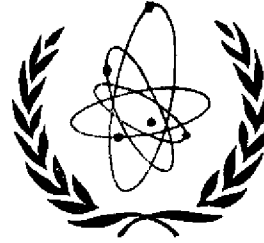




REFERENCE

IC/66/90



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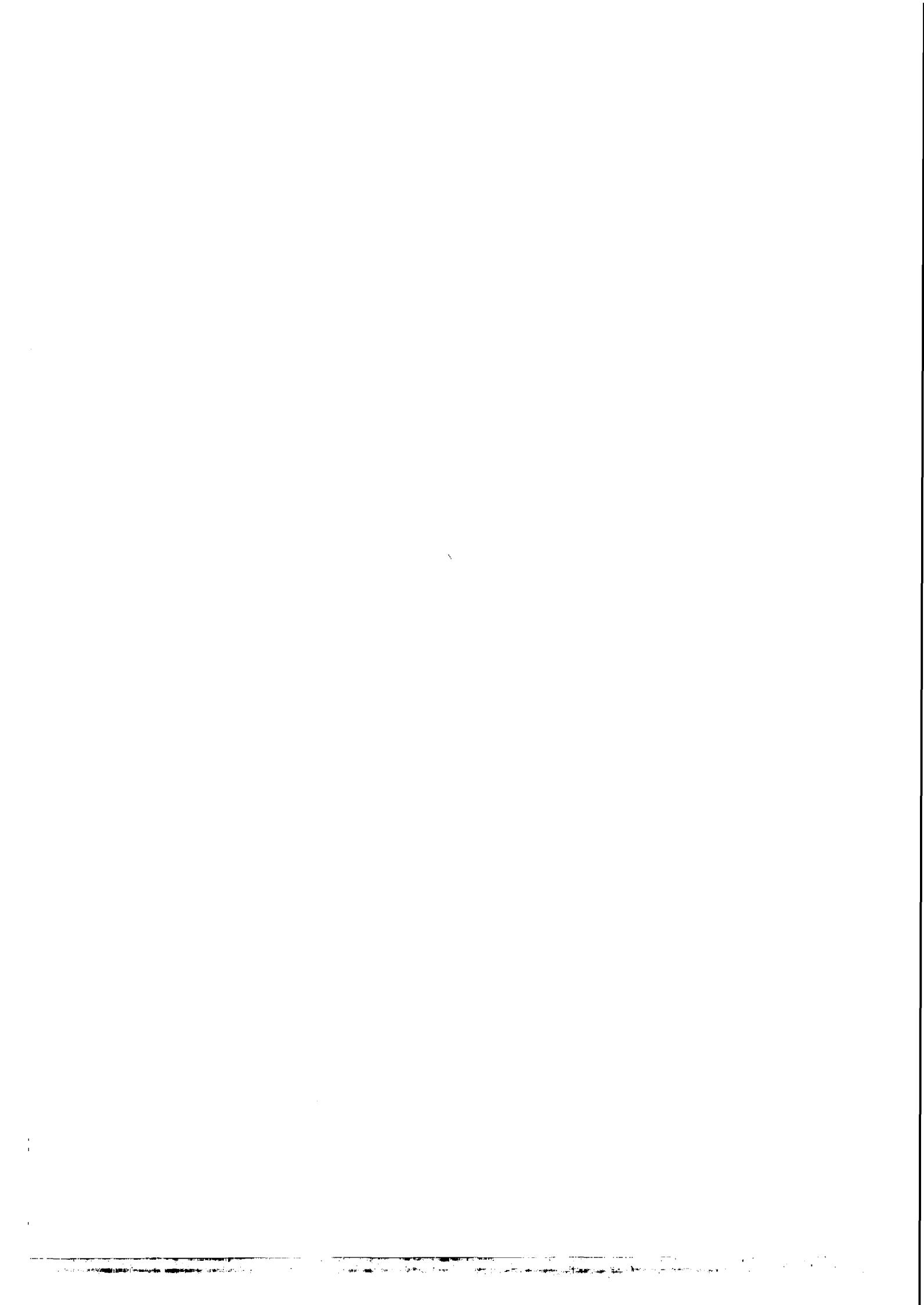
VELOCITY SPACE DIFFUSION  
FROM WEAK PLASMA TURBULENCE  
IN A MAGNETIC FIELD

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AND  
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1966

PIAZZA OBERDAN

TRIESTE



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(This is a revised version of IC/66/17 issued in March 1966)

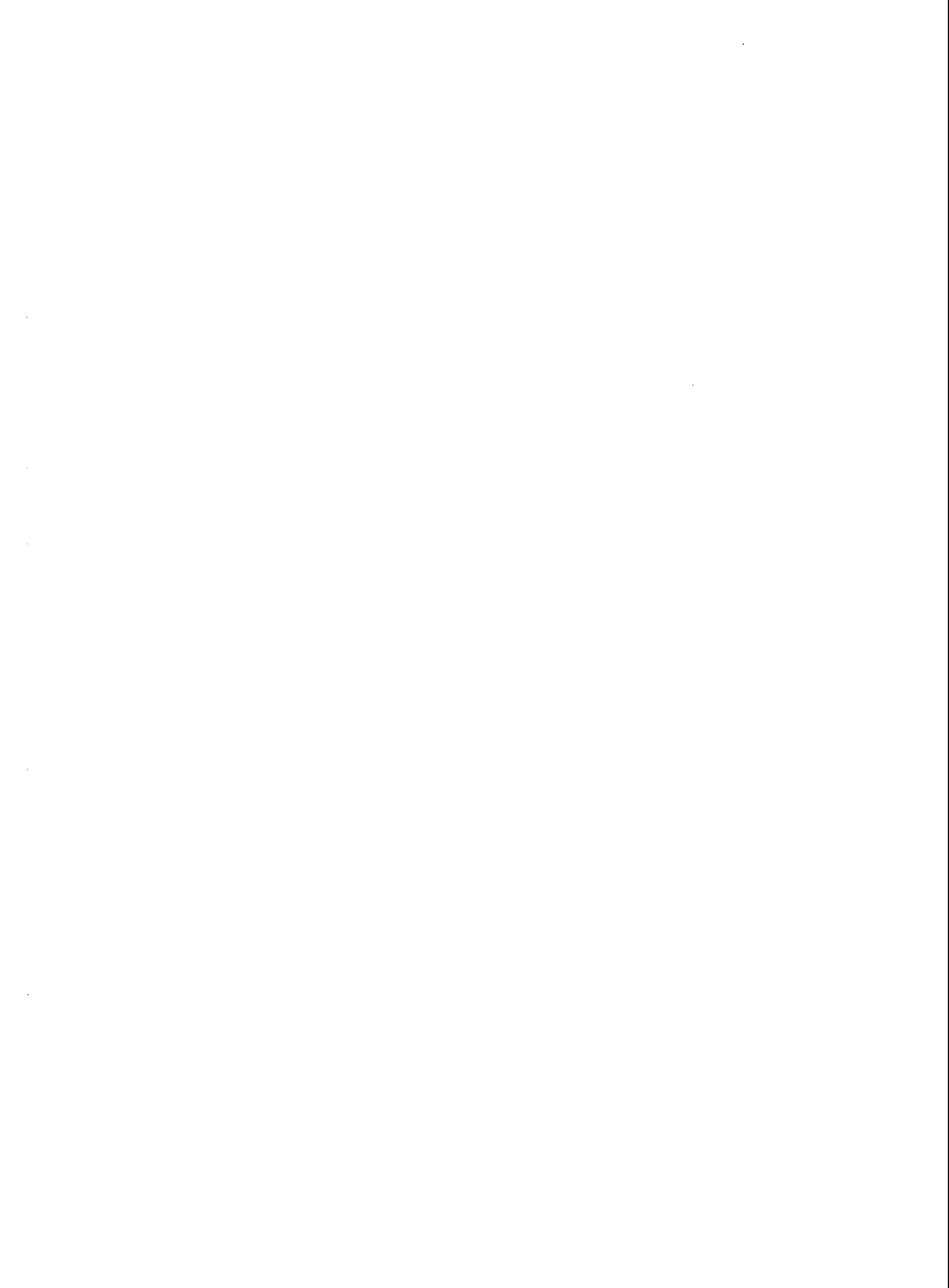
TRIESTE

July 1966

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## ABSTRACT

We consider quasi-linear velocity space diffusion for waves of any oscillation branch propagating at an arbitrary angle to a uniform magnetic field in a spatially uniform plasma. The space-averaged distribution function is assumed to change slowly compared to a gyroperiod and characteristic times of the wave motion. We neglect non-linear mode coupling. An H-like theorem shows that both resonant and non-resonant quasi-linear diffusion forces the particle distributions towards marginal stability. Creation of the marginally stable state in the presence of a sufficiently broad wave spectrum in general involves diffusing particles to infinite energies, and so the marginally stable plateau is not accessible physically, except in special cases.

