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Vector Two-Point Functions in Maximally Symmetric Spaces

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Abstract. We obtain massive and massless vector two-point functions in maximally symmetric spaces (and vacua) of any number of dimensions. These include de Sitter space and anti-de Sitter space, and their Euclidean analogs S^n and H^n. Our method is based on a simple way of constructing every possible maximally symmetric bitensor $T_{a \dots bc' \dots d'}(x, x')$ which carries tangent-space indices $a \dots b$ at x and $c' \dots d'$ at x'.

Introduction

A great deal has been written about two-point functions for scalar fields in de Sitter space [1–8]. In de Sitter-invariant states, two-point functions like $G(x, x') = \langle \Phi(x) \Phi(x') \rangle$ are functions only of the geodesic distance $\mu(x, x')$ between x and x'. Hence the wave equation $(\Box_x - m^2) G(x, x') = 0$ can be written as an ordinary differential equation in the variable μ and its solution $G(\mu)$ can be expressed in terms of hypergeometric functions.

Our intuition is that the two-point function for a vector field $Q^{ab'}(x, x') = \langle A^a(x) A^{b'}(x') \rangle$ in a maximally symmetric state should be a function only of the geodesic distance $\mu(x, x')$. But how does one incorporate the tangent space indices a and b'? Clearly $Q^{ab'}(x, x')$ has to transform as a vector under coordinate transformations at x and x'. How can it be expressed in some simple way?

In the first section we answer that question. There is a fundamental set of three bitensors, obtained by differentiating $\mu(x, x')$, in terms of which any maximally symmetric bitensor can be expressed. In order to solve wave equations one needs to know the derivatives of these bitensors. Formulae which express each of these derivatives in terms of the original set of three bitensors are derived. They have previously been obtained by Peters [27].

In Sect. IIa these methods are applied to find the scalar two-point functions $G(\mu(x, x'))$ by solving the wave equation for $G(\mu)$. The principles by which appropriate solutions are selected in de Sitter and anti-de Sitter spaces are

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explained in Sects. IIb and IIc respectively and used ad nauseam in the treatment of the vector two-point functions.

Sections III and IV treat the massive and massless vector cases, and are similarly divided into parts a–c. The wave equation $(-\Box+\kappa)Q_{ab'}=0$ is imposed on a maximally symmetric bitensor ansatz $Q_{ab'}$ (supplemented with $\nabla^a Q_{ab'}=0$ in the massive case and corresponding to Feynman gauge in the massless case). This yields coupled ordinary second order differential equations for the coefficient functions in $Q_{ab'}$. In the massive case the equations decouple, whereas in the massless case we must solve the full fourth order system.

Section V contains a summary and discussion of our results.

There are five appendices. Appendix B contains a brief discussion of maximally symmetric bitensors. Appendix E gives the massless Feynman gauge vector propagators for the "Euclidean" vacuum state of $(2\leqq n\leqq 12)$-dimensional de Sitter space. The remaining three appendices contain technical details.

We conclude this introduction with a brief review of the literature. Scalar two-point functions in de Sitter space may be found in [1–8, 15, 16] and in anti-de Sitter space in [18, 20, 30]. Spin 1/2 in de Sitter space [1, 3, 5, 26], spin 1/2 in anti-de Sitter space [25], and spin-1 in CP^n [9] have also been treated. The massless spin-1 Feynman function in Adler gauge on S^n is given in [26], and the massless spin-1 retarded propagator in "conformal" gauge on any conformally flat spacetime is given in [27]. Finally an expression for the *traced* two-point function for arbitrary spin on S^4 is given in [28].

I. Maximally Symmetric Bitensors

A maximally symmetric space is an n-dimensional manifold with metric which has as many global Killing fields as is possible [10, 11, 29]. This type of space "looks the same" in every direction and at every point. The simplest examples are flat space and spheres, each of which has $\frac{1}{2}n(n+1)$ independent Killing fields. For S^n these generate all rotations, and for R^n they include both rotations and translations.

Consider the expectation value of a vector field $A^a(x)$ in some maximally symmetric state $|\psi\rangle$,

$$\langle\psi|A^a(x)|\psi\rangle\,. \tag{1.1}$$

This must vanish, since otherwise it would define a preferred direction at x, thus breaking the maximal symmetry of the space.

For a two-point expectation value

$$\langle\psi|A^a(x)A^{b'}(x')|\psi\rangle \tag{1.2}$$

the situation is quite different, since the pair of points x and x' determine preferred geometric objects. These objects are the distance $\mu(x,x')$ along the shortest geodesic from x to x', the unit tangents $n^a(x,x')$ and $n^{a'}(x,x')$ to the geodesic at x and x', and the parallel propagator $g^a{}_{b'}(x,x')$ along the geodesic (see Fig. 1). In a maximally symmetric space, other geodesics do not determine new geometric objects. For example, on a unit sphere, the length of the longer geodesic $2\pi-\mu(x,x')$ is a function of $\mu(x,x')$. In the pseudo-Riemannian case not all pairs of points can be connected by a geodesic. These geometric objects have unique analytic extensions to such pairs, however, as shown in Appendix A.

Tensors such as these that depend on two points x and x' are called *bitensors* [12]. They may carry unprimed or primed indices that live in the tangent space at x or x' respectively.

Bitensor manipulations may be carried out exactly as for tensors. Unprimed indices are raised with g^{ab} and primed ones with $g^{a'b'}$. It follows from the definition of the parallel propagator $g_{ab'}$ that $g_{ab}(x) = g_a{}^{c'}(x,x')\, g_{c'b}(x',x)$ and $g_{a'b'}(x') = g_{a'}{}^c(x',x)\, g_{cb'}(x,x')$. Since (un)primed indices always go with (un)primed points, the arguments can be omitted from these equations, viz $g_a{}^{c'} g_{c'b} = g_{ab}$, etc.

One can differentiate a bitensor function of x and x' with respect to either argument. To distinguish the two possibilities we write ∇_a and $\nabla_{a'}$. For example $\nabla_{a'} g_b{}^{c'}$ is the bitensor obtained by (1) fixing x and regarding $g_b{}^{c'}$ as a vector function of x' and then (2) taking its covariant derivative with respect to x'.

The world function $\mu(x,x')$ is the distance along the geodesic $x^a(\lambda)$ connecting x and x'. If $\cdot \equiv \dfrac{\partial}{\partial\lambda}$ then

$$\mu(x,x') = \int_0^1 [g_{ab}\dot{x}^a(\lambda)\, \dot{x}^b(\lambda)]^{1/2}\, d\lambda, \qquad x^a(0) = x, \qquad x^a(1) = x'. \tag{1.3}$$

In the literature $\frac{1}{2}[\mu(x,x')]^2$ is often denoted by $\sigma(x,x')$. The vectors defined by

$$n_a(x,x') = \nabla_a \mu(x,x') \quad \text{and} \quad n_{a'}(x,x') = \nabla_{a'}\mu(x,x') \tag{1.4}$$

have unit length, since μ is the proper distance along a geodesic. Note that they point away from each other, so that $g_a{}^{b'} n_{b'} = -n_a$, as shown in Fig. 1.

Expectation values such as (1.2) define *maximally symmetric bitensors*. These are bitensors invariant under any isometry of the manifold. In Appendix B, we define the action of diffeomorphisms on bitensors, and prove that any maximally

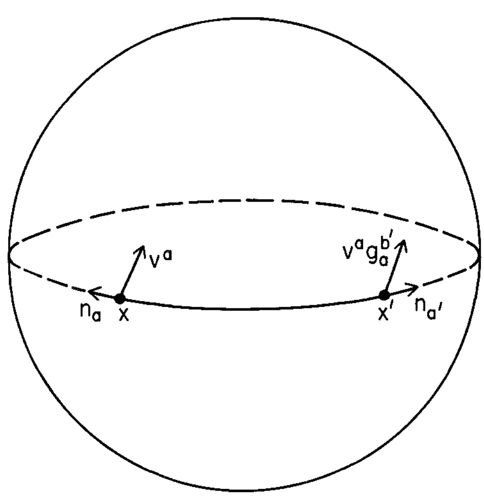

Fig. 1. Two points x and x' in a maximally symmetric space determine preferred directions at those points. They are tangent vectors to the geodesics joining x and x'. The unit vectors n_a at x and $n_{a'}$ at x' point *away* from the geodesic. $V^a g_a{}^{b'}$ is the vector at x' obtained by parallel transport of V^a along the geodesic from x to x'. Hence $-n_a = g_a{}^{b'} n_{b'}$

symmetric bitensor can be expressed as a sum of products of g_{ab}, $g_{a'b'}$, μ, n_a, $n_{a'}$, and $g_{ab'}$. The coefficients of these terms are functions only of the geodesic distance $\mu(x,x')$. For example if $T_{abc'}(x,x')$ is a maximally symmetric bitensor then

$$T_{abc'} = f_1(\mu) g_{ab} n_{c'} + f_2(\mu) g_{ac'} n_b + f_3(\mu) g_{bc'} n_a + f_4(\mu) n_a n_b n_{c'}, \tag{1.5}$$

where the $f_i(\mu)$ are functions of $\mu(x,x')$.

