

# Vector Space Decomposition of Reactive Power for Periodic Nonsinusoidal Signals

Niels LaWhite and Marija D. Ilić

**Abstract**—A vector space is used to represent periodic voltage and current signals in a single port electrical network. This representation is convenient for expressing time average power quantities, such as average power, apparent power, and certain definitions of reactive power. The vector representation for reactive power provides insight into the difficulty of finding a consistent definition for a scalar measure of reactive power when harmonics are present. Instead, a reactive power vector is defined and shown to obey conservation according to Tellegen's theorem. Projections of the reactive power vector are shown to obey conservation and can be used as signed, scalar measures of reactive power.

## I. INTRODUCTION

CONFUSION over reactive power for periodic signals with harmonics dates to 1927, when Budeanu introduced an orthogonal decomposition of apparent power into active, reactive, and distortion power components [1]. While these components were observed to add in quadrature to equal the apparent power  $S$  neither reactive nor distortion power components could be assigned any physical significance. Furthermore, unlike the active and reactive components, distortion power disobeys conservation, making it counterintuitive as a measure of energy transfer.

Seeing the limitations of Budeanu's two-component decomposition, Fryze introduced a reactive power definition for a single component [2]. Here, reactive power  $Q$  is the single orthogonal component accounting for the difference in apparent and average power. The magnitude of  $Q$  was seen as a useful quantity, because  $Q = 0$  corresponds to  $S = P$ , or unity power factor.  $Q$  was also given a sign convention to account for the difference between capacitive and inductive reactive power. Despite the sign convention, however, Fryze's definition does not obey conservation, meaning that a circuit with  $Q = +1$  does not, in general, compensate a circuit with  $Q = -1$ .

Many other authors [3]–[10], have proposed decompositions of reactive power quantities, but none have succeeded in defining a decomposition that not only accounts for the total inactive power  $\sqrt{S^2 - P^2}$  but also satisfies conservation. The reason for the difficulty becomes apparent when the problem is mapped to a vector space. If the periodic voltage and current are represented as  $n$  orthogonal components, the inactive

power can be shown to consist of  $m = \frac{1}{2}n(n-1)$  orthogonal components. Linear combinations of these components obey conservation but only represent one of the  $m$  underlying dimensions. Quadrature, or rms combinations of these components can reflect the magnitude contribution from multiple dimensions but violate conservation.

## II. VECTOR REPRESENTATION OF PERIODIC SIGNALS

We begin by considering periodic voltage and current waveforms of an arbitrary single port circuit element. If the voltage and current waveforms are periodic with period  $T$ , such that  $v(t+T) = v(t)$  and  $i(t+T) = i(t)$ , the waveforms can be considered members of the  $L_2$  function space with a scalar product and induced norm given in (1):

$$\begin{aligned} \langle x(t), y(t) \rangle &= \frac{1}{T} \int_0^T x(t)y(t) dt \\ \|x(t)\| &= \langle x(t), x(t) \rangle^{1/2} \\ &= \left[ \frac{1}{T} \int_0^T x^2(t) dt \right]^{1/2}. \end{aligned} \quad (1)$$

This function space is convenient for expressing time average power components into the port. The average power  $P$  is a scalar product, and the apparent power  $S$  is a product of norms.

$$\begin{aligned} P &= \overline{v(t) i(t)} \\ &= \langle v(t), i(t) \rangle \\ S &= v_{\text{rms}} i_{\text{rms}} \\ &= \|v(t)\| \|i(t)\|. \end{aligned} \quad (2)$$

The function space is then mapped to an  $\ell_2$  vector space  $\mathfrak{R}^n$  by expressing the periodic signals as a linear combination of  $n$  orthonormal basis functions  $\phi_j(t)$ .

$$x(t) = \sum_{j=1}^n x_j \phi_j(t). \quad (3)$$

This expression is written in vector notation by stacking the coefficients,  $x_j$  into a constant vector  $X$ , and the basis functions,  $\phi_j(t)$  into a vector of time functions  $\Phi(t)$ .

$$x(t) = X^T \Phi(t) \quad (4)$$

where  $X^T$  denotes the transpose of  $X$ .

The constant vector  $V$  then fully represents the signal waveform,  $v(t)$ , in the  $\ell_2$  vector space associated with a

Manuscript received March 15, 1995. This work was supported by Second Wind Inc., the National Renewable Energy Laboratory, and the U.S. Department of Energy. This paper was recommended by Associate Editor S. Mori.

The authors are with the Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

Publisher Item Identifier S 1057-7122(97)02723-2.

particular basis  $\Phi(t)$ . The orthonormality of  $\Phi(t)$  is expressed in the  $L_2$  function space.

$$\langle \phi_i(t), \phi_j(t) \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (5)$$

or in vector notation

$$\begin{aligned} \langle \Phi(t), \Phi^T(t) \rangle &= \frac{1}{T} \int_0^T \Phi(t) \Phi^T(t) dt \\ &= \mathbf{I} \end{aligned} \quad (6)$$

where  $\mathbf{I}$  is the identity matrix.

The most common orthonormal basis for representing periodic signals is the Fourier basis. Here, the basis functions are a constant followed by sine and cosine pairs at multiples of the fundamental frequency  $\omega = 2\pi/T$ .

$$\Phi(t) = \begin{bmatrix} 1 \\ \sqrt{2} \cos(\omega t) \\ \sqrt{2} \sin(\omega t) \\ \sqrt{2} \cos(2\omega t) \\ \sqrt{2} \sin(2\omega t) \\ \sqrt{2} \cos(3\omega t) \\ \sqrt{2} \sin(3\omega t) \\ \vdots \end{bmatrix}. \quad (7)$$

While the length of this basis is infinite, for band-limited signals,  $\Phi(t)$  can be truncated to an odd length,  $n$ , with sufficient accuracy. The resulting vector space is  $\mathfrak{R}^n$  and has coordinate axes corresponding to rms frequency components. Using this vector space, a periodic waveform  $x(t)$  is represented by the constant,  $n$ -length vector  $X$ .  $X$  has an odd number of elements, the first of which is the d.c. or average value of  $x(t)$ . Subsequent elements are in pairs of rms components for each harmonic frequency in the basis.

