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VECTOR-VALUED LAPLACE TRANSFORMS AND CAUCHY PROBLEMS

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ABSTRACT
Linear differential equations in Banach spaces are systematically treated with the help of Laplace transforms. The central tool is an "integrated version" of Widder's theorem (characterizing Laplace transforms of bounded functions). It holds in any Banach space (whereas the vector-valued version of Widder's theorem itself holds if and only if the Banach space has the Radon–Nikodým property). The Hille–Yosida theorem and other generation theorems are immediate consequences. The method presented here can be applied to operators whose domains are not dense.

Introduction

The theory of linear differential equations in Banach spaces and one-parameter semigroups of operators has been stimulated to a large extent by the theory of Laplace transforms. In 1934 Widder [27] had proved the following characterization of Laplace transforms of real-valued bounded functions.

Let $r \in C^\infty(0, \infty)$. There exists $f \in L^\infty(0, \infty)$ such that

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt \qquad (\lambda > 0)$$

if and only if

$$\sup\{|\lambda^{n+1} r^{(n)}(\lambda)/n!| : \lambda > 0, n \in \mathbf{N}\} < \infty.$$

The Hille–Yosida theorem can be reformulated by saying that Widder's theorem holds for the resolvent of a densely defined operator (cf. Section 2).

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However, Hille's as well as Yosida's proof are of operator theoretical nature and make no use of Widder's theorem.

In the present paper we investigate vector-valued versions of Widder's theorem and their applications to linear differential equations in Banach spaces.

We show that the canonical extension of Widder's theorem to Banach space valued functions holds if and only if the Banach space has the Radon–Nikodým property. However, an "integrated version" of Widder's theorem always holds (Theorem 1.1). This theorem turns out to be extremely useful to treat linear differential equations.

In fact, an operator A is the generator of a C_0-semigroup if and only if $(\lambda - A)^{-1}$ is the Laplace transform of a strongly continuous function $T: [0, \infty) \to \mathscr{L}(E)$ (which then is the semigroup generated by A). Thus, the Hille–Yosida theorem can be obtained as an immediate consequence of the integrated version of Widder's theorem (see Section 4).

More generally, for each $n \in \mathbf{N} \cup \{0\}$ we consider the class of operators A for which $(\lambda - A)^{-1}/\lambda^n$ is a Laplace transform. An operator belonging to this class is called generator of an n-times integrated semigroup. The Cauchy problem associated with such an operator admits a unique solution for each initial value in $D(A^n)$.

Once integrated semigroups had been introduced in [1] in the context of resolvent positive operators, Neubrander [21] investigated n-times integrated semigroups and obtained a characterization of their generators in the case when the domain is dense.

In the present paper we obtain this generation theorem as an immediate consequence of the integrated version of Widder's theorem. It is remarkable that by our approach one also obtains generation theorems for not densely defined operators (this seems to be new even if $n = 0$; i.e., when the norm condition for the resovent is just the Hille–Yosida condition).

There are many examples of generators of n-times integrated semigroups.

Every resolvent positive operator generates a twice integrated semigroup and a once integrated semigroup if in addition its domain is dense.

Among the densely defined operators the class of all operators which generate an n-times integrated semigroup for some $n \in \mathbf{N} \cup \{0\}$ coincides with the class of all generators of exponential distribution semigroups in the sense of Lions [16]. But our approach seems to be technically simpler and allows a more detailed analysis (classifying the Cauchy problem by the parameter $n \in \mathbf{N} \cup \{0\}$). An approach due to Sova [26] is closely related: he characterizes this class of operators by a notion of well-posedness of the Cauchy problem where the nth primitives of the solutions depend continuously on the initial data.

Finally, we come back to the starting point, the Hille–Yosida theorem. It can now be seen as a special case of the integrated version of Widder's theorem. In addition, the role of the density of the domain is clarified. It is related to the structure of the Banach space in the following way. Suppose that a linear operator A satisfies the Hille–Yosida condition

$$\| \lambda(\lambda - A)^{-1} \| \leqq 1 \qquad (\lambda > 0)$$

(but is not densely defined, *a priori*). Then A is the generator of a once integrated semigroup, but not of a semigroup in general. However, if the Banach space has the Radon–Nikodým property, then A generates a contraction semigroup $(T(t))_{t>0}$ which is strongly continuous for $t > 0$ (but not a C_0-semigroup in general). Finally, if E is reflexive, then A has dense demain and thus generates a C_0-semigroup.

1. The vector-valued version of Widder's theorem

Let G be a Banach space and f: $[0, \infty) \to G$ be a measurable function. If $\| f(t) \| \leqq Me^{wt}$ $(t \geqq 0)$ for some $w \in \mathbf{R}$, $M \geqq 0$, then the Laplace transform of f is given by

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt \qquad (\lambda > w), \tag{1.1}$$

where the integral is understood in the sense of Bochner. Widder's classical theorem [28, 6.8] and [27] characterizes those real-valued functions which are Laplace transforms of bounded functions. On arbitrary Banach spaces the following "*integrated version of Widder's theorem*" holds.

THEOREM 1.1. *Let r: $(0, \infty) \to G$ be a function. Let $M \geqq 0$. The following assertions are equivalent.*

(i) *r is infinitely differentiable and*

$$\sup\{ \| \lambda^{n+1} r^{(n)}(\lambda)/n! \| : \lambda > 0, n = 0, 1, \ldots \} \leqq M.$$

(ii) *There exists a function F: $[0,\infty) \to G$ satisfying*

$$F(0) = 0 \quad \textit{and} \quad \| F(t+h) - F(t) \| \leqq Mh \qquad (t \geqq 0, h \geqq 0)$$

such that

(1.1) $$r(\lambda)=\int_0^\infty \lambda e^{-\lambda t}F(t)dt \qquad (\lambda>0.$$

PROOF. Assume that (i) holds. Let $x'\in G'$. By Widder's classical theorem (see [28, 6.8] and [27]) there exists $f(.,x')\in L^\infty[0,\infty)$ satisfying $\| f(.,x')\|_\infty \leqq M\|x'\|$ such that

$$\langle r(\lambda),x'\rangle=\int_0^\infty e^{-\lambda t}f(t,x')dt \qquad (\lambda>0).$$

Define $F(.,x')\colon[0,\infty)\to G$ by $F(t,x')=\int_0^t f(r,x')dr$. Integrating by parts one obtains

$$\langle r(\lambda),x'\rangle=\int_0^\infty \lambda e^{-\lambda t}F(t,x')dt \qquad (\lambda>0)$$

Since $F(.,x')$ is continuous, by the uniqueness theorem for Laplace transforms [28, 5.7 Corollary 7.2] one obtains that $F(t,x')$ is linear in $x'\in G'$. Moreover,

$$|F(t+h,x')-F(t,x')|\leqq M\|x'\|h \qquad (h\geqq 0, t\geqq 0, x'\in G').$$

Consequently, for every $t\geqq 0$ there exists $F(t)\in G''$ such that $F(t,x')=\langle F(t),x'\rangle$ for all $x'\in G'$.

Assertion (ii) will be proved if we show that $F(t)\in G$ (we identify G with a closed subspace of G'' via evaluation). Denote by $q\colon G''\to G''/G$ the quotient mapping. Since $r(\lambda)\in G$ we have

$$0=q(r(\lambda)/\lambda)=\int_0^\infty e^{-\lambda t}q(F(t))dt \qquad (\lambda>0).$$

It follows from the uniqueness theorem for Laplace transforms that $q(F(t))=0$; i.e., $F(t)\in G$ for all $t\geqq 0$.

The converse implication is proved as easily as in the numerical case. □

By the same proof one obtains the following more general result:

COROLLARY 1.2. *Let $a\geqq 0$ and $r\colon(a,\infty)\to G$ be an infinitely differentiable function. For $M\geqq 0$, $w\in(-\infty,a]$ the following assertions are equivalent.*

(i) $\|(\lambda-w)^{n+1}r(\lambda)^{(n)}/n!\|\leqq M$, $\lambda>a$, $n=0,1,2,\ldots$.