Resonant particles with velocities much larger than typical phase velocities in the excited spectrum are scattered primarily in pitch angle about the magnetic field. Only particles with velocities the order of the wave phase velocities or less are scattered in energy at a rate comparable with their pitch angle scattering rate.

# VELOCITY SPACE DIFFUSION FROM WEAK PLASMA TURBULENCE IN A MAGNETIC FIELD

## 1. INTRODUCTION

The theory of weak plasma turbulence has recently undergone considerable development<sup>1-13</sup>. Much of this extensive development clarified the formal nature of the theory. The present paper makes no contribution to the basic understanding of quasi-linear theory, but attempts to draw general conclusions assuming the correctness of the present theoretical approach with the aim of possibly being useful in the interpretation of experimental phenomena. Therefore, we remove some restrictions which were unimportant in investigations of the formal structure of weak turbulence theory but which greatly simplified the algebraic manipulations. These restrictions involve mainly the nature of the waves considered, and are not basic mathematical restrictions inherent in the weak turbulence approximation.

One such limitation was the usual but by no means universal consideration of turbulence only in electrostatic wave modes. Because of the very low particle energy densities (relative to magnetic) commonly encountered in laboratory devices, electrostatic waves may play the dominant role in the laboratory. However, the investigations of the Earth's immediate environment in space using artificial Earth satellites have created a new class of plasma observations in which electromagnetic waves such as whistlers and ion cyclotron waves most certainly play a role. Thus we consider electromagnetic as well as electrostatic wave turbulence.

In addition, special propagation directions, in general parallel to the magnetic field, were usually chosen for analytic simplicity. Clearly, waves propagating at an angle to the magnetic field will be present in actual experiments. The removal of the above two restrictions are the basic effects we investigate.

The basic plasma is assumed non-relativistic, collisionless, infinite and spatially uniform in the absence of wave excitation, and immersed in a static magnetic field of constant magnitude with no curvature. We do not restrict the excitation spectrum, except to require that wave amplitudes be small. However, we investigate only the form of the lowest order turbulent velocity diffusion, disregarding

the direct non-linear coupling between waves. This may enter essentially into the determination of the spectrum, and, as previous work<sup>7,11</sup> indicates, it can be very important for how the system develops and finally is stabilized.

There are physical systems for which non-linear mode couplings can be neglected. For instance, convectively unstable waves may propagate out of finite systems before mode-mode couplings can get effective. This is the case for the whistler turbulence in the magnetosphere<sup>14</sup>. Another example is provided by situations where resonant mode coupling vanishes as in the case of Alfvén waves propagating parallel to the external magnetic field. The diffusion equation derived here will adequately describe the evolution of the distribution function when the excitation spectrum is known. This corresponds to a complete solution of the problem, if the motion of the waves can be described in linear approximation; otherwise it forms one part of a more complex picture.

In order to derive the quasi-linear diffusion equation (Section 2), it is necessary to assume that the distribution function averaged over space changes slowly in the time scales associated with the motion of the waves. For magnetic fields so strong that the gyration frequencies of the particles are larger than the frequencies and growth rates of the waves, this implies that there is no strong dependence of the averaged distribution function on the azimuth of velocity around the magnetic field. Only this case will be considered in the sequel. It comprises, on the one hand, all situations in which the spectrum of the excited modes is likewise approximately axially symmetric. On the other hand, it also contains cases of strongly anisotropic spectra, provided that the time variation of the averaged distribution is small during one particle gyration. This requirement is similar to that used in Chew-Goldberger-Low theory.

We leave the diffusion equation in terms of the polarizations of the waves in the excited spectrum. Finding the appropriate expression for a given turbulent mode requires a solution of the linear dispersion relation to find the polarizations. However, the general diffusion equation for arbitrary polarizations is remarkably simple. In particular, the strings of Bessel functions, which usually plague the analysis for waves propagating at an angle to the magnetic field,

have been reduced to a positive definite form.

Interestingly enough, this apparently indiscriminating approach leads to several interesting conclusions, which are, albeit of a qualitative nature, quite general because both the diffusion operator and the diffusion coefficient, when expressed in terms of the electric field polarization amplitudes, do have simple forms. In Section 3, we use an H-theorem to demonstrate that unstable plasma waves force the particles' velocity distributions to a marginally stable state. Since we do not specify the magnitude of the growth rate, this is true for both resonant and non-resonant instabilities.

In Section 4, we discuss the resonant limit, where the growth rates are all taken small. Here the diffusion is even simpler, being characterized by a single velocity space operator, for a given wave number, throughout velocity space. The characteristics of this operator are just the orbits of single particles interacting with single waves. Qualitative arguments based on our knowledge of these characteristics suggest that reconstruction of the particles' initial velocity distributions in the form of a "plateau", ordinarily requires that particles diffuse from finite to infinite velocities, and that the rigorous plateau defined by the H-theorem is therefore not physically accessible, except in special circumstances.

In Section 5, we investigate the conditions under which an arbitrary turbulent wave spectrum scatters particles in pitch angle relative to the magnetic field and when it scatters particles in energy. Particles with velocities much larger than typical phase velocities in the excited spectrum are scattered primarily in pitch angles, whereas only particles with velocities of the order of phase wave velocities or less are energized. At the same rate they are scattered in pitch angle. We also suggest that quasi-plateaus, characterized by slow diffusion rates, may exist.

Because it is difficult to draw rigorous conclusions at this level of generality, many of our results are qualitative. However, we hope these may serve as a guideline for future more specific theoretical work and, possibly, for the interpretation of experiments.



## 2. DERIVATION OF THE QUASI-LINEAR DIFFUSION EQUATION

We outline only briefly the relatively standard derivation of the quasi-linear diffusion equation. The Vlasov equations describe the evolution of the one-particle distributions  $f^\pm(\underline{x}, \underline{v}, t)$  for a two-component plasma of electrons and singly charged ions, denoted by - and +, respectively:

$$\frac{\partial f^\pm}{\partial t} + \underline{v} \cdot \nabla f^\pm + \frac{e}{M_\pm} \left[ \underline{E} + \frac{\underline{v} \times \underline{B}}{c} \right] \cdot \frac{\partial f^\pm}{\partial \underline{v}} = 0. \quad (2.1)$$

$\underline{x}$  and  $\underline{v}$  denote configuration and velocity space coordinates, respectively, and  $t$  the time,  $e$  is the elementary charge,  $M_\pm$  the mass either of ions (+) or electrons (-), and  $c$  the speed of light.  $\underline{E}(\underline{x}, t)$  and  $\underline{B}(\underline{x}, t)$  are the electric and magnetic field vectors. Maxwell's equations using the current and electric charge moments of the particle velocity distributions as source terms complete the set of equations. The generalization to a plasma consisting of more than two components or with multiply charged ions is straightforward.

Quasi-linear theory separates  $f^\pm$ ,  $\underline{E}$  and  $\underline{B}$  into space independent components and small, rapidly fluctuating parts due to waves. Denoting the fluctuating components of the distribution function by  $\delta f^\pm$  with  $\int d^3x \delta f^\pm = 0$  and Fourier-analysing in space, we obtain

$$f^\pm(\underline{x}, \underline{v}, t) = g^\pm(\underline{v}, t) + \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \delta f_{\mathbf{k}}^\pm(\underline{v}, t) \quad (2.2a)$$

and analogously,

$$\underline{B}(\underline{x}, t) = B_0 \underline{e}_z + \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \underline{B}_{\mathbf{k}}(t), \quad (2.2b)$$

$$\underline{E}(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \underline{E}_{\mathbf{k}}(t).$$

$g^\pm$  is explicitly defined by  $\lim_{V \rightarrow \infty} \frac{1}{V} \int d^3x f^\pm(\underline{x}, \underline{v}, t)$  where  $V$  is the volume of integration. The plasma under consideration is infinite, spatially homogeneous in the absence of wave excitation, and immersed in a uniform static magnetic field  $B_0$  aligned along the  $z$ -axis of a Cartesian coordinate system. There is no static electric

field. The condition that the time dependence of  $f^{\pm}$ ,  $\underline{E}$  and  $\underline{B}$  split up into a rapid variation due to waves, and a slow variation due to the reaction of the wave distribution back on the space-averaged velocity distribution is that all wave amplitudes remain small.

We first discuss the linear fast time scale properties of  $\delta f_{\underline{k}}^{\pm}$ , depending upon the instantaneous value of  $g^{\pm}(\underline{v}, t)$ , and then derive the quasi-linear diffusion equation for the slow evolution of  $g^{\pm}$ . It is convenient to transform to cylindrical coordinate systems in velocity and wave number space, using the magnetic field direction as the axis:

$$\begin{aligned} v_x &= v_{\perp} \cos \phi, & k_x &= k_{\perp} \cos \psi, \\ v_y &= v_{\perp} \sin \phi, & k_y &= k_{\perp} \sin \psi, \\ v_z &= v_{\parallel}, & k_z &= k_{\parallel}. \end{aligned} \quad (2.3)$$

Since there are only two basic directions for the linear wave problem, those of the static magnetic field,  $B_0 \underline{e}_z$ , and of the wave vector,  $\underline{k}$ , the dispersion properties must be independent of  $\psi$ . However, due, for instance, to unsymmetric initial conditions or wave sources, the wave distribution may be unsymmetrically distributed about the magnetic field direction.

