{GM}{r^3} \left[3 \frac{\vec{r} \cdot \vec{v}}{r^2} x^\ell - v^\ell \right] \left[(1 + \omega\phi + \eta\phi) \left(\frac{1}{4} \alpha_1 \right) - \frac{1}{2} \eta \phi (d + d') \right] = 0.$$

Solving for α_1 and using $\omega = \frac{1}{2}\eta + 1$

$$\alpha_2 = \frac{1}{2} \alpha_1 = \frac{4\eta}{4\phi^{-1} + 4 + 6\eta + \eta^2}. \quad (\text{A.24})$$

Also

$$d = d' = \frac{1}{2} \frac{4\phi^{-1} + 4 + 6\eta}{4\phi^{-1} + 4 + 6\eta + \eta^2}.$$

The only PPN parameters remaining involve second order static terms in the metric. The static equations to second order are II.2,

$$R_{00}(1 + \omega\phi + \eta\phi) - \frac{1}{2}R(1 - \omega\phi) - (2\omega + \eta) \frac{1}{2}(K_0^0 K_{1\ell m}^0) g^{\ell m} - F_{0\ell} F_{0ni} g^{\ell m} = 0; \quad (\text{A.25})$$

II.3,

$$R + (6\omega + 3\eta)K_0^0 K_{100}^0 + (6\omega + \eta) \frac{1}{2}(K_0^0 K_{1\ell m}^0) g^{\ell m} + (6\omega + \eta)K_{\ell 10} K_{m10} g^{\ell m} \\ + 2\eta K_{01\ell} K_{m10} g^{\ell m} + 2\eta K_{\ell 10} K_{m10} g^{\ell m} + \eta R_{00} K^0 K^0 = 0; \quad (\text{A.26})$$

and II.4 (multiplied by K_0),

$$\omega\phi R + \omega\phi R_{00} + (K_0 K_0)_{\ell m} g^{\ell m} - 2K_{0\ell} K_{0m} g^{\ell m} - 2K_0 K_{\ell 10 m} g^{\ell m} = 0. \quad (\text{A.27})$$

Adding $\frac{1}{2}$ of A.26 and $-\frac{1}{2}$ of A.27 to A.25, and using $\omega = \frac{1}{2}n + 1$,

$$R_{00} \left(1 + \frac{3}{2}n\phi + \phi\right) + (3n + 3)K_0 K^0_{100} - (2n + 2)K_{\ell 10} K_{\ell 10} \\ + (n + 1)K_0 K^{\ell}_{10\ell} = 0$$

where we have also used the fact that to first order the only non-zero derivative of K_μ is $K_{\ell 10}$. To second order,

$$R_{00} = (2 - 2\beta) \frac{G^2 M^2}{r^4}$$

$$K_0 K^0_{100} = K_{\ell 10} K_{\ell 10} = \frac{G^2 M^2}{r^4}$$

$$K_0 K^{\ell}_{10\ell} = (2\beta - 3) \frac{G^2 M^2}{r^4}.$$

Using these gives the relation

$$(2 - 2\beta) \left(1 + \frac{1}{2}n\phi\right) = 0,$$

or $\beta = 1$ as in general relativity. To get δ , which only appears in R, we solve equations A.25 and A.27 for R. Multiplying A.27 by $\omega + \frac{1}{2}n$ and adding to A.25

$$\frac{1}{2}R[-1 + \omega\phi(2\omega + \eta + 1) - F_{0\lambda}F_{0m}g^{\lambda m} - (2\omega + \eta)K_{0\lambda}K_{\lambda 0m}g^{\lambda m}] = 0$$

where we have used the fact that $R_{00} = 0$ when $\beta = 1$. Solving and using

$$\eta = 2\omega - 2,$$

$$\frac{1}{2}R = \frac{(4\omega - 1)\phi \frac{G^2 M^2}{r^4}}{1 - \omega\phi(4\omega - 1)}.$$

Now,

$$R = (-4\beta - 4\delta + 8)\frac{G^2 M^2}{r^4} = (4 - 4\delta)\frac{G^2 M^2}{r^4},$$

so the δ term can also be found:

$$\delta = 1 + \frac{1 - 4\omega}{2\phi^{-1} + 2\omega(1 - 4\omega)}.$$

The ζ_2 parameter appearing in the two mass interaction term can be found by means of a convenient trick. The configuration of a point mass m inside a spherical shell of mass M and radius $R \gg r$ is solved by two approaches and the results are compared. First, M is included as a source of the metric and we write

$$\begin{aligned} g_{00} &= 1 - 2G\frac{M}{R} - 2G\frac{m}{r} + 4\beta G\frac{mM}{rR} - (2 - 4\beta + 2\zeta_2)G^2\frac{mM}{rR} + O(m^2, M^2) \\ g_{ss} &= -1 - 2\gamma G\frac{M}{R} - 2\gamma G\frac{m}{r}. \end{aligned} \tag{A.28}$$

This has the asymptotic limit (as $\frac{m}{r} \rightarrow 0$ but always $R \gg r$)

$$\begin{aligned} g_{00} &= 1 - 2\frac{GM}{R} \\ g_{ss} &= -1 - 2\gamma\frac{GM}{R} . \end{aligned} \tag{A.29}$$

Second, we note that the post-Newtonian limit is valid inside the shell, so one must be able to write to linear order.

$$\begin{aligned} g'_{00} &= 1 - 2G^* \frac{m}{r'} \\ g'_{ss} &= -1 - 2\gamma G^* \frac{m}{r'} \end{aligned} \tag{A.30}$$

where it is recognized that the presence of the mass shell may affect G , and that the coordinates will be different to allow a Minkowskian asymptotic limit. Comparison of A.29 and A.30 shows that the coordinate transformation must be

$$\begin{aligned} \frac{\partial x^0}{\partial x'^0} &= 1 + \frac{GM}{R} \\ \frac{\partial r}{\partial r'} &= 1 - \gamma \frac{GM}{R} . \end{aligned}$$

Applying this to A.28 gives

$$g'_{00} = 1 - 2\frac{GM}{R} + (8\beta - 6 - 2\zeta_2)G^2 \frac{mM}{rR} .$$

But $r = (1 - \gamma\frac{GM}{R})r'$, so

$$g'_{00} = 1 - 2G \frac{m}{r} + (8\beta - 6 - 2\gamma - 2\zeta_2) G^2 \frac{mM}{rR}.$$

Comparison with A.30 shows the effect of M on G,

$$G^* = G \left[1 - (4\beta - 3 - \gamma - \zeta_2) \frac{GM}{R} \right].$$

However, the effect of M on G is well known from previous analysis.

Equation A.14 gave

$$G^* = \frac{G_0}{1 + \frac{1}{2} \eta g_{00} K_0 K_0} = G_0 \left[1 + \frac{1}{2} \eta \left(1 + 2 \frac{M}{R} \right) \phi \left(1 - \frac{M}{R} \right) \left(1 - \frac{M}{R} \right) \right]^{-1}$$

$$G^* = \frac{G_0}{1 + \frac{1}{2} \eta \phi} = G.$$

Therefore $4\beta - 3 - \gamma - \zeta_2 = 0$, and since $\beta = \gamma = 1$, this implies

$$\zeta_2 = 0.$$

APPENDIX B

The substitution of the leading terms of IV.6,

$$g_{00} = ax^s$$

$$g_{rr} = -bx^t$$

$$K_0 = cx^u,$$

into the field equations has produced the following equations (equations IV.7, IV.8, and IV.9):

$$\lambda(u-s)[(w-1)x^{w-2} + 2x^{w-1}] - \frac{a}{c}x^{p-1}(s+t) + 2\omega x^w(bx^{t-1}\frac{w}{x}) = 0 \quad (\text{B.1})$$

$$(u+\frac{1}{4}\lambda s)[(w-1)x^{w-2} + 2x^{w-1}] - \frac{a}{c}x^p(bx^{t-1}\frac{s}{x}) - 2\omega x^w(bx^{t-1}\frac{w}{x}) = 0 \quad (\text{B.2})$$

$$\lambda x^{w-2} \frac{1}{2}s(u-\frac{1}{2}s) + x^{w-2}u^2 - \frac{a}{c}x^p(bx^{t-1}\frac{s}{x}) + \omega x^w(bx^{t-1}\frac{s}{x} - 4\frac{u}{x}) = 0. \quad (\text{B.3})$$

These are polynomials in x whose powers depend on the values of s , t , and u . For various ranges of values of these parameters, certain terms in the polynomials will be dominant near the singularity. The method of solution will be to assume some range of exponents, s , t , and u , keep only lowest order terms in x , and examine the resulting simplified equations to see if a consistent solution can be found. The criteria for inconsistency are as follows:

1. a , b , or $c = 0$. This is inconsistent with the assumption that ax^s , for example, is the leading term in the power series.