(ii) *There exists a function $F\colon[0,\infty)\to G$ satisfying $F(0)=0$ and*

$$\limsup_{h\downarrow 0}(1/h)\|F(t+h)-F(t)\|\leqq Me^{wt} \qquad (t\geqq 0)$$

such that

$$r(\lambda)=\int_0^\infty \lambda e^{-\lambda t}F(t)dt \qquad (\lambda > a). \tag{1.2}$$

Moreover, r has an analytic extension to $\{\lambda\in\mathbf{C}: \operatorname{Re}\lambda > w\}$ *which is given by* (1.2) *if* $\operatorname{Re}\lambda > 0$.

DEFINITION 1.3. Let G be a Banach space. We say, *Widder's theorem holds in G*, if every $r\in C^\infty([0,\infty), G)$ satisfying

$$\sup\{\,\|\lambda^{n+1}r^{(n)}(\lambda)/n!\,\|:\lambda>0, n=0,1,2,\ldots\}<\infty$$

can be represented in the form

$$r(\lambda)=\int_0^\infty e^{-\lambda t}f(t)dt \quad (\lambda>0) \quad \text{where } f\in L^\infty([0,\infty), G).$$

Here $L^\infty([0,\infty), G)$ denotes the Banach space of all classes of measurable functions f: $[0,\infty)\to G$ satisfying $\|f\|_\infty := \sup_{t\geq 0}\|f(t)\| < \infty$.

Now we want to characterize those Banach spaces G in which Widder's theorem holds. The fact that $G=\mathbf{R}$ belongs to this class is Widder's classical theorem.

THEOREM 1.4. *A Banach space G has the Radon–Nikodým property if and only if Widder's theorem holds.*

The Radon–Nikodým property has been investigated extensively, and the results are well documented in the treatise by Diestel–Uhl [6] to which we refer for further information. Every reflexive space and every separable dual space has the Radon–Nikodým property. The spaces $C[0,1]$, $L^1[0,1]$ and c_0 do not possess the Radon–Nikodým property.

REMARK. Miyadera [18] proved that Widder's theorem holds in reflexive spaces; he showed by an example that it does not hold in $\mathscr{L}(C[0,\infty])$.

Our proof of Theorem 1.4 is based on the following characterization of the Radon–Nikodým property, which is of proper interest.

THEOREM 1.5. *A Banach space G has the Radon–Nikodým property if and only if every Lipschitz continuous map f*: $[0,1]\to G$ *is differentiable a.e.*

Before proving Theorem 1.5 we observe the following useful consequence of the property formulated in the theorem.

LEMMA 1.6. *Let G be a Banach space, $M \geqq 0$ and $f: [0, 1] \to G$ a function satisfying*

$$\| f(s) - f(t) \| \leqq M |s - t| \qquad (s, t \in [0, 1]).$$

If f is differentiable a.e., then $f' \in L^\infty([0, 1], G]$ and

$$f(t) = \int_0^t f'(s)ds \qquad (t \in [0, 1]). \tag{1.3}$$

PROOF. There exists a set N of measure zero such that $f'(s)$ exists for all $s \in [0, 1] - N$. Set $f'(s) = 0$ for $s \in N$. Then f' is weakly measurable and $\| f'(s) \| \leqq M$ for all $s \in [0, 1]$. Since f is separably valued, also f' is separably valued. Thus $f' \in L^\infty([0, 1], G)$ by Pettis' theorem [6, p. 42]. Then (1.3) holds, since it holds weakly. □

PROOF OF THEOREM 1.5. If G has the Radon–Nikodým property, then G is a Gelfand space [6, IV, 3 (p. 106)] and so every Lipschitz continuous function with values in G is differentiable almost everywhere.

Conversely, assume that this condition holds. By [6, V.3 Corollary 8 (p. 138)] it suffices to show that G has the Radon–Nikodým property with respect to the Lebesgue measure on [0, 1]. We use [6, III.1 Theorem 5 (p. 63)]. Let $T: L^1[0, 1] \to G$ be a bounded linear operator. We have to show that there exists $h \in L^\infty([0, 1], G)$ such that $Tg = \int_0^1 g(t)h(t)dt$ for all $g \in L^1[0, 1]$. For $t \in [0, 1]$ we denote by $1_{[0,t]}$ the characteristic function of $[0, t]$. Let $f: [0, 1] \to G$ be defined by $f(t) = T1_{[0,t]}$. Then

$$\| f(t) - f(s) \| = \| T1_{[s,t]} \| \leqq \| T \| \, |t - s| \qquad \text{for } 0 \leqq s \leqq t \leqq 1.$$

It follows from the hypothesis that the derivative f' of f exists a.e. Moreover, by Lemma 1.6, $f' \in L^\infty[(0, 1], G)$ and $f(t) = \int_0^t f'(s)ds$ $(t \in [0, 1])$. This means that

$$T1_{[0,t]} = \int_0^1 f'(s)1_{[0,t]}(s)ds$$

for all $t \in [0, 1]$. Since the set $\{1_{[0,t]}: t \in [0, 1]\}$ is total in $L^1[0, 1]$, it follows that

$$Tg = \int_0^1 g(s)f'(s)ds$$

for all $g \in L^1[0, 1]$. □

PROOF OF THEOREM 1.4. Assume that G has the Radon–Nikodým property. Let $r \in C^\infty([0, \infty), G)$ such that

$$M := \sup\{\ \| \lambda^{n+1} r(\lambda)/n! \| : n = 0, 1, 2, \ldots, \lambda > 0\} < \infty.$$

By Theorem 1.1 there exists $F: [0, \infty) \to G$ satisfying $F(0) = 0$, $\| F(s) - F(t) \| \leqq M |s - t|$ $(s, t \geqq 0)$ such that

$$r(\lambda) = \int_0^\infty \lambda e^{-\lambda t} F(t) dt \qquad (\lambda > 0). \tag{1.4}$$

By Theorem 1.5 and Lemma 1.6 the derivative $f(s) = F'(s)$ exists a.e. and $f \in L^\infty([0, \infty), G)$. Integrating (1.4) by parts one obtains

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt \qquad (\lambda > 0).$$

This proves one implication. Assume now that Widder's theorem holds in G. We show that G has the Radon–Nikodým property. For that we make use of Theorem 1.5. Let $F: [0, 1] \to G$ satisfy $\| F(s) - F(t) \| \leqq M |s - t|$ $(s, t \in [0, 1])$. We have to show that F is differentiable a.e. Considering $F - F(0)$ if necessary, we can assume that $F(0) = 0$. Extend F to F^* or $[0, \infty)$ by letting $F^*(t) = F(1)$ for $t > 1$. Define $r \in C^\infty[(0, \infty), G)$ by

$$r(\lambda) = \int_0^\infty \lambda e^{-\lambda t} F^*(t) dt \qquad (\lambda > 0). \tag{1.5}$$

Then $\| \lambda^{n+1} r^{(n)}(\lambda)/n! \| \leqq M$ for all $\lambda > 0$, $n = 0, 1, 2, \ldots$. Since Widder's theorem holds in G by assumption, we find $f \in L^\infty([0, \infty), G)$ such that

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt = \int_0^\infty \lambda e^{-\lambda t} f(s) ds dt \qquad (\lambda > 0).$$

This together with (1.5) implies that $F^*(t) = \int_0^t f(s) ds$ for all $t \geqq 0$ by the uniqueness theorem for Laplace transforms. It follows from [6, II Theorem 9 (p. 49)] that F is differentiable a.e. □

2. Resolvents as Laplace transforms

Let E be a Banach space and $R: (w, \infty) \to \mathscr{L}(E)$ a function (where $w \in \mathbf{R}$). We say that R is a *Laplace tansform* if there exists a strongly continuous function $S: [0, \infty) \to \mathscr{L}(E)$ satisfying $\| S(t) \| \leqq Me^{wt}$ $(t \geqq 0)$ for some $M \geqq 0$ such that

$$R(\lambda) = \int_0^\infty e^{-\lambda t} S(t) dt \qquad (\lambda > w).$$

(If we want to be more specific we say that R is the Laplace transform of S.)