Another useful basis consists of a set of delayed periodic sync functions. As above, an odd length  $n$  is chosen as one plus twice the number of the highest harmonic to be represented. The basis can then be written as

$$\Phi(t) = \begin{bmatrix} \psi(t) \\ \psi\left(t - \frac{T}{n}\right) \\ \psi\left(t - \frac{2T}{n}\right) \\ \vdots \\ \psi\left\{t - (n-1)\frac{T}{n}\right\} \end{bmatrix} \quad (8)$$

where

$$\psi(t) = \frac{\sin\left(\frac{n\pi}{T}t\right)}{\sqrt{n} \sin\left(\frac{\pi}{T}t\right)}.$$

Fig. 1 shows an example of the periodic sync function basis, which has the desirable property that at  $n$  evenly spaced points in time during a single period, only one of the basis functions is nonzero. According to (3), if at one of these points, only  $\phi_j$  is

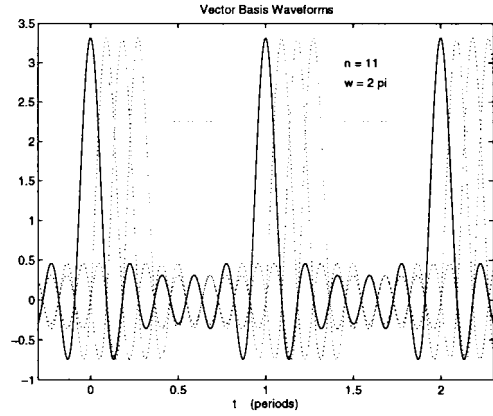


Fig. 1. Periodic sync functions for use as a vector basis.

nonzero, then the constant coefficient  $x_j$  must be proportional to the value of  $x(t)$  sampled at that instant. The coordinate axes in this vector basis, then, correspond to rms-normalized time samples, and a vector  $X$  consists of  $n$  evenly spaced samples of the corresponding waveform  $x(t)$ .

The periodic sync function basis is the time domain equivalent of the Fourier basis above. The transformation between these bases is performed by a matrix multiplication that is equivalent to the discrete Fourier transform. The constraint used to choose  $n$  can be viewed as a result of the Nyquist Sampling Theorem [11].

Regardless of the choice of basis, the orthonormal property of  $\Phi(t)$  provides an equivalence between the scalar product in the  $L_2$  function space and the  $\ell_2$  vector space. Substituting the vector expression (5) into (2) gives

$$\begin{aligned} \langle x(t), y(t) \rangle &= \frac{1}{T} \int_0^T X^T \Phi(t) Y^T \Phi(t) dt \\ &= X^T \left[ \frac{1}{T} \int_0^T \Phi(t) \Phi^T(t) dt \right] Y \\ &= X^T Y \\ &= \langle X, Y \rangle. \end{aligned} \quad (9)$$

The function-space induced norm in (2) is then equivalent to the vector-space 2-norm

$$\begin{aligned} \|x(t)\| &= \sqrt{X^T X} \\ &= \|X\|. \end{aligned} \quad (10)$$

This norm equivalence applies to any orthonormal basis and allows the expressions for time average and apparent power in (2) to be written in terms of the vector representation of voltage and current

$$\begin{aligned} S &= \|V\| \|I\| \\ P &= V^T I. \end{aligned} \quad (11)$$

Here,  $V$  and  $I$  are constant  $n$ -vectors,  $P$  is a signed scalar quantity, and  $S$  is a nonnegative scalar. The power expressions in (11) are independent of the orthonormal basis  $\Phi(t)$ . These vector expressions can also be applied to multiphase circuits, as shown in the Appendix.

### III. VECTOR REPRESENTATION OF REACTIVE POWER

Fryze's definition of reactive power in periodic nonsinusoidal systems is given in [2]

$$Q = \pm \sqrt{S^2 - P^2}, \quad (12)$$

This equation defines  $Q$  as orthogonal component to time average power  $P$  comprising apparent power  $S$  according to the Pythagorean Theorem. As the sign convention in (12) is arbitrary and misleading, only the positive instance is used.  $Q$ , then, is a nonnegative scalar quantity, which can also be expressed in terms of the vectors  $V$  and  $I$

$$\begin{aligned} Q &= [S^2 - P^2]^{1/2} \\ &= [V^T V I^T I - (V^T I)^2]^{1/2} \\ &= [V^T (V I^T - I V^T) I]^{1/2}, \end{aligned} \quad (13)$$

The vector equation (13) can be written in terms of the elements of  $V$  and  $I$ , leading to a scalar equation for  $Q$

$$Q = \left[ \sum_{j=1}^n \sum_{k=j+1}^n (V_j I_k - V_k I_j)^2 \right]^{1/2} \quad (14)$$

where  $V_i$  and  $I_i$  are the  $i$ th elements of the vectors  $V$  and  $I$ .

In the Fourier basis, (14) expresses  $Q$  in terms of the rms values of the orthogonal cosine and sine frequency components of  $v(t)$  and  $i(t)$ . Equations (13) and (14), however, are general and hold for any orthonormal decomposition.

While it is not immediately obvious, (14) suggests that  $Q$  is the norm of a vector product of  $V$  and  $I$ . This vector product, or cross product, of the two  $n$ -dimensional vectors is formed by cross multiplying voltage components  $V_k$  with current components  $I_j$  for each unique permutation of  $j \neq k$ . Each componentwise cross product is projected in an orthogonal direction  $\vec{l}$  according to an arbitrary rule, such as the right-hand-rule commonly used for three dimensions

$$\begin{aligned} \vec{l} &= (\vec{j} \times \vec{k}) \\ &= -(\vec{k} \times \vec{j}). \end{aligned} \quad (15)$$

The form of this cross product is called a two-form and is the exterior product of two 1-forms, as discussed in detail in [12]. For the purposes of this paper, however, the cross product has  $m$  orthogonal elements given by  $(V_j I_k - V_k I_j) \vec{l}$  equal to the projected area of the parallelogram  $(V, I)$  on coordinate axes  $\vec{j}$  and  $\vec{k}$ .  $m = n(n-1)/2$ , corresponding to the unique permutations of  $j \neq k$ . These elements appear in the upper triangle of an antisymmetric matrix,  $\mathbf{R}$ , which will be referred to as the reactive power matrix.

$$\mathbf{R} = (V I^T - I V^T). \quad (16)$$

As  $\mathbf{R}$  is antisymmetric, there are zeros on the main diagonal, and the negative of the upper triangle appears in the lower triangle.