2. $s \leq 0$. This is inconsistent with the assumption that $r = r_0$ is an event horizon.

[1. $w = 1$]

First assume $w = 1$. Then B.1 to B.3 become

$$2\lambda(u-s) - \frac{a}{c}x^{p-1}(s+t) + 2\omega(bx^{t+1}-1) = 0 \quad (\text{B.4})$$

$$2(u+\frac{1}{4}\lambda s) - \frac{a}{c}x^p(bx^{\frac{t-s}{x}}) - 2\omega(bx^{t+1}-1) = 0 \quad (\text{B.5})$$

$$\frac{\lambda}{2x}s(u-\frac{1}{2}s) + \frac{1}{x}u^2 - \frac{a}{c}x^p(bx^{\frac{t-s}{x}}) + \omega(bx^{t+1}-s-4u) = 0. \quad (\text{B.6})$$

If $t > -1$, the leading parts become

[1a. $t > -1$]

$$2\lambda(u-s) - \frac{a}{c}x^{p-1}(s+t) - 2\omega = 0 \quad (\text{B.7})$$

$$2(u+\frac{1}{4}\lambda s) + \frac{a}{c}x^{p-1}s + 2\omega = 0 \quad (\text{B.8})$$

$$\frac{\lambda}{2x}s(u-\frac{1}{2}s) + \frac{1}{x}u^2 + \frac{a}{c}x^{p-1}s = 0. \quad (\text{B.9})$$

If $p < 1$, the second equation reduces to

[1a(i). $p < 1$]

$$\frac{as}{c} = 0.$$

So either $a = 0$ or $s = 0$ and we have a contradiction. Now assuming that $p > 1$, the three equations become

[1a(ii). $p > 1$]

$$\lambda(u-s) - \omega = 0$$

$$u + \frac{1}{4}\lambda s + \omega = 0$$

$$\lambda s(u - \frac{1}{2}s) + 2u^2 = 0.$$

The first equation gives

$$u = s + \frac{\omega}{\lambda}$$

which, when substituted into the second equation gives

$$s = -\frac{4\omega(1+\lambda)}{\lambda(4+\lambda)} \quad ; \quad u = \frac{-3\omega}{4+\lambda}.$$

These values, substituted into the third equation lead to an inconsistency. Therefore, it can be concluded that $p = 1$.

[1a(iii). $p = 1$]

Now it should be remembered that

$$w \equiv 2u - \frac{1}{2}s - \frac{1}{2}t = 1$$

$$p = \frac{1}{2}s - \frac{1}{2}t = 1$$

which can be subtracted to give

$$u = \frac{1}{2}s.$$

Substituting this into equation B.9 with $p = 1$, gives

$$\frac{s}{4x^2} = 0$$

which is again inconsistent. Thus no value of p gives a solution and we are forced to conclude that the assumption leading to B.7-B.9 is not valid. Thus $t \leq -1$, and it is still assumed that $w = 1$.

If $t < -1$, B.4 and B.5 become

[1b. $t < -1$]

$$-\frac{a}{c^2}x^{p-1}(s+t) + 2\omega bx^{t+1} = 0$$

$$-\frac{a}{c^2}x^p(bx^t) - 2\omega bx^{t+1} = 0.$$

Adding these and dividing by x^p gives

$$\frac{r}{x}(s+t) + bx^t = 0.$$

But $t < -1$ implies $b = 0$ which is inconsistent. Therefore, the only remaining value of t is $t = -1$.

In this case equations B.4 to B.6 become

[1c. $t = -1$]

$$2\lambda(u-s) - \frac{a}{c^2}(s+t)x^{p-1} + 2\omega(b-1) = 0$$

$$2(u + \frac{1}{4}s) - \frac{a}{c^2}(b-s)x^{p-1} - 2\omega(b-1) = 0.$$

$$\frac{\lambda}{2x}s(u - \frac{1}{2}s) + \frac{1}{x}u^2 - \frac{a}{c^2}x^{p-1}(b-s) = 0,$$

and $w = 2u - \frac{1}{2}s - \frac{1}{2}t = 1$ implies $s = 4u - 1$. Now if $p < 1$, the first equation becomes

[1c(i). $p < 1$]

$$\frac{a}{c^2}(s-1) = 0$$

or $s = 1$. This however, along with $t = -1$, means $p = \frac{1}{2}s - \frac{1}{2}t = 1$, which contradicts $p < 1$. $p = 1$ is also quickly ruled out. In that case, $t = -1$, $s = 1$, $u = \frac{1}{2}$, and the third equation is

[1c(ii). $p = 1$]

$$\frac{1}{x}u^2 = 0$$

which is inconsistent. The only remaining possibility is $p > 1$. In this case the three equations are (using $s = 4u - 1$):

[1c(iii). $p > 1$]

$$\lambda(1-3u) - \omega(b-1) = 0$$

$$u - \lambda u + \frac{1}{4}\lambda - \omega(b-1) = 0$$

$$\lambda(4u-1)\left(-u + \frac{1}{2}\right) + 2u^2 = 0.$$

Adding the first two and solving for u gives

$$u = \frac{5\lambda}{4(4\lambda-1)}.$$

This result, used in the third equation, leads to an inconsistency.

All work thus far has been to show that $w = 1$ is not consistent with the field equations. It may now be assumed that $w \neq 1$, and the same type of analysis will be performed, this time leading to a solution. If $w \neq 1$, then x^{w-1} may be dropped with respect to x^{w-2} , and equations B.1 through B.3 become

[2. $w \neq 1$]

$$\lambda(u-s)(w-1) - \frac{a}{c^2}x^{q-1}(s+1) + 2\omega bx^{t+2} = 0 \quad (\text{B.10})$$

$$\left(u + \frac{1}{4}\lambda s\right)(w-1) - \frac{a}{c^2}x^q\left(bx^t - \frac{s}{x}\right) - 2\omega bx^{t+2} = 0 \quad (\text{B.11})$$

$$\lambda \frac{1}{2}s\left(u - \frac{1}{2}s\right) + u^2 - \frac{a}{c^2}x^q\left(bx^t - \frac{s}{x}\right) + \omega bx^{t+2} = 0, \quad (\text{B.12})$$

where we have defined

$$p-w+2 = s+2-2u \equiv q,$$

and where terms like $2\omega wx$ have been dropped in favor of the constant first terms.

Assuming that $t > -1$, equations B.10 through B.12 become

[2a. $t > -1$]

$$\lambda(u-s)(w-1) - \frac{a}{c^2} x^{q-1} (s+t) = 0$$

$$(u + \frac{1}{4}\lambda s)(w-1) + \frac{a}{c^2} x^{q-1} s = 0$$

$$\frac{\lambda}{2}s(u - \frac{1}{2}s) + u^2 + \frac{a}{c^2} x^{q-1} s = 0.$$

[2a(i). $q < 1$]

The additional assumption that $q < 1$ leads to $\frac{as}{c^2} = 0$ in the second equation, which is inconsistent with the basic conditions. If $q > 1$, the first equation is just

[2a(ii). $q > 1$]

$$(u-s)(w-1) = 0$$

which has solution ($w \neq 1$), $u = s$. Inserted in the second equation, this leads to

$$s(1 + \frac{1}{4}\lambda) = 0$$

which is again inconsistent. The assumption that $q = 1$ requires more effort. In this case the three equations reduce to

[2a(iii). $q = 1$]

$$\lambda(s-1)(w-1) + \frac{2a}{c}(s+t) = 0$$

$$(2s+2+\lambda s)(w-1) + \frac{4a}{c}s = 0$$

$$\lambda s + (s+1)^2 + \frac{4a}{c}s = 0$$

where we have used the fact that $q \equiv s+2-2u = 1$ implies $2u = s+1$.

Solving the first two equations for s and t yields

$$s = -1 - \frac{1}{2}\lambda - \frac{2a}{c}$$

which is used in the third equation, leading to an inconsistency. Therefore one must have $t \leq -1$.

[2b. $t = -1$]

The assumption that $t = -1$ in equations B.10, B.11, and B.12, gives

$$\lambda(u-s)(w-1) - \frac{a}{c}x^{q-1}(s-1) = 0$$

$$(u + \frac{1}{4}\lambda s)(w-1) - \frac{a}{c}x^{q-1}(b-s) = 0$$

$$\frac{\lambda s}{2}(u - \frac{1}{2}s) + u^2 - \frac{a}{c}x^{q-1}(b-s) = 0.$$

[2b(i). $q > 1$]

$q > 1$ reduces the first equation to

$$u-s = 0$$

which can be used in the second equation to give

$$(1 + \frac{1}{4}\lambda)s = 0,$$

an inconsistency.