Here we denote by $\int_0^\infty e^{-\lambda t}S(t)xdt$ $(x \in E, \lambda > w)$ the Bochner integral which coincides with the improper Riemann integral. By $\int_0^\infty e^{-\lambda t}S(t)dt \in \mathscr{L}(E)$ we understand the operator $x \to \int_0^\infty e^{-\lambda t}S(t)xdt$.

THEOREM 2.1. *A linear operator A or E is the infinitesimal generator of a C_0-semigroup if and only if there exists $w \in \mathbf{R}$ such that $(w, \infty) \subset \rho(A)$ (the resolvent set of A) and $R: (w, \infty) \to \mathscr{L}(E)$ defined by $R(\lambda) = (\lambda - A)^{-1}$ is a Laplace transform. In that case R is the Laplace transform of the semigroup generated by A.*

The proof of Theorem 2.1 is based on the fact that the semigroup property corresponds precisely to the resolvent equation via Laplace transformation.

PROPOSITION 2.2. *Let $T: [0, \infty) \to \mathscr{L}(E)$ be strongly continuous such that $\| T(t) \| \leqq Me^{wt}$ $(t \geqq 0)$ for some $M, w \in \mathbf{R}$. Let $R(\lambda) = \int_0^\infty e^{-\lambda t}T(t)dt$ $(\lambda > w)$. Then $(R(\lambda))_{\lambda > w}$ is a pseudoresolvent if and only if*

$$T(s)T(t) = T(s+t) \qquad (s, t \geqq 0). \tag{2.1}$$

PROOF. Let $\lambda, \mu > w$. Then

$$
\begin{aligned}
&(R(\lambda) - R(\mu))/\mu - \lambda \\
&= \int_0^\infty e^{(\lambda-\mu)t}R(\lambda)dt - \int_0^\infty 1/(\mu - \lambda)e^{(\lambda-\mu)t}e^{-\lambda t}T(t)dt \\
&= \int_0^\infty e^{(\lambda-\mu)t}\int_0^\infty e^{-\lambda s}T(s)dsdt - \int_0^\infty e^{(\lambda-\mu)t}\int_0^t e^{-\lambda s}T(s)dsdt \\
&= \int_0^\infty e^{(\lambda-\mu)t}\int_t^\infty e^{-\lambda s}T(s)dsdt \\
&= \int_0^\infty e^{-\mu t}\int_t^\infty e^{-\lambda(s-t)}T(s)dsdt \\
&= \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}T(s+t)dsdt.
\end{aligned}
$$

On the other hand,

$$R(\mu)R(\lambda) = \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}T(s)T(t)dsdt.$$

So the claim follows from the uniqueness theorem for Laplace transforms. □

PROOF OF THEOREM 2.1. Suppose that $R(\lambda) = (\lambda - A)^{-1}$ $(\lambda > w)$ is the Laplace transform of $T: [0, \infty) \to \mathscr{L}(E)$. Then $T(s+t) = T(s)T(t)$ $(s, t \geqq 0)$

by Proposition 2.2. Consequently, $T(0)$ is a projection. If $T(0)x = 0$, then $T(t)x = T(t)T(0)x = 0$ for all $t > 0$ and so $R(\lambda)x = 0$ $(\lambda > w)$. This implies $x = 0$. Hence $T(0) = I$. We have shown that $(T(t))_{t \geqq 0}$ is a C_0-semigroup. Denote by B its generator. Then

$$(\lambda - B)^{-1} = \int^{\infty} e^{-\lambda t} T(t)dt = (\lambda - A)^{-1} \qquad (\lambda > w).$$

Hence $A = B$. This proves one implication. The other is well known. □

The Hille–Yosida theorem (or more precisely its extension due to Feller [9], Miyadera [17] and Phillips [23]) asserts that a densely defined operator A is the generator of a C_0-semigroup if and only if $(w, \infty) \subset \rho(A)$ for some $w \in \mathbf{R}$ and

$$\| (\lambda - w)^{m+1} R(\lambda, A)^{(m)}/m! \| \leqq M \tag{2.2}$$

for all $\lambda > w$, $m = 0, 1, \ldots$ and some $M \geqq 0$. (Observe that $R(\lambda, A)^{(m)}/m! = (-1)^m R(\lambda, A)^{m+1}$.) So in view of Theorem 2.1, the Hille–Yosida theorem can be reformulated by saying that Widder's theorem holds for resolvents of densely defined linear operators. This is surprising, since Widder's theorem does not hold for arbitrary functions in general as we saw. The reason why it holds for resolvents will be made clear in the following (where a proof of the Hille–Yosida theorem based on the integrated version of Widder's theorem (Theorem 1.1) is given (see Theorem 4.2)).

3. Integrated semigroups and their generators

The main reason why generators of C_0-semigroups are of great interest is that the associated Cauchy problem admits a unique solution for a large set of initial values. For that, however, it is not essential that $R(\lambda, A)$ is a Laplace transform; we will consider the weaker condition that $R(\lambda, A)/\lambda^n$ is a Laplace transform for some $n = 0, 1, 2, \ldots$. At first we analyze this condition and give some consequences. In Section 4, as an extension of the Hille–Yosida theorem, a characterization of those operators verifying the condition is given as an immediate consequence of Theorem 1.1 (the integrated version of Widder's theorem). The Cauchy problem is considered in Section 5.

Let $(T(t))_{t \geqq 0}$ by a C_0-semigroup with generator A. There exist $M, w \in \mathbf{R}$ such that $\| T(t) \| \leqq Me^{wt}$ $(t \geqq 0)$. For $n \in \mathbf{N}$ let

$$S^n(t) = \int_0^t (t - s)^{n-1}/(n-1)! T(s)ds \qquad (t \geqq 0). \tag{3.1}$$

Then for $\lambda > w$,

$$R(\lambda, A)/\lambda^n = \int_0^\infty e^{-\lambda t} S^n(t)dt \qquad (\lambda > \max\{w, 0\}). \tag{3.2}$$

[In fact, $R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)dt$ $(\lambda > w)$. Integrating by parts n-times yields (3.2).]

We are going to consider the class of operators for which $\lambda \to R(\lambda, A)/\lambda^n$ is a Laplace transform of some function S: $[0, \infty) \to \mathscr{L}(E)$. Similar to Proposition 2.7 one expects that the resolvent equation corresponds to some functional equation for S.

THEOREM 3.1. *Let S: $[0, \infty) \to \mathscr{L}(E)$ be strongly continuous such that $\| S(t) \| \leqq Me^{wt}$ $(t \geqq 0)$ for some $M, w \in \mathbf{R}$. Let $n \in \mathbf{N}$ and*

$$R(\lambda) = \int_0^\infty \lambda^n e^{-\lambda t} S(t)dt \qquad (\lambda > w). \tag{3.3}$$

Then $(R(\lambda, A))_{\lambda > w}$ is a pseudoresolvent if and only if

$$S(t)S(s) = 1/(n-1)! \left[\int_t^{s+t} (s+t-r)^{n-1} S(r)dr - \int_0^s (s+t-r)^{n-1} S(r)ds \right] \qquad (s, t \geqq 0). \tag{3.4}$$

PROOF. Let $\lambda, \mu > w$, $\lambda \neq \mu$. Since

$$R(\lambda)/\lambda^n R(\mu)/\mu^n = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} S(t)S(s)dsdt,$$

it suffices to prove that

$$\begin{aligned} &1/(\mu - \lambda)\lambda^{-n}\mu^{-n}(R(\lambda) - R(\mu)) \\ &= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} 1/(n-1)! \int_t^{s+t} (s+t-r)^{n-1} S(r)drdsdt \\ &- \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} 1/(n-1)! \int_0^s (s+t-r)^{n-1} S(r)drdsdt. \end{aligned} \tag{3.5}$$

Then the claim follows from the uniqueness theorem for Laplace transforms. But

$$\begin{aligned} (\mu - \lambda)^{-1}\lambda^{-n}\mu^{-n}(R(\lambda) - R(\mu)) &= \mu^{-n}(\mu - \lambda)^{-1}(R(\lambda)/\lambda^n - R(\mu)/\mu^n) \\ &+ (\mu - \lambda)^{-1}(\mu^{-n} - \lambda^{-n})R(\mu)/\mu^n. \end{aligned} \tag{3.6}$$