$Q$  in (14) is the root mean square of the  $m$  orthogonal elements of the cross product, and can be written as a norm. In matrix form,  $Q$  is the Frobenius norm of  $\mathbf{R}$  divided by  $\sqrt{2}$ . Alternately, for mathematical convenience, the  $m$  elements of

cross product can be assembled into an  $m$ -vector  $\vec{R}$  called the reactive power vector, and  $Q$  is the two-norm of  $\vec{R}$ .

Regardless of the notation, the cross product maps the  $n$ -vectors  $V$  and  $I$  to an  $m$ -dimensional space, and  $Q$  is a norm in this space. Each component represents the interaction of two orthogonal frequency components of voltage and current. Each pair of orthogonal frequency components does not contribute to average power, but does contribute, orthogonally, to  $Q$ . While each component of  $Q$  is signed, according to the rule (15),  $Q$  cannot have a single sign convention, because it is the norm of  $m$  orthogonal signed components.

For the special case of sinusoidal signals, voltage and current waveforms can be expressed as two-dimensional vectors or phasors. In this case,  $n = 2$ , so  $m = 1$ , and the cross product has a single component. This component is signed and corresponds to the classical definition of reactive power for sinusoidal signals. The single sign convention is only possible when  $m = 1$  and is not possible when harmonics are present.

### IV. CONSERVATION OF REACTIVE POWER COMPONENTS

Conservation of power implies that the sum of the power into all elements of a network is zero.

$$\sum_k \rho^k = 0 \quad (17)$$

where  $\rho^k$  is the power at element  $k$ .

Conservation of energy implies that instantaneous power is conserved at each instant of time. As the time-average operator is linear, it commutes with the summation in (17), so average power is also conserved

$$\begin{aligned} \sum_k P^k &= \sum_k \overline{p^k(t)} \\ &= \overline{\sum_k p^k(t)} \\ &= 0. \end{aligned} \quad (18)$$

Other power definitions, however, may not obey conservation.  $S$  and  $Q$ , for example, are nonnegative by definition and cannot, therefore, sum to zero unless all are zero. Tellegen's theorem can be used to prove conservation for a set of power definitions called generalized powers. As shown in [13]–[15], a particular power is a generalized power and will obey conservation if it can be expressed in the following form

$$\rho^k = \Lambda_v(v^k) \Lambda_i(i^k) \quad (19)$$

where  $v^k$  and  $i^k$  are the voltage and current at the  $k$ th circuit element, and  $\Lambda_v$  and  $\Lambda_i$  are linear operators, or more generally, Kirchhoff voltage and current operators. As in (18), applying a linear operator  $\beta$  to (19) preserves conservation and leads to a more general form for generalized powers

$$\rho^k = \beta[\Lambda_v(v^k) \Lambda_i(i^k)]. \quad (20)$$

The definition for  $Q$  in (17) cannot be written in the form (20), and Fryze's definition for  $Q$  does not obey conservation. Each of the elements of the cross product, however, do take the required form, because the elements of the vectors  $V$  and

$I$  can be expressed as linear transforms of the voltage and current waveforms.

$$V = \frac{1}{T} \int_0^T v(t) \Phi(t) dt. \quad (21)$$

Each element in the reactive power vector  $R$  is a component-wise cross product,  $V_j I_k - V_k I_j$ , which is a linear combination of products of transformed voltage and current, as in (20). Therefore, the components of the cross product are generalized powers and obey conservation. Equation (18) can then be written in vector form

$$\sum_k R^k = \underline{0} \quad (22)$$

where  $\underline{0}$  is the zero  $m$ -vector. This classification is independent of the specific decomposition of voltage and current, and will hold for any orthonormal basis  $\Phi(t)$ .

## V. COORDINATE ROTATION

We now consider an arbitrary single port network with specific periodic voltage and current waveforms,  $v(t)$  and  $i(t)$  at the port. For the analysis,  $v(t)$  and  $i(t)$  are assumed to be nonzero and not proportional, so  $Q \neq 0$ . Choosing a truncated Fourier basis of dimension  $n$ , we can express the port voltage and current as constant  $n$ -vectors,  $V$  and  $I$ . Using the same basis, we can express the voltage and current across the  $k$ th network element as the vectors  $V^k$  and  $I^k$ .

The average power into the port  $P$  and into the  $k$ th element  $P^k$  can be expressed using the vector-space inner product, as in (11)

$$\begin{aligned} P &= V^T I \\ P^k &= V^{kT} I^k. \end{aligned} \quad (23)$$

Time average power is a generalized power and obeys conservation, so  $P = \sum P^k$ . Thus there is an intuitive mechanism for showing where power flowing into the circuit is being dissipated. However,  $Q$  is not a generalized power, and  $Q \neq \sum Q^k$ . It is not obvious how  $Q^k$  at each element contributes to the total  $Q$  at the port.

The following section shows how a simple coordinate rotation provides an alternate vector decomposition of  $Q$  such that  $R$  of the port has only one nonzero element,  $q$ , which is positive, by definition. As the other elements are zero,  $Q = \sqrt{q^2} = q$ , but while  $Q$  does not obey conservation,  $q$  is an element of  $R$  and is a generalized power. Conservation, then, implies that  $Q = q = \sum q^k$ , where  $q^k$  is the same element of  $R^k$ , the reactive power vector of the  $k$ th circuit element. As with the classical definition of reactive power in sinusoidal systems, this single signed component of reactive power provides a measure that obeys conservation and can be used to indicate where the port reactive power flows in the circuit.

The coordinate rotation is performed using a symmetric  $n \times n$  matrix  $M$ , which is orthonormal in that  $MM^T = \mathbf{I}$ . The port voltage waveform,  $v(t)$ , can be expressed as

$$\begin{aligned} v(t) &= V^T M M^T \Phi(t) \\ &= V'^T \Phi'(t), \end{aligned} \quad (24)$$

$\Phi'(t) = M^T \Phi(t)$  can be recognized as a rotated orthonormal basis, and  $V' = M^T V$  is the vector representation in the rotated vector space.