[2b(ii). $q < 1$]

$q < 1$ implies a solution when

$$s = 1$$

$$b = 1,$$

and in order to have $q < 1$, it is necessary that $x+2-2u < 1$, or

$$u > 1.$$

This appears to be a valid solution. However, substituting these results into the μ equation (equation IV.1) gives¹

¹The reason for doing this is that during the algebraic simplification which led to equations IV.5 and IV.6 (B.1 and B.2), we introduced into the μ equation elements which were lower order in x than the original. Thus, the satisfying of this lowest order part gives the illustration of satisfying all three equations; but it is the lowest order part of the original three equations which must in fact be satisfied.

$$\left\{ \left(u + \frac{1}{4}\lambda\right) \left[(2u-1)x^{-2} + 2x^{-1} \right] - w - \frac{\lambda}{2x^2} \left(u - \frac{1}{2}\right) + \frac{u^2}{x^2} \right\} x^{2u} = 0,$$

whose leading terms are

$$\left(u + \frac{1}{4}\lambda\right)(2u-1) - \frac{\lambda}{2} \left(u - \frac{1}{2}\right) + u^2 = 0,$$

with solution

$$u = 0 \quad \text{or} \quad u = 1.$$

This is inconsistent with the requirement of $u > 1$, so this is in fact no solution.

[2b(iii). $q = 1$]

A solution is found when $q \equiv s+2+2u = 1$. Then $2u = s+1$, and the equations B.10, B.11, and B.12 become

$$\lambda(s-1)(w-1) + \frac{2a}{c^2}(s-1) = 0$$

$$(2s+2+\lambda s)(w-1) - \frac{4a}{c^2}(b_0s) = 0$$

$$\lambda s + (s+1)^2 - \frac{4a}{c^2}(b-s) = 0.$$

The first equation is solved by $s = 1$. Using this in both the second and third equations gives

$$\lambda + 4 - \frac{4a}{c^2}(b-1) = 0$$

or

$$b = 1 + \frac{c^2}{4a}(\lambda+4).$$

Here, then, is a solution

$$g_{00} = a \frac{r-r_0}{r_0}$$

$$g_{rr} = -\left[1 + \frac{1}{4} \frac{c^2}{a}(\lambda+4)\right] \frac{r_0}{r-r_0}$$

$$k_0 = c \frac{r-r_0}{r}$$

Substitution of this solution into the original three equations gives consistency, with a , c , and r_0 still arbitrary.

Finally it is necessary to make the assumption of $t < -1$ in equations B.10, B.11, and B.12. First, we try $t < -2$. Then the three equations become

[2c. $t < -2$]

$$-\frac{a}{c^2} x^{q-1} (s+t) + 2\omega b x^{t+2} = 0$$

$$-\frac{a}{c^2} x^q b x^t - 2\omega b x^{t+2} = 0$$

$$\frac{a}{c^2} b^{q+t} + \omega b x^{t+2} = 0.$$

Adding the second and third gives

$$-\omega b = 0$$

which is inconsistent with the basic conditions. Therefore $t \geq -2$.

[2d. $t = -2$]

The assumption that $t = -2$ produces the three equations

$$\lambda(u-s)(w-1) - \frac{a}{c^2} x^{q-1} (s-2) + 2\omega b = 0 \quad (\text{B.13})$$

$$(u + \frac{1}{4}\lambda s)(w-1) - \frac{ab}{c^2} x^{q-2} - 2\omega b = 0 \quad (\text{B.14})$$

$$\frac{\lambda}{2}s(u - \frac{1}{2}s) + u^2 - \frac{ab}{c^2} x^{q-2} + \omega b = 0. \quad (\text{B.15})$$

[2d(i). $q < 2$]

If $q < 2$ the second equation is

$$\frac{ab}{c^2} = 0$$

which is inconsistent.

[2d(ii). $q > 2$]

If $q > 2$, the three equations are

$$\lambda(u-s)(w-1) + 2\omega b = 0$$

$$(u + \frac{1}{4}\lambda s)(w-1) - 2\omega b = 0$$

$$\frac{\lambda}{2}s(u - \frac{1}{2}s) + u^2 + \omega b = 0.$$

Adding the first and second equations together and remembering that $w \neq 1$ gives

$$u = \frac{3\lambda s}{4(1+\lambda)}.$$

Using this, the second and third equations reduce to

$$\lambda s \frac{2(\lambda+4)(4\lambda-2)}{(4\lambda+4)^2} - 2\omega b = 0$$

$$\lambda s \frac{2(\lambda+4)(4\lambda-2)}{(4\lambda+4)^2} + 2\omega b = 0$$

which are inconsistent with $s > 0$. Now if $q = 2$, $s = 2u$ and equations B.13, B.14, and B.15 become

[2d(iii). $q = 2$]

$$-\lambda u^2 + 2\omega b = 0$$

$$u^2 + \frac{1}{2}\lambda u^2 - \frac{ab}{c^2} - 2\omega b = 0$$

$$u^2 - \frac{ab}{c^2} + \omega b = 0.$$

Subtracting the third from the second and adding one-half of the first yields

$$-2\omega b = 0$$

which is inconsistent. We have now exhausted all possibilities with $t = -2$, and have limited a possible second solution to the range $-2 < t < -1$.

In this case, equations B.10, B.11, and B.12 reduce to

[2e. $-2 < t < -1$]

$$\lambda(u-s)(w-1) - \frac{a}{c} 2^x q^{-1} (s+t) = 0$$

$$(u + \frac{1}{4}\lambda s)(w-1) - \frac{ab}{c} 2^x q^{t+1} = 0$$

$$\frac{\lambda}{2}s(u - \frac{1}{2}s) + u^2 - \frac{ab}{c} 2^x q^{t+1} = 0.$$

[2e(i). $q \leq 1$]

If $q \leq 1$, then $t < -1$ implies that $q+t < 0$, and the second equation is just

$$\frac{ab}{c} = 0$$

which is inconsistent. The remaining possibility ($q > 1$) reduces the first equation to

[2e(ii). $q > 1$]

$$\lambda(u-s)(w-1) = 0$$

or, since $w \neq 1$,

$$u = s.$$

Putting this into the second and third equations gives

$$\begin{aligned} (1 + \frac{1}{4}\lambda)s(w-1) - \frac{ab}{c}x^{q+t} &= 0 \\ (1 + \frac{1}{4}\lambda)s^2 - \frac{ab}{c}x^{q+t} &= 0 \end{aligned} \tag{B.16}$$

which can only be solved if

$$s = w-1 \equiv 2u - \frac{1}{2}s - \frac{1}{2}t - 1$$

or

$$t = s-2.$$

If these results are substituted into equation B.1, there results

$$\lambda(s-s)[sx^{s-1} + 2x^s] - \frac{a}{c^2}(2s-2) + 2\omega[bx^{2s-1} - x^{s+1} - (s+1)x^s] = 0 \tag{B.17}$$

Now the limits on t ($-2 < t < -1$) give limits on s ($0 < s < 1$), so it is possible to compare the orders of some of the terms in B.17.

$$s-1 < 2s-1 < s$$

$$s-1 < 0 < s$$

The lowest order term (x^{s-1}) vanishes. However, it must be remembered that the basic expansion of the fields was a power series, and if one

goes beyond the leading term in the power series, the next order contribution will be at least one power higher, or x^s . Therefore, the second lowest order terms in B.17,

$$-\frac{a}{c^2}(2s-2) + 2\omega bx^{2s-1}, \quad (\text{B.18})$$

must also vanish, since any terms arising from continuing the power series expansion must be of different (greater) order than B.18. It is therefore necessary that

$$-\frac{2a}{c^2}(s-1) + 2\omega bx^{2s-1} = 0.$$

If $s \neq \frac{1}{2}$, this reduces to

$$s = 1$$

$$b = 0$$

which is inconsistent. If $s = \frac{1}{2}$, then

$$b = \frac{-a}{2\omega c^2} \quad (\text{B.19})$$

and

$$u = \frac{1}{2}$$

$$t = -\frac{3}{2}$$

Now, if these values are used in equation B.2, there results

$$\left[\frac{1}{4}\left(1+\frac{1}{4}\lambda\right) - \frac{ab}{c^2}\right]x^{-1/2} + \frac{a}{c^2} - 2\omega b = 0.$$

The $x^{-1/2}$ term vanishes because of equation B.16, and an argument like the one leading to the vanishing of B.18, leads to

$$\frac{a}{c^2} - 2\omega b = 0$$

which is inconsistent with B.19.