Replacing $R(\lambda)$ by $R(\lambda)/\lambda^n$ in the proof of Proposition 2.2 one obtains that

$$(\mu-\lambda)^{-1}(R(\lambda)/\lambda^n - R(\mu)/\mu^n) = \int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(t+s)dsdt.$$

Integrating n times by parts one obtains

$$\begin{aligned}
&\mu^{-n}(\mu-\lambda)^{-n}(R(\lambda)/\lambda^n - R(\mu)/\mu^n)\\
&\quad = \int_0^\infty e^{-t}\int_0^\infty e^{-\mu s}\int_0^s 1/(n-1)!(s-r)^{n-1}S(r+t)drdsdt\\
&\quad = \int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}\int_t^{t+s} 1/(n-1)!(s+t-r)^{n-1}S(r)drdsdt.
\end{aligned}$$

This is the first term on the right side of (3.5). It remains to compute the second term in (3.6).

$$\begin{aligned}
&(\mu-\lambda)^{-1}[\mu^{-n}-\lambda^{-n}]R(\mu)/\mu^n\\
&= -\left(\sum_{k=0}^{n-1}\lambda^{-(k+1)}\mu^{-(n-k)}\right)R(\mu)/\mu^n\\
&= -\sum_{k=0}^{n-1}\lambda^{-(k+1)}\int_0^\infty \mu^{k-n}e^{-\mu s}S(s)ds\\
&= -\sum_{k=0}^{n-1}\lambda^{-(k+1)}\int_0^\infty e^{-\mu s}\int_0^s (s-r)^{n-k-1}/(n-k-1)!S(r)drds\\
&= -\sum_{k=0}^{n-1}\int_0^\infty e^{-\lambda t}t^k/k!dt\int_0^\infty e^{-\mu s}\int_0^s (s-r)^{n-k-1}/(n-k-1)!S(r)drds\\
&= -\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}\int_0^s 1/(n-1)!\sum_{k=0}^{n-1}\binom{n-1}{k}(s-r)^{n-k-1}t^kS(r)drdsdt\\
&= -\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}1/(n-1)!\int_0^s (t+s-r)^{n-1}S(r)drdsdt.
\end{aligned}$$

This is the second term on the right side of (3.5). □

Note that condition (3.4) implies

$$S(t)S(s) = S(s)S(t) \qquad (s, t \geqq 0) \tag{3.7}$$

and

$$S(t)S(0) = 0 \qquad (t \geqq 0). \tag{3.8}$$

DEFINITION 3.2. Let $n \in \mathbf{N}$. A strongly continuous family $(S(t))_{t \geqq 0} \subset$

$\mathcal{L}(E)$ is called *n-times integrated semigroup* if (3.4) is satisfied and $S(0)=0$. Moreover, $(S(t))_{t\geqq 0}$ is called *non-degenerate* if $S(t)x=0$ for all $t\geqq 0$ implies $x=0$. Finally, $(S(t))_{t\geqq 0}$ is called *exponentially bounded* if there exist M, $w\in\mathbf{R}$ such that $\|S(t)\|\leqq Me^{wt}$ for all $t\geqq 0$.

For convenience we call a C_0-semigroup also *0-times integrated semigroup*.

Let $(S(t))_{t\geqq 0}$ be an n-times integrated semigroup, where $n\in\mathbf{N}$. Assume in addition that $(S(t))_{t\geqq 0}$ is exponentially bounded. Define $R(\lambda)$ by (3.3) ($\lambda > w$). Then ker $R(\lambda)$ is independent of $\lambda > w$ (by the resolvent equation). Hence by the uniqueness theorem $R(\lambda)$ is injective if and only if $(S(t))_{t\geqq 0}$ is non-degenerate. In that case there exists a unique operator A satisfying $(w,\infty)\subset\rho(A)$ such that $R(\lambda)=(\lambda-A)^{-1}$ for all $\lambda > w$. This operator is called the *generator of* $(S(t))_{t\geqq 0}$.

Usually, the given object is the operator. Thus for $n\in\mathbf{N}\cup\{0\}$, an operator A is the generator of an n-times integrated semigroup if and only if $(a,\infty)\subset\rho(A)$ for some $a\in\mathbf{R}$ and the function $\lambda\to(\lambda-A)^{-1}/\lambda^n$ is a Laplace transform. By the result of Section 2, for the case $n=0$ this is consistent with the definition of the generator of a C_0-semigroup.

PROPOSITION 3.3. *Let A be the generator of an n-times integrated semigroup $(S(t))_{t\geqq 0}$ (where $n\in\mathbf{N}\cup\{0\}$). Then for all $x\in D(A)$, $t\geqq 0$,*

$$S(t)x\in D(A) \quad \textit{and} \quad AS(t)x=S(t)Ax \tag{3.9}$$

and

$$S(t)x=(t^n/n!)x+\int_0^t S(s)Axds. \tag{3.10}$$

Moreover, $\int_0^t S(s)xds\in D(A)$ for all $x\in E$, $t\geqq 0$ and

$$A\int_0^t S(s)xds=S(t)x-(t^n/n!)x. \tag{3.11}$$

PROOF. There exist $w, M\geqq 0$ such that $\|S(t)\|\leqq Me^{wt}$ ($t\geqq 0$) and $R(\lambda,A)=\int_0^\infty \lambda^n e^{-\lambda t}S(t)dt$ ($\lambda > w$). Fix $\mu\in\rho(A)$. Then

$$\begin{aligned}\int_0^\infty e^{-\lambda t}S(t)R(\mu,A)x\,dt&=\lambda^{-n}R(\lambda,A)R(\mu,A)x\\&=\lambda^{-n}R(\mu,A)R(\lambda,A)x\\&=\int_0^\infty e^{-\lambda t}R(\mu,A)S(t)x\,dt \qquad \text{for all } \lambda > w,\quad x\in E.\end{aligned}$$

By the uniqueness theorem it follows that

$$R(\mu, A)S(t) = S(t)R(\mu, A) \qquad (\mu \in \rho(A),\, t \geqq 0). \tag{3.12}$$

This implies (3.9). Let $x \in D(A)$. Then for all $\lambda > w$,

$$\begin{aligned}\int_0^\infty \lambda^{n+1} e^{-\lambda t} t^n/n! x\, dt = x \\ &= \lambda R(\lambda, A)x - R(\lambda, A)Ax \\ &= \int_0^\infty \lambda^{n+1} e^{-\lambda t} S(t)x\, dt - \int_0^\infty \lambda^n e^{-\lambda t} S(t)Ax\, dt \\ &= \int_0^\infty \lambda^{n+1} e^{-\lambda t} S(t)x\, dt \\ &\quad - \int_0^\infty \lambda^{n+1} e^{-\lambda t} \int_0^t S(s)Ax\, ds dt.\end{aligned}$$

Thus (3.10) follows from the uniqueness theorem.

In order to prove (3.11) let $x \in E$, $t \geqq 0$, $\lambda > w$. Then by (3.9), (3.10) and (3.12),

$$\begin{aligned}\int_0^t S(s)x ds &= \lambda R(\lambda, A) \int_0^t S(s)x\, ds - \int_0^t S(s)AR(\lambda, A)x\, ds \\ &= \lambda R(\lambda, A) \int_0^t S(s)x\, ds - S(t)R(\lambda, A)x + (t^n/n!)R(\lambda, A)x.\end{aligned}$$

Hence $\int_0^t S(s)xds \in D(A)$ and

$$(\lambda - A) \int_0^t S(s)x\, ds = \lambda \int_0^t S(s)x\, ds - S(t)x + (t^n/n!)x.$$

This implies (3.11). □

Corollary 3.4. *For all $x \in E$ one has $S(t)x \in \overline{D(A)}$ ($t \geqq 0$). Moreover, let $x \in E$. Then $S(\,.\,)x$ is right-sided differentiable in $t \geqq 0$ if and only if $S(t)x \in D(A)$. In that case*

$$(d/ds_{|s=t})S(s)x = \begin{cases} AS(t)x + (t^{n-1}/(n-1)!)x & \text{if } n > 0, \\ AS(t)x & \text{if } n = 0. \end{cases} \tag{3.13}$$

Proof. For all $x \in E$, $t \geqq 0$ one has

$$S(t)x = \lim_{h \downarrow 0} (1/h) \int_t^{t+h} S(s)x ds \in \overline{D(A)}$$

(by Proposition 3.3). The second assertion directly follows from (3.11) since A is closed. □

4. Characterization of generators of integrated semigroups

The following theorem immediately follows from Corollary 1.2.