The first column in  $M$  is chosen as a unit vector in the direction of  $V$ . The second column is chosen as the orthogonal unit vector closest to  $I$  in the plane of  $V$  and  $I$ .

$$\begin{aligned} M_1 &= \frac{V}{\|V\|} \\ M_2 &= \frac{\left( I - \frac{V V^T I}{V^T V} \right)}{\left\| I - \frac{V V^T I}{V^T V} \right\|}, \end{aligned} \quad (25)$$

The other columns of  $M$  are not important to this analysis and can be generated by a Gram–Schmidt orthogonalization procedure.

The coordinate rotation matrix  $M$  selects the first basis function  $\phi'_1(t)$  proportional to the port voltage waveform  $v(t)$  and the second basis function  $\phi'_2(t)$  proportional to orthogonal, or nonpower producing, component of the port current. In the new vector space, the port voltage vector  $V'$  will be nonzero only in the first element  $V'_a$ . The port current vector  $I'$  will be nonzero only in the first two elements  $I'_a$  and  $I'_b$ . Because of the choice of unit vectors  $V'_a$  and  $I'_b$  are positive.

Expressed in the new vector space, average power into the port is the scalar product  $P = V'^T I' = V'_a I'_a$ . The port reactive power vector  $R'$  has only one nonzero element

$$q = V'_a I'_b - V'_b I'_a. \quad (26)$$

$q$  is positive, because  $V'_a$  and  $I'_b$  are positive and  $V'_b$  is zero. Consequently,  $q = Q$ , the reactive power of the port.

The definition for  $q$  is a projection of  $R$  in a particular direction.  $q$  of the port is the projection of  $R$  in the direction of  $R$ , and is therefore equal to the norm of  $R$ , which equals  $Q$ . However,  $q^k$  of the  $k$ th circuit element is the projection of  $R^k$  in the direction of the port  $R$ .

$$q^k = V_a'^k I_b'^k - V_b'^k I_a'^k. \quad (27)$$

This projection has a sign and can be interpreted as a generalized measure of how the element  $k$  contributes to the total  $Q$  at the port. The sign convention is such that  $q^k$  is positive if element  $k$  contributes to the total  $Q$  and negative the element helps compensate  $Q$ .  $q^k$  is always zero for resistive elements, because resistor voltage and current are proportional. Energy storage elements can have either positive or negative  $q^k$ , depending on whether the energy transfer happens to increase or decrease the  $Q$  at the port.

The expression for  $q$  can be simplified by expressing the projection in the nonrotated vector space. In terms of  $M_1$  and  $M_2$ , the first two columns of  $M$ , (26) and (27) become

$$\begin{aligned} q &= M_1^T V M_2^T I - M_2^T V M_1^T I \\ &= V^T (M_1 M_2^T - M_2 M_1^T) I \end{aligned} \quad (28)$$

$$q^k = V^{kT} (M_1 M_2^T - M_2 M_1^T) I^k. \quad (29)$$

Substituting  $M_1$  and  $M_2$  from (26) yields an expression for  $q^k$  in terms of  $V^k$  and  $I^k$  of element  $k$ , and  $V$ ,  $I$ , and  $Q$ ,

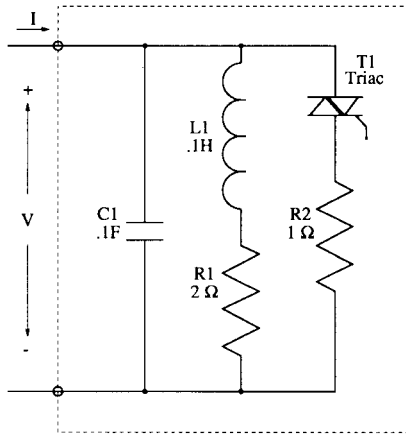


Fig. 2. Example circuit.

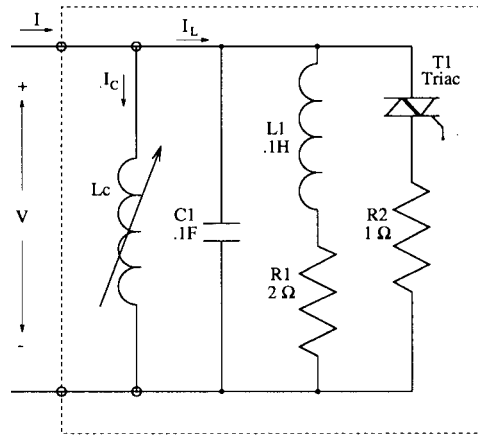


Fig. 3. Example compensation circuit.

of the port.

$$\begin{aligned}
 q^k &= V^{k^T} \left( \frac{VI^T - IV^T}{\sqrt{V^T V I^T I - V^T I V^T I}} \right) I^k \\
 &= V^{k^T} \left( \frac{VI^T - IV^T}{Q} \right) I^k \\
 &= V^{k^T} \left( \frac{\mathbf{R}}{Q} \right) I^k
 \end{aligned} \tag{30}$$

where  $\mathbf{R}/Q$  is the  $n \times n$  reactive power matrix for the port normalized by the reactive power at the port.

Equation (30) can be mapped to the time domain by expressing the inner product as a definite integral, as in (9)

$$\begin{aligned}
 q^k &= \frac{1}{Q} (V^{k^T} V I^{k^T} I - V^{k^T} I I^{k^T} V) \\
 &= \frac{\int_0^T v_k v dt \int_0^T i_k i dt - \int_0^T v_k i dt \int_0^T i_k v dt}{T \sqrt{\int_0^T v^2 dt \int_0^T i^2 dt - \left( \int_0^T v i dt \right)^2}}
 \end{aligned} \tag{31}$$

where  $v_k$ ,  $i_k$ ,  $v$ , and  $i$ , are the voltage and current waveforms of the  $k$ th element and of the port.