If $T_{a\ldots bc'\ldots d'}$ is any maximally symmetric bitensor, then the derivatives $\nabla_e T_{a\ldots bc'\ldots d'}$ and $\nabla_{e'} T_{a\ldots bc'\ldots d'}$ also define maximally symmetric bitensors (see Appendix B). The derivatives of n_a and $g_{ab'}$ can therefore be expressed in terms of our fundamental set:

$$\begin{aligned} \nabla_a n_b &= A(\mu) g_{ab} + B(\mu) n_a n_b, \\ \nabla_a n_{b'} &= C(\mu) g_{ab'} + D(\mu) n_a n_{b'}, \\ \nabla_a g_{bc'} &= E(\mu) g_{ab} n_{c'} + F(\mu) g_{ac'} n_b + G(\mu) g_{bc'} n_a + H(\mu) n_a n_b n_{c'}. \end{aligned} \tag{1.6}$$

The idea is that on the right-hand side of (1.6) we have written down every possible maximally symmetric bitensor with the correct index structure. In the remainder of this section the functions $A, \ldots, G$ are determined for the three cases of zero, positive and negative curvature. They are given in Table 1, which contains the only results used in the following sections. The contents of Table 1 may also be found in formulae (3.23) and (3.26) of Peters [27].

We begin by finding the coefficient functions A, B in the expression (1.6) for $\nabla_a n_b$. Since $n^b n_b = 1 \Rightarrow n^b \nabla_a n_b = 0 \Rightarrow A + B = 0$, we have

$$\nabla_a n_b = A(g_{ab} - n_a n_b). \tag{1.7}$$

Now contracting (1.7) yields $\nabla^a n_a = (n-1)A$, where n is the dimension of the maximally symmetric space. From (1.4) it then follows that

$$A = (n-1)^{-1} \Box \mu. \tag{1.8}$$

The biscalar $\Box \mu(x,x')$ is again maximally symmetric, hence it must depend only on the geodesic distance μ from x to x'. We need to evaluate it in three special cases, R^n, S^n, and H^n, which are maximally symmetric spaces of constant (zero, positive and negative) scalar curvature.

To evaluate $\Box \mu$ in R^n note that in spherical coordinates $ds^2 = dr^2 + r^2\, d\Omega_{n-1}^2$ centered around x' we have $\mu = r$, and the Laplacian on a function of μ is

$$\Box = \frac{1}{r^{n-1}} \frac{d}{dr} r^{n-1} \frac{d}{dr}. \tag{1.9}$$

Hence $\Box \mu = (n-1)\mu^{-1}$ and therefore $A = \mu^{-1}$.

To evaluate $\Box \mu$ on S^n, the sphere is assumed to have radius $R = 1$ (the units can be restored by dimensional analysis). In spherical coordinates centered about x', the metric is

$$ds^2 = d\theta^2 + \sin^2\theta\, d\Omega_{n-1}^2. \tag{1.10}$$

Here $d\Omega_{n-1}^2$ is the metric on a unit S^{n-1}. The geodesic distance is $\mu = \theta$, and the Laplacian on a function of μ is

$$\Box = (\sin\theta)^{1-n} \frac{d}{d\theta} (\sin\theta)^{n-1} \frac{d}{d\theta}. \tag{1.11}$$

Hence $\Box\mu=(n-1)\cot\mu$ and therefore $A=\cot\mu$. Restoring the units yields $A=R^{-1}\cot(\mu/R)$. Notice that for $\mu/R\ll 1$, the effects of local curvature are negligible, and $A(\mu)$ looks the same as in flat space.

A similar computation for H^n of (hyperbolic) radius $|R|$ yields $A=|R|^{-1}\coth(\mu/|R|)$. This is also obtained from the result for S^n by simply letting $R=i|R|$. For this reason, we will henceforth give formulae only for S^n. The corresponding formulae for H^n are obtained by letting R have pure imaginary values and for R^n by letting $R\to\infty$.

To evaluate C and D in (1.6) note that $n^{b'}n_{b'}=1\Rightarrow n^{b'}\nabla_a n_{b'}=0\Rightarrow -C+D=0$ since $g_{ab'}n^{b'}=-n_a$. Hence

$$\nabla_a n_{b'}=C(g_{ab'}+n_a n_{b'})\,. \tag{1.12}$$

We can obtain C by applying $n^e\nabla_e$ to (1.12), and then integrating out along a geodesic.

Consider a geodesic γ through a fixed point x', and let x be any point on γ. The derivative with respect to the proper distance $\mu(x,x')$ along γ is $d/d\mu=n^e\nabla_e$. Thus we have $n^e\nabla_e g_{ab'}=n^e\nabla_e n_a=n^e\nabla_e n_{b'}=0$, i.e., the tensors $g_{ab'}$, n_a, and n_b are by construction covariantly constant along the geodesic.

Applying $n^e\nabla_e$ to the left hand side of (1.12) we find

$$n^e\nabla_e\nabla_a n_{b'}=n^e\nabla_a\nabla_e n_{b'} \tag{1.13}$$

$$=(\nabla_a n^e)(\nabla_e n_{b'}) \tag{1.14}$$

$$=-AC(g_{ab'}+n_a n_{b'}) \tag{1.15}$$

We can commute derivatives in the first step since $n_{b'}$ is a scalar at x (it has no unprimed indices), and the third step follows using (1.7) and (1.12). Now applying $n^e\nabla_e$ to the right hand side of (1.12) we have

$$n^e\nabla_e C(g_{ab'}+n_a n_{b'})=(dC/d\mu)(g_{ab'}+n_a n_{b'})\,. \tag{1.16}$$

Together (1.12), (1.15), and (1.16) yield the ordinary differential equation along γ

$$\frac{d}{d\mu}C(\mu)=-A(\mu)\,C(\mu) \tag{1.17}$$

whose solution is

$$C(\mu)=\text{const}\cdot\exp\left(-\int A(\mu)\,d\mu\right). \tag{1.18}$$

We will shortly determine the constant of integration in (1.18) from the short distance (or flat-space) behavior of $C(\mu)$. It is convenient to find E, F, G, and H in (1.6) before doing this.

To determine E, F, G, and H first note that $n^a\nabla_a g_{bc'}=0$ and (1.6) imply

$$(E-F+H)n_b n_{c'}-Gg_{bc'}=0\,. \tag{1.19}$$

Hence $(E-F+H)=0$ and $G=0$ [contract (1.19) alternately with $n^b n^{c'}$ and $g^{bc'}$]. Furthermore $\nabla_a(g_{bb'}n^b)=-\nabla_a n_{b'}$ implies that $n^b\nabla_a g_{bb'}=-\nabla_a n_{b'}-g_{bb'}\nabla_a n^b$. The left-hand side can be evaluated from (1.6) and the right-hand side from (1.7) and (1.12).

Table 1. This table gives expressions for the derivatives of the elementary maximally symmetric bitensors n_b, $n_{b'}$, and $g_{bc'}$, in terms of sums of products of these same bitensors. Since $n_a = \nabla_a \mu$ and $n_{a'} = \nabla_{a'} \mu$, higher derivatives can be found by repeated application of these formulae. The curvature and the functions $A(\mu)$ and $C(\mu)$ are identical for S^n and H^n. However R is imaginary for H^n and real for S^n

Definition of tangent vectors:

$n_a = \nabla_a \mu$
$n_{a'} = \nabla_{a'} \mu$

Definition of $A(\mu)$ and $C(\mu)$:

$\nabla_a n_b = A(g_{ab} - n_a n_b)$
$\nabla_a n_{b'} = C(g_{ab'} + n_a n_{b'})$
$\nabla_a g_{bc'} = -(A+C)(g_{ab} n_{c'} + g_{ac'} n_b)$

Relations between $A(\mu)$ and $C(\mu)$:

$dA/d\mu = -C^2$
$dC/d\mu = -AC$

$C^2 - A^2 = 1/R^2$

$A(\mu)$ and $C(\mu)$:

Space:	Scalar curvature:	$A(\mu)$:	$C(\mu)$:
R^n	0	μ^{-1}	$-\mu^{-1}$
S^n or H^n	$\dfrac{n(n-1)}{R^2}$	$\dfrac{1}{R}\cot(\mu/R)$	$\dfrac{-1}{R}\csc(\mu/R)$

The process can be repeated for $n^{b'} \nabla_a g_{bb'}$. One obtains

$$\begin{Bmatrix} n^b \\ n^{b'} \end{Bmatrix} \nabla_a g_{bb'} = \begin{Bmatrix} F g_{ab'} + (E+H) n_a n_{b'} \\ E g_{ab} - (F-H) n_a n_b \end{Bmatrix} = -(A+C) \begin{Bmatrix} g_{ab'} + n_a n_{b'} \\ g_{ab} - n_a n_b \end{Bmatrix}. \tag{1.20}$$

The unique solution to these equations is

$$E = F = -(A+C) \quad \text{and} \quad G = H = 0, \tag{1.21}$$

so one finds from (1.6) that

$$\nabla_a g_{bc'} = -(A+C)(g_{ab} n_{c'} + g_{ac'} n_b). \tag{1.22}$$

In flat space $A+C$ must vanish. Hence the constant of integration in (1.18) must be chosen so that $\lim_{\mu \to 0}(A(\mu) + C(\mu)) = 0$. The resulting functions A and C for S^n, H^n, and R^n are shown in Table 1.