THEOREM 4.1. *Let $n \in \mathbf{N} \cup \{0\}$, $w \in \mathbf{R}$, $M \geqq 0$. A linear operator A is the generator of an $(n+1)$-times integrated semigroup $(S(t))_{t \geqq 0}$ satisfying*

$$\limsup_{h \downarrow 0} = (1/h) \| S(t+h) - S(t) \| \leqq Me^{wt} \qquad (t \geqq 0) \tag{4.1}$$

if and only if there exists $a \geqq \max\{w, 0\}$ such that $(a, \infty) \subset \rho(A)$ and

$$\| (\lambda - w)^{k+1} [R(\lambda, A))/\lambda^n]^{(k)} \| \leqq M \quad \text{for all } \lambda > a, \quad k = 0, 1, 2, \ldots . \tag{4.2}$$

Now let the equivalent conditions (4.1) and (4.2) be satisfied. Because of (4.1), the set F_1 of all $x \in E$ such that $S(\,.\,)x \in C^1[(0, \infty), E)$ is a closed subspace of E. It follows from Proposition 3.3 that $F := \overline{D(A)} \subset F_1$. For $x \in F$ let $T(x)x := d/dt\ S(t)x$ $(t \geqq 0)$. Then $T(t)x \in F$ (as a consequence of Corollary 3.4). Thus we obtain a strongly continuous family $(T(t))_{t \geqq 0}$ of linear operators on F. It follows from (4.1) that $\| T(t) \| \leqq Me^{wt}$ $(t \geqq 0)$. Let A_F be the part of A in F (i.e., $A_F x = Ax$ for $x \in D(A_F) := \{x \in D(A) : Ax \in F\}$). Then $(a, \infty) \subset \rho(A_F)$ and $R(\lambda, A)_{|F}$ $(\lambda > a)$. Moreover,

$$R(\lambda, A_F)x = \int_0^\infty \lambda^{n+1} e^{-\lambda t} S(t)x\, dt = \int_0^\infty \lambda^n e^{-\lambda t} T(t)x\, dt \qquad (\lambda > a, x \in F).$$

We have proved the following.

COROLLARY 4.2. *If A satisfies the equivalent conditions of Theorem 4.1, then the part of A in $\overline{D(A)}$ is the generator of an n-times integrated semigroup.*

In the case when $D(A)$ is dense, we obtain the following characterization.

THEOREM 4.3. *Let A be a densely defined operator such that $(a, \infty) \subset \rho(A)$ for some $a \geqq 0$. Let $n \in \mathbf{N} \cup \{0\}$, $M \geqq 0$, $w \in (-\infty, a]$. The following assertions are equivalent.*

(i) *A generates an n-times integrated semigroup* $(T(t))_{t \geqq 0}$ *satisfying* $\| T(t) \| \leqq Me^{wt}$ $(t \geqq 0)$.

(ii) $\| (\lambda - w)^{k+1}[R(\lambda, A)/\lambda^n]^{(k)}/k! \| \leqq M$ *for all* $\lambda > a$, $k = 0, 1, 2, \ldots$.

REMARK. The case $n = 0$ in Theorem 4.3 is the Hille–Yosida theorem.

COROLLARY 4.4. *If a densely defined operator A generates an n-time integrated semigroup, then its adjoint A' generates an $(n + 1)$-times integrated semigroup.*

PROOF. This follows immediately from Theorems 4.1 and 4.3 since $R(\lambda, A)' = R(\lambda, A')$. □

REMARK. Theorem 4.3 has been obtained by Neubrander [21] by a different proof (namely by extending Kisyński's proof of the Hille–Yosida theorem [13, p. 358] to the case $n > 0$ (see also the proof of [7, Theorem 2.1.1])).

REMARKS. (a) *Relation to distribution semigroups*. Densely defined operators satisfying condition (ii) of Theorem 4.3 have also been considered by Sova [26]. In particular, it follows from [26, Theorem 3.2] that a densely defined operator A is the generator of an exponential distribution semigroup (in the sense of Lions [16]) if and only if A generates an n-times integrated semigroup $(S(t))_{t \geqq 0}$ for some $n \in \mathbf{N} \cup \{0\}$. One can show that in that case the distribution semigroup T is given by

$$T(\phi) = (-1)^n \int_0^\infty \phi^{(n)}(t) S(t) dt \qquad \text{for all } \phi \in \mathcal{D}(0, \infty).$$

(b) *Complex characterization*. It follows from Lions [16, Théorème 6.1] that a densely defined operator A is the generator of an n-times integrated semigroup for some $n \in \mathbf{N} \cup \{0\}$ if and only if there exist $w \geqq 0$, $m \in \mathbf{N}$, $M \geqq 0$ such that $\{\lambda : \operatorname{Re} \lambda > w\} \subset \rho(A)$ and $\| R(\lambda, A) \| \leqq M(1 + |\lambda|^m)$ whenever $\operatorname{Re} \lambda > w$.

(c) *Examples*. Besides the examples for exponential distribution semigroups given in the literature (see Fattorini [7] and Krein–Khazan [14] for further references) interesting new examples have recently been given by Kellermann [12] using explicitly the integrated semigroup for the construction. For example, the operator $i\Delta$ (where Δ denotes the Laplace operator) with maximal (distributional) domain generates a 3-times integrated semigroup on the spaces $L^p(\mathbf{R}^N)$ $(1 \leqq p \leqq \infty)$, $C^b(\mathbf{R}^N)$ and $C_0(\mathbf{R}^N)$ if $N \leqq 3$ and a once integrated semigroup if $N = 1$. However, it is well known that $i\Delta$ does not generate a C_0-semigroup on any of these spaces other than $L^2(\mathbf{R}^N)$.

Further examples are given by Neubrander [21]. For every $n \in \mathbf{N}$ he constructs an operator A which generates an n-times integrated semigroup but not an $(n-1)$-times integrated semigroup.

We conclude this section by considering resolvent positive operators (cf. [1]).

COROLLARY 4.5. *Let E be an ordered Banach space with normal and generating cone (in particular, E may be a Banach lattice or a C^*-algebra). Let A be an operator on E such that $(a, \infty) \subset \rho(A)$ for some $a \in \mathbf{R}$ and $R(\lambda, A) \geqq 0$ for all $\lambda > a$.*

Then A is the generator of a twice integrated semigroup. If $D(A)$ is dense, then A generates a once integrated semigroup.

The proof is based on the following lemma.

LEMMA 4.6. *Let A be an operator and $\lambda \in \rho(A)$. Then for all $m \in \mathbf{N}$,*

$$(4.3) \qquad (-1)^m \lambda^{m+1} [R(\lambda, A)/\lambda]^{(m)}/m! = \sum_{k=0}^{m} \lambda^k R(\lambda, A)^{k+1}.$$

PROOF. This is immediate by developing $[R(\lambda, A)/\lambda, A)/\lambda]^{(m)}$ and using that $(-1)^k R(\lambda, A)^{(k)}/k! = R(\lambda, A)^{k+1}$. □

PROOF OF COROLLARY 4.5. Considering $A - a$ instead of A if necessary, we may assume that $[0, \infty) \subset \rho(A)$ and $R(\lambda, A) \geqq 0$ for all $\lambda \geqq 0$. Then for all $m \in \mathbf{N}$,

$$(4.4) \qquad \sum_{k=0}^{m} \lambda^k R(\lambda, A)^{k+1} = R(0, A) - \lambda^m R(\lambda, A)^m R(0, A)$$

as one easily deduces from the resolvent equation by induction. Consequently, $0 \leqq \Sigma_{k=0}^{m} \lambda^k R(\lambda, R)^{k+1} \leqq R(0, A)$. We obtain by (4.3) that condition (4.2) is satisfied for $n = 1$ and $w = 0$. So the claim follows from Theorems 4.1 and 4.2. □

Using a vetor-valued version of Bernstein's theorem, it had been shown in [1] that a resolvent positive operator (i.e., an operator satisfying the hypotheses of Corollary 4.5) on certain spaces generates a once integrated semigroup even if its domain is not dense. The class of spaces in question includes all

Banach lattices with order continuous norm but not the space $C(K)$ (K compact). This is shown by the following example which we owe to H. Kellermann.