There is, therefore, only one solution to the three equations, B.1, B.2, and B.3, which is consistent with the assumptions of an event horizon; and that is

$$g_{00} = a \frac{r-r_0}{r_0}$$

$$g_{rr} = -\left[1 + \frac{1}{4} \frac{c^2}{a^2} (\lambda+4)\right] \frac{r_0}{r-r_0}$$

$$K_0 = c \frac{r-r_0}{r_0}$$

with a , c , and r_0 arbitrary.

APPENDIX C

We first derive the linearized form of the field equations. The fields are written

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$K_{\mu} = \phi_{\mu} + A_{\mu}$$

where $h_{\mu\nu} \ll \eta_{\mu\nu}$, $A_{\mu} \ll \phi_{\mu}$, and $\eta_{\mu\nu}$ and ϕ_{μ} are constant cosmological values of the fields (coordinates are chosen so that $\eta_{\mu\nu}$ is the Minkowski metric tensor). Then the curvature tensors are

$$R_{\mu\alpha\nu\beta} = \frac{1}{2}(h_{\mu\nu,\alpha\beta} + h_{\alpha\beta,\mu\nu} - h_{\mu\beta,\alpha\nu} - h_{\alpha\nu,\mu\beta})$$

$$R_{\mu\nu} = \frac{1}{2}(\square^2 h_{\mu\nu} + h_{,\mu\nu} - 2h^{\alpha}_{\mu,\alpha\nu})$$

$$R = \square^2 h - h^{\alpha\beta}_{,\alpha\beta}$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$. The derivatives appearing in the field equations are (remember $K \equiv K_{\mu} K_{\nu} g^{\mu\nu}$)

$$K_{|\mu\nu} = 2\phi^{\alpha} A_{\alpha,\mu\nu} - \phi^{\alpha}\phi^{\beta} h_{\alpha\beta,\mu\nu}$$

$$\square^2 K = 2\phi^{\alpha}\square^2 A_{\alpha} - \phi^{\alpha}\phi^{\beta}\square^2 h_{\alpha\beta}$$

$$(K^{\alpha}K^{\beta})_{|\alpha\beta} = 2\phi^{\alpha}\phi^{\beta}_{,\beta\alpha} + \phi^{\alpha}\phi^{\beta}R_{\alpha\beta} - \phi^{\alpha}\phi^{\beta}\square^2 h_{\alpha\beta}$$

$$\begin{aligned} \square^2(K_{\mu}K_{\nu}) &= \phi_{\mu}\square^2 A_{\nu} + \phi_{\nu}\square^2 A_{\mu} - \frac{1}{2}\phi_{\mu}\phi^{\alpha}(\square^2 h_{\nu\alpha} + h^{\beta}_{\alpha,\beta\nu} - h^{\beta}_{\nu,\beta\alpha}) \\ &\quad - \frac{1}{2}\phi_{\nu}\phi^{\alpha}(\square^2 h_{\mu\alpha} + h^{\beta}_{\alpha,\beta\mu} - h^{\beta}_{\mu,\beta\alpha}) \end{aligned}$$

$$\begin{aligned}
(K_\mu K^\alpha)_{|\nu\alpha} + \nu(K_\nu K^\alpha)_{|\mu\alpha} &= \phi^\alpha A_{\mu,\nu\alpha} + \phi^\alpha A_{\nu,\mu\alpha} + \phi_\mu A^{\alpha}_{,\nu\alpha} + \phi_\nu A^{\alpha}_{,\mu\alpha} \\
&+ 2\phi^\alpha \phi^\beta R_{\mu\alpha\nu\beta} - \phi^\alpha \phi^\beta h_{\alpha\beta,\mu\nu} \\
&- \phi_\mu \phi^\alpha \frac{1}{2} (h^\beta_{\alpha,\nu\beta} - h^\beta_{\nu,\alpha\beta} + \square^2 h_{\nu\alpha}) \\
&- \phi_\nu \phi^\alpha \frac{1}{2} (h^\beta_{\alpha,\mu\beta} - h^\beta_{\mu,\alpha\beta} + \square^2 h_{\mu\alpha})
\end{aligned}$$

in which case, one can also write the combination

$$\begin{aligned}
\square^2 (K_\mu K_\nu) - (K_\mu K^\alpha)_{|\nu\alpha} - (K_\nu K^\alpha)_{|\mu\alpha} &= \phi_\mu F_{\nu,\alpha}^\alpha + \phi_\nu F_{\mu,\alpha}^\alpha - 2\phi^\alpha \phi^\beta R_{\mu\alpha\nu\beta} \\
&+ \phi^\alpha \phi^\beta h_{\alpha\beta,\mu\nu} - \phi^\alpha (A_{\mu,\nu\alpha} + A_{\nu,\mu\alpha}).
\end{aligned}$$

Thus the $g_{\mu\nu}$ equation (II.2) can be written,

$$\begin{aligned}
R_{\mu\nu} (1 + \omega\phi^2) + R(\omega\phi_\mu\phi_\nu - \frac{1}{2}g_{\mu\nu} - \frac{1}{2}\omega\phi^2 g_{\mu\nu}) + 2\omega\phi^\alpha (A_{\alpha,\mu\nu} - g_{\mu\nu}\square^2 A_\alpha) \\
- (\omega + \frac{1}{2}\eta)\phi^\alpha\phi^\beta (h_{\alpha\beta,\mu\nu} - g_{\mu\nu}\square^2 h_{\alpha\beta}) + \eta\phi^\alpha (\phi_\mu R_{\nu\alpha} + \phi_\nu R_{\mu\alpha}) \\
+ \eta\phi^\alpha\phi^\beta (R_{\mu\alpha\nu\beta} - g_{\mu\nu}R_{\alpha\beta}) - \eta g_{\mu\nu}\phi^\alpha A^{\beta}_{,\beta\alpha} + \frac{1}{2}\eta\phi^\alpha (A_{\mu,\nu\alpha} + A_{\nu,\mu\alpha}) \\
- \frac{1}{2}\eta (\phi_\mu F_{\nu,\alpha}^\alpha + \phi_\nu F_{\mu,\alpha}^\alpha) = 0 \quad (C.1)
\end{aligned}$$

where $\phi^2 \equiv \phi_\alpha\phi^\alpha$. The g equation (II.3) is

$$R + \eta\phi^\alpha\phi^\beta R_{\alpha\beta} + 3\square^2 P - 2\eta\phi^\alpha F_{\alpha,\beta}^\beta = 0 \quad (C.2)$$

where we have defined $P \equiv (2\omega + \eta)(\phi^\alpha A_\alpha - \frac{1}{2}\phi^\alpha\phi^\beta h_{\alpha\beta})$ for notational ease.

The K_μ equation (II.4) is

$$\omega\phi_{\mu} R + \eta\phi^{\alpha} R_{\mu\alpha} + 2F_{\mu,\alpha}^{\alpha} = 0. \quad (\text{C.3})$$

Based on C.3 alone, the following equation can also be written.

$$-2\omega\phi_{\mu}\phi_{\nu} R - \eta\phi^{\alpha}(\phi_{\mu} R_{\nu\alpha} + \phi_{\nu} R_{\mu\alpha}) - 2(\phi_{\mu} F_{\nu,\alpha}^{\alpha} + \phi_{\nu} F_{\mu,\alpha}^{\alpha}) = 0.$$

Multiplying equation C.2 by $\frac{1}{2}g_{\mu\nu}$ yields

$$\frac{1}{2}g_{\mu\nu} R + \frac{1}{2}\eta g_{\mu\nu} \phi^{\alpha} \phi^{\beta} R_{\alpha\beta} + \frac{3}{2}g_{\mu\nu} \square^2 P - \eta\phi^{\alpha} F_{\alpha,\beta}^{\beta} = 0.$$

Dotting equation with ϕ^{μ} and multiplying by $\frac{1}{2}g_{\mu\nu}$ gives

$$\frac{1}{2}\omega g_{\mu\nu} \phi^2 R + \frac{1}{2}\eta g_{\mu\nu} \phi^{\alpha} \phi^{\beta} R_{\alpha\beta} + g_{\mu\nu} \phi^{\alpha} F_{\alpha,\beta}^{\beta} = 0.$$

Adding these last three equations to C.1 results in a simplified version of the $g_{\mu\nu}$ equation.