II. Scalar Two-Point Function

a)

In this section we calculate the scalar two-point function

$$G(x, x') = \langle \psi | \varphi(x) \varphi(x') | \psi \rangle. \tag{2.1}$$

This simple and well-known example serves as a model for the spin-1 case. We assume that the state $|\psi\rangle$ is maximally symmetric, so that for spacelike separated

x, x', $G(x, x')$ depends only upon the geodesic distance $\mu(x, x')$. For timelike separated points, the symmetric and Feynman functions also depend only on μ, whereas (2.1) and the commutator function depend additionally on the time ordering. We will obtain all of these two point functions by analytic continuation from spacelike $\mu^2 > 0$ to timelike $\mu^2 < 0$.

Denoting the derivative $dG/d\mu$ by G', we have

$$\begin{aligned} \Box\, G(\mu) &= \nabla^a \nabla_a G(\mu) = \nabla^a (G'(\mu) n_a) \\ &= G''(\mu) n^a n_a + G'(\mu) \nabla^a n_a \\ &= G''(\mu) + (n-1)\, A(\mu)\, G'(\mu)\,, \end{aligned} \tag{2.2}$$

where the formulae of Table 1 have been used. The equation of motion $(-\Box + m^2)\varphi = 0$ therefore implies

$$G''(\mu) + (n-1)\, A(\mu)\, G'(\mu) - m^2 G = 0\,, \tag{2.3}$$

as long as $x \neq x'$. This equation can be converted into a hypergeometric equation [13, 14] by a simple change of variable.

Defining a new variable

$$z = \cos^2(\mu/2R)\,, \tag{2.4}$$

the Eq. (2.3) for G becomes

$$H(a_0, b_0, c_0)\, G(z) = 0\,. \tag{2.5}$$

Here $H(a, b, c)$ is the hypergeometric operator

$$H(a, b, c) = z(1-z)\frac{d^2}{dz^2} + [c - (a+b+1)z]\frac{d}{dz} - ab\,, \tag{2.6}$$

and the parameters in the present case take the values (subscript 0 denotes scalar)

$$\begin{aligned} a_0 &= \tfrac{1}{2}[n - 1 + \sqrt{(n-1)^2 - 4m^2R^2}]\,, \\ b_0 &= \tfrac{1}{2}[n - 1 - \sqrt{(n-1)^2 - 4m^2R^2}]\,, \\ c_0 &= \tfrac{1}{2} n\,. \end{aligned} \tag{2.7}$$

There are two linearly independent solutions $G(z)$ to Eq. (2.5). The choice of a particular solution is determined by which maximally symmetric state $|\psi\rangle$, and which 2-point function $G(x, x')$ is desired. In the next two subsections, we find appropriate solutions in de Sitter and anti-de Sitter space by examining the following properties of G:

i) short distance behavior as $\mu \to 0$
ii) long distance behavior as $\mu \to \infty$
iii) location of singular points
iv) location of branch cuts.

b) de Sitter Space (R Real)

Equation (2.5) is invariant under $z \to 1-z$, since the parameters satisfy $a_0 + b_0 + 1 = 2c_0$. Two independent solutions are therefore [13, 14]

$$F(a_0, b_0; c_0; z) \quad \text{and} \quad F(a_0, b_0; c_0; 1-z). \tag{2.8}$$

Here $F(a, b; c; z)$ is the hypergeometric function defined by a convergent power series for $|z| < 1$ and by analytic continuation elsewhere. These solutions are singular at $z = 1$ and $z = 0$ respectively, since $F(a_0, b_0; c_0; z) \sim (1-z)^{-n/2}$ as $z \to 1$ and $F(a_0, b_0; c_0; z) \to 1$ as $z \to 0$. They must be linearly independent, because they have different singular points, so the general solution is some linear combination of them.

To locate the first possible singularity in $G(\mu(x, x'))$, notice that $z = 1$ when $\mu(x, x') = 0$. In Riemannian space this implies $x = x'$, while in Lorentzian space it implies x and x' are null-related. On the other hand, if x and $\bar{x}$ are antipodal points (see Fig. 2) then $\mu(x, \bar{x}) = \pi R$ and $z = 0$. This locates the second possible singularity of G.

In de Sitter space (R real), the spacelike intervals $\mu^2 > 0$ correspond to the range $0 \leqq z < 1$. In the Riemannian case, this encompasses the entire space. In the Lorentzian case, we have in addition the timelike intervals $\mu^2 < 0$, corresponding to the range $1 < z < \infty$. Thus in the Riemannian case $z \in [0, 1]$, and in the Lorentzian case $z \in [0, \infty)$. In the domain $|z| < 1$, we have the same two-point functions $G(z)$ in the two cases. By analytic continuation into the complex z-plane,

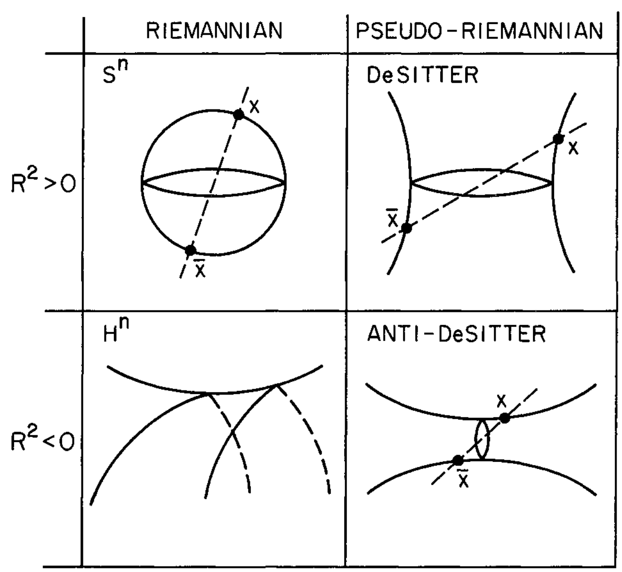

Fig. 2. The geometry of maximally symmetric spaces. The top two figures are spaces of constant positive curvature, and the bottom two figures are constant negative curvature. The Riemannian spaces (with positive definite metric) are shown to the left and the pseudo-Riemannian spaces to the right. In three of the four cases a point x in the space (time) has a unique antipodal point $\bar{x}$ located directly across from it

the two-point functions for $\mu^2>0$ can be extended to points with $\mu^2<0$. This should become clear in the example that follows.

There exists a one complex parameter family of de Sitter-invariant Fock vacuum states $|\psi\rangle$ [15, 16]. Each one determines a particular solution $G(z)$. There is a special member of this family called the "Euclidean" or "Bunch-Davies" vacuum [8, 17]. It is the *only* one whose two-point function $G(x,x')$ [15]

a) has only *one* singular point, at $\mu(x,x')=0$ [and is therefore regular at $\mu(x,x')=\pi R$] and

b) has the same strength $\mu\to 0$ singularity as in flat space.

It appears that this is the "most reasonable" vacuum state [17]. Moreover, one may obtain the two-point functions for any other de Sitter invariant Fock vacuum from $G_{\text{Euclidean}}$. Therefore now, and in the spin-1 case to follow, we will obtain the two-point function *only* in the Euclidean vacuum, and denote it by G.

Condition (a) for the Euclidean vacuum selects the solution to be

$$G(z)=qF(a_0,b_0;c_0;z)\,. \tag{2.9}$$

Condition (b) determines the constant q as follows. As $\mu\to 0$ $G(\mu)$ must approach the flat space form,

$$G_{\text{flat}}(\mu)\sim\frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}}\mu^{-n+2}\,. \tag{2.10}$$

Furthermore, near $z=1$ we have

$$F(a,b;c;z)\sim\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}(1-z)^{c-a-b}\,, \tag{2.11}$$

and $(1-z)\sim(\mu/2R)^2$, so we find that

$$q=\frac{\Gamma(a_0)\,\Gamma(b_0)}{\Gamma(n/2)\,2^n\pi^{n/2}}R^{-n+2}\,. \tag{2.12}$$

Equation (2.9) gives the two-point function for spacelike intervals $0\leqq z<1$. Generically the hypergeometric function $F(a,b;c;z)$ has a branch cut for $z>1$ along the real axis, i.e., for timelike intervals. The Feynman function is the limiting value $G(z+i0)$ approaching the branch cut from above. The symmetric function is the average value across the cut, $G(z+i0)+G(z-i0)$. The commutator function is given by $\varepsilon(x,x')\Delta G(z)$. Here $\varepsilon(x,x')=(+1,-1,0)$ if x,x' are (future, past, spacelike) separated, and $\Delta G(z)=G(z+i0)-G(z-i0)$ is the discontinuity of $G(z)$ across the branch cut. (Note that the commutator function is thus symmetric only under the orthochronous components of the isometry group of the manifold; cf. Appendix B.)