EXAMPLE 4.7. Let $E = C[-1, 0] \times \mathbf{R}$ and A be given by $D(A) = C^1[-1, 0] \times \{0\}$ and $A(f, 0) = (f', -f(0))$. Then $\rho(A) = \mathbf{C}$ and $R(\lambda, A)(f, c) = (g, 0)$ with $g(x) = e^{\lambda x}[\int_x^0 e^{-\lambda y} f(y)dy + c]$ ($\lambda \in \mathbf{R}$). Hence A is resolvent positive. Let $e_\lambda \in C[-1, 0]$ be given by $e_\lambda(x) = e^{\lambda x}$ ($\lambda > 0$, $x \in [-1, 0]$). Then $(e_\lambda, 0) = R(\lambda, A)(0, 1)$. One has $e_\lambda = \int_0^\infty \lambda^2 e^{-\lambda t} k_t dt$ where $k_t \in C[-1, 0]$ is given by $k_t(x) = 0$ if $x + t \leqq 0$ and $k_t(x) = x + t$ otherwise. If A were the generator of a once integrated semigroup, than e_λ/λ would be a Laplace transform. Hence $k: [0, \infty) \to C[-1, 0]$ would be differentiable. But $(d/dt)k_t(x)$ is not continuous in x if $t \in (-1, 0)$.

5. The Cauchy problem

Let A be an operator on E, $u_0 \in E$, $f \in C([0, b], E)$ (where $b > 0$). By a *solution* of

$$P(u_0, f) \qquad \begin{cases} u'(t) = Au(t) + f(t) & (t \in [0, b]) \\ u(0) = u_0 \end{cases}$$

we understand a function $u \in C^1([0, b], E)$ satisfying $u(t) \in D(A)$ ($t \in [0, b]$) such that the equations in $P(u_0, f)$ hold.

Assume now that A is the generator of an n-times integrated semigroup $(S(t))_{t \geqq 0}$, where $n \in \mathbf{N} \cup \{0\}$. We first show that there exists at most one solution of $P(u_0, f)$. Consider the function $v \in C([0, b], E)$ given by

$$v(t) = S(t)u_0 + \int_0^t S(s) f(t - s) ds. \tag{5.1}$$

PROPOSITION 5.1. *If there exists a solution u of $P(u_0, f)$, then $v \in C^{n+1}[(0, b], E)$ and $u = v^{(n)}$.*

PROOF. Let $0 \leqq t \leqq b$. For $s \in [0, t]$ let $w(s) = S(t - s)u(s)$. Since $u(s) \in D(A)$ by hypothesis, we obtain from (3.10) that

$$\begin{aligned} w'(s) &= -(t - s)^{n-1}/(n - 1)!u(s) - S(t - s)Au(s) + S(t - s)u'(s) \\ &= -(t - s)^{n-1}/(n - 1)!u(s) + S(t - s)f(s) \qquad (s \in [0, t]). \end{aligned}$$

Hence

$$\begin{aligned}S(t)u_0 &= w(0) - w(t)\\ &= -\int_0^t w'(s)ds\\ &= \int_0^t (t-s)^{n-1}/(n-1)!u(s)ds - \int_0^t S(t-s)f(s)ds\\ &= \int_0^t (t-s)^{n-1}/(n-1)!u(s)ds - \int_0^t S(s)f(t-s)ds \qquad (t\in[0,b]).\end{aligned}$$

Consequently,

$$v(t) = \int_0^t (t-s)^{n-1}/(n-1)!u(s)ds \qquad (t\in[0,b]). \qquad \square$$

In order to show that a solution exists one merely has to verify that $v\in C^{n+1}[0,b]$.

THEOREM 5.2. *If $v\in C^{n+1}[0,b]$, then $u := v^{(n)}$ is a solution of $P(u_0,f)$.*

REMARK. We do not require that $u_0\in D(A)$. This follows automatically from the hypothesis.

The proof of Theorem 5.2 is based on the following lemma.

LEMMA 5.3. *For every $t\geqq 0$ one has $\int_0^t v(s)ds\in D(A)$ and*

$$A\int_0^t v(s)ds = v(t) - (t^n/n!)u_0 - \int_0^t (t-r)^n/n!\,f(r)dr \quad (t\in[0,b]). \tag{5.2}$$

PROOF. By Fubini's theorem we have

$$\begin{aligned}\int_0^t\int_0^r S(s)f(r-s)dsdr &= \int_0^t\int_s^t S(s)f(r-s)drds\\ &= \int_0^t\int_0^{t-s} S(s)f(r)drds\\ &= \int_0^t\int_0^{t-r} S(s)f(r)dsdr.\end{aligned}$$

The integrand is in $D(A)$ by Proposition 3.3, and using (3.11) one obtains

$$A\int_0^t v(s)ds = S(t)u_0 - (t^n/n!)u_0 + \int_0^t [S(t-r)f(r) - (t-r)^n/n!\,f(r)]dr. \ \square$$

PROOF OF THEOREM 5.2. Assume that $v\in C^{n+1}$. Since A is closed, we may differentiate (5.2) $(n+1)$-times and obtain

$$
(5.3) \qquad \begin{aligned} Av^{(k)}(t) &= v^{(k+1)}(t) - t^{n-k-1}/(n-k-1)!u_0 \\ &\quad - \int_0^t (t-r)^{n-k-1}/(n-k-1)!\, f(r)dr \end{aligned}
$$

for $k = 0, 1, \ldots, n-1$ and $v^{(n)}(t) \in D(A)$ and

$$
(5.4) \qquad Av^{(n)}(t) = v^{(n+1)}(t) - f(t) \qquad (t \in [0, b]).
$$

Relation (5.4) shows that $u := v^{(n)}$ satisfies the differential equation. Moreover, if $n = 0$, then $S(0) = I$ and so $u(0) = v(0) = u_0$. If $n > 0$, then $S(0) = 0$ and so $v(0) = 0$. It follows from (5.3) that $v^{(k)}(0) = 0$ for $k < n$, and for $k = n-1$ one obtains $v^{(n)}(0) = u_0 + Av^{(n-1)}(0) = u_0$. So $u(0) = u_0$ in any case. □

Now we obtain the following sufficient condition on f and u_0 for the existence of a solution of $P(u_0, f)$.

Proposition 5.4. *If $f \in C^{n+1}([0, b], E)$ and $u_0 \in D(A)$, $u_1 := Au_0 + f(0) \in D(A)$, $u_2 := Au_1 + f'(0) \in D(A), \ldots, u_{k+1} = Au_k + f^{(k)}(0) \in D(A), \ldots, u_n = Au_{n-1} + f^{(n)}(0) \in D(A)$, then $P(u_0, f)$ has a unique solution.*

Proof. By (3.10) we obtain that $v \in C^1$ and

$$
v'(t) = t^{n-1}/(n-1)!u_0 + S(t)Au_0 + S(t)f(0) + \int_0^t S(s)f'(t-s)ds.
$$

Using the hypothesis $Au_0 + f(0) \in D(A)$ we obtain by (3.10) that $v \in C^2$. Repeating the argument we finally obtain that $v \in C^{n+1}[0, b]$ and the claim follows from Theorem 5.2. □

Remark. If $n > 0$, merely a regularity assumption on f is not sufficient to obtain solutions even if $u_0 = 0$. For example, assume that $n = 1$ but A is not the generator of a C_0-semigroup. Then there exists $x \in E$ such that $S(\,.\,)x$ is not continuously differentiable. Let $u_0 = 0$ and $f(t) = x$ $(t \in [0, b])$. Then $v(t) = \int_0^t S(s)xds$ $(t \in [0, b])$. But $u(t) = v'(t) = S(t)x$ is not continuously differentiable.