Because the fast oscillation frequencies can be of the order of the gyro-frequencies, it is useful to express the wave electromagnetic field component perpendicular to  $B_0$  in terms of complex polarization vectors denoting right and left-handed rotation ( $r, l$ ) about  $B_0$ , and of plane components parallel ( $\parallel$ ) to  $B_0$ , since in the limit of parallel wave propagation the linear eigenmodes involve pure right-hand, left-hand, or parallel components alone:

$$\begin{aligned} \underline{E}_{\underline{k}}^r &= \left( \frac{E_x + iE_y}{\sqrt{2}} \right)_{\underline{k}}, & \underline{B}_{\underline{k}}^l &= \left( \frac{B_x + iB_y}{\sqrt{2}} \right)_{\underline{k}}, \\ \underline{E}_{\underline{k}}^l &= \left( \frac{E_x - iE_y}{\sqrt{2}} \right)_{\underline{k}}, & \underline{B}_{\underline{k}}^r &= \left( \frac{B_x - iB_y}{\sqrt{2}} \right)_{\underline{k}}, \\ \underline{E}_{\underline{k}}^{\parallel} &= (E_z)_{\underline{k}}, & \underline{B}_{\underline{k}}^{\parallel} &= (B_z)_{\underline{k}}. \end{aligned} \quad (2.4)$$

We define  $\epsilon_{\underline{k}}^l$  and  $\epsilon_{\underline{k}}^r$  so that they respectively represent left-hand and right-hand components, for positive real part of the frequency. If the real part of the frequency is negative, the polarization properties are reversed. Because of the axial symmetry of the wave properties,  $\epsilon_{\underline{k}}^l$  and  $\epsilon_{\underline{k}}^r$  can be simply rotated by the angle  $\psi$  and the basic eigenvector  $\underline{E}_{\underline{k}}$  for a single wave  $(k_{\perp}, k_{\parallel}, \psi)$  has the form

$$\underline{E}_{\underline{k}} = \begin{pmatrix} \epsilon_{\underline{k}}^e e^{-i\psi} \\ \epsilon_{\underline{k}}^r e^{+i\psi} \\ \epsilon_{\underline{k}}^H \end{pmatrix} \quad (2.5)$$

In this way the eigenvectors and dispersion relation for  $\psi \neq 0$  can be obtained directly from the two-dimensional case,  $\psi = 0$ , usually treated.

The Vlasov equation (2.1), after Fourier analysis and transformation to cylindrical velocity and wave number space coordinate systems, may be symbolically written as follows:

$$\hat{L}_{\underline{k}}^{\pm} \delta f_{\underline{k}}^{\pm} = \hat{P}_{\underline{k}}^{\pm} q^{\pm} + \frac{1}{2} \int \frac{d^3 k'}{(2\pi)^3} \left( \hat{P}_{\underline{k}-\underline{k}'}^{\pm} \delta f_{\underline{k}'}^{\pm} + \hat{P}_{\underline{k}}^{\pm} \delta f_{\underline{k}-\underline{k}'}^{\pm} \right)$$

where  $\hat{L}_{\underline{k}}^{\pm}$  and  $\hat{P}_{\underline{k}}^{\pm}$  denote the operators

$$\hat{L}_{\underline{k}}^{\pm} = \frac{\partial}{\partial t} + i \left( k_{\parallel} v_{\parallel} + k_{\perp} v_{\perp} \cos(\phi - \psi) \right) - \Omega_{\pm} \frac{\partial}{\partial \phi}$$

(2.7)

$$\hat{P}_{\underline{k}}^{\pm} = \mp \frac{e}{m_{\pm}} \left\{ \underline{E}_{\underline{k}} + \frac{\underline{v} \times \underline{B}_{\underline{k}}}{c} \right\} \cdot \frac{\partial}{\partial \underline{v}}$$

$\Omega_{\pm} = \pm e B_0 / m_{\pm} c$  is the gyrofrequency for each species. Note that  $\Omega_{\pm}$  contains the sign of the charge.

The convolution term in (2.6), which has been explicitly symmetrized, is formally second order in the wave amplitude and represents non-linear mode-mode couplings, which will be neglected in this analysis. We solve now for the linear waves on the fast time scale by treating  $q^{\pm}$  as constant in time, a procedure formally identical to that in ordinary linear theory. Then  $\partial/\partial t$  may be replaced by  $-i\nu_{\underline{k}}$ , where  $\nu_{\underline{k}}$  is the complex wave frequency,  $\nu_{\underline{k}} = \omega_{\underline{k}} + i\gamma_{\underline{k}}$ , and  $\omega_{\underline{k}}$  and  $\gamma_{\underline{k}}$

are both real.  $\hat{L}_{\underline{k}}^{\pm}$  becomes, to lowest order

$$\hat{L}_{\underline{k}, \underline{v}_k}^{\pm} = -i \left( \lambda_{\underline{k}} - k_{\perp} v_{\perp} \cos(\phi - \psi) \right) - \Omega_{\pm} \frac{\partial}{\partial \phi}$$

$$= -\Omega_{\pm} \exp \left\{ \frac{-i(\tau_{\underline{k}} \phi - k_{\perp} v_{\perp} \sin(\phi - \psi))}{\Omega_{\pm}} \right\} \frac{\partial}{\partial \phi} \left[ \exp \left\{ + \frac{i(\tau_{\underline{k}} \phi - k_{\perp} v_{\perp} \sin(\phi - \psi))}{\Omega_{\pm}} \right\} \right] \quad (2.8)$$

with  $\lambda_{\underline{k}} = \underline{v}_k - k_{\parallel} v_{\parallel}$ .

The relation between  $\underline{v}_k$  and  $\underline{k}$  must be found from the linear dispersion relation for the wave mode of interest. The condition that all physical quantities be real will then lead to the relation  $\underline{v}_k = -\underline{v}_{-k}^*$ .

Inverting (2.6), we find, to lowest order,

$$\delta f_{\underline{k}}^{\pm} = \left( \hat{L}_{\underline{k}, \underline{v}_k}^{\pm} \right)^{-1} \hat{P}_{\underline{k}}^{\pm} g^{\pm} = -\frac{1}{\Omega_{\pm}} \exp \left\{ \frac{-i(\tau_{\underline{k}} \phi - k_{\perp} v_{\perp} \sin(\phi - \psi))}{\Omega_{\pm}} \right\} \int d\phi' \exp \left\{ \frac{i(\tau_{\underline{k}} \phi' - k_{\perp} v_{\perp} \sin(\phi' - \psi))}{\Omega_{\pm}} \right\} \cdot \hat{P}_{\underline{k}}^{\pm} g^{\pm} \quad (2.9)$$

where  $\left( \hat{L}_{\underline{k}, \underline{v}_k}^{\pm} \right)^{-1}$  is easily found from the second form of Eq. (2.8).  $\underline{v}_k$  is supposed to have at least a small positive imaginary part which ensures the convergence of  $\left( \hat{L}_{\underline{k}, \underline{v}_k}^{\pm} \right)^{-1}$ , and causal solutions of the linearized Vlasov equation. This means that only unstable ( $\gamma_{\underline{k}} > 0$ ) modes are explicitly considered, though damped modes could be included by proper analytic continuation of the results for  $\gamma_{\underline{k}} < 0$ . (See also I.B. BERNSTEIN AND F. ENGELMANN<sup>13</sup>). The Bessel function identity

$$\exp \left\{ i \frac{k_{\perp} v_{\perp}}{\Omega_{\pm}} \sin(\phi - \psi) \right\} = \sum_{n=-\infty}^{n=+\infty} J_n \left( \frac{k_{\perp} v_{\perp}}{\Omega_{\pm}} \right) \exp \left\{ i n (\phi - \psi) \right\} \quad (2.10)$$

further simplifies  $\left( \hat{L}_{\underline{k}, \underline{v}_k}^{\pm} \right)^{-1}$  to

$$\left( \hat{L}_{\underline{k}, \underline{v}_k}^{\pm} \right)^{-1} = \sum_{n,m} \frac{J_n J_m}{\Omega_{\pm}} \exp \left\{ i(n-m)\psi - i \frac{(\tau_{\underline{k}} - m\Omega_{\pm})\phi}{\Omega_{\pm}} \right\} \cdot \int d\phi' \left[ \exp \left\{ i \frac{(\tau_{\underline{k}} - n\Omega_{\pm})\phi'}{\Omega_{\pm}} \right\} \right] \quad (2.11)$$

$J_n$  is an ordinary Bessel function of integral order, with argument  $\frac{k_{\perp} v_{\perp}}{\Omega_{\pm}}$ . Henceforth the arguments will be suppressed; which particle species is being referred to will usually be clear from the context.