### VI. CIRCUIT EXAMPLE

Fig. 2 shows a one-port circuit example consisting of a linear  $R$ - $L$ - $C$  circuit in parallel with a switched resistive load. The triac switch is self commutating with a  $90^\circ$  firing angle, so it is on for the latter half of each half-cycle. The voltage at the circuit port is given and contains a 1 Hz fundamental component as well as small components of the third and fifth harmonics. The port current contains these harmonics as well as additional harmonics introduced by the switching discontinuity.

For the analysis, the voltage and current waveforms are decomposed with a Fourier basis consisting of normalized cosine and sine pairs of only the odd harmonics. For this

example, the basis is truncated to length  $n = 20$ .

$$\Phi(t) = \begin{bmatrix} \sqrt{2} \cos(\omega t) \\ \sqrt{2} \sin(\omega t) \\ \sqrt{2} \cos(3\omega t) \\ \sqrt{2} \sin(3\omega t) \\ \vdots \\ \sqrt{2} \cos(19\omega t) \\ \sqrt{2} \sin(19\omega t) \end{bmatrix}. \tag{32}$$

For the linear circuit elements, the voltage and current relationships can be expressed as an impedance relationship analogous to Ohm's law

$$V = ZI \text{ or } I = Z^{-1}V \tag{33}$$

where  $Z$  is an impedance matrix for the linear element.  $Z_R$  for resistive elements is the resistance  $R$  times the identity matrix. In the Fourier basis  $Z_L$  and  $Z_C$  for inductive and capacitive elements are block diagonal, antisymmetric matrices with zeros on the main diagonal  $Z_L = L J_\omega$  and  $Z_C = (C J_\omega)^{-1}$  where  $L$  and  $C$  are the inductance and capacitance, and

$$J_\omega = \begin{bmatrix} 0 & \omega & 0 & 0 & 0 & 0 & \cdot \\ -\omega & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 3\omega & 0 & 0 & \cdot \\ 0 & 0 & -3\omega & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 5\omega & \cdot \\ 0 & 0 & 0 & 0 & -5\omega & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The impedance matrix for the linear  $R$ - $L$ - $C$  circuit is calculated as a parallel combination  $Z = [Z_{C1}^{-1} + (Z_{R1} + Z_{L1})^{-1}]^{-1}$ , and the current vector is then calculated from (33).

To calculate the current in the switched resistive load, the vector for the  $R2$  voltage is calculated using the Fourier transform.  $V_{R2} = (1/T) \int_0^T v_{R2}(t)\Phi(t) dt$ . The current is then calculated from (33) using  $Z_{R2}$ .

Fig. 3 shows the steady state voltage and current waveforms of the circuit port. Fig. 3 also shows current waveforms of the  $R$ - $L$ - $C$  load and the switched resistive load. The vector

TABLE I  
VECTOR REPRESENTATION OF VOLTAGE AND CURRENT

$V$	$I_{RLC}$	$I_{SW}$	$I$
0.0000	0.4853	-0.3491	0.1363
1.0000	0.4551	0.4982	0.9533
0.0000	0.1635	0.3109	0.4744
0.1000	0.0265	0.0518	0.0783
0.0000	-0.0292	-0.0736	-0.1028
-0.0100	-0.0014	-0.0068	-0.0082
0.0000	0.0000	0.0965	0.0965
0.0000	0.0000	0.0018	0.0018
0.0000	0.0000	-0.0526	-0.0526
0.0000	0.0000	-0.0018	-0.0018
0.0000	0.0000	0.0580	0.0580
0.0000	0.0000	0.0018	0.0018
0.0000	0.0000	-0.0387	-0.0387
0.0000	0.0000	-0.0018	-0.0018
0.0000	0.0000	0.0413	0.0413
0.0000	0.0000	0.0018	0.0018
0.0000	0.0000	-0.0305	-0.0305
0.0000	0.0000	-0.0018	-0.0018
0.0000	0.0000	0.0320	0.0320
0.0000	0.0000	0.0018	0.0018

representation of the voltage waveform is given in Table I, along with the vectors for the linear and switched load current components and the total load current. The linear load current contains only those frequencies that appear in the voltage, while the switched load current contains all frequencies, as generated by the switching discontinuity. For the  $n = 20$  basis used here, the reactive power vector has length  $m = 190$  corresponding to all possible cross product terms between the voltage and current.

Table II summarizes the power in the circuit, listing  $P$ ,  $S$ ,  $Q$ , and  $q$ , for each element in the circuit and for the port. Average power is dissipated in the resistive elements, with a small dissipation in the switch introduced by errors in truncating the Fourier basis. Unlike  $S$  and  $Q$ ,  $P$  is conserved, so the average power into the port equals the total dissipation in the circuit. As shown in the data,  $q$  is also conserved, because it is a projection of the reactive power vector.

Specifically,  $q$  is the projection of reactive power in the direction of the port reactive power, so  $q$  provides a measure of the contribution of each circuit element to the  $Q$  seen at the port. For this example, the capacitor is contributing to the port reactive power, while the inductor is providing compensation. While the resistive elements do not contribute reactive power, the triac switch does. Even though the switch is not an energy storage device, it generates reactive power, because the nonlinear switching characteristic generates current harmonics.

## VII. OPTIMIZATION EXAMPLE

The next example shows how a projection of the reactive power vector can be used to optimize a compensation circuit for maximum power factor. Using the previous circuit example as a load, the problem is to find the value of inductor  $L_C$  that when placed in parallel with the load, maximizes the power factor seen by the voltage source. This example is a constrained optimization in that compensation with a single inductor can improve the power factor but cannot achieve unity power factor.

TABLE II  
AVERAGE, APPARENT, AND REACTIVE POWER, AND THE PROJECTION  $q$

	$P$	$S$	$Q$	$q$
C1	0.0000	0.6600	0.6600	0.3416
L1	0.0000	0.1471	0.1471	-0.0598
R1	0.4577	0.4577	0.0000	0.0000
T1	0.0080	0.4971	0.4970	0.2454
R2	0.4955	0.4955	0.0000	0.0000
Total	0.9612	2.2575	1.3042	0.5272
Port	0.9612	1.0963	0.5272	0.5272

The power factor is defined as the magnitude of average power divided by the apparent power.

$$PF = \frac{|P|}{S} = \frac{|P|}{\sqrt{P^2 + Q^2}}. \quad (34)$$

Because the compensator is lossless, and the voltage source is constant,  $P$  is not affected by compensation and remains constant. Maximizing the power factor, then, corresponds to minimizing  $Q$  at the port.