$$\begin{aligned} R_{\mu\nu}(1 + \omega\phi^2) + \eta\phi^{\alpha} \phi^{\beta} R_{\mu\alpha\nu\beta} - \omega\phi_{\mu}\phi_{\nu} R + \frac{1}{2}g_{\mu\nu} \square^2 P + P_{,\mu\nu} + g_{\mu\nu} \phi^{\alpha} F_{\alpha,\beta}^{\beta} \\ - \frac{1}{2}\eta\phi^{\alpha} (F_{\alpha\mu,\nu} + F_{\alpha\nu,\mu}) - (\frac{1}{2}\eta + 2)(\phi_{\mu} F_{\nu,\alpha}^{\alpha} + \phi_{\nu} F_{\mu,\alpha}^{\alpha}) = 0. \end{aligned} \quad (\text{C.4})$$

A solution is now assumed in which the perturbations ($h_{\mu\nu}$ and A_{μ}) are plane waves moving in the z direction.

$$h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik_{\mu} x^{\mu}}$$

$$A^{\mu} = a^{\mu} e^{ik_{\mu} x^{\mu}}.$$

k_μ is a propagation 4-vector whose components are given by

$$k_\mu = (ck, 0, 0, -k) \quad (C.5)$$

where c is not necessarily the speed of light (which is equal to one in our units), but is the speed of propagation of gravity waves.

There are fifteen non-zero components of the Riemann tensor, not counting permutations of the subscripts, but of these, only six are independent. These six can be taken to be

$$R_{0101} , R_{0102} , R_{0202} , R_{0103} , R_{0203} , R_{0303}.$$

The remaining nine are given by

$$R_{1313} = \frac{1}{c^2} R_{0101}$$

$$R_{1323} = \frac{1}{c^2} R_{0102}$$

$$R_{2323} = \frac{1}{c^2} R_{0202}$$

$$R_{0131} = -\frac{1}{c} R_{0101}$$

$$R_{0132} = R_{0231} = -\frac{1}{c} R_{0102}$$

$$R_{0232} = -\frac{1}{c} R_{0202}$$

$$R_{0313} = \frac{1}{c} R_{0103}$$

$$R_{0323} = \frac{1}{c} R_{0203}.$$

The components of $R_{\mu\nu}$ are also linear combinations of the first six.

$$R_{00} = -R_{0101} - R_{0202} - R_{0303}$$

$$R_{01} = -\frac{1}{c}R_{0103}$$

$$R_{02} = -\frac{1}{c}R_{0203}$$

$$R_{03} = \frac{1}{c}(R_{0101} + R_{0202})$$

$$R_{11} = (1 - \frac{1}{c^2})R_{0101}$$

$$R_{12} = (1 - \frac{1}{c^2})R_{0102}$$

$$R_{22} = (1 - \frac{1}{c^2})R_{0202}$$

$$R_{13} = R_{0103}$$

$$R_{23} = R_{0203}$$

$$R_{33} = -\frac{1}{c^2}(R_{0101} + R_{0202}) + R_{0303}$$

and R is

$$R = -2(1 - \frac{1}{c^2})(R_{0101} + R_{0202}) - 2R_{0303}$$

We have already chosen coordinates so that the background metric is Minkowskian, and oriented the three space coordinates so that the z-axis is along the direction of propagation of the wave; but there still exists

the freedom of choice of inertial frame, since any boost will leave $\eta_{\mu\nu}$ unchanged. To simplify calculations, we choose to work in the rest frame of the universe. In this frame

$$\phi_0 \equiv \phi \neq 0$$

$$\phi_k = 0$$

The vanishing or non-vanishing of various components of $R_{\mu\alpha\nu\beta}$ will depend on this choice of reference frame, but once the linearized components are found in one frame, they may be determined in any frame by the application of Lorentz transformations.

In the rest frame of the universe, the K_ℓ equation (C.3) is

$$\eta\phi R_{0\ell} = -2F_{\ell,\alpha}^\alpha$$

(We assume for the time being that $\eta \neq 0$). When $\ell = 1$, there results

$$R_{0103} = \frac{2c_F}{\eta\phi} A_{1,\alpha}^\alpha = \frac{2c_F}{\eta\phi} A_1, \quad (C.6)$$

and $\ell = 2$ gives

$$R_{0103} = \frac{2c_F}{\eta\phi} A_{2,\alpha}^\alpha = \frac{2c_F}{\eta\phi} A_2, \quad (C.7)$$

Setting $\ell = 3$ yields

$$R_{0101} + R_{0202} = -\frac{2c_F}{\eta\phi} A_{3,\alpha}^\alpha \quad (C.8)$$

The K_0 equation is

$$\omega\phi R + \eta\phi R_{00} + 2F_{0,\alpha}^\alpha = 0,$$

which, using the expressions for R and R_{00} and the result C.8, gives

$$(2\omega + \eta)\phi R_{0303} = 2F_{0,\alpha}^\alpha + (2\omega + \eta - \frac{2\omega}{c^2})\frac{2c}{\eta}F_{3,\alpha}^\alpha.$$

However $F_{0,\alpha}^\alpha$ and $F_{3,\alpha}^\alpha$ are not independent.

$$F_{0,\alpha}^\alpha = F_{0,3}^3 = -F_{03,3} = \frac{1}{c}F_{03,0} = -\frac{1}{c}F_{30,0} = -\frac{1}{c}F_{3,\alpha}^\alpha.$$

So R_{0303} is given by

$$R_{0303} = \frac{2(1 - c^2)}{\eta\phi}F_{0,\alpha}^\alpha. \quad (C.9)$$

This is all the information available from the four K_μ equations.

There remain ten $g_{\mu\nu}$ equations which will determine $R_{0101} = R_{0202}, R_{0303}, A_\mu$, and c . These equations are (equation C.4)

$$(12) \quad R_{12}(1 + \omega\phi^2) + \eta\phi^2 R_{0102} = 0$$

$$(11) \quad R_{11}(1 + \omega\phi^2) + \eta\phi^2 R_{0101} - \frac{1}{2}\square^2 P - \phi F_{0,\alpha}^\alpha = 0$$

$$(22) \quad R_{22}(1 + \omega\phi^2) + \eta\phi^2 R_{0202} - \frac{1}{2}\square^2 P - \phi F_{0,\alpha}^\alpha = 0$$

$$(13) \quad R_{13}(1 + \omega\phi^2) + \eta\phi^2 R_{0103} - \frac{1}{2}\eta\phi F_{01,3} = 0$$

$$(23) \quad R_{23}(1 + \omega\phi^2) + \eta\phi^2 R_{0203} - \frac{1}{2}\eta\phi F_{02,3} = 0$$

$$(01) \quad R_{01}(1 + \omega\phi^2) - \frac{1}{2}\eta\phi F_{01,0} - \left(\frac{1}{2}\eta + 2\right)\phi F_{1,\alpha}^\alpha = 0$$

$$(02) \quad R_{02}(1 + \omega\phi^2) - \frac{1}{2}\eta\phi F_{02,0} - \left(\frac{1}{2}\eta + 2\right)\phi F_{2,\alpha}^\alpha = 0$$

$$(03) \quad R_{03}(1 + \omega\phi^2) + P_{,03} - 2F_{3,\alpha}^\alpha = 0$$

$$(00) \quad R_{00}(1 + \omega\phi^2) - \omega\phi^2 R + \frac{1}{2}\square^2 P + P_{,00} - (\eta + 3)\phi F_{0,\alpha}^\alpha = 0$$

$$(33) \quad R_{33}(1 + \omega\phi^2) + \eta\phi^2 R_{0303} - \frac{1}{2}\square^2 P + P_{,33} + (\eta - 1)\phi F_{0,\alpha}^\alpha = 0$$

We now rewrite these ten equations, incorporating the expressions for $R_{\mu\nu}$ and the information gleaned from the K_μ equations.

$$(12) \quad R_{0102} \left[\left(1 - \frac{1}{2}\right)(1 + \omega\phi^2) + \eta\phi^2 \right] = 0 \quad (C.10)$$

$$(11) \quad R_{0101} \left[\left(1 - \frac{1}{2}\right)(1 + \omega\phi^2) + \eta\phi^2 \right] - \frac{1}{2}\square^2 P - \phi F_{0,\alpha}^\alpha = 0$$

$$(22) \quad R_{0202} \left[\left(1 - \frac{1}{2}\right)(1 + \omega\phi^2) + \eta\phi^2 \right] - \frac{1}{2}\square^2 P - \phi F_{0,\alpha}^\alpha = 0.$$