Using Faraday's Law to express wave magnetic fields in terms of wave electric fields, and considering only the fast time scale, where  $\partial/\partial t \rightarrow -i\nu_{\mathbf{k}}$ , we arrive at the following form for  $\hat{P}_{\mathbf{k}}^{\pm}$

$$\hat{P}_{\mathbf{k}}^{\pm} = \begin{cases} \mp \frac{e}{M_{\pm}} e^{i(\phi-\psi)} \left\{ \frac{\epsilon_{\mathbf{k}}^r e^{i\psi}}{\sqrt{2}} \left( \hat{G}_{\mathbf{k}} + \frac{i\nu_{\mathbf{k}}}{v_{\mathbf{k}} v_{\perp}} \frac{\partial}{\partial \phi} \right) + \frac{k_{\perp} \epsilon_{\mathbf{k}}^{\parallel}}{2\nu_{\mathbf{k}}} \left( \hat{H} + i \frac{v_{\parallel}}{v_{\perp}} \frac{\partial}{\partial \phi} \right) \right\} \\ \pm \frac{e}{M_{\pm}} e^{-i(\phi-\psi)} \left\{ \frac{\epsilon_{\mathbf{k}}^l e^{-i\psi}}{\sqrt{2}} \left( \hat{G}_{\mathbf{k}} - \frac{i\nu_{\mathbf{k}}}{v_{\mathbf{k}} v_{\perp}} \frac{\partial}{\partial \phi} \right) + \frac{k_{\perp} \epsilon_{\mathbf{k}}^{\parallel}}{2\nu_{\mathbf{k}}} \left( \hat{H} - i \frac{v_{\parallel}}{v_{\perp}} \frac{\partial}{\partial \phi} \right) \right\} \\ \mp \frac{e}{M_{\pm}} \left\{ \frac{i k_{\perp}}{v_{\mathbf{k}}} \left( \frac{\epsilon_{\mathbf{k}}^e e^{-i\psi} - \epsilon_{\mathbf{k}}^r e^{+i\psi}}{\sqrt{2}} \right) \frac{\partial}{\partial \phi} + \epsilon_{\mathbf{k}}^{\parallel} \frac{\partial}{\partial v_{\parallel}} \right\} \end{cases} \quad (2.12)$$

where we have abbreviated the following differential velocity space operators

$$\hat{G}_{\mathbf{k}} = \left( 1 - \frac{k_{\parallel} v_{\parallel}}{v_{\mathbf{k}}} \right) \frac{\partial}{\partial v_{\perp}} + \frac{k_{\perp} v_{\perp}}{v_{\mathbf{k}}} \frac{\partial}{\partial v_{\parallel}} = \frac{\partial}{\partial v_{\perp}} - \frac{k_{\parallel}}{v_{\mathbf{k}}} \hat{H} \quad (2.13)$$

$$\hat{H} = v_{\parallel} \frac{\partial}{\partial v_{\perp}} - v_{\perp} \frac{\partial}{\partial v_{\parallel}}$$

Notice that  $\hat{H}$  is a gradient in pitch angle relative to the magnetic field. Thus, we have solved for  $\delta f_{\mathbf{k}}^{\pm}$  in terms of  $q^{\pm}$ , and the wave polarizations. Now, we find an expression for the rate of change of  $q^{\pm}$ . Writing the space-averaged Vlasov equation, and introducing Fourier transforms, one has

$$\begin{aligned}
\frac{\partial g^\pm}{\partial t} - \Omega_\pm \frac{\partial g^\pm}{\partial \phi} &= \lim_{V \rightarrow \infty} \frac{1}{2V} \int \frac{d^3 k}{(2\pi)^3} \left\{ \hat{P}_{-k}^\pm (\hat{L}_{k, \nu_k}^\pm)^{-1} \hat{P}_{+k}^\pm \right. \\
&\quad \left. + \hat{P}_{+k}^\pm (\hat{L}_{-k, \nu_{-k}}^\pm)^{-1} \hat{P}_{-k}^\pm \right\} g^\pm \\
&= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left\{ \hat{D}_k^\pm + \hat{D}_{-k}^\pm \right\} g^\pm
\end{aligned} \tag{2.14}$$

where we have substituted the linear expression for  $\delta f_{\pm k}^\pm$  in the convolution term.

Observe the formal similarity between Eq. (2.14) above and those for the distribution functions in the so-called Chew-Goldberger-Low, or small Larmor radius approximation<sup>15</sup>. In that theory, all changes are considered small over space scales comparable with particle Larmor radii or time scales comparable with typical gyroperiods. Therefore, the Larmor radius and gyroperiod are convenient small expansion parameters. The Larmor orbiting of particles about the magnetic field is so rapid that all inhomogeneities in the  $\phi$ -distribution of particles smooth out on the macroscopic scales, and the distribution functions are independent of  $\phi$  to lowest order. Here the actual distribution may have rapid variations, due to the waves. However, the space-averaged distribution functions  $g^\pm$  have been assumed to be effectively constant over times of the order of gyroperiods and characteristic wave periods in accordance with the assumed low excitation of waves. Consequently, for consistency, the  $\phi$  dependence of  $g^\pm$  must also be weak and it is reasonable to look for an expansion of  $g^\pm$  in powers of  $1/\Omega_\pm^2$ ,

$$g^\pm = g_0^\pm + \frac{1}{\Omega_\pm^2} g_2^\pm + o\left(\frac{1}{\Omega_\pm^2}\right) \tag{2.15}$$

Substituting Eq. (2.15) into Eq. (2.14) and using the assumption that the wave background creates only a slow time variation in  $g_0^\pm$ , we find to lowest order

$$\frac{\partial g_0^\pm}{\partial \phi} = 0 \tag{2.16}$$

Thus the lowest order spatially averaged distribution function is independent of Larmor phase  $\phi$ . Basically, a particle has time to make many gyro-orbits about the magnetic field before it diffuses a significant amount. By the same reasoning, even if the turbulent interaction between waves and particles were limited to a given localized region of  $\phi$  and  $\psi$ , all particles would gyrate many times through the turbulent interaction region and the turbulence would therefore affect particles with all values of  $\phi$  almost equally. Thus the corresponding diffusion is two-dimensional, since  $\partial g_0^\pm / \partial \phi$  is always zero. The condition that the expansion (2.15) be valid here is that the mean diffusion time for all particles be long relative to the particles' gyroperiods. Hence, in this case the theory is useful in practice only if there are no waves with growth rates much larger than the gyrofrequency, since these reach large amplitudes, causing a considerable  $\phi$ -dependent diffusion, before the particles have a chance to complete one gyro-orbit. On the other hand, when the excited wave spectrum is axially symmetric, the diffusion is two-dimensional in any case and thus preserves an initial axial symmetry of  $g_0^\pm$  for all times.

The time dependence of  $g_0^\pm$  is still undetermined by Eq. (2.16) and we must go to next order to find the variations in  $g_0^\pm$  due to the wave excitation:

$$\frac{1}{\Omega_\pm} \frac{\partial g_0^\pm}{\partial t} - \frac{\partial g_1^\pm}{\partial \phi} = \frac{1}{2\Omega_\pm} \int \frac{d^3 k}{(2\pi)^3} \left\{ \hat{D}_{+\mathbf{k}}^\pm + \hat{D}_{-\mathbf{k}}^\pm \right\} g_0^\pm. \quad (2.17)$$

The physical requirement that all  $g_n^\pm$  be periodic in  $\phi$ , so that averaging from 0 to  $2\pi$  in  $\phi$  annihilates the unwanted higher order term, leads to the following Larmor-phase-averaged equation:

$$\frac{\partial g_0^\pm}{\partial t} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int \frac{d^3 k}{(2\pi)^3} \left\{ \hat{D}_{+\mathbf{k}}^\pm + \hat{D}_{-\mathbf{k}}^\pm \right\} g_0^\pm. \quad (2.18)$$

Henceforth, we work only with the lowest order  $\phi$ -independent part of the distribution, and therefore drop the 0 subscript notation. Then  $\delta f_{\mathbf{k}}^\pm$  simplifies considerably; from Eqs. (2.9), (2.11) and (2.12) we find

$$\delta f_{\underline{k}}^{\pm} = \mp \frac{ie}{m_{\pm}} \sum_{n,m} \frac{J_m e^{i(m-n)(\phi-\psi)}}{\tau_{\pm} - n\Omega_{\pm}} \left\{ \epsilon_{n,\underline{k}}^{\pm} \hat{G}_{\underline{k}} + J_n \epsilon_{\underline{k}}'' \hat{K}_{n,\underline{k}}^{\pm} \right\} \mathcal{F}^{\pm}. \quad (2.19)$$

Here  $\epsilon_{n,\underline{k}}^{\pm}$  is the following composition of wave fields and Bessel functions:

$$\epsilon_{n,\underline{k}}^{\pm} = \frac{\epsilon_{\underline{k}}^r e^{+i\psi} J_{n+1} + \epsilon_{\underline{k}}^l e^{-i\psi} J_{n-1}}{\sqrt{2}} \quad (2.20)$$

and  $\hat{K}_{n,\underline{k}}^{\pm}$  is the following operator:

$$\hat{K}_{n,\underline{k}}^{\pm} = \frac{\partial}{\partial v_{\parallel}} + \frac{n\Omega_{\pm}}{v_{\underline{k}} v_{\perp}} \hat{H} = \frac{v_{\parallel}}{v_{\perp}} \hat{G}_{\underline{k}} - \frac{\tau_{\pm} - n\Omega_{\pm}}{v_{\underline{k}} v_{\perp}} \hat{H} \quad (2.21)$$

where in the second form we have explicitly separated terms with and without the resonance factor  $\tau_{\pm} - n\Omega_{\pm}$ .