For some values of mass and spacetime dimension the branch cut is absent. There is nevertheless a pole at $z=1$, and the same $i0$ limiting prescription holds.

c) *Anti-de Sitter Space* ($R=\text{Im}(R)$)

Since anti-de Sitter space is not globally hyperbolic, the Cauchy problem is not well posed. Boundary conditions, controlling the flow of information through a timelike surface at spatial infinity, are therefore required to define a quantum field theory [18]. These boundary conditions will determine the possible states and hence the two-point functions. We will select a vacuum by requiring that the two-point function $G(\mu(x,x'))=\langle\varphi(x)\,\varphi(x')\rangle$

(a) falls off as fast as possible at spatial infinity $\mu^2\to\infty$, and

(b) has the same strength $\mu^2\to 0$ singularity as in flat space.

These correspond to the "reflecting" Dirichlet boundary conditions of [18] in the conformally invariant scalar case. For spins greater than zero, condition (a) seems to be required in order that the state be stable against small fluctuations, and for other reasons found in [19]. In the scalar case, however, condition (a) is not the only possibility.

A convenient set of solutions for $G(z)$ in the present case is [13, 14]

$$\begin{aligned}&(1/z)^{a_0}F(a_0, a_0-c_0+1; a_0-b_0+1; 1/z)\\&(1/z)^{b_0}F(b_0, b_0-c_0+1; b_0-a_0+1; 1/z),\end{aligned} \tag{2.13}$$

which fall off as z^{-a_0} and z^{-b_0} respectively, as $z\to\infty$. In anti-de Sitter space ($R=\mathrm{Im}(R)$) we have $z=\cos^2(\mu/2R)=\cosh^2(\mu/2|R|)$. Thus timelike intervals correspond to $0\leqq z<1$ on the real z axis, and spacelike intervals correspond to $z>1$. Furthermore spatial infinity $\mu\to\infty$ corresponds to $z\to\infty$. Hence since $F(a,b,c,0)=1$, and $0<b_0<a_0$, condition (a) selects the solution

$$G(z)=rz^{-a_0}F(a_0, a_0-c_0+1; a_0-b_0+1; z^{-1}) \tag{2.14}$$

and condition (b) determines r to be

$$r=\frac{\Gamma(a_0)\,\Gamma(a_0-c_0+1)}{\Gamma(a_0-b_0+1)\pi^{n/2}2^n}|R|^{2-n}. \tag{2.15}$$

The standard branch of the hypergeometric function is cut along the real z axis from 1 to ∞. The function z^{-a} is defined as $\exp(-a\log z)$ where $\log z$ is cut along the negative real axis. Hence $G(z)$ is cut for $z\leqq 1$. In particular, it is cut along the timelike region $0\leqq z<1$. As before, the Feynman function is obtained as the limiting value $G(z+i0)$ above this cut. The symmetric function is the average value $G(z+i0)+G(z-i0)$ across the cut. The commutator function is $\varepsilon(x,x')\,[G(z+i0)-G(z-i0)]$. Although anti-de Sitter space has closed timelike curves, the Feynman function can still be interpreted as the expectation value of a "time ordered" product as explained in [18]. The two-point function (2.14) has also been obtained [20] by doing a mode sum over the "regular" modes.

Anti-de Sitter space (adS) has topology $S^1\times R^{n-1}$, with the S^1 periodicity occurring in the timelike dimension. To eliminate closed timelike curves, one may "unwrap" the S^1 by considering the universal covering space CadS with topology R^n. In CadS all values of the mass m are allowed. The allowed mass values in adS, and the form of $G(x,x')$ on CadS can be determined by noting that the Feynman function is obtained by adding O^+ to μ (remember $\mu=i|\mu|$ for timelike intervals). Only those discrete values of m for which $G(z(\mu+O^+))$ is periodic as $\mu\to\mu+2\pi R$ are allowed. This is because although we said originally that $|\mu(x,x')|$ is the length of the *shortest* geodesic joining x and x', our derivation of Eq. (2.3) for the two point function holds for $|\mu|$ the length of *any* such geodesic. These discrete values of m are precisely the mass values for which a complete set of mode functions $\varphi_n(x)$ [orthonormal, satisfying $(\Box - m^2)\varphi_n = 0$, periodic in S^1 direction] exists on adS [18].

Since $R = i|R|$ for adS, $z(\mu + 0^+) = \cos^2\left(\frac{\mu + 0^+}{2R}\right) = \cos^2(\mu/2R) + i0^+ \sin(\mu/R)$. Thus as μ increases from 0 to $2\pi Rj$, z follows a contour which loops j times counterclockwise around the interval [0, 1] in the complex z-plane. Continuing analytically around this contour, we discover from (2.14) (by deforming the contour to large $|z|$) that

$$G(z(\mu + 2\pi Rj + 0^+)) = \exp(-j2\pi i a_0)\, G(z(\mu + 0^+)). \tag{2.16}$$

Thus in adS the allowed values of m are given by a_0 = integer. In CadS, (2.16) gives an explicit expression for $G(x, x')$ when x, x' are separated by j sheets. See also [30] for a careful discussion of the various scalar two-point functions in CadS. Similar considerations apply for the vector case to follow.

$G(z)$ (2.14) can be expressed as a linear combination $C_1 F(a_0, b_0; c_0; z) + C_2 F(a_0, b_0; c_0; 1 - z)$ in either the upper half plane *or* the lower half plane since these two functions provide a complete set of solutions to the hypergeometric equation [14, Eq. (2.9) (25)]. The coefficients C_1 and C_2 are in fact different in the two half planes, since the two sides of this equation have different branch cuts.

III. Massive Spin-1 Two-Point Functions

a)

The Euclidean action for a massive vector field A_a is

$$S = \int \left(\tfrac{1}{4} F_{ab} F^{ab} + \tfrac{1}{2} m^2 A_a A^a\right) dV, \tag{3.1}$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$. This is the action in the Proca theory of a massive spin-1 particle. It also arises in gauge theories, in the unitary ($\xi \to 0$) limit of the 't Hooft background field gauge [21]. There, the gauge potentials A_a have a lowest order (non-interacting) effective action of this form, with m^2 a matrix in the internal space. In the massless case $m^2 = 0$ one must add a gauge fixing term to the action (3.1) [21]. We shall consider this case separately in Sect. IV.

The equation of motion arising from $\delta S = 0$ is

$$\nabla_a F^{ab} = m^2 A^b. \tag{3.2}$$

The antisymmetry of F^{ab} implies $\nabla_a \nabla_b F^{ab} = 2R_{ab} F^{ab} = 0$. Hence taking the divergence of (3.2) we obtain $m^2 \nabla_b A^b = 0$. Thus if $m^2 \neq 0$ we have a supplementary equation

$$\nabla_a A^a = 0. \tag{3.3}$$

Using the definition of F_{ab} together with (3.2) and (3.3) one obtains the wave equation $W_{ab} A^b = 0$, with the operator

$$W_{ab} = (g_{ab}\,\Box - R_{ab} - m^2 g_{ab}). \tag{3.4}$$

The Laplacian operator is $\Box = \nabla_a \nabla^a$ and R_{ab} is the Ricci tensor. In a maximally symmetric space $R_{ab} = (n-1) R^{-2} g_{ab}$, where as before we take R pure real (imaginary) for $S^n(H^n)$ of radius $|R|$. Thus in a maximally symmetric space, defining

$$\kappa = m^2 + (n-1)/R^2 \tag{3.5}$$

we have $W_{ab} = g_{ab}(\Box - \kappa)$.

The expectation value of a product of two quantum fields

$$Q_{ab'}(x, x') = \langle \psi | A_a(x) A_{b'}(x') | \psi \rangle \tag{3.6}$$

defines a bitensor $Q_{ab'}$. This tensor satisfies the wave equation $W_{ab}Q^{bc'} = 0$ for $\mu(x, x') \neq 0$ (there is a singularity when $\mu = 0$) and the supplementary condition $\nabla^a Q_{ab'} = 0$, since the field operator A^a satisfies these equations. In a maximally symmetric state $|\psi\rangle$ the two point function $Q_{ab'}$ will be a maximally symmetric bitensor, which can be written as

$$Q_{ab'}(x, x') = \alpha(\mu) g_{ab'} + \beta(\mu) n_a n_{b'}, \tag{3.7}$$

where $\alpha(\mu)$ and $\beta(\mu)$ are two arbitrary functions of the geodesic distance $\mu(x, x')$. The wave equation then takes the form

$$W_a{}^b Q_{bc'} = F(\mu) g_{ac'} + G(\mu) n_a n_{c'} = 0, \tag{3.8}$$

yielding $F = G = 0$.

Using (3.7), (3.4) and the formulae of Table 1, and denoting $\dfrac{df(\mu)}{d\mu}$ by $f'(\mu)$, one finds

$$F = \alpha'' + (n-1)A\alpha' - [(A+C)^2 + \kappa]\alpha + 2AC\beta = 0, \tag{3.9}$$

$$G = \beta'' + (n-1)A\beta' + [2AC - (n-1)(A^2 + C^2) - \kappa]\beta + (n-2)(A+C)^2\alpha = 0. \tag{3.10}$$

The supplementary condition $\nabla_a Q^{ab'} = H(\mu) n^{b'} = 0$ yields

$$H = -\alpha' + \beta' - (n-1)(A+C)\alpha + (n-1)A\beta = 0. \tag{3.11}$$

Various identities useful for obtaining these and other equations are given in Appendix C.