Subtracting these last two equations gives

$$(R_{0101} - R_{0202}) \left[\left(1 - \frac{1}{2}\right)(1 + \omega\phi^2) + \eta\phi^2 \right] = 0. \quad (C.11)$$

Adding (11) and (22) gives

$$(R_{0101} + R_{0202}) \left[\left(1 - \frac{1}{2}\right)(1 + \omega\phi^2) + \eta\phi^2 \right] = \square^2 P + 2\phi F_{0,\alpha}^\alpha. \quad (C.12)$$

Substitution of C.8 for $R_{0101} + R_{0202}$ gives,

$$\square^2 P = \frac{c^2 - 1}{\eta\phi} (1 + \omega\phi^2 + \eta\phi^2) 2F_{0,\alpha}^\alpha. \quad (C.12)$$

Continuing, using equations C.6 and C.7

$$(13) \quad \left[\frac{2c}{\eta\phi} (1 + \omega\phi^2 + \eta\phi^2) - \frac{1}{2} \frac{\eta\phi c}{c^2 - 1} \right] F_{1,\alpha}^\alpha = 0 \quad (C.13)$$

$$(23) \quad \left[\frac{2c}{\eta\phi} (1 + \omega\phi^2 + \eta\phi^2) - \frac{1}{2} \frac{\eta\phi c}{c^2 - 1} \right] F_{2,\alpha}^\alpha = 0 \quad (C.14)$$

where we have also used the fact that

$$F_{02,3} = -\frac{1}{2} F_{02,0} = \frac{1}{c} F_{2,0}^0 = \frac{c}{c^2 - 1} F_{2,\alpha}^\alpha$$

and a similar expression for $F_{01,3}$. The same type of conversion is used in the (01) and (02) equations to give

$$(01) \quad \left[\frac{-2}{\eta\phi} (1 + \omega\phi^2) - 2 + \frac{1}{2} \frac{\eta\phi}{c^2 - 1} \right] F_{1,\alpha}^\alpha = 0 \quad (C.15)$$

$$(02) \quad \left[\frac{-2}{\eta\phi} (1 + \omega\phi^2) - 2 + \frac{1}{2} \frac{\eta\phi}{c^2 - 1} \right] F_{2,\alpha}^\alpha = 0. \quad (C.16)$$

In writing the (03) equation, $R_{03} = \frac{1}{c} (R_{0101} + R_{0202})$ is eliminated via equation C.8 and the fact that $F_{3,\alpha}^\alpha = -cF_{0,\alpha}^\alpha$ is also used.

$$(03) \quad \left[\frac{2c}{\eta\phi} (1 + \omega\phi^2) + 2c \right] F_{0,\alpha}^\alpha - \frac{c}{c^2 - 1} \square^2 P = 0,$$

$$\text{or } \square^2 P = \frac{c^2 - 1}{\eta\phi} (1 + \omega\phi^2 + \eta\phi^2) 2F_{0,\alpha}^\alpha \quad (C.17)$$

in agreement with C.12. Before writing down the (00) and (33) equations, we note that, based on the K equations alone,

$$R_{00} = -R_{0303} - (R_{0101} + R_{0202}) = -\frac{2}{\eta\phi} F_{0,\alpha}^{\alpha}$$

$$R_{33} = R_{0303} - \frac{1}{c^2}(R_{0101} + R_{0202}) = -\frac{2c}{\eta\phi} F_{0,\alpha}^{\alpha}$$

$$R = -2R_{0303} - 2\left(1 - \frac{1}{c^2}\right)(R_{0101} + R_{0202}) = 0.$$

Then, using C.17 for $\square^2 P$, the (00) and (33) equations are both

$$\begin{aligned} (00) \quad & \frac{1}{\eta\phi} [3(c^2 - 1)(1 + \omega\phi^2 + \eta\phi^2) - \eta\phi^2(\eta + 1)] F_{0,\alpha}^{\alpha} = 0. \\ (33) \quad & \end{aligned} \quad (C.18)$$

These are all of the $g_{\mu\nu}$ equations, and we proceed to solve them.

Equation C.10 is satisfied if

$$c^2 = 1 - \frac{\eta\phi^2}{1 + \omega\phi^2 + \eta\phi^2} \quad (C.19)$$

So R_{0102} must obey

$$R_{0102} = \epsilon_1 e^{ik_{\mu} x^{\mu}} \quad (C.20)$$

with k_{μ} representing a wave with speed given by C.19. Equation C.11 is also satisfied if

$$R_{0101} - R_{0202} = \epsilon_2 e^{ik'_\mu x^\mu} \quad (\text{C.21})$$

with the same k'_μ .

Equations C.13, C.14, C.15, and C.16 are all satisfied if

$$c^2 = 1 + \frac{1}{4} \frac{\eta_\phi^2 \phi^2}{1 + \omega\phi^2 + \eta_\phi^2} \quad (\text{C.22})$$

which means that (using C.6 and C.7)

$$R_{0103} = \epsilon_3 e^{ik'_\mu x^\mu}$$

$$R_{0203} = \epsilon_4 e^{ik''_\mu x^\mu}$$

$$F_{1,\alpha}^\alpha = \frac{\eta_\phi}{2c} \epsilon_3 e^{ik'_\mu x^\mu}$$

$$F_{2,\alpha}^\alpha = \frac{\eta_\phi}{2c} \epsilon_4 e^{ik''_\mu x^\mu}$$

where k'_μ is the propagation vector for a wave with speed given by C.22. Finally C.12, C.17, and C.18 are satisfied if

$$c^2 = 1 + \frac{1}{3} \frac{\eta_\phi^2 \phi^2 + \eta_\phi^2}{1 + \omega\phi^2 + \eta_\phi^2} \quad (\text{C.23})$$

and

$$R_{0101} + R_{0202} = \epsilon_5 e^{ik''_\mu x^\mu}$$

$$R_{0303} = \frac{1 - c^2}{c^2} \epsilon_5 e^{ik'_\mu x^\mu}$$

$$F_{0,\alpha}^\alpha = \frac{\eta\phi}{2c^2} \epsilon_5 e^{ik''_\mu x^\mu}$$

$$\square^2 P = \frac{c^2 - 1}{c^2} (1 + \omega\phi^2 + \eta\phi^2) \epsilon_5 e^{ik''_\mu x^\mu}$$

where k''_μ is the propagation vector for a wave moving at speed given by C.23.

To sum up, all of the fourteen equations are satisfied by

$$R_{0102} = \epsilon_1 e^{ik'_\mu x^\mu}$$

$$R_{0101} - R_{0202} = \epsilon_2 e^{ik'_\mu x^\mu}$$

$$R_{0103} = \epsilon_3 e^{ik'_\mu x^\mu}$$

$$R_{0203} = \epsilon_4 e^{ik'_\mu x^\mu}$$

$$R_{0101} + R_{0202} = \epsilon_5 e^{ik''_\mu x^\mu}$$

$$R_{0303} = \frac{1 - c''^2}{c''^2} \epsilon_5 e^{ik''_\mu x^\mu}$$

$$F_{1,\alpha}^\alpha = \frac{\eta\phi}{2c^2} \epsilon_3 e^{ik'_\mu x^\mu}$$

$$F_{2,\alpha}^\alpha = \frac{\eta\phi}{2c^2} \epsilon_4 e^{ik'_\mu x^\mu}$$

$$F_{0,\alpha}^\alpha = \frac{\eta\phi}{2c''^2} \epsilon_5 e^{ik''_\mu x^\mu}$$

$$p = \frac{1 + \omega\phi^2 + \eta\phi^2}{k_0''^2} \epsilon_5 e^{ik''x^\mu}$$

where

$$c^2 \equiv \left(\frac{k_0}{k_3}\right)^2 = 1 - \frac{\eta\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$$

$$c'^2 \equiv \left(\frac{k_0'}{k_3'}\right)^2 = 1 + \frac{1}{4} \frac{\eta^2\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$$

$$c''^2 \equiv \left(\frac{k_0''}{k_3''}\right)^2 = 1 + \frac{1}{3} \frac{\eta\phi^2 + \eta^2\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$$

In all that has been done until now, it has been assumed that $\eta \neq 0$. The other case $\eta = 0$ must be taken separately. The K_ℓ equations (C.3) are then

$$F_{\ell, \alpha}^\alpha = 0 \quad (C.24)$$

For $\ell = 1$ and $\ell = 2$, this becomes

$$\square^2 A_1 = \square^2 A_2 = 0$$

For $\ell = 3$, one has

$$F_{3, \alpha}^\alpha = -cF_{0, \alpha}^\alpha = 0 \quad (C.25)$$

Using the last equation in the K_0 equation gives

$$\omega\phi R = -2F_{0, \alpha}^\alpha = 0 \quad (C.26)$$

Then the g equation (C.2) gives the additional result

$$3\Box^2 Q = -R = 0 \quad (\text{C.27})$$

where $Q = 2\omega(\phi^\alpha A_\alpha - \frac{1}{2}\phi^\alpha \phi^\beta h_{\alpha\beta})$.