Noting that in cylindrical coordinates  $-\underline{k} = (k_{\perp}, -k_{\parallel}, \psi + \pi)$ , and that wave fields are real, so that  $\underline{E}_{-\underline{k}} = \underline{E}_{+\underline{k}}^*$  and  $\underline{B}_{-\underline{k}} = \underline{B}_{+\underline{k}}^*$ , we find

$$\epsilon_{-\underline{k}}^r e^{i(\psi+\pi)} = - (\epsilon_{\underline{k}}^r)^* e^{i\psi} = - (\epsilon_{\underline{k}}^l e^{-i\psi})^*,$$

$$\epsilon_{-\underline{k}}^l e^{-i(\psi+\pi)} = - (\epsilon_{\underline{k}}^r e^{+i\psi})^*, \quad (2.22)$$

$$\epsilon_{-\underline{k}}'' = (\epsilon_{+\underline{k}}'')^*.$$

Using these relations and  $v_{-\underline{k}} = -v_{\underline{k}}^*$ , we find  $\hat{P}_{-\underline{k}}^{\pm} = (\hat{P}_{+\underline{k}}^{\pm})^*$  as well as  $\delta f_{-\underline{k}}^{\pm} = (\delta f_{+\underline{k}}^{\pm})^*$  and  $(\hat{D}_{-\underline{k}}^{\pm}) = (\hat{D}_{+\underline{k}}^{\pm})^*$  so that the quasi-linear diffusion term in Eq. (2.18) is pure real. Since Eq. (2.18) involves an average from 0 to  $2\pi$  in the Larmor phase  $\phi$ , the terms  $i \partial / \partial \phi$  in  $\hat{P}_{-\underline{k}}^{\pm}$  may be integrated by parts; this is equivalent to replacing  $e^{+i\phi} i \partial / \partial \phi$  by  $\pm e^{+i\phi}$  and  $i \partial / \partial \phi$  by 0. The modified operator  $\hat{Q}_{-\underline{k}}^{\pm}$  resulting from these replacements is



$$\hat{Q}_{-k}^{\pm} = \left\{ \begin{array}{l} + \frac{e}{M_{\pm}} e^{i(\phi-\psi)} \left[ \frac{\epsilon_k^+ e^{-i\psi}}{\sqrt{z}} \left( \hat{G}_k + \frac{\Gamma_k}{v_k v_{\perp}} \right) + \frac{k_{\perp} \epsilon_k''}{2v_k} \left( \hat{H} + \frac{v_{\parallel}}{v_{\perp}} \right) \right]^* \\ + \frac{e}{M_{\pm}} e^{-i(\phi-\psi)} \left[ \frac{\epsilon_k^+ e^{+i\psi}}{\sqrt{z}} \left( \hat{G}_k + \frac{\Gamma_k}{v_k v_{\perp}} \right) + \frac{k_{\perp} \epsilon_k''}{2v_k} \left( \hat{H} + \frac{v_{\parallel}}{v_{\perp}} \right) \right]^* \\ + \frac{e}{M_{\pm}} (\epsilon_k'')^* \frac{\partial}{\partial v_{\parallel}} \end{array} \right. \quad (2.23)$$

where all quantities on the r.h.s. above refer to  $+k$ . Since the integral  $\int_0^{2\pi} e^{i(n-m)\phi} d\phi = 2\pi$  only if  $n=m$  and zero otherwise, the double sum of Bessel functions reduces to a single sum and Eq. (2.18) becomes

$$\frac{\partial g^{\pm}}{\partial t} = \left\{ \begin{array}{l} \lim_{V \rightarrow \sigma} \sum_n \frac{e^2}{M_{\pm}^2} \int \frac{d^3 k}{V(2\pi)^3} \cdot \\ \left\{ \left( \hat{G}_k + \frac{\Gamma_k}{v_k v_{\perp}} \right)^* (\epsilon_{n,k}^{\pm})^* + \hat{K}_{n,k}^{\pm} J_n (\epsilon_k'')^* \right\} \\ \frac{i}{v_k - k_{\parallel} v_{\parallel} - n\Omega_{\pm}} \left\{ \epsilon_{n,k}^{\pm} \hat{G}_k + \epsilon_k'' J_n \hat{K}_{n,k}^{\pm} \right\} g^{\pm} \end{array} \right. \quad (2.24)$$

In vector notation this may be written as

$$\frac{\partial g^{\pm}}{\partial t} = \frac{\partial}{\partial \underline{v}} \cdot \left( \underline{\underline{D}}^{\pm} \cdot \frac{\partial g^{\pm}}{\partial \underline{v}} \right) \quad (2.25)$$

with the diffusion tensor  $\underline{\underline{D}}^{\pm}$  defined by

$$\mathbb{D}^{\pm} = \lim_{V \rightarrow \infty} \sum_n \frac{e^2}{m_{\pm}^2} \int \frac{d^3 k}{(2\pi)^3 V} \frac{i}{\gamma_k - k_{\parallel} v_n - n\Omega_{\pm}} (\underline{a}_{n, \underline{k}}^{\pm})^* (\underline{a}_{n, \underline{k}}^{\pm}) \quad (2.26)$$

where

$$\underline{a}_{n, \underline{k}}^{\pm} = \mathcal{E}_{n, \underline{k}}^{\pm} \frac{k_{\parallel}}{v_k} \left[ \left( \frac{v_k}{k_{\parallel}} - v_{\parallel} \right) \underline{e}_{v_{\perp}} + v_{\perp} \underline{e}_{z'} \right] + \mathcal{E}_{\underline{k}}^{\pm} J_n \left[ \underline{e}_z + \frac{n\Omega_{\pm}}{v_y v_{\perp}} (v_{\parallel} \underline{e}_{v_{\perp}} - v_{\perp} \underline{e}_z) \right]. \quad (2.27)$$

Here  $\underline{e}_{v_{\perp}}$  is the unit vector in the direction of  $v_{\perp}$ . In deducing the preceding results, we noted that the first and the second term in Eq. (2.18), after summing over  $\underline{k}$  and  $n$ , are real and, hence, equal. If the wave distribution is independent of the  $\underline{k}$ -space azimuth  $\Psi$ , Eqs. (2.24) and (2.25) may be obtained alternatively by summing over  $\Psi$ , since the integral on the r.h.s. of Eq. (2.14) depends only on the combination  $\phi - \Psi$ . When turbulence in more than one branch of oscillations is excited, the diffusion equation will be a sum of terms of the above form, one for each branch, since different branches are uncorrelated in lowest order.

To find specific information about diffusion rates, the above formal diffusion equation must be coupled with the solution of the linear dispersion relation, which, for a specific branch of waves, relates the wave polarization components to one another. In several particularly simple circumstances, the wave polarizations are a priori known. For instance, for a distribution of waves all with  $k_{\perp} = 0$ , the eigenmodes are pure right- or left-hand or pure longitudinal, corresponding to excitation in the whistler, ion cyclotron-Alfvén and electrostatic branches. The previously obtained results of VEDENOV et al.,<sup>6</sup> CHANG and PEARLSTEIN,<sup>16</sup> ENGEL,<sup>17</sup> KENNEL and PETSCHKE,<sup>14</sup> DRUMMOND<sup>(18)</sup> and others for these modes may be easily obtained.