To solve for α and β it is convenient to define

$$\gamma(\mu) = \alpha(\mu) - \beta(\mu). \tag{3.12}$$

Then $H = 0$ implies

$$-(n-1)C\alpha = \gamma' + (n-1)A\gamma, \tag{3.13}$$

while $F - G = 0$ implies

$$\gamma'' + (n-1)A\gamma' - [(n-1)(A^2 + C^2) + \kappa]\gamma - 2(n-1)AC\alpha = 0. \tag{3.14}$$

Now substituting (3.13) into (3.14) to eliminate α one finds

$$\gamma'' + (n+1)A\gamma' - [\kappa + (n-1)/R^2]\gamma = 0, \tag{3.15}$$

where the identity $C^2 - A^2 = 1/R^2$ has been used. It is sufficient to solve (3.15) for γ, since then α can be found from (3.13) and β can be found from (3.12).

By the same change of variables (2.4) as in the scalar case Eq. (3.15) for γ can be converted to hypergeometric form,

$$H(a_1, b_1, c_1)\gamma(z)=0\,, \tag{3.16}$$

where now (subscript 1 denotes spin-1)

$$\begin{aligned} a_1 &= \frac{n}{2}+\frac{1}{2}+\frac{1}{2}[(n-3)^2-4m^2R^2]^{1/2}\,, \\ b_1 &= \frac{n}{2}+\frac{1}{2}-\frac{1}{2}[(n-3)^2-4m^2R^2]^{1/2}\,, \\ c_1 &= \frac{n}{2}+1\,. \end{aligned} \tag{3.17}$$

The complete solution for $Q_{ab'}$ in the massive case is given from (3.7), (3.12), and (3.13) by

$$\begin{aligned} Q_{ab'}(x,x') &= \alpha(\mu)g_{ab'}+\beta(\mu)n_a n_{b'}\,, \\ \alpha(\mu) &= [(n-1)^{-1}R\sin(\mu/R)(d/d\mu)+\cos(\mu/R)]\,\gamma(\mu)\,, \\ \beta(\mu) &= [(n-1)^{-1}R\sin(\mu/R)(d/d\mu)+\cos(\mu/R)-1]\,\gamma(\mu)\,. \end{aligned} \tag{3.18}$$

b) de Sitter Space (R Real)

We select a two-point function $Q_{ab'}$ following the principles explained in Sect. IIb. Thus we take

$$\gamma(z)=qF(a_1, b_1; c_1; z) \tag{3.19}$$

with

$$q=\frac{(1-n)\,\Gamma(a_1)\,\Gamma(b_1)}{2^{n+1}\pi^{n/2}\Gamma\left(\frac{n}{2}+1\right)}m^{-2}R^{-n}\,. \tag{3.20}$$

Above the branch cut along the real axis $z \geqq 1$, this together with (3.18) yields the Feynman function.

c) Anti-de Sitter Space $(R=\mathrm{Im}(R))$

Following Sect. IIc we take

$$\gamma(z)=rz^{-a_1}F(a_1, a_1-c_1+1; a_1-b_1+1; z^{-1})\,, \tag{3.21}$$

with

$$r=\frac{(1-n)\,\Gamma(a_1)\,\Gamma(a_1-c_1+1)}{2^{n+1}\pi^{n/2}\Gamma(a_1-b_1+1)}m^{-2}|R|^{-n}\,. \tag{3.22}$$

Above the branch cut along the real axis $z<1$, this together with (3.18) yields the Feynman function.

IV. Massless Spin-1 Two-Point Functions

a)

The action for a massless vector field A^a with a convenient gauge-fixing term is [21]

$$S=\int\left(\frac{1}{4}F_{ab}F^{ab}+\frac{\lambda}{2}(\nabla_a A^a)^2\right)dV, \tag{4.1}$$

where $F_{ab}=\nabla_a A_b-\nabla_b A_a$, and λ is any positive real number. The equation of motion is $W_{ab}A^b=0$, where the wave operator is now

$$W_{ab}=g_{ab}\,\Box-R_{ab}+(\lambda-1)\nabla_a\nabla_b\,. \tag{4.2}$$

As before, we consider the two-point function $Q_{ab'}=\langle A_a A_{b'}\rangle$ in a maximally symmetric state, so that

$$Q_{ab'}(x,x')=\alpha(\mu)g_{ab'}+\beta(\mu)n_a n_{b'}\,. \tag{4.3}$$

As before, the equation of motion $W_{ab}A^b$ then implies

$$W_{ab}Q^{bc'}=0\,. \tag{4.4}$$

Note that if one sets $\lambda=1$, this equation is identical to the massive case for $m=0$. However since $m=0$, the supplementary equation $m^2\nabla_a A^a=0$ is vacuous and no longer implies a first order equation like (3.11) for α and β. Therefore one must solve (4.4) differently than before.

It is convenient to introduce the maximally symmetric bitensor

$$Q_{ab}{}^{a'b'}(x,x')=4\nabla_{[a}\nabla^{[a'}Q_{b]}{}^{b']}=\sigma(\mu)g_{[a}^{[a'}g_{b]}^{b']}+\tau(\mu)n_{[a}g_{b]}^{[b'}n^{a']}\,, \tag{4.5}$$

which is the gauge invariant expectation value $\langle F_{ab}F^{a'b'}\rangle$. We will see shortly that it is independent of λ. From the definition of $Q_{ab'}$ in terms of α and β, it follows that σ and τ in (4.5) are

$$\sigma=4C[\alpha'+(A+C)\alpha-C\beta]\,, \tag{4.6}$$

$$\tau=C^{-1}[\sigma'+2(A+C)\sigma]\,. \tag{4.7}$$

The equation of motion $W_{ab}Q^{bc'}=0$ implies that $\nabla_a Q^{ab}{}_{a'b'}=-\lambda\nabla^b(\nabla_a\nabla_{[a'}Q_{b']}{}^a)$, and the maximally symmetric bitensor in parentheses can be expanded as $\varrho g_{[a'b']}+\nu n_{[a'}n_{b']}$, which vanishes identically. Hence the equation of motion for $Q_{ab}{}^{a'b'}$ doesn't involve λ and is

$$\nabla^a Q_{ab}{}^{a'b'}=0\,. \tag{4.8}$$

This equation is easily solved to find σ and τ. Once σ is known, Eq. (4.6) then provides a first order "supplementary" equation which we can use later to solve (4.4) for α and β.

The equation of motion for $Q_{ab}{}^{a'b'}$ (4.8) can be written

$$\nabla^a Q_{ab}{}^{a'b'}=Jg_b{}^{[a'}n^{b']}=0\,, \tag{4.9}$$

where from (4.5), J is

$$J=\sigma'-\tfrac{1}{2}\tau'+(n-2)(A+C)\sigma-\tfrac{1}{2}(n-2)A\tau=0\,. \tag{4.10}$$

Using Eq. (4.7) which defines τ in terms of σ, one can eliminate τ from Eq. (4.10) to obtain a second-order equation for σ,

$$\sigma'' + (n+1)A\sigma' - (2(n-1)/R^2)\sigma = 0. \tag{4.11}$$

This equation is *identical* to the hypergeometric Eq. (3.15) for γ when $m=0$. Hence the parameters (3.17) are now $a_1 = n-1$, $b_1 = 2$, and $c_1 = (n/2)+1$.

As in the massive case, the equation for σ has two linearly independent solutions, and as before the choice of solution is related to the choice of maximally symmetric state. Once σ is chosen, $Q_{ab}{}^{a'b'}$ is then determined using (4.7) and (4.5).

We can now find $Q_{ab'} = \langle A_a A_{b'} \rangle$ by using the above result for σ. The bitensor $Q_{ab'}$ is not gauge-invariant and depends upon λ. We choose $\lambda = 1$, because the equations for α and β are then simpler. This choice of λ is called Feynman gauge [21]. With $\lambda = 1$, the wave operator W_{ab} (4.2) is identical to W_{ab} in the massive case (3.4) with m set to zero. Hence we again have the coupled equations $F=0$ (3.9) and $G=0$ (3.10) for α and β, where now $\kappa = (n-1)/R^2$.

The equation $F=0$ is

$$\alpha'' + (n-1)A\alpha' - [(A+C)^2 + (n-1)/R^2]\alpha + 2AC\beta = 0. \tag{4.12}$$

Eliminating β from Eqs. (4.6) and (4.12), and using $C^2 - A^2 = 1/R^2$, one obtains

$$\alpha'' + (n+1)A\alpha' - (n/R^2)\alpha = (A/2C)\sigma. \tag{4.13}$$

The same change of variable as before (2.4) converts this into the *inhomogeneous* hypergeometric equation

$$H(a_1+1, b_1-1, c_1)\,\alpha(z) = \tfrac{1}{2}R^2(1-2z)\,\sigma(z), \tag{4.14}$$

where we have used $A/C = 1-2z$.