$Q$  is not a generalized power and does not obey conservation, so the port  $Q$  is not simply the sum of load reactive power,  $Q_L$ , and compensator reactive power,  $Q_C$ . The reactive power vector, however, is a generalized power, so the port  $R$  vector is the sum of  $R_L$  and  $R_C$ , and the port  $Q$  is the norm of this vector sum

$$Q = \|R_L + R_C\|. \quad (35)$$

$R_L$  and  $R_C$  are, in general, neither collinear nor orthogonal, but (35) can be simplified by decomposing the  $R$  vectors into collinear and orthogonal parts. For this example,  $R_C$  is decomposed into  $R_{C1}$  and  $R_{C2}$ , which are collinear with and orthogonal to  $R_L$ , respectively. This decomposition yields a quadrature decomposition of the compensator reactive power

$$Q_C^2 = Q_{C1}^2 + Q_{C2}^2 = \|R_{C1}\|^2 + \|R_{C2}\|^2. \quad (36)$$

For the total  $Q$  seen by the source,  $Q_{C1}$  adds linearly with  $Q_L$ , and the sum adds in quadrature with  $Q_{C2}$

$$Q^2 = (Q_L + Q_{C1})^2 + Q_{C2}^2. \quad (37)$$

Equation (37) can be simplified by expanding the square and substituting from (36)

$$Q^2 = Q_L^2 + 2Q_L Q_{C1} + Q_{C1}^2 + Q_{C2}^2 = Q_L^2 + 2Q_L Q_{C1} + Q_C^2 \quad (38)$$

where  $Q_L$  is the constant load reactive power,  $Q_{C1}$  is the projection of  $R_C$  in the direction of  $R_L$ , and  $Q_C$  is the total compensator reactive power. The projection,  $Q_{C1}$ , can be expressed in terms of the load reactive power matrix  $\mathbf{R}_L$  as in (30)

$$Q_{C1} = V^T \left( \frac{\mathbf{R}_L}{Q_L} \right) I_C. \quad (39)$$

As the compensator is lossless,  $P_C = 0$ , and  $Q_C^2 = S_C^2$

$$Q_C^2 = V^T V I_C^T I_C. \quad (40)$$

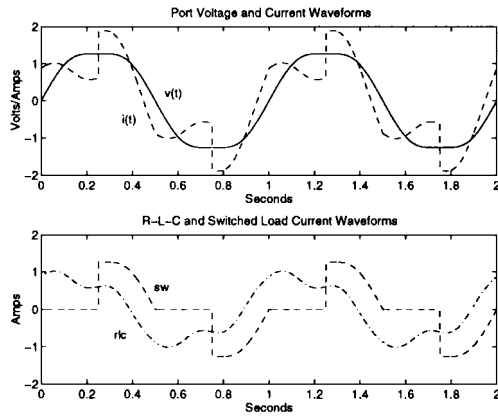


Fig. 4. Example circuit waveforms.

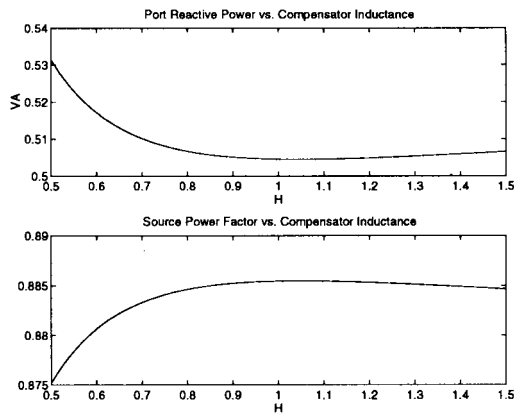


Fig. 5. Optimum compensation.

Using the impedance relationship, (33),  $I_C$  can be expressed in terms of the impedance matrix for the compensator inductance. This impedance matrix is proportional to  $L_C$ , the inductance.

$$I_C = Z_{L_C}^{-1} V = \frac{1}{L_C} J_{\omega}^{-1} V. \quad (41)$$

Now substituting (39)–(41) into (38) yields a scalar quadratic equation in  $1/L_C$ .

$$Q^2 = Q_L^2 + 2V^T \mathbf{R}_L J_{\omega}^{-1} V \frac{1}{L_C} + V^T V V^T J_{\omega}^{-1 T} J_{\omega}^{-1} V \frac{1}{L_C^2}. \quad (42)$$

Inserting the values for the example circuit gives.

$$Q^2 = 0.0256 \frac{1}{L_C^2} - 0.0490 \frac{1}{L_C} + 0.2779. \quad (43)$$

This equation has a single minimum, at  $L_C = 1.05H$ , where  $Q = 0.505$ . Fig. 5 shows the port reactive power and power factor as a function of  $L_C$  in the vicinity of this minimum.

The compensated power factor is 0.885, improved from 0.877 by  $L_C$ . Due to the extreme harmonic content of the waveforms, the power factor can not be improved beyond 0.885 without a more complicated compensator designed to better match harmonics in current to those in voltage. The third and fifth harmonics, which exist in the voltage, could

be adjusted by a linear network of tuned circuits, but nulling the higher harmonics in current would require a nonlinear or active compensator. In any case, (38)–(40) would still apply, but  $Z_{L_C}$  in (41) would be a complicated function of control parameters. As (42) would no longer be quadratic, finding the global minima in closed form would be difficult. Still, for a small number of control parameters, the global minimum could be found numerically from (38)–(40).

### VIII. COMPARISON TO OTHER REACTIVE POWER DEFINITIONS

The expression of  $Q$  as the norm of a cross product leads to a decomposition of  $Q$  into  $m$  orthogonal elements. As with the vector decomposition of the voltage and current, the reactive power vector depends on the choice of basis functions, but the dimensionality  $m$  is fundamental to the cross product. Many other reactive power decompositions have been suggested, but the definitions suffer because they attempt to represent the  $m$  orthogonal conserved power components with fewer than  $m$  quantities.