Similarly, the polarizations in the low  $\beta$  purely longitudinal approximation are known in advance, and the diffusion equation follows

immediately. With  $\underline{E}_{\underline{k}} = E_{\underline{k}} \frac{\underline{k}}{k}$  and

$$\xi_{\underline{k}}^{(e)} = \frac{1}{(2\pi)^3 V} \frac{|E_{\underline{k}}|^2}{8\pi} \quad (2.28)$$

for the electrostatic energy density associated with the mode  $\underline{k}$ , one obtains

$$\frac{\partial g_{\pm}^{\pm}}{\partial t} = 8\pi i \sum_n \frac{e^2}{M_{\pm}^2} \int d^3k \frac{\xi_{\underline{k}}^{(e)}}{k^2} \left[ \frac{n\Omega_{\pm}}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} + k_{\parallel} \frac{\partial}{\partial v_{\parallel}} \right] \quad (2.29)$$

$$\frac{J_n^2}{v_{\underline{k}} - k_{\parallel} v_{\parallel} - n\Omega_{\pm}} \left[ \frac{n\Omega_{\pm}}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} + k_{\parallel} \frac{\partial}{\partial v_{\parallel}} \right] g_{\pm}^{\pm};$$

in the vector notation of Eq. (2.25) this may be expressed by a diffusion tensor

$$\underline{D}_{\pm}^{\pm} = 8\pi i \sum_n \frac{e^2}{M_{\pm}^2} \int d^3k \frac{\xi_{\underline{k}}^{(e)} J_n^2}{k^2 [v_{\underline{k}} - k_{\parallel} v_{\parallel} - n\Omega_{\pm}]} \underline{b}_{n,\underline{k}} \underline{b}_{n,\underline{k}} \quad (2.30)$$

with

$$\underline{b}_{n,\underline{k}} = \frac{n\Omega_{\pm}}{v_{\perp}} \underline{e}_{v_{\perp}} + k_{\parallel} \underline{e}_{v_{\parallel}} \quad (2.31)$$

In Eqs. (2.29) and (2.30) the factor  $i[v_{\underline{k}} - k_{\parallel} v_{\parallel} - n\Omega_{\pm}]^{-1}$  may be replaced by its real part  $\gamma_{\underline{k}} / [(\beta_{\underline{k}} - k_{\parallel} v_{\parallel})^2 + \gamma_{\underline{k}}^2]$ , since the contribution from the imaginary part vanishes.

However, in general, the solution of the dispersion relation is quite difficult, since it is a determinant each of whose elements is a transcendental function. Meaningful approximations are often crucial to this procedure. The above unapproximated but formal diffusion equation (2.24) allows some conclusions to be drawn independent of the particular mode of excitation and approximation scheme.

### 3. QUASI H-THEOREM

To investigate stabilization of growing modes for a two-dimensional velocity distribution in the framework of quasi-linear theory we first define the positive definite functional H:

$$H = \frac{1}{2} \sum_{+, -} \int d^3 v (g^{\pm})^2. \quad (3.1)$$

$dH/dt$ , upon integration by parts and using Eqs. (2.21) and (2.24), is

$$\frac{dH}{dt} = - \lim_{V \rightarrow \infty} \sum_{+, -} \frac{e^2}{m_s^2} \sum_n \int d^3 v \int \frac{d^3 k}{V^{(2\pi)^3}} \left| \mathcal{D}_{n, \underline{k}}^{\pm} \hat{G}_{\underline{k}} g^{\pm} - \frac{v_{\underline{k}} - k_{\parallel} v_{\parallel} - n \Omega_s}{v_{\underline{k}} v_{\perp}} J_n E_{\underline{k}}^{\parallel} \hat{H} g^{\pm} \right|^2$$

$$\frac{\gamma_{\underline{k}}}{(\omega_{\underline{k}} - k_{\parallel} v_{\parallel} - n \Omega_s)^2 + \gamma_{\underline{k}}^2} \leq 0 \quad (3.2)$$

where  $\mathcal{D}_{n, \underline{k}}^{\pm}$  is

$$\mathcal{D}_{n, \underline{k}}^{\pm} \equiv E_{n, \underline{k}}^{\pm} + \frac{v_{\parallel}}{v_{\perp}} J_n E_{\underline{k}}^{\parallel}. \quad (3.3)$$

Since H is positive definite and  $dH/dt$  negative definite, H decreases monotonically with time to an asymptotic steady limit. This is given by a zero of  $dH/dt$ . It is clear that the zeros of  $dH/dt$  correspond to a marginally stable state for all the waves. Suppose  $\gamma_{\underline{k}}$  were positive in a certain domain of  $\underline{k}$ -space, but  $dH/dt = 0$ . Then, since  $dH/dt$  is a summation of positive definite terms,

$$\left( \mathcal{D}_{n, \underline{k}}^{\pm} \hat{G}_{\underline{k}} - \frac{v_{\underline{k}} - k_{\parallel} v_{\parallel} - n \Omega_s}{v_{\underline{k}} v_{\perp}} J_n E_{\underline{k}}^{\parallel} \hat{H} \right) g^{\pm} \equiv 0 \quad (3.4)$$

identically in  $\underline{v}$  for all  $\underline{k}$  and within the  $\underline{k}$ -domain considered.

It may easily be seen that the perturbed distribution functions  $\delta f_{\underline{k}}(\underline{v})$  would then be identically zero also, and we reach the contradictory conclusion that no waves are excited in the above domain. Therefore, the asymptotic state must be one of marginal stability to all waves.

Notice that in the above argument, the limit  $\gamma_k \rightarrow 0^+$  was not taken first. Therefore, there is no distinction between non-resonant adiabatic diffusion and the resonant diffusion of those particles which satisfy  $\omega_k - k_{\parallel} v_{\parallel} - n\Omega_{\pm} = 0$ . Therefore, as well as quasi-linear stabilization of resonant particle instabilities, the case ordinarily treated in the literature, there is stabilization of non-resonant instabilities, with any ratio of  $\gamma_k / \omega_k$ , provided that the quasi-linear assumptions are satisfied. Thus many weakly growing fluid-type instabilities will saturate in the non-linear regime. The extension to spatially inhomogeneous plasmas is particularly interesting for these waves.

An example of quasi-linear stabilization of non-resonant instabilities, concerning magnetohydrodynamic waves being unstable by the firehose mechanism, has been discussed by SHAPIRO and SCHEVCHENKO<sup>19</sup>. Here, even though  $\gamma_k > 0$  and  $\omega_k = 0$ , quasi-linear theory applies if  $\gamma_k \ll k_{\parallel} v_{th}$  (where  $v_{th}$  is a typical thermal velocity) and leads to stabilization of firehose unstable Alfvén waves.

Since the growth rates of all unstable waves decrease monotonically to zero, the final asymptotically steady state may be treated in the resonant diffusion limit where  $\gamma_k \rightarrow 0^+$ . Therefore, we devote our attention to resonant diffusion in the remainder of this paper.

#### 4. LIMIT OF RESONANT DIFFUSION

In this section we perform the limit  $\gamma_k \rightarrow 0^+$  first, and then discuss the development of the quasi-linear plateau. In this limit, Eq. (2.24) becomes

$$\frac{\partial g^{\pm}}{\partial t} = \lim_{V \rightarrow \infty} \sum_n \frac{\pi c^2}{M_i^2} \int \frac{d^3 k}{V (2\pi)^3} \left\{ \left( \hat{G}_k + \frac{\gamma_k}{v_k v_{\perp}} \right)^* \delta(\omega_k - k_{\parallel} v_{\parallel} - n\Omega_{\pm}) \left| \hat{\sigma}_{n,k}^{\pm} \right|^2 \hat{G}_k \right\} g^{\pm} \quad (4.1)$$

where the Bessel function-polarization composition  $\hat{\sigma}_{n,k}^{\pm}$  is defined in Eq. (3.3). In deriving (4.1), we noted<sup>in the limit</sup> in the limit  $\gamma_k \rightarrow 0^+$  the second term  $\frac{\lambda_k - n\Omega_{\pm}}{v_k v_{\perp}} \hat{A}$  of  $\hat{R}_{n,k}^{\pm}$  (see Eq. 2.21) will be non-zero only if there are zero frequency oscillations excited. These we rule out on

physical grounds. Thus  $\lim_{\gamma_k \rightarrow 0^+} \hat{K}_{n,k}^{\pm} = v_{\perp} / \omega_k \hat{G}_k$ .

The above resonant diffusion Eq. (4.1) is easier to interpret than the non-resonant form (2.24). The function  $\mathcal{G}_{n,k}^{\pm}$  may be thought of as weighting the strength of the wave-particle coupling in various velocity space regions; however, the kinematics of the wave-particle scattering, described by the operator  $\hat{G}_k$ , are now independent of the wave properties.