The general solution to this equation is of the form

$$\alpha(z) = k_1\alpha_1(z) + k_2\alpha_2(z) + \tilde{\alpha}(z), \tag{4.15}$$

where α_1 and α_2 are independent solutions to the equation

$$H(a_1+1, b_1-1, c_1)\alpha_i = 0, \qquad i = 1, 2, \tag{4.16}$$

and $\tilde{\alpha}(z)$ is any particular solution to the inhomogeneous equation (4.14). Note that once σ is given, there is a 2-parameter family (k_1, k_2) of solutions α. Together with the freedom in σ, this makes 4 parameters in all: Four conditions are required to determine a unique solution to the coupled system of second order equations $F=G=0$ for α and β.

Having chosen a solution $\sigma(\mu)$ and $\alpha(\mu)$, $\beta(\mu)$ is then determined by Eq. (4.6).

b) de Sitter Space (R Real)

We select a solution $Q_{ab'}$ in two steps, following the principles explained in Sect. IIb. First we choose the particular solution for σ

$$\sigma(z) = pF(n-1, 2; n/2+1; z) \tag{4.17}$$

with

$$p = \frac{\Gamma(n-1)}{\Gamma(n/2+1)2^{n-1}\pi^{n/2}} R^{-n}. \tag{4.18}$$

In Appendix D it is shown that for $n \neq 3$,

$$\tilde{\alpha}(z) = \tfrac{1}{4} p R^2 (n-3)^{-2} \left[(2-n) F + (n-4) F(a_1 - 1) + 2(3-n)(\partial/\partial a - \partial/\partial b) F \Big|_{\substack{a_1+1 \\ b_1-1}} \right] \tag{4.19}$$

is a particular solution to (4.14) with source (4.17) [the parameters of $F(a_1, b_1; c_1; z)$ are omitted unless they differ from (a_1, b_1, c_1)]. The unique solution for α with the correct short distance and singular behavior is then [cf. (4.15)]

$$\alpha(z) = q F(n, 1; n/2+1; z) + \tilde{\alpha}(z) \tag{4.20}$$

with

$$q = \tfrac{1}{4} p R^2 (n-3)^{-2} [(n-1)(n-2) - 2(n-3)(\psi(n) - \psi(1))], \tag{4.21}$$

where $\psi(z) = \dfrac{d}{dz} \log \Gamma(z)$ is the psi function [13], and $\psi(1+z) = \psi(z) + 1/z$.

The other homogeneous solution to (4.16) is absent in (4.20) because it is singular at $z=0$. The constant q in (4.20) is chosen to cancel the $(1-z)^{-n/2}$ short distance pole in $\tilde{\alpha}$. The residue at order $(1-z)^{-n/2+1}$ is correctly fixed by the earlier choice (4.18) of p.

If the number of dimensions n is an integer then this solution takes a simple form. In Appendix E we give a table of solutions in dimensions $2 \leqq n \leqq 12$. In four dimensions it is

$$\begin{aligned}
\alpha(z) &= (48\pi^2 R^2)^{-1} [-3(z-1)^{-1} + z^{-1} + (2z^{-1} + z^{-2}) \log(1-z)], \\
\beta(z) &= (24\pi^2 R^2)^{-1} [1 - z^{-1} + (z^{-1} - z^{-2}) \log(1-z)], \\
\sigma(z) &= (8\pi^2 R^4)^{-1} (z-1)^{-2}, \\
\tau(z) &= (2\pi^2 R^4)^{-1} (z-1)^{-2}.
\end{aligned} \tag{4.22}$$

The reader can easily verify that these are (1) solutions to the α, β equations, (2) regular at $z=0$, (3) flat-space singular at $z=1$, and (4) cut along $(1, \infty)$ in the complex z plane.

c) *Anti-de Sitter Space* ($R = \operatorname{Im} R$)

Following Sect. IIc we first choose the particular solution for σ,

$$\sigma(z) = p z^{1-n} F(n-1, n/2-1; n-2; z^{-1}) \tag{4.23}$$

with

$$p = 2^{2-n} \pi^{-n/2} \Gamma(n/2) |R|^{-n}. \tag{4.24}$$

It is shown in Appendix D that for $n \neq 3$

$$\tilde{\alpha}(z) = s \left[(2-n) F_{\text{ads}} + (n-4) F_{\text{ads}}(a_1 - 1) + 2(3-n)(\partial/\partial a - \partial/\partial b) F_{\text{ads}} \Big|_{\substack{a_1+1 \\ b_1-1}} \right], \tag{4.25}$$

with

$$s=\frac{e^{-i\pi(n/2+1)}\Gamma(n-1)}{2^{n+1}\pi^{n/2}(n-3)^2\,\Gamma(n/2+1)}|R|^{2-n} \tag{4.26}$$

is a particular solution to (4.14) with (4.23) as source. The parameters of $F_{\mathrm{adS}}(a,b;c;z)$ are omitted unless they differ from (a_1,b_1,c_1). The function F_{adS} is defined by

$$F_{\mathrm{adS}}(a,b;c;z)=F_{\mathrm{new}}(a,b;c;z)+F(a,b;c;1-z)\,. \tag{4.27}$$

$F_{\mathrm{new}}(a,b;c;z)$ is the unique analytic function (1) equal to $F(a,b;c;z)$ in the upper half z-plane and (2) cut for $z<1$ along the real z axis. The reason for defining F_{new} is that the standard branch of the hypergeometric function is cut along $(1,\infty)$, yet we need a solution which has its cut only in the *timelike* region $(0,1)$, as explained in Sect. IIc. One may express F_{new} in terms of F by [14, Eq. (2.9) (25)]

$$\begin{aligned}F_{\mathrm{new}}(a,b;c;z)=&\left\{\frac{e^{i\pi b}\Gamma(c)\,\Gamma(a+1-c)}{\Gamma(c-b)\,\Gamma(a+b+1-c)}\right\}F(a,b;a+b+1-c;1-z)\\&+\left\{\frac{e^{i\pi(a+b-c)}\Gamma(c)\,\Gamma(a+1-c)}{\Gamma(b)\,\Gamma(a+1-b)}\right\}z^{-a}F(a,a+1-c;a+1-b;z^{-1})\,.\end{aligned} \tag{4.28}$$

The unique solution for α with the correct short distance limit and the fastest possible falloff as $z\to\infty$ is then [cf. (4.15)]

$$\alpha(z)=\tilde{\alpha}(z)+qF_{\mathrm{new}}(n,1;n/2+1;z)+rF(n,1;n/2+1;1-z) \tag{4.29}$$

with

$$q=s[(n-1)(n-2)+2(n-3)(\psi(1)-\psi(n))]\,, \tag{4.30}$$

$$r=2\pi i(n-3)s+q\,. \tag{4.31}$$

The constant q in (4.29) is chosen to cancel the $(1-z)^{-n/2}$ short distance pole in $\tilde{\alpha}$. The residue at order $(1-z)^{-n/2+1}$ is correctly fixed by the earlier choice (4.24) of p. The constant r in (4.29) is chosen to cancel the z^{-1} (slowest falloff) term in $\tilde{\alpha}+qF_{\mathrm{new}}$. This solution for $\sigma(z)$ and $\alpha(z)$, and the corresponding solutions for $\beta(z)$ and $\tau(z)$ [determined from (4.6) and (4.7)], fall off as

$$\begin{aligned}&\alpha\sim z^{-n+2} \qquad \sigma\sim z^{-n+1}\\&\beta\sim z^{-n+2} \qquad \tau\sim z^{-n+2}\end{aligned} \tag{4.32}$$

as $z\to\infty$.

If the number of dimensions n is an integer then this solution takes a simple form. In four dimensions it is

$$\begin{aligned}\alpha(z)=&(48\pi^2|R|^2)^{-1}\{4(z-1)^{-1}-4z^{-1}\\&+[2(z-1)^{-1}-(z-1)^{-2}]\log z-[2z^{-1}+z^{-2}]\log(z-1)\}\,,\\\beta(z)=&(24\pi^2|R|^2)^{-1}\{2(z-1)^{-1}-2z^{-1}\\&+[(z-1)^{-1}-2(z-1)^{-2}]\log z+[-z^{-1}+z^{-2}]\log(z-1)\}\,,\\\sigma(z)=&(8\pi^2|R|^4)^{-1}\{(z-1)^{-2}-z^{-2}\}\,,\\\tau(z)=&(2\pi^2|R|^4)^{-1}(z-1)^{-2}\,.\end{aligned} \tag{4.33}$$

V. Summary and Discussion

We have shown that any maximally symmetric bitensor can be expanded in terms of a minimal set of bitensors consisting of n_a, $n_{a'}$, and $g_{ab'}$. The coefficients of the expansion are scalar functions of μ, the geodesic distance between x and x'. The derivatives of this minimal set are themselves maximally symmetric bitensors, and can thus be expanded in terms of n_a, $n_{a'}$, and $g_{ab'}$. In Sect. I we obtained the coefficient scalar functions of these derivatives. That knowledge permits one to compute arbitrary derivatives of arbitrary maximally symmetric bitensors. The two-point expectation values of field operators such as $\langle\varphi(x)\varphi(x')\rangle$, $\langle A^a(x)A^{a'}(x')\rangle$, and $\langle h^{ab}(x)h^{a'b'}(x')\rangle$ are bitensors, and they are maximally symmetric if the state and gauge fixing terms are. Since the two-point functions satisfy wave equations, the results of Sect. I yield ordinary (coupled) differential equations for the coefficient scalar functions.