Some definitions consist of linear combinations of the elements of  $R$ , which, like  $q$ , are projections of  $R$  and obey conservation. Each projection accounts for only one of the  $m$  dimensions in  $R$ ; the other  $m - 1$  components must be accounted for if the total decomposition is to reflect the total inactive power. Other definitions combine orthogonal projections of  $R$  in quadrature, thus representing the magnitude contribution from several of the  $m$  components. The sign convention, however, is lost when components are combined in quadrature, and the resulting definition is not conserved.

The original Budeanu definition of reactive power is given in [1]

$$Q_b = \sum_i E_i I_i \sin(\theta_i) \quad (44)$$

where  $E_i$ ,  $I_i$ , and  $\theta_i$  are the rms voltage and current and the phase angle difference of the  $i$ th harmonic. In the Fourier basis, each harmonic is represented with a cosine/sine pair, and each term in (44) represents the interaction of the cosine and sine components at a particular frequency. These terms appear as every other entry in the first upper diagonal in the reactive power matrix,  $\mathbf{R}$ . Equation (44) adds these terms linearly and ignores all other components, so Budeanu's definition is a projection of  $R$  in a fixed direction, and this projection has a sign and obeys conservation. The direction of the projection, however, is arbitrary and does not reflect any particular useful quantity, such as power factor or  $\sqrt{S^2 - P^2}$ .

In order to account for the other components of  $R$ , Budeanu introduced Distortion Power.

$$D_b = \sqrt{S^2 - P^2 - Q_b^2} \quad (45)$$

$D_b$ , then, is the norm of  $m - 1$  orthogonal projections of  $R$ , which are also orthogonal to the  $Q_b$  projection in (44). As  $D_b$  adds components in quadrature, the definition does not obey conservation. Because the direction of the  $Q_b$  projection is arbitrary, the  $D_b$  component has no generally useful interpretation.

Perhaps Budeanu intended that  $Q_b$  reflect the individual interactions of the  $n/2$  frequencies, and that  $D_b$  represent the cross terms between different frequencies. If so,  $Q_b$  should have been defined as the projection of  $R$  in the direction pointed by the  $n/2$  components in the first upper diagonal of  $\mathbf{R}$ , and  $D_b$  should have been the projection of  $R$  in the direction pointed by the all the other elements in  $\mathbf{R}$ . The other  $m-2$  orthogonal projections would then have been zero, so the definition would have properly decomposed the total inactive power with two conserved quantities.

The Fryze reactive power, shown in (13), represents the total inactive power in a single signed quantity. The magnitude is the norm of all the elements of  $R$ , which is useful because minimizing  $Q$  minimizes  $S$ , maximizing power factor. The sign definition, however, is misleading in that the definition is a norm and is not conserved. Presumably, the sign was added to reflect the sign of fundamental reactive power component, which would be dominant in the nearly sinusoidal case. If the intention was to obtain a single conserved quantity to represent  $\sqrt{S^2 - P^2}$ , the definition should have been the projection of  $R$  in the direction of  $R$ , which is positive and equal to  $\sqrt{S^2 - P^2}$ . This is, in fact, the definition of  $q$  given in this paper.

Page, in [5], defined capacitive reactive power and inductive reactive power,  $Q_C$  and  $Q_L$ , the two components of  $Q$  that could be compensated by a parallel capacitor and inductor. These powers are, in fact, projections of the  $R$  vector in the directions of  $R_C$  and  $R_L$ , the reactive power vectors of a pure capacitance and pure inductance. Page correctly noted that these quantities can have a sign and are conserved, so a negative  $Q_C$  can be compensated by adding a capacitor in parallel. Page also noticed that the  $Q_C$  and  $Q_L$  are not generally orthogonal or parallel, and introduced a refinement to account for the cross terms.

Page's projections failed to account for the other  $m-2$  orthogonal components of  $R$  that must be considered if the compensation is to minimize the total  $Q$ . His projections help to solve for the best parallel  $L$  and  $C$ , but do not adapt to other compensation topologies, such as a series  $L$  and  $C$ , or a linear network of  $L$ 's and  $C$ 's. As shown in Example 2, a single element parallel compensator does not necessarily provide substantial power factor improvement.

Wyatt and Ilić, in [9], discusses the merits of an instantaneous expression of reactive power.

$$p_{\text{react},2}(t) = v(t) \frac{d}{dt} i(t) - i(t) \frac{d}{dt} v(t). \quad (46)$$

The time average of this quantity can also be interpreted as a measure of reactive power, and can be expressed in vector notation. In the Fourier basis,  $(d/dt)\Phi(t) = J_\omega^T \Phi(t)$ , where  $J_\omega$  is the block diagonal, antisymmetric matrix given in Example 1. As in (9), the time average of (46) simplifies by taking the constants outside the integral.

$$\begin{aligned} \frac{1}{T} \int_0^T p_{\text{react},2}(t) dt &= V^T J_\omega I - V^T J_\omega^T I \\ &= 2V^T J_\omega I. \end{aligned} \quad (47)$$

While (47) is not normalized, it has the form of  $q^k$  in (30) and is proportional to a projection of  $R$ . Like Budeanu's  $Q_b$ ,

the projection contains only interactions between the sine and cosine components of individual harmonics, but unlike  $Q_b$ , each component is scaled by its frequency.

In linear sinusoidal systems, reactive power can also be interpreted in terms of average stored electric and magnetic energy.

$$Q = 2\omega [\overline{W_M(t)} - \overline{W_E(t)}]. \quad (48)$$

Because  $W_M(t)$  and  $W_E(t)$  both oscillate as  $\cos(2\omega t)$  but are  $180^\circ$  out of phase, (48) can also be written in terms of the total stored energy,  $W(t) = W_M(t) + W_E(t)$ .

$$Q = \text{rms} \left\{ \frac{d}{dt} [W(t)] \right\}. \quad (49)$$

This expression has the nice interpretation that  $Q$ , in the sinusoidal case, is an rms measure of the oscillatory transfer of stored energy. This fact, however, is a result of the  $180^\circ$  phase difference between  $W_M(t)$  and  $W_E(t)$ , and no such relationship holds when harmonics are present or when nonlinear circuits are considered.