In the resonant limit, the vectors  $\underline{a}_{n,k}$  entering into the diffusion dyadic reduce to

$$\underline{a}_{n,k} = \mathcal{G}_{n,k}^{\pm} \frac{k_{\parallel}}{\omega_k} \left[ \left( \frac{\omega_k}{k_{\parallel}} - v_{\parallel} \right) \underline{e}_v + v_{\perp} \underline{e}_z \right] \quad (4.2)$$

If we complete Eq. (4.1) by the first-order terms of the expansion of  $i [v_k - k_{\parallel} v_{\parallel} - n\Omega_{\pm}]^{-1}$  in powers of  $\gamma_k$ , we may derive a necessary condition for the resonant approximation to be valid. This reads

$$\left| \lim_{\gamma \rightarrow 0} \mathcal{P} \int \frac{d^3 k}{V(2\pi)^3} \frac{\gamma_k}{v_{\parallel}} \sum_n \frac{\partial / \partial k_{\parallel} (\underline{a}_{n,k}^{\pm} \underline{a}_{n,k}^{\pm*})}{\omega_k - k_{\parallel} v_{\parallel} - n\Omega_{\pm}} \right| \quad (4.3)$$

$$\ll \pi \left| \lim_{\gamma \rightarrow 0} \int \frac{d^3 k}{V(2\pi)^3} \sum_n \underline{a}_{n,k}^{\pm} \underline{a}_{n,k}^{\pm*} \delta(\omega_k - k_{\parallel} v_{\parallel} - n\Omega_{\pm}) \right|$$

where the modulus is meant only with respect to the sign of the different tensor components. Eq. (4.3) may be taken as a requirement on the smallness of  $\gamma_k$  or on the smoothness of the fluctuation spectrum as a function of  $k_{\parallel}$ .

The growth rate in the resonant limit also involves only the weighting function  $\mathcal{G}_{n,k}^{\pm}$  and the operator  $\hat{G}_k$ , and so the correspondence with the quasi-linear diffusion equation is clear:

$$\frac{\gamma_k}{|\omega_k|} = \frac{\pi}{16N} \left| \frac{\omega_k}{k_{\parallel}} \right| \sum_{\pm} \left( \frac{\omega_{\pm}}{\omega_k} \right)^2 \sum_n \int_0^{\infty} v_{\perp}^2 dv_{\perp} \int_{-\infty}^{+\infty} dv_{\parallel} \delta \left( \frac{v_{\parallel} - \omega_k - n\Omega_{\pm}}{k_{\parallel}} \right) \quad (4.4)$$

$$\frac{|\mathcal{G}_{n,k}^{\pm}|^2}{W_k} \hat{G}_k q^{\pm}$$

where  $\omega_{p\pm} = \left( \frac{4\pi N e^2}{M_{\pm}} \right)^{1/2}$  is the plasma frequency for each species, and  $W_k$  is the energy, electromagnetic plus kinetic, of the given wave mode.

In the resonant limit, the H-theorem becomes

$$\left( \frac{dH}{dt} \right)_{res} = - \lim_{V \rightarrow \infty} \sum_{\pm} \frac{\pi e^2}{M_{\pm}^2} \sum_n \int \frac{d^3k}{V(2\pi)^3} \int d^3v \left| \mathcal{D}_{n,k}^{\pm} \right|^2 \delta(\omega_k - k \cdot v_{\parallel} - n\Omega_{\pm}) (G_k g^{\pm})^2 \leq 0 \quad (4.5)$$

In the following, we discuss the nature of the asymptotically steady, marginally stable state. In particular, we investigate the conditions under which a situation is possible, whereby a steady non-zero wave excitation and steady particle distributions co-exist; such a state will be referred to as a "plateau". A plateau will not exist, of course, if there is insufficient "free" energy contained in the initially unstable distribution to reconstruct, via quasi-linear diffusion, the velocity distribution in such a way that a broad spectrum of waves becomes marginally stable without causing particle diffusion. In particular, a plateau will not be possible if particles can increase their energy by diffusion without limit. In these cases, the particles will diffuse part way towards a plateau, and then the wave energy will die out. We investigate here the conditions for which a plateau is possible in principle, in other words, when only a finite energy is required for reconstruction of the particle distributions.

For resonant diffusion, a sufficient but not necessary condition that  $dH/dt$  be zero is

$$\mathcal{D}_{n,k}^{\pm} G_k g^{\pm} = 0 \quad (4.6)$$

for both particle species, for all resonances  $n$ , and all wave numbers  $k$  in the excited spectrum. A trivial solution,  $\mathcal{D}_{n,k}^{\pm} = 0$ , corresponds to the absence of wave excitation to drive diffusion. For steady particle distributions to co-exist with non-zero wave energy, a sufficient condition is

$$\frac{\omega_k}{k_{\parallel}} \hat{G}_k g^{\pm} = \left( \frac{\omega_k}{k_{\parallel}} - v_{\parallel} \right) \frac{\partial g^{\pm}}{\partial v_{\perp}} + v_{\perp} \frac{\partial g^{\pm}}{\partial v_{\parallel}} = 0 \quad (4.7)$$

for all regions of velocity space where the diffusion coefficient is non-zero. Let us consider the satisfaction of this condition for one value of  $\underline{k}$  and discuss the extension to all  $\underline{k}$  afterwards. Eq. (4.7) will be satisfied if  $g^{\pm}$  is constant along the characteristics of  $\hat{G}_k$ . These single-wave characteristics are given by

$$\left( v_{\parallel} - \frac{\omega_k}{k_{\parallel}} \right)^2 + v_{\perp}^2 = \text{constant} \quad (4.8)$$

and are curves of constant particle energy measured in the coordinate system moving with the parallel component of the wave phase velocity, i.e., circles displaced away from the origin along the  $v_{\parallel}$  axis by  $\omega_k/k_{\parallel}$ .

A simple physical argument suggests that a single particle interacting with a single wave is constrained to move on such a characteristic surface. If a particle interacting with a fluctuation quantum in the turbulent background gains energy  $\Delta E$  from the wave, the wave loses energy  $\Delta E = -\hbar \omega_k$ . Similarly, the gain in parallel energy is  $\Delta E = M_{\pm} v_{\parallel} \Delta v_{\parallel} = -\hbar k_{\parallel} v_{\parallel}$ . Noting that  $\Delta E = M_{\pm} (v_{\perp} \Delta v_{\perp} + v_{\parallel} \Delta v_{\parallel})$  and taking the ratio  $\Delta E / \Delta E_{\parallel} = \omega_k / k_{\parallel} v_{\parallel}$  we find that  $v_{\perp} \Delta v_{\perp} + (v_{\parallel} - \omega_k / k_{\parallel}) \Delta v_{\parallel} = 0$ . This integrates to  $v_{\perp}^2 + (v_{\parallel} - \omega_k / k_{\parallel})^2 = \text{constant}$ . Thus, the particle is physically constrained to move in the direction of the single-wave characteristic.

Were it not for the delta-function selection rule, corresponding to the limitation to resonant wave-particle interactions, a given particle would interact with all waves in the excited spectrum; Eq.(4.7) would then be a necessary as well as a sufficient condition for the existence of a plateau. However, only statements following from the properties of the single-wave characteristics can be made with any generality, since the introduction of the resonance selection rule inevitably involves the dispersion relation, and the choice of a particular branch of oscillations. However, it seems physically



reasonable that many conclusions based on the interactions of particles with an arbitrary component of the excited wave spectrum will also be true when the interactions are restricted to the resonant subset.

If only a single wave vector  $\underline{k}$  were excited, the single wave characteristics would be a set of concentric circles in the  $v_{\perp}, v_{\parallel}$  plane with their centers displaced by the given  $\omega_{\underline{k}}/k_{\parallel}$  along the  $v_{\parallel}$  axis. Since a given particle could not change surfaces, it could not reach infinite velocities through diffusion. However, quasi-linear theory does not apply to a single wave mode, and so we must consider broader wave spectra. As the spread in  $\omega_{\underline{k}}/k_{\parallel}$  increases due to the broadening wave spectrum, the characteristic circles no longer are concentric, since their centers cover a finite piece of the  $v_{\parallel}$  axis. Moreover, the characteristic surfaces for different  $\underline{k}$  intersect. As the spectrum becomes even broader, the angle of intersection between characteristics becomes large, and the condition that  $q^{\dagger}$  be constant along all single-wave characteristics is difficult to satisfy, except for the completely flat distribution  $q^{\dagger}(v_{\perp}, v_{\parallel}) = \text{constant}$ , which is unphysical. (Other special solutions may exist as well in any particular case.) In addition, a particle can now find a path to infinite velocities consisting of intersecting pieces of single-wave characteristics. Thus, the sufficient condition (4.6) appears difficult to satisfy.

However, the delta function selection rule can restrict the diffusion to a finite part of velocity space in certain cases. For instance, at the Landau resonance, we must have  $v_{\parallel} = \omega_{\underline{k}}/k_{\parallel}$ , independent of the particular mode of excitation, so that the particles are constrained to move on surfaces of constant  $v_{\perp}$ . Thus, Landau diffusion involves only the parallel velocity component  $v_{\parallel}$ . If then the range of  $\omega_{\underline{k}}/k_{\parallel}$  in the excited spectrum is finite, ranging from  $|\omega_{\underline{k}}/k_{\parallel}|_{\text{min}} \leq |\omega_{\underline{k}}/k_{\parallel}| \leq |\omega_{\underline{k}}/k_{\parallel}|_{\text{max}}$  where "min" and "max" denote minimum and maximum, respectively, a particle can only increase its  $|v_{\parallel}|$  up to  $|\omega_{\underline{k}}/k_{\parallel}|_{\text{max}}$  by Landau diffusion. Thus, a particle originally at finite energies cannot diffuse to infinite energies by Landau resonance diffusion alone. In the "infinite" magnetic field case, where  $\beta_0$  is so large that we can neglect all cyclotron particles, there is only one-dimensional Landau diffusion

for which a plateau exists (cf. Fig. 1).