In Sects. III and IV these equations are solved for the massive and massless spin-1 case. Because the ordinary differential equations admit a two or four parameter family of solutions, some principle was needed to select a particular solution. Although we have not constructed a Fock vacuum state, we believe that Feynman two-point functions are obtained via the following considerations:

1. They are cut in the complex μ plane for timelike separations.
2. They have flat-space short distance ($\mu\to 0$) singularities.
3a. In de Sitter space they have only one singular point.
3b. In anti-de Sitter space, they fall off as fast as possible at spatial ∞.

In Sect. II we showed that considerations 1–3 select the "Euclidean" or "Bunch-Davies" vacuum in the scalar de Sitter case and the "regular reflecting" vacuum in the anti-de Sitter case. In both the scalar and vector cases these two-point functions have the Hadamard form [22, 23].

We now intend to use these methods to calculate the spin-2 Feynman function. Another possible application is to calculate quantities such as the correlation function of the stress-energy tensor $\langle T_{ab}(x)\,T_{a'b'}(x')\rangle$ in maximally symmetric states. However this quantity contains five undetermined functions $f_i(\mu)$ and the field equations $\nabla^a T_{ab}=0$ impose only three relations among the f_i. Hence this correlation function still contains two undetermined functions, which can be obtained only from knowledge of what fields are creating the stress-tensor.

The definition of maximally symmetric bitensors can be generalized to bi-spintensors, and a spinor parallel propagator $D_A{}^{B'}(x,x')$ can be introduced which is related to $g_a{}^{b'}(x,x')$ by $g_a{}^{b'}=D_A{}^{B'}\bar{D}_{\dot{A}}{}^{\dot{B}'}$. It is then possible to express all maximally symmetric bi-spintensors in terms of $n_{A\dot{A}}$, $n_{B'\dot{B}'}$, and $D_A{}^{B'}$, and thus to generalize our results to spin-1/2 and 3/2 [32].

Finally we should note that it is not surprising that the coefficient functions $f_i(\mu)$ generally satisfy equations of the hypergeometric type. This is because the hypergeometric functions occur naturally as the matrix elements of irreducible representations of the rotation groups $O(n,m)$ which are the symmetry groups of maximally symmetric spaces [24].

Appendix A. If There is no Geodesic from x to x'

In a pseudo-Riemannian space there is not always a geodesic from x to x'. We thus require an extended definition of the maximally symmetric bitensors $\mu(x,x')$, $n_a(x,x')$, and $g_{ab'}(x,x')$. For example, let M denote Lorentzian de Sitter space and define the point set

$$J_x = \{x' \in M \text{ such that } \exists \text{ geodesic from } x \text{ to } x'\}. \tag{A1}$$

Then $M - J_x$ is nonempty: it includes the interior of the (past and future) light cone of $\bar{x}$ ($\bar{x}$ is the antipodal point to x). We can nonetheless define $\mu(x,x')$ by analytic continuation in the coordinates of x and x'.

Consider de Sitter space to be the set of points $Y^a \in R^{n+1}$ which satisfy

$$Y^a Y^b \eta_{ab} = R^2, \tag{A2}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$, and the induced metric is $ds^2 = \eta_{ab}\, dY^a\, dY^b$. Let $Y^a(x)$ denote the vector in the R^{n+1} embedding space which "points to x." Then for $x' \in J_x$ one has

$$\cos\left(\frac{\mu(x,x')}{R}\right) = \frac{\eta_{ab} Y^a(x)\, Y^b(x')}{R^2}, \tag{A3}$$

and consequently that $z = \cos^2(\mu/2R) = \frac{1}{2}[1 + \cos(\mu/R)]$ is

$$z = \frac{1}{2}\left[1 + \frac{\eta_{ab} Y^a(x)\, Y^b(x')}{R^2}\right]. \tag{A4}$$

The right-hand side equals $\cos^2(\mu/2R)$ *inside* J_x. However it is also well defined *outside* J_x, and it is an analytic (polynomial) function of the coordinates Y^a. Hence (A4) serves to define $z(x,x')$ *everywhere*, and we can define $\mu(x,x') \equiv 2R\cos^{-1}(z^{1/2})$ as the limiting value above the standard branch cut of $\cos^{-1}$.

Now that $\mu(x,x')$ is globally defined, one can define n_a and $n_{a'}$ everywhere by

$$n_a(x,x') = \nabla_a \mu(x,x') \qquad n_{a'}(x,x') = \nabla_{a'} \mu(x,x'). \tag{A5}$$

Once again, the right-hand side is an analytic function of the coordinates Y^a, and equals the "geometrical" n_a and $n_{a'}$ everywhere inside J_x.

To define $g_a{}^{b'}$ everywhere, one may invert Eq. (1.12) to obtain

$$g_{ab'} = C(\mu)^{-1} \nabla_a n_{b'} - n_a n_{b'}. \tag{A6}$$

The right-hand side is an analytic function of the coordinates Y^a and is well defined everywhere. As before it reduces to the "geometrical" $g_{ab'}$ inside J_x but serves also to define it outside that region.

The formulae (A4)–(A6), defining μ, n_a, $n_{a'}$, and $g_{ab'}$ outside J_x apply also in anti-de Sitter space with R pure imaginary.

Appendix B. Maximally Symmetric Bitensors

The action $\varphi^* T_{a\ldots b}{}^{c\ldots d}{}_{a'\ldots b'}{}^{c'\ldots d'}(x,x')$ of a diffeomorphism φ on a bitensor T is defined by direct extension of its action on ordinary tensor fields [10]. Similarly, the Lie derivative is defined by $L_X T = \lim_{\lambda\to 0} \lambda^{-1}[\varphi_\lambda^* T - T]$, where $\{\varphi_\lambda\}$ is the one-parameter family of diffeomorphisms generated by the vector field X^a. For example, if $T(x,x')$ is a biscalar then $L_X T = X^a(x)\nabla_a T + X^{a'}(x')\nabla_{a'} T$.

A *maximally symmetric bitensor* T is one for which $\sigma^* T = T$ for any isometry σ of the maximally symmetric space. It follows in particular that $L_K T = 0$ for any Killing vector K. The converse is not true however, since there are isometries which are not continuously connected to the identity.

A maximally symmetric biscalar $f(x,x')$ is in fact a function only of $\mu(x,x')$. To see this, note that for all isometries σ that fix x, we have $f(x,x') = f(\sigma(x),\sigma(x')) = f(x,\sigma(x'))$. The orbit $B_\mu = \{\sigma(x')\}$ of x' under the isometries that fix x is precisely the set of points at distance $\mu(x,x')$ from x. Thus $f(x,x')$ is really a function of the orbit, i.e., $f(x,x') = f(\mu(x,x'))$. As an aside, note that for timelike intervals $\mu^2 < 0$ in Minkowski space, B_μ consists of two disconnected pieces, the future and past sheets. $f(x,x')$ must be invariant under the *entire* isometry group (including any disconnected component) for the above argument to hold.

The covariant derivative $\nabla_a T$ of a maximally symmetric bitensor is again maximally symmetric. Indeed, since parallel transport commutes with isometries, we have $\sigma^*(\nabla_a T) = \nabla_a(\sigma^* T) = \nabla_a T$. Thus $n_a \equiv \nabla_a \mu(x,x')$, $n_{a'} \equiv \nabla_{a'}\mu(x,x')$, and $g_{ab'} \equiv C^{-1}\nabla_a n_{b'} - n_a n_{b'}$ are all maximally symmetric.

We now show that these three bitensors provide a complete set, in the sense that any maximally symmetric bitensor can be expressed as a sum of products of n_a, $n_{a'}$, and $g_{ab'}$, with coefficients that are functions only of $\mu(x,x')$. It is sufficient to prove the proposition for bitensors with all indices at one point, since one can always write $T_{a\ldots bc'\ldots d'}(x,x') = g_{c'}{}^{c} \ldots g_{d'}{}^{d}\, T_{a\ldots bc\ldots d}(x,x')$. The argument now proceeds by induction on the number of indices.