## IX. CONCLUSIONS

The work in this paper uses an  $n$ -dimensional vector-space decomposition of periodic nonsinusoidal voltage and current waveforms to show that reactive power  $Q$  consists of  $m = \frac{1}{2}n(n-1)$  orthogonal conserved components. If these components are stacked into the  $m$ -vector  $R$ , then the scalar  $Q$  is the norm of  $R$  and does not obey conservation. Projections of  $R$ , however, do obey conservation and can provide useful measures when analyzing or compensating a circuit. One such measure is  $q$ , the projection of  $R$  in the direction of  $R$ , which is positive and equal to  $Q$ . While this definition seems trivial, Example 1 shows how such a projection can be used to analyze flow of reactive power in a circuit. Example 2 shows how a similar projection is used to solve a constrained power factor optimization problem.

The vector-space expressions for voltage and current provide a convenient notation for solving minimization problems, but the dimensionality does not arise purely from the notation. The underlying dimensionality arises from the terms in the expression  $Q^2 = S^2 - P^2$ .  $S^2$  is the product of two  $n$ -term equations and has  $n^2$  terms.  $P^2$  is the square of one  $n$ -term equation and has  $\frac{1}{2}n(n+1)$  terms. The difference has  $n^2 + \frac{1}{2}n(n+1)$  terms, but  $2n$  cancel, leaving  $\frac{3}{2}n(n-1)$  terms. These remaining terms are group in triples to form  $m = \frac{1}{2}n(n-1)$  perfect squares, corresponding to the  $m$  orthogonal components of  $Q$ .

The components of  $Q$ , then, do not correspond to the  $2n$  orthogonal frequency components of instantaneous power, but instead arise from the definition of  $S$ .  $Q$ , then, is not a quantity bearing physical interpretation in terms of instantaneous power, but is simply the  $m$  remaining orthogonal components of  $S$  not contained in  $P$ . Similarly, the usefulness of  $Q$  is not as a reflection of actual circulating powers, but as a measure of the overall useless component of  $S$ .



## APPENDIX

## VECTOR REPRESENTATION FOR MULTIPHASE CIRCUITS

Multiphase periodic waveforms can also be represented as vectors given an appropriate set of basis functions. By combining the vectors representing voltage and current in each phase into collective vectors, the entire set of waveforms can be represented as two constant vectors  $V$  and  $I$ . While the results are general, this analysis will focus on the three phase case.

The set of phase voltage waveforms is assembled as a row-vector, and expressed in terms of  $V$  and the basis functions.

$$\begin{aligned} [v_1(t) \ v_2(t) \ v_3(t)] &= V^T \underline{\Phi}(t) \\ [i_1(t) \ i_2(t) \ i_3(t)] &= I^T \underline{\Phi}(t), \end{aligned} \quad (50)$$

Here  $V$  is length  $3n$ , and  $\underline{\Phi}(t)$  is a  $3n \times 3$  matrix of basis functions.

$$\underline{\Phi}(t) = \begin{bmatrix} \Phi(t) & \vec{0} & \vec{0} \\ \vec{0} & \Phi(t) & \vec{0} \\ \vec{0} & \vec{0} & \Phi(t) \end{bmatrix}. \quad (51)$$

Where  $\vec{0}$  is the  $n$ -vector of zeros. Although  $\underline{\Phi}$  is a rectangular matrix, the basis is still orthonormal.

$$\frac{1}{T} \int_0^T \underline{\Phi}(t) \underline{\Phi}^T(t) dt = \mathbf{I}. \quad (52)$$

The total instantaneous power is the sum of instantaneous power in each phase.

$$\begin{aligned} p(t) &= [v_1(t) \ v_2(t) \ v_3(t)] \begin{bmatrix} i_1(t) \\ i_2(t) \\ i_3(t) \end{bmatrix}^T \\ &= V^T \underline{\Phi}(t) \underline{\Phi}^T(t) I. \end{aligned} \quad (53)$$

Because  $\underline{\Phi}$  is orthonormal, the time average total power simplifies to a single scalar product, as in the single phase case.

$$\begin{aligned} P &= \frac{1}{T} \int_0^T V^T \underline{\Phi}(t) \underline{\Phi}^T(t) I dt \\ &= V^T I. \end{aligned} \quad (54)$$

Similarly, the multiphase rms value is again the two-norm, and the definition for total apparent power  $S = \frac{\|V\| \|I\|}{\sqrt{2}}$  is the same as in the single phase case.  $Q = \sqrt{S^2 - P^2}$  has the same form as (14), and can be interpreted as the norm of a cross product vector  $R$ . Therefore, it is possible to construct a reactive power measure,  $q$ , that is a projection of  $R$  specific to the set of phase voltage and current waveforms at the port. This measure is signed, obeys conservation, and indicates how circuit elements contribute or compensate for the reactive power at the port.

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**Niels LaWhite** received the S.B. degree in 1987 and the S.M. and E.E. degrees in 1995, all in electrical engineering from the Massachusetts Institute of Technology, Cambridge.

Between the S.B. and S.M. degrees, he worked in the wind power industry, designing instrumentation and control systems for large scale wind turbine generators. More recently, he has been working as a consultant and conducting research in digital signal processing algorithms for measuring power and other electrical quantities. His other research interests include wind turbine control, modeling, and vibration analysis.



**Marija D. Ilić** received the Dipl. Eng., and M.E.E. degrees from the University of Belgrade, Yugoslavia, in 1974 and 1977, respectively, and the M.Sc. and D.Sc. degrees in systems science and mathematics from Washington University, St. Louis, MO, in 1979 and 1980, respectively.

She has since been actively involved in teaching and research in the area of large scale electric power systems at three major universities, Cornell University, the University of Illinois at Urbana-Champaign, and M.I.T. She is presently a member of the M.I.T. Department of Electrical Engineering and Computer Science, where she is a Senior Research Scientist. Her research interests are in the areas of control and network theory applications to large scale power systems, with the specialty in voltage modeling and control.

Dr. Ilić is the recipient of the First Presidential Young Investigator (PVI) award for the area of power systems.