The (01) equation is

$$-\frac{1}{c}R_{0103}(1 + \omega\phi^2) = 2\phi F_{1,\alpha}^\alpha = 0$$

where we have used C.24. Therefore,

$$R_{0103} = 0,$$

which also satisfies the (13) equation.

Similarly the (02) and (23) equations are satisfied by

$$R_{0203} = 0$$

$$F_{2,\alpha}^\alpha = 0$$

The remaining equations are (using C.25 and C.27)

$$(12) \quad R_{0102}(1 + \omega\phi^2)(1 - \frac{1}{c^2}) = 0$$

$$(11) \quad R_{0101}(1 + \omega\phi^2)(1 - \frac{1}{c^2}) = 0$$

$$(22) \quad R_{0202}(1 + \omega\phi^2)(1 - \frac{1}{c^2}) = 0$$

(C.28)

and

$$(03) \quad \frac{1}{c}(R_{0101} + R_{0202})(1 + \omega\phi^2) + Q_{,03} = 0$$

$$(00) \quad -R_{0303}(1 + \omega\phi^2) - (R_{0101} + R_{0202})(1 + \omega\phi^2) + Q_{,00} = 0 \quad (C.29)$$

$$(33) \quad R_{0303}(1 + \omega\phi^2) - \frac{1}{c^2}(R_{0101} + R_{0202})(1 + \omega\phi^2) + Q_{,33} = 0$$

If $c \neq 1$ then R_{0101} and R_{0202} must vanish due to equations C.28, and the only solution is the trivial one of no radiation. If $c = 1$, then the three C.28 equations are satisfied identically and the three C.29 equations reduce to

$$R_{0101} + R_{0202} = \frac{Q_{,00}}{1 + \omega\phi^2}$$

$$R_{0303} = 0.$$

The satisfying of C.28 implies that $R_{0101} - R_{0202}$ and R_{0102} are unspecified by the field equations. Also

$$F_{0,\alpha}^{\alpha} = 0$$

means that there is no physical longitudinal wave,¹ but

$$F_{1,\alpha}^{\alpha} = 0 \quad \text{and}$$

$$F_{2,\alpha}^{\alpha} = 0$$

do allow non-zero A_1, A_2 , which produce physical transverse waves.

¹See Appendix D.

APPENDIX D

It is here shown that $F_{0,\alpha}^\alpha = 0$ does not represent physical waves.

$$F_{0,\alpha}^\alpha = 0$$

is solved if

$$A_3 = -\frac{1}{c}A_0.$$

Thus, it would appear that $F_{0,\alpha}^\alpha$ allows the propagation of a non-zero vector wave with the above relationship between its components. However, we now show that this is not a physical wave at all, but a "coordinate ripple" which can be created by a coordinate transformation, and that this wave has no observable effects.

Consider a coordinate system (\bar{x}^μ) in which there is no wave disturbance to the vector field,

$$\bar{K}^\mu = (\phi_0, 0, 0, 0),$$

and effect a coordinate transformation $\bar{x}^\mu \rightarrow x^\mu$ where

$$x^0 = \bar{x}^0 + \frac{a_0}{\omega\phi_0} \sin(\omega x^0 - Kx^3)$$

$$x^1 = \bar{x}^1$$

$$x^2 = \bar{x}^2$$

$$x^3 = \bar{x}^3$$

In the new coordinate system

$$K_{\mu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\mu}} \bar{K}_{\nu} = \frac{\partial \bar{x}^0}{\partial x^{\mu}} \phi_0$$

or

$$K_0 = \phi_0 + a_0 \cos(\omega x^0 - kx^3) \equiv \phi_0 + A_0$$

$$K_3 = -\frac{k}{\omega} a_0 \cos(\omega x^0 - kx^3) = -\frac{1}{c} A_0$$

This is just the type of wave required for $F_0^{\alpha}{}_{,\alpha} = 0$.

There are two ways this wave could be observed. It could couple to the metric via the field equations or it could affect the gravitational constant as in Chapter III. In the field equations it only appears in three expressions

$$F_0^{\alpha}{}_{,\alpha}, \quad F_3^{\alpha}{}_{,\alpha}, \quad \text{and} \quad \phi^0(A_0 - \frac{1}{2}\phi^0 h_{00})$$

The first two are obviously zero; the third can be shown to be zero.

The coordinate transformation that induced the vector wave will also induce a metric wave in an otherwise flat background

$$g_{\mu\nu} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} \bar{g}_{\mu\nu}$$

$$g_{00} = \frac{\partial \bar{x}^{\alpha}}{\partial x^0} \frac{\partial \bar{x}^{\beta}}{\partial x^0} \eta_{\mu\nu} = (1 + A_0/\phi_0)(1 + A_0/\phi_0)\eta_{00}$$

$$g_{00} = 1 + 2A_0/\phi_0 \equiv 1 + h_{00}$$

Then

$$\phi^0(A_0 - \frac{1}{2}\phi^0 h_{00}) = 0$$

and the wave can have no observable effect on the metric.

Finally, the coordinate ripple could affect $K_\mu K^\mu$; which, as was seen in Chapter III, determines the effective gravitational coupling constant. However,

$$\begin{aligned} K_\mu K^\mu &= (\phi_{,\mu} + A_{,\mu})(\phi_{,\nu} + A_{,\nu})(\eta^{\mu\nu} + h_{\mu\nu}) \\ &= \phi_{,0}^2 + 2\phi_{,0}A_{,0}\eta^{00} + \phi_{,0}^2 h^{00} \end{aligned}$$

Noting that $h^{00} = -h_{00} = -2A_{,0}/\phi_{,0}$, it is seen that $K_\mu K^\mu$ is unchanged (as a scalar must be).

Therefore the wave

$$\begin{aligned} A_0 &= a_0 \cos k_\mu x^\mu \\ A_3 &= \frac{a_0}{c} \cos k_\mu x^\mu \end{aligned}$$

is an unphysical coordinate wave, and may not be counted as a mode of propagation.

APPENDIX E

The action integral is

$$A = \int d^4x [L_1 + L_2 + L_3 + L_4 + L_5],$$

with

$$L_1 = \sqrt{-g} 16\pi G_0 L_m(g_{\mu\nu}, \text{matter variables}),$$

$$L_2 = \sqrt{-g} R,$$

$$L_3 = \omega \sqrt{-g} K_\mu^\mu K^\mu R,$$

$$L_4 = \eta \sqrt{-g} K^\mu K^\nu R_{\mu\nu},$$

$$L_5 = -\sqrt{-g} F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta},$$

with

$$F_{\mu\nu} \equiv K_{\mu|\nu} - K_{\nu|\mu}.$$

We require that the action be invariant under variation of $g_{\mu\nu}$. Variation of L_1 defines the stress-energy tensor for matter.

$$\frac{1}{\sqrt{-g}} \frac{\delta L_1}{\delta g^{\mu\nu}} \equiv 8\pi G_0 T_{\mu\nu} \quad (\text{E.1})$$

L_2 also occurs in general relativity, from which the variation is known

to be¹

$$\frac{1}{\sqrt{-g}} \frac{\delta L_2}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (\text{E.2})$$

L_3 looks much like a Brans-Dicke Lagrangian term, and

$$\frac{\delta L_3}{\delta g^{\mu\nu}} = \omega \sqrt{-g} K_{\mu\nu} K R + \omega K_{\alpha}^{\alpha} K \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} R).$$

The last term is the same as the variation Brans and Dicke performed to get their field equations. $K_{\mu}^{\mu} \equiv K$ corresponds to the scalar ϕ , and we can use their result to write²

$$\frac{1}{\sqrt{-g}} \frac{\delta L_3}{\delta g^{\mu\nu}} = \omega (K_{\mu\nu} K R + K R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} K R + K_{\mu\nu} - g_{\mu\nu} \square^2 K) \quad (\text{E.3})$$

L_5 is analogous to the free field energy of the Maxwell field of electrodynamics, whose stress-energy tensor has been found to be³

$$\frac{1}{\sqrt{-g}} \frac{\delta L_5}{\delta g^{\mu\nu}} = 2 F_{\mu\alpha} F_{\nu}^{\alpha} + \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (\text{E.4})$$

¹See, for example, James L. Anderson, Principles of Relativity Physics, Academic Press (1967), p. 345.