If A is an operator satisfying condition (4.1), then A generates an $(n+1)$-times integrated semigroup on E and an n-times integrated semigroup on $F = \overline{D(A)}$ So if $F \neq E$ one may improve Proposition 5.4. We merely consider the case $n = 0$.

Proposition 5.5. *Let A be a linear operator such that $(w, \infty) \subset \rho(A)$ for some $w \in \mathbf{R}$ and*

$$\sup\{\ \|\ [(w-\lambda)R(\lambda,A)]^n\ \|: \lambda>0, n=0,1,2,\ldots\}<\infty. \tag{5.5}$$

Let $f\in C^2[0,b]$. If $u_0\in D(A)$ and

$$Au_0+f(0)\in\overline{D(A)}, \tag{5.6}$$

then the problem $P(u_0,f)$ has a unique solution.

PROOF. We have to show that $v\in C^2$, where v is given by (6.1). By (3.10) we have

$$v'(t)=S(t)Au_0+u_0+S(t)f(0)+\int_0^t S(s)f'(t-s)ds.$$

Since A generates a C_0-semigroup on $\overline{D(A)}$, $S(\,.\,)(Au_0+f(0))$ is continuously differentiable. Consequently, $v'\in C^1$. □

REMARK. (a) If u is a solution of $P(u_0,f)$, then

$$Au_0+f(0)=u'(0)=\lim_{t\downarrow 0}t^{-1}(u(t)-u(0))\in\overline{D(A)}.$$

So condition (5.6) is also necessary.

(b) Proposition 5.5 has been proved by Da Prato–Sinestrari [5] with different methods.

(c) If A is an operator satisfying $\rho(A)\neq\emptyset$ such that the homogeneous probem has a unique solution for every $u_0\in D(A)$ (and not only those $u_0\in D(A)$ satisfying $Au_0\in\overline{D(A)}$), then $D(A)$ is dense (see [20, A–II, Corollary 1.2]).

6. The Hille–Yosida theorem on spaces with Radon–Nikodým property

Let A be a not densely defined operator satisfying the Hille–Yosida condition

$$\|\ (\lambda R(\lambda,A))^m\ \|\leqq M \qquad (\lambda>a, m=0,1,2,\ldots) \tag{6.1}$$

where we assume that $0\leqq a$, $(a,\infty)\subset\rho(A)$. Then by Theorem 4.1 A is the generator of a once integrated semigroup $(S(t))_{t\geqq 0}$. The part A_F of A in $F:=\overline{D(A)}$ is the generator of a C_0-semigroup $(T(t))_{t\geqq 0}$ on F. For $x\in F$ we have $S(t)x=\int_0^t T(s)xds$ $(t\geqq 0)$. However, for $x\in E\setminus F$ the function $S(\,.\,)x$ is not differentiable at 0 [in fact, if $(d/dt)_{|t=0}\ S(t)x$ exists, then by (3.13), $x=(d/dt)_{|t=0}S(t)x\in\overline{D(A)}$]. It can even happen, that $S(\,.\,)x$ is not differentiable at any $t\geqq 0$ for all $x\in E\setminus F$.

EXAMPLE 6.1. Let $E = C_0(-\infty, 0]$ and A be given by

$$D(A) = \{f \in E \cap C^1(-\infty, 0]: f(0) = 0\}, \qquad Af = f'.$$

Then $(0, \infty) \subset \rho(A)$ and

$$(R(\lambda, A)f)(x) = e^{\lambda x} \int_x^0 e^{-\lambda s} f(s)ds \qquad (\lambda > 0).$$

Hence $\| \lambda R(\lambda, R) \| \leqq 1$ $(\lambda > 0)$. It is easy to see that A generates the integrated semigroup $(S(t))_{t \geqq 0}$ given by

$$(S(t)f)(x) = \begin{cases} \int_x^{x+t} f(s)ds & \text{for } t \leqq -x, \\ \int_x^0 f(s)ds & \text{for } t > -x. \end{cases}$$

Hence for $t > 0$ we have

$$(d/ds)_{|s=t}(S(s)f)(x) = \begin{cases} f(x+t) & \text{for } x \leqq -t, \\ 0 & \text{for } x > -t, \end{cases}$$

This is a continuous function in x only if $f(0) = 0$. Hence for any $t \geqq 0$, $S(\,.\,)f$ is differentiable at t if and only if $f \in \overline{D(A)} = C_0(-\infty, 0)$.

However, if E has the Radon–Nikodým property, then $S(\,.\,)x$ is differentiable at every $t > 0$ for all $x \in E$. In fact, the following holds.

THEOREM 6.2. *Let E be a Banach space with Radon–Nikodým property. Let A be a (not densely defined) operator on E such that $(a, \infty) \subset \rho(A)$ for some $a \geqq 0$ and such that the Hille–Yosida condition* (6.1) *holds.*

Then there exists a semigroup $(T(t))_{t>0}$, which is strongly continuous for $t > 0$ and satisfies $\| T(t) \| \leqq M$ $(t > 0)$ such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)dt \qquad (\lambda > a). \tag{6.2}$$

Moreover, the once integrated semigroup $(S(t))_{t \geqq 0}$ generated by A is given by

$$S(t) = \int_0^t T(s)ds. \tag{6.3}$$

REMARKS. (a) The integrals (6.2) and (6.3) have to be understood strongly, as Bochner integrals or equivalently Riemann integrals which are improper in 0 and in ∞ in (6.2) and in 0 in (6.3).

(b) We have $\lim_{t\downarrow 0} T(t)x = x$ if and only if $x \in \overline{D(A)}$. In fact, if $\lim_{t\downarrow 0} T(t)x = x$, then $(d/dt)_{|t=0}\, S(t)x = x$ which implies $x \in \overline{D(A)}$ (see above). The converse is clear since $(T(t)_{|F})_{t\geqq 0}$ is a C_0-semigroup on $F = \overline{D(A)}$.

For the proof we need the following lemma.

LEMMA 6.3. *Let $a > 0$ and $N \subset (0, a]$ be a set of measure zero such that $s, t \notin N$, $s + t \leqq a$ imply $s + t \notin N$. Then $N = \emptyset$.*

PROOF. Suppose that there exists $b \in N$. Without loss of generality we can assume that $b = a$. Let $x \in (0, a] \setminus N$. Then $a - x \in N$ by assumption. Consequently $(0, a] \setminus N \subset a - N$. Hence $(0, a] = (0, a] \setminus N \cup N \subset (a - N) \cup N$. This is impossible since $(a - N) \cup N$ has measure 0. □

PROOF OF THEOREM 6.2. By Theorem 4.1 there exists a strongly continuous integrated semigroup $(S(t))_{t\geqq 0}$ satisfying

$$\| S(t) - S(s) \| \leqq M \,|\, t - s \,|$$

$$\text{such that } R(\lambda, A) = \int_0^\infty \lambda e^{-\lambda t} S(t)dt \quad (\lambda > w).$$

We show that $S(t)x$ is continuously differentiable at every $t > 0$ for every $x \in E$.

Let $x \in E$. Consider $E_0 := \overline{\operatorname{span}}\{S(t)x : t \geqq 0\}$. Then E_0 is separable. Let $\{x_n : n \in \mathbf{N}\}$ be total in E_0. Since E_0 has the Radon–Nikodým property, the set

$$N := \{t \in [0, \infty) : S(\,.\,)x_n \text{ is not differentiable at } t \text{ for some } n \in \mathbf{N}\}$$

has measure zero. Since the difference quotients of $S(t)$ are uniformly bounded, a 3ε-argument shows that $[0, \infty) \subset N = \{t \in [0, \infty) : S(\,.\,)y$ is differentiable at t for all $y \in E_0\}$.

For $t \in N$ we let $T_0(t)y = (d/dr)_{|r=t} S(r)y$ $(y \in E_0)$. Clearly, $y \to T_0(t)y$ defines a linear operator $T_0(t)$ on E_0 satisfying $\| T_0(t) \| \leqq M$ $(t \notin N)$. We claim that

(6.4) $\quad s, t \notin N$ implies $s + t \notin N$ and $T_0(s + t)y = T_0(s)T_0(t)y \quad (y \in E.)$.