The biscalar case (0 indices) is proved in the preceding paragraph. Now suppose the proposition is true for bitensors with p indices, and let $T_{a\ldots b}(x,x')$ be a maximally symmetric bitensor with $p+1$ indices. We have

$$\begin{aligned} T_{a\ldots b}(x,x') &= (h_a{}^c + n_a n^c) \ldots (h_b{}^d + n_b n^d)\, T_{c\ldots d}(x,x') \\ &= h_a{}^c \ldots h_b{}^d\, T_{c\ldots d}(x,x') + (\text{terms with } n^e), \end{aligned} \tag{B1}$$

where $h_a{}^c \equiv g_a{}^c - n_a n^c$ is the projector onto the hyperplane V orthogonal to n^a. All the terms with at least one n_e are the product of an n_e with a p-index maximally symmetric bitensor, and hence by induction have the required form. The term with no n_e's is a $p+1$ tensor lying entirely in V. This term is invariant in particular under all isometries which leave both x and x' fixed, i.e., under the full orthogonal group of the hyperplane V. It must therefore be decomposable into sums of products of the projected metric $h_{ac}(x,x')$, with scalar coefficients $c(x,x')$. Contracting all indices shows that the $c(x,x')$ must be maximally symmetric as well, so that $c(x,x') = c(\mu(x,x'))$. This completes the proof.

Appendix C. Useful Formulae

The following are useful formulae derived from Table 1.

$$n^a \nabla_a n_b = 0\,, \tag{C1}$$

$$n^a \nabla_a g_{bc'} = 0\,, \tag{C2}$$

$$n^a \nabla_a n_{b'} = 0\,, \tag{C3}$$

$$n^{b'} \nabla_a n_{b'} = 0\,, \tag{C4}$$

$$\nabla^a n_a = (n-1)A\,, \tag{C5}$$

$$\nabla^a g_{ab'} = (1-n)(A+C)n_{b'}\,, \tag{C6}$$

$$\nabla^a (n_a n_{b'}) = (n-1)A n_{b'}\,, \tag{C7}$$

$$\Box\, n_a = (1-n)A^2 n_a\,, \tag{C8}$$

$$\Box\, n_{a'} = (1-n)C^2 n_{a'}\,, \tag{C9}$$

$$\Box\, g_{ab'} = -(A+C)^2 [g_{ab'} + (2-n) n_a n_{b'}]\,, \tag{C10}$$

$$\Box\, (n_a n_{b'}) = 2AC g_{ab'} + [2AC + (1-n)(A^2+C^2)] n_a n_{b'}\,. \tag{C11}$$

Appendix D. The Inhomogeneous Solution $\tilde{\alpha}$ for the Massless Case

1. de Sitter Space (R Real)

The equation to be solved (4.14) is

$$H(a_1+1, b_1-1, c_1)\tilde{\alpha} = \tfrac{1}{2} R^2 (1-2z)\sigma\,, \tag{D1}$$

with $\sigma(z) = pF(a_1, b_1; c_1; z)$ (4.17). One can express the source term as a sum of three hypergeometric functions using formulae (15.2.14/19) of [13]. Denoting $F(a_1, b_1+18; c_1; z)$ by $F(b_1+18)$ and so on, one finds

$$(1-2z)F = (n-3)^{-1}[(2-n)F + (n-4)F(a_1-1) + (n-1)F(a_1+1, b_1-1)]\,. \tag{D2}$$

Using formulae (15.2) from [13] one can show that

$$H(a_1+1, b_1-1, c_1)F = (n-2)F\,, \tag{D3}$$

$$H(a_1+1, b_1-1, c_1)(F(a_1-1)+F) = 2(n-3)F(a_1-1)\,. \tag{D4}$$

Furthermore, for arbitrary values a, b, and c, $HF = 0$ implies

$$0 = \frac{\partial}{\partial a}(HF) = H\left(\frac{\partial}{\partial a}F\right) - z\frac{d}{dz}F - bF\,, \tag{D5}$$

$$0 = \frac{\partial}{\partial b}(HF) = H\left(\frac{\partial}{\partial b}F\right) - z\frac{d}{dz}F - aF\,. \tag{D6}$$

Subtracting these two equations, and letting $a=a_1+1$ and $b=b_1-1$, one finds

$$H(a_1+1,b_1-1,c)\left[\frac{\partial}{\partial a}-\frac{\partial}{\partial b}\right]F\Big|_{\substack{a=a_1+1\\ b=b_1-1}}=(b_1-a_1-2)\,F(a_1+1,b_1-1)\,. \quad \text{(D7)}$$

Putting together the results of Eqs. (D2)–(D4) and (D7) one can construct a solution $\tilde{\alpha}$ to the inhomogeneous equation (D1)

$$\tilde{\alpha}=\frac{1}{4}pR^2(n-3)^{-2}\left[(2-n)F+(n-4)\,F(a_1-1)+2(3-n)\left[\frac{\partial}{\partial a}-\frac{\partial}{\partial b}\right]F\Big|_{\substack{a=a_1+1\\ b=b_1-1}}\right]$$

as long as $n\neq 3$. (D8)

2. Anti-de Sitter Space ($R=\mathrm{Im}\,R$)

In the upper half plane the source term $\frac{1}{2}R^2(1-2z)\sigma$ with σ given by (4.23) can be expressed as [14],

$$\sigma=\mathrm{const}\cdot[F(a_1,b_1;c_1;z)-F(a_1,b_1;c_1;1-z)]\,. \quad \text{(D9)}$$

The operator $H(a_1+1,b_1-1,c_1)$ which appears on the left-hand side of (D1) is invariant under the substitution $z\to 1-z$. The right-hand side transforms into $-\frac{1}{2}R^2(1-2z)\,\sigma(1-z)$. Hence one can generate a solution to (D1) with (D9) as source, from the de Sitter solution $\tilde{\alpha}_{ds}$, by taking

$$\tilde{\alpha}_{\mathrm{adS}}(z)=\mathrm{const}\cdot[\tilde{\alpha}_{\mathrm{dS}}(z)+\tilde{\alpha}_{\mathrm{dS}}(1-z)] \quad \text{(D10)}$$

in the upper half plane. Since $F_{\mathrm{new}}=F$ in the upper half plane, the solution (4.25) is of exactly this form in view of (4.27) and the definition of F_{new}. The solution as given in (4.25) is therefore the analytic continuation of (D10) from the upper half plane to the whole complex plane cut along the negative real axis $z\in(-\infty,1)$.

Appendix E. Exact Massless Vector Propagators in n-Dimensional de Sitter Space

In the massless case the function $\alpha(z)$ (4.20) simplifies if n is integer. It may be expressed as

$$\alpha(z)=d(n)\sum_k[a_k z^{k-n/2}\log(1-z)+b_k(1-z)^{k-n/2}+c_k z^{-k}] \quad \text{(E1)}$$

if n is even and $n\geqq 2$, and as

$$\alpha(z)=d(n)\sum_k[a_k(z^{-k}+(1-z)^{-k})\,\omega(z)-b_k(z^{-k}-(1-z)^{-k})] \quad \text{(E2)}$$

if n is odd and $n\geqq 5$. The function $\omega(z)$ in (E2) is

$$\omega(z)=z^{-1/2}(1-z)^{-1/2}\arcsin(z^{1/2})\,, \quad \text{(E3)}$$

and the overall coefficient is

$$d(n)=\frac{\Gamma(n-1)}{2^{n-1}\pi^{n/2}\Gamma(n/2+1)}R^{2-n}\,. \quad \text{(E4)}$$

Table 2. The nonzero constants a_k, b_k, and c_k appearing in formulae (E1) and (E2), which give the (Feynman gauge) massless de Sitter propagator in dimensions $n \leqq 12$

	Dimension n									
	2	4	5	6	7	8	9	10	11	12
a_0	$-1/2$	1/6	$-5/32$	1/60	$-7/96$	1/350	$-3/64$	1/1764	$-11/320$	1/8316
a_1		1/3	5/32	1/20	7/128	2/175	85/2048	5/1764	77/2048	1/1386
a_2				1/10	21/512	1/35	117/4096	5/588	99/4096	1/396
a_3						2/35	45/4096	5/252	715/65536	2/297
a_4								5/126	385/131072	1/66
a_5										1/33
b_1	$-1/2$	1/2	5/32	1/8	21/256	1/30	17/256	1/112	187/3072	1/420
b_2				1/8	21/512	1/12	147/4096	11/336	3245/98304	19/1680
b_3						1/10	45/4096	71/1008	1265/98304	17/540
b_4								97/1008	385/131072	109/1680
b_5										27/280
c_1		1/6		7/120		37/1050		533/21168		1627/83160
c_2				1/60		9/700		107/10584		25/3024
c_3						1/350		11/3528		73/24948
c_4								1/1764		13/16632
c_5										1/8316

The values of the nonzero constants a_k, b_k, c_k are given in Table 2 for $n \leqq 12$. They were obtained using a MACSYMA program, which may be obtained from one of the authors (B.A.). The program uses the result that if $a-b$ is a positive integer, then one has

$$[\partial/\partial a - \partial/\partial b]\, F(a,b;c;z) = \sum_{k=1}^{a-b} \left\{ -(a-k)^{-1} F(a,b;c;z) + z^{k-a} \int_0^z x^{a-k-1} F(a,b;c;x)\, dx \right\}. \quad \text{(E5)}$$

Differentiating these results for $\alpha(z)$, one obtains $\beta(z)$ from (4.12). One may also obtain $\sigma(z)$ and $\tau(z)$ from (4.6) and (4.7). Despite appearances, these functions are nonsingular at $z=0$, as they were constructed to be. They may be useful in supergravity, superstring and Kaluza-Klein theories, which can have maximally symmetric internal spaces.

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