²C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

³See, for example, Adler, Bazin, and Schiffer, Introduction to General Relativity, McGraw-Hill (1965), p. 326.

In L_4 , variation of $g^{\mu\nu}$ gives

$$\begin{aligned} \delta L_4 = & -\frac{1}{2}g_{\mu\nu}\sqrt{-g}K^\alpha K^\beta R_{\alpha\beta}\delta g^{\mu\nu} + \sqrt{-g}K^\alpha(K_{\mu\alpha\nu}R_{\alpha\nu} + K_{\nu\alpha\mu}R_{\alpha\mu})\delta g^{\mu\nu} \\ & + \sqrt{-g}K^\alpha K^\beta \delta R_{\alpha\beta} \end{aligned} \quad (E.5)$$

The last term's contribution to the action is

$$\delta I = \int d^4x \sqrt{-g} K^\alpha K^\beta \delta R_{\alpha\beta}$$

From the definition of $R_{\alpha\beta}$, one can write

$$\delta R_{\alpha\beta} = (\delta\Gamma_{\alpha\mu}^{\mu})_{|\beta} - (\delta\Gamma_{\alpha\beta}^{\mu})_{|\mu},$$

where it should be noted that $\delta\Gamma_{\alpha\beta}^{\mu}$ is a tensor even though $\Gamma_{\alpha\beta}^{\mu}$ isn't.⁴

Therefore

$$\begin{aligned} \delta I &= \int d^4x \sqrt{-g} [K^\alpha K^\mu (\delta\Gamma_{\alpha\beta}^{\beta})_{|\mu} - K^\alpha K^\beta (\delta\Gamma_{\alpha\beta}^{\mu})_{|\mu}] \\ &= \int d^4x \sqrt{-g} [(K^\alpha K^\mu \delta\Gamma_{\alpha\beta}^{\beta} - K^\alpha K^\beta \delta\Gamma_{\alpha\beta}^{\mu})_{|\mu} + (K^\alpha K^\beta)_{|\mu} \delta\Gamma_{\alpha\beta}^{\mu} - (K^\alpha K^\mu)_{|\mu} \delta\Gamma_{\alpha\beta}^{\beta}] \end{aligned}$$

Writing the expression in parentheses as a single vector,

$$\xi^\mu = K^\alpha K^\mu \delta\Gamma_{\alpha\beta}^{\beta} - K^\alpha K^\beta \delta\Gamma_{\alpha\beta}^{\mu},$$

we collect together a perfect divergence,

⁴Ibid., p. 307.

$$\sqrt{-g}\xi^{\mu}_{|\mu} = (\sqrt{-g}\xi^{\mu})_{,\mu},$$

whose contribution to the integral vanishes when the boundary of the region of integration is taken at infinity. The integral then reduces to

$$\delta I = \int d^4x \sqrt{-g} [(K^{\alpha\beta})_{|\mu} \delta\Gamma^{\mu}_{\alpha\beta} - (K^{\alpha\beta})_{|\beta} \delta\Gamma^{\mu}_{\alpha\mu}],$$

Now

$$\sqrt{-g}\Gamma^{\mu}_{\alpha\mu} = (\sqrt{-g})_{,\alpha},$$

so

$$\begin{aligned} \sqrt{-g}\delta\Gamma^{\mu}_{\alpha\mu} &= \delta(\sqrt{-g})_{,\alpha} - \Gamma^{\beta}_{\alpha\beta} \delta\sqrt{-g} \\ &= (\delta\sqrt{-g})_{,\alpha} - \frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} \delta\sqrt{-g} \\ &= \left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right)_{,\alpha} + \frac{1}{2}(\sqrt{-g})_{,\alpha}g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu})_{,\alpha} \end{aligned}$$

The second term's contribution to the integral is then

$$\delta I_2 = +\frac{1}{2}\int d^4x \sqrt{-g}(K^{\alpha\beta})_{|\beta}(g_{\mu\nu}\delta g^{\mu\nu})_{,\alpha}.$$

Integrating by parts gives

$$\delta I_2 = -\frac{1}{2} \int d^4x \sqrt{-g} (K^\alpha K^\beta)_{|\alpha\beta} g_{\mu\nu} \delta g^{\mu\nu} \quad (E.6)$$

where $(\sqrt{-g}\xi^\alpha)_{,\alpha} = \sqrt{-g}\xi^\alpha_{;\alpha}$ has again been used.

The remaining term is

$$\delta I_1 = \int d^4x \sqrt{-g} (K^\mu K^\nu)_{|\alpha} \delta \Gamma_{\mu\nu}^{\alpha}$$

It can be shown that for any tensor $M^{\mu\nu}$

$$\int d^4x \sqrt{-g} M^{\mu\nu}{}_{|\alpha} \delta \Gamma_{\mu\nu}^{\alpha} = \frac{1}{4} \int d^4x \sqrt{-g} [M_{\alpha\mu\nu} + M_{\alpha\nu\mu} + M_{\mu\alpha\nu} - M_{\mu\nu\alpha} + M_{\nu\alpha\mu} - M_{\nu\mu\alpha}] g^{\alpha\beta} \delta g^{\mu\nu}$$

Applying this to $(K^\mu K^\nu)_{|\alpha}$ gives

$$\delta I_1 = \frac{1}{2} \int d^4x \sqrt{-g} [(K^\alpha K^\mu)_{|\nu} + (K^\alpha K^\nu)_{|\mu} - (K^\mu K^\nu)_{|\alpha}] g^{\alpha\beta} \delta g^{\mu\nu}$$

or

$$\delta I_1 = \frac{1}{2} \int d^4x \sqrt{-g} [(K^\alpha K^\mu)_{|\nu\alpha} + (K^\alpha K^\nu)_{|\mu\alpha} - \square^2 (K^\mu K^\nu)] \delta g^{\mu\nu} \quad (E.7)$$

Putting E.6 and E.7 into E.5 and gathering E.1 through E.5 yields the $g_{\mu\nu}$ field equation:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \omega (K^\mu K^\nu R + K R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} K R + K_{|\mu\nu} - g_{\mu\nu} \square^2 K) \\ + \eta K^\alpha K^\beta (g_{\mu\alpha} R_{\nu\beta} + g_{\nu\beta} R_{\mu\alpha} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta}) - \frac{1}{2} \eta [g_{\mu\nu} (K^\alpha K^\beta)_{|\alpha\beta} + \square^2 (K^\mu K^\nu) \\ - (K^\mu K^\alpha)_{|\nu\alpha} - (K^\nu K^\alpha)_{|\mu\alpha}] + 2F_{\mu\alpha} F^\alpha{}_\nu + \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = -8\pi G_0 T_{\mu\nu} \end{aligned}$$

The K equation is straight forward. K_{μ} does not appear in either L_1 or L_2 ; in L_3 and L_4 the variation is simply

$$\frac{1}{\sqrt{-g}} \frac{\delta L_3}{\delta K^{\mu}} = 2\omega K_{\mu} R$$

$$\frac{1}{\sqrt{-g}} \frac{\delta L_4}{\delta K^{\mu}} = 2\eta K^{\nu} R_{\mu\nu}.$$

L_5 can be written

$$L_5 = -\sqrt{-g}(K_{\mu,\nu} - K_{\nu,\mu})(K_{\alpha,\beta} - K_{\beta,\alpha})g^{\mu\alpha}g^{\nu\beta}.$$

Then

$$\delta L_5 = -4\sqrt{-g}(K_{\alpha,\beta}\delta K_{\mu,\nu})(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}).$$

Integration of $\int d^4x L_5$ by parts yields

$$\delta L_5 = 4[\sqrt{-g}K_{\alpha,\beta}(g^{\mu\alpha}g^{\nu\beta} - g^{\xi\beta}g^{\nu\alpha})]_{,\nu}\delta K_{\mu},$$

or

$$\delta L_5 = 4\sqrt{-g}[(K_{\alpha,\beta} - K_{\beta,\alpha})g^{\mu\alpha}g^{\nu\beta}]_{,\nu}\delta K_{\mu}.$$

So the contribution to the field equation is

$$\frac{1}{\sqrt{-g}} \frac{\delta L_5}{\delta K^\mu} = 4F_{\mu\beta} g^{\nu\beta}.$$

The K_μ field equation is therefore

$$\omega K_\mu R + \eta K^\nu R_{\mu\nu} + 2F_{\mu\alpha} = 0$$

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