In fact, $S(s)S(t)y = \int_0^{s+t} S(r)y\,dr - \int_0^s S(r)dr - \int_0^t S(r)dr$ $(y \in E_0)$ by Theorem 3.1. Differentiating with respect to t yields

$$S(s)T_0(t)y = S(s + t)y \qquad (t \notin N, y \in E_0).$$

Consequently, for $t \notin N$, $S(r+t)y = S(r)T_0(t)y + S(t)y$ is differentiable at $r = s$ whenever $s \notin N$ and $(d/dr)_{|r=s+t}S(r)y = T_0(s)T_0(t)y\,(t, s \notin N, y \in E_0)$. We have shown (6.4) to hold. It follows from the preceding lemma that $N \subset \{0\}$.

Since $x \in E$ was arbitrary, we now know that $S(t)x$ is differentiable at all $t > 0$ for all $x \in E$ and $T(t)x := (d/dr)_{|r=t}S(r)x$ defines a semigroup $(T(t))_{t>0}$ on E. Moreover, $T(\,.\,)x$ is weakly measurable and separable-valued $(x \in E)$. Hence $(T(t))_{t>0}$ is strongly continuous at $t > 0$ by [11, 10.2.3]. Furthermore, $\| T(t) \| \leqq M\ (t > 0)$ and $S(t)x = \int_0^t T(s)xds\ (x \in E,\ t \geqq 0)$. Thus

$$R(\lambda, A)x = \int_0^\infty \lambda e^{-\lambda t}S(t)xdt = \int_0^\infty e^{-\lambda t}T(t)xdt$$

(by integration by parts). □

We point out that on a reflexive space every operator satisfying the Hille–Yosida condition (6.1) automatically has dense domain (see e.g. [29, VII.4 Corollary 1′ (p. 218)]). On spaces with Radon–Nikodým property this is no longer true, in general. The following example is due to H. P. Lotz.

Example 6.4. The James space J consists of all sequences $x = (x_k)_{k \in \mathbf{N}}$ in c_0 for which there exists a constant $d \geqq 0$ such that

$$(6.5) \qquad (|x_{p_1} - x_{p_2}|^2 + |x_{p_2} - x_{p_3}|^2 + \cdots + |x_{p_n} - x_{p_1}|^2)^{1/2} \leqq d$$

for all $n \in \mathbf{N}$ and all natural numbers $1 \leqq p_1 < p_2 < \cdots < p_n$. It is a Banach space if the norm $\| x \|$ is defined as the infimum of all constants d such that (6.5) holds (see [15, Example 1.d.2 (p. 25)]).

Let $E = J + \mathbf{R}e \subset c$, where c denotes the space of all convergent sequences and e the constant-1 sequence. Then E, with respect to the product norm, is a Banach space with the Radon–Nikodým property.

Define A on E by $(Ax)_k = -kx_k\ (k \in \mathbf{N})$ with maximal domain; i.e., $D(A) = \{x \in E: (-kx_k)_{k \in \mathbf{N}} \in E\}$. Then $(0, \infty) \subset \rho(A)$, the Hille–Yosida condition (6.1) is satisfied, but $D(A)$ is not dense.

Proof. Since the space of all finite sequences is dense in J, the space E is separable. Moreover, E is isomorphic to J'' (see [15]), and so E has the Radon–Nikodým property as a separable dual space.

It is obvious from the definition that $l^2 \subset J$ and $\| x \| \leqq 2 \| x \|_2$ for all $x \in l^2$.

We define an auxiliary operator A_c on c given by $(A_c x)_k = -kx_k$ with domain $D(A_c) = \{x \in c: \lim_{k \to \infty} -kx_k$ exists). Then $(-1, \infty) \subset \rho(A_c)$ and

$(R(\lambda, A_c)x)_k = (\lambda + k)^{-1}x_k$ $(k \in \mathbf{N}, \lambda > -1)$. Consequently, $R(\lambda, A_c)E \subset l^2 \subset J$ $(\lambda > -1)$. The operator A is the part [22, Definition 10.3] of A_c in E. Since $R(\lambda, A_c)$ leaves E invariant, it follows that $(-1, \infty) \subset \rho(A)$ and $R(\lambda, A) = R(\lambda, A_c)_{|E}$ $(\lambda > -1)$. Similarly, let A_J be the part of A_c in J. Then $(-1, \infty) \subset \rho(A_J)$ and $R(\lambda, A_J) = R(\lambda, A_c)_{|J}$ $(\lambda > -1)$. We show that A_J is the generator of a bounded C_0-semigroup on J. This implies

$$\sup\{\, \| (\lambda R(\lambda, A)^n_{|J} \| : \lambda > 0, n = 0, 1, 2, \ldots\} < \infty. \tag{6.6}$$

In fact, for $x \in J$, $t \geqq 0$ let $T_J(t)x = (e^{-tn}x_n)_{n \in \mathbf{N}}$. Then for $t > 0$, $T_j(t)x \in l^2 \subset J$ for all $x \in J$ and $T_j(t)$ is a bounded operator on J (as is easy to see by the closed graph theorem). Let $\phi \in J'$. Then $\phi \in l^2$ and $\sum_{n=1}^{\infty} x_n\phi_n = \langle x, \phi \rangle$ exists for all $x \in J$ (since the unit vectors form a Schauder basis in J). Moreover,

$$\sum_{n=1}^{\infty} e^{-tn}x_n\phi_n = \langle T_J(t)x, \phi \rangle \qquad (t > 0).$$

By Abel's classical theorem it follows that $\lim_{t \to 0} \langle T_J(t)x, \phi \rangle = \langle x, \phi \rangle$. Thus $(T_J(t))_{t \geqq 0}$ is weakly continuous in 0 and so a C_0-semigroup [2, Proposition 1.23].

Since for $\varepsilon > 0$, $\sup_{t \geqq \varepsilon} \| T_J(t)x \| \leqq 2 \sup_{t \geqq \varepsilon} \| T_J(t)x \|_2 < \infty$ for all $x \in J$, it follows from the uniform boundedness principle that $(T_J(t))_{t \geqq 0}$ is bounded.

Let B be the generator of $(T_J(t))_{t \geqq 0}$. Let $x \in J$. Then for $\lambda > 0$ one has

$$(R(\lambda, B)x)_k = \int_0^{\infty} e^{-\lambda t}e^{-kt}x_k dt = (\lambda + k)^{-1}x_k = (R(\lambda, A_J)x)_k \qquad (k \in \mathbf{N}).$$

Hence $R(\lambda, B) = R(\lambda, A_J)$. This implies that $B = A_J$.

In view of (6.6) it remains to show that $\sup\{\, \| (\lambda R(\lambda, A))^n e \| : \lambda > 0, n = 0, 1, 2, \ldots\} < \infty$ in order to obtain (6.5).

It follows immediately from the definition of J that every decreasing sequence $x \in c_0$ is in J and $\| x \| \leqq 2x_1$. Hence

$$\| (\lambda R(\lambda, A))^n e \| = \| (\lambda/(\lambda + k))^n_{k \in \mathbf{N}} \| \leqq 2.$$

Finally, since $D(A) = R(1, A)E \subset l^2 \subset J$ and $e \notin \bar{J} = J$, the domain of A is not dense in E. □

Remark. It is not difficult to verify Theorem 6.2 in the concrete example. Of course, here $T(t)$ is given by $T(t)x = (e^{-tn}x_n)_{n \in \mathbf{N}}$ for all $x \in E$.

7. Concluding remarks

The technique developed here can also be used for the Cauchy problem of second order. In fact, it is not difficult to show that an operator A is the generator of a cosine function ([8], [10, Chap. 2, Sec. 7, 8] if and only if $(a, \infty) \subset \rho(A)$ for some $a \geqq 0$ and the function $\lambda \to R(\lambda^2, A)/\lambda: (a^{1/2}, \infty) \to \mathscr{L}(E)$ is a Laplace transform. By similar arguments as in the proof of Theorem 4.3 one obtains the characterization theorem due to Sova (see [8] or [10, 8.3]) from Corollary 1.2.

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