Unlike the Landau resonance, the resonance characteristics for the cyclotron harmonic resonances require knowledge of the dispersion relation, and so no completely general statements can be made. We construct the equation for the resonant diffusion characteristics by substituting in  $\hat{G}_k$  for  $\omega_k/k_{\parallel}$  from the resonance condition:

$$\frac{dv_{\parallel}}{dv_{\perp}} = \frac{v_{\perp}}{v_{\parallel}} \frac{\omega_k - n\Omega_{\pm}}{n\Omega_{\pm}} \quad (4.9)$$

where  $k_{\parallel}$  is related to  $v_{\parallel}$ ,  $k_{\perp}$  and  $n$  through the subsidiary condition

$$k_{\parallel}(k_{\perp}, v_{\parallel}, n) = \frac{\omega(k_{\perp}, k_{\parallel}) - n\Omega_{\pm}}{v_{\parallel}} \quad (4.10)$$

In general, this subsidiary condition is quite complex.

A necessary and sufficient condition for the existence of a steady plateau with finite wave energy is that wherever  $\mathcal{G}_{n,k}^{\pm} \delta(\omega_k - k_{\parallel} v_{\parallel} - n\Omega_{\pm}) \neq 0$ ,  $g^{\pm}$  be constant along all resonance characteristics (4.9). If no point of the domain where the diffusion coefficient is non-zero can be connected without leaving the diffusion domain to the point  $v = \infty$  by a sequence of intersecting pieces of resonance characteristics as defined by (4.9), then the reconstructed plateau distribution has finite energy and is therefore physically possible.

Notice that the diffusion coefficient, for a given  $v_{\parallel}$ , is non-zero for all  $v_{\perp}$ . Similarly, when  $k_{\perp} \neq 0$ , there are many cyclotron resonances which cover much of  $v_{\parallel}$  space, even to large velocities, with a non-zero diffusion coefficient. Therefore, it is not possible that the non-zero diffusion coefficient covers only a small region of velocity space.

In certain circumstances reconstruction of the cyclotron resonance distribution will not require infinite energy. For instance, for an excited spectrum of electromagnetic waves with  $k_{\perp} = 0$ , the subsidiary condition uniquely relates  $k_{\parallel}$  to  $v_{\parallel}$ , since then there is only one resonance with non-zero weighting, the  $n = \pm 1$  cyclotron harmonic depending upon whether the pure left- or pure right-hand mode

is excited, and since the dependence upon  $k_{\perp}$  is fixed. There is no problem of overlap of resonances with different  $n$ . VEDENOV et al.,<sup>6</sup> ANDRONOV and TRAKHTENGERTS,<sup>21</sup> KENNEL and PETSCHKE,<sup>14</sup> ROWLANDS et al.<sup>22</sup> and others have considered this problem. The resonance diffusion is restricted to two-dimensional nested surfaces in velocity space since the resonance condition excludes all wave-particle interactions which could spread the particles out over a volume.

If for the excited waves, the dependence of  $\omega_{k_{\perp}, k_{\parallel}}$  on  $k_{\parallel}$  can be neglected, then the subsidiary equation (4.10) is decoupled from the resonance characteristic equation (4.9), which may be solved explicitly, yielding

$$\left( \frac{n \Omega_{\pm}}{n \Omega_{\pm} - \omega_{k_{\perp}}} \right) v_{\parallel}^2 + v_{\perp}^2 = \text{constant} \quad (4.11)$$

For  $n \neq 0$  and  $|\omega_{k_{\perp}}| < |n \Omega_{\pm}|$  the above resonance characteristics are all closed. Notice that when  $|\omega_{k_{\perp}}/n \Omega_{\pm}| \ll 1$ , these are very close to surfaces of constant particle energy.

In general, however, the particles are not restricted to closed nested surfaces, because there is ordinarily a distribution of  $k_{\perp}$  for a given  $k_{\parallel}$  in the excited spectrum; moreover, one particle can interact simultaneously with different cyclotron harmonics  $n$  if the excited spectrum is broad enough. In general, each point in velocity space will have an intersection of more than one resonance characteristic. In physical terms, a given particle may be scattered in more than one direction in velocity space since it resonates with more than one wave component. Then a volume, rather than a surface, is accessible to that particle.

Since we do not pretend to solve simultaneously the equations for the time development of the wave and particle distributions, we cannot guarantee that a broad, initially disordered wave distribution will never develop towards a state where the resonance characteristics are closed and do not cross. However, this seems very unlikely. For we can guarantee that there is always a combination of single-wave diffusion paths, defined by Eq. (4.8), which permits particles to diffuse to infinite velocities. Then, since resonance diffusion occurs on some subset of the single wave paths, a wave distribution of finite width will, in general, also allow cyclotron particles access to

infinity using resonant paths. On these grounds, it is highly probable that rigorous plateaux in general do not exist. However, this does not exclude the existence of "quasi-plateaux" whose diffusion in velocity space is extremely slow, and which take for all physically practical purposes the role of a plateau.

## 5. QUALITATIVE NATURE OF QUASI-LINEAR DIFFUSION

The existence of a mathematically rigorous, unique plateau is less important than some insight into how quasi-linear diffusion modifies the particles' distribution functions. Since the modification is always in the direction of the plateau, the discussion of the previous section was useful in this respect, even if the question of the existence of a mathematically rigorous plateau is academic.

We again assume that for the excited modes  $|\omega_k/k_u| \ll |\omega_k/k_u|_{\max}$  and that  $|\omega_k/k_u|_{\min}$  is not too close to  $|\omega_k/k_u|_{\max}$ . The nature of resonant Landau diffusion was already discussed in the previous Section 4. There it was shown that  $v_u$ -diffusion of particles beyond  $|v_u| \gg |\omega_k/k_u|_{\max}$  was not possible, due to Landau-resonant waves. When  $|v_u| \gg |\omega_k/k_u|_{\max}$ , diffusion for that particle can only occur at a cyclotron resonance. Here we demonstrate that when the speed  $v = (v_\perp^2 + v_u^2)^{1/2} \gg |\omega_k/k_u|_{\max}$ , cyclotron resonance diffusion is primarily in pitch angle. This may be seen from (4.8) where for very large particle speeds, all the single-wave characteristics converge to a surface of approximately constant energy. It is also clear that the angles at which single-wave characteristics cross become smaller at large speeds.

From the arguments immediately following Eq. (4.8), we may roughly estimate the small changes in pitch angle  $\Delta\alpha$  and particle energy  $\Delta E$  resulting from an interaction between a given particle and an arbitrary component of the wave spectrum. Here  $\alpha = \tan^{-1} v_\perp/v_u$  and  $E = \frac{1}{2} M_\perp (v_\perp^2 + v_u^2)$ . For a turbulent spectrum with  $|\omega_k| \ll |\Omega_\perp|$  the moduli of all cyclotron resonance velocities are larger than

$|\omega_k/k_{||}|_{\max}$  and we have

$$\left| \frac{\Delta E}{E} \right| \approx \left| \frac{\omega_k}{n\Omega_{\pm}} \right| \frac{|v_{||} v_{\perp}|}{v_{||}^2 + v_{\perp}^2} |\Delta\alpha| < \left| \frac{\omega_k}{2n\Omega_{\pm}} \right| |\Delta\alpha| \quad (5.1)$$

Then, constructing the phenomenological diffusion coefficients

$D_E \sim (\Delta E)^2 / (2\Delta t)$  and  $D_{\alpha} \sim (\Delta\alpha)^2 / (2\Delta t)$ , where  $\Delta t$  is the "duration" of the interaction, we may compare the time  $T_{\alpha}$  to diffuse a radian in pitch angle to the time  $T_E$  to random walk an energy  $E$ :

$$\frac{T_E}{T_{\alpha}} > \left( \frac{2n\Omega_{\pm}}{\omega_k} \right)^2 \gg 1 \quad (5.2)$$

Since these arguments are true for all single-wave characteristics, they also hold for the resonance characteristics. This may be seen from the expression for the angle  $\delta$  between a resonance characteristic and the surfaces of constant energy, which follows from Eq. (4.9)

$$\tan \delta = \frac{v_{||}}{v_{\perp}} \frac{\omega_k / n\Omega_{\pm}}{\left( \frac{v_{||}}{v_{\perp}} \right)^2 + 1 - \omega_k / n\Omega_{\pm}} \quad (5.3)$$

When  $v_{\perp} \rightarrow \infty$ ,  $\tan \delta \rightarrow \omega_k / n\Omega_{\pm}$ . Similarly, particles with  $|v_{||}| \rightarrow \infty$  must resonate with waves  $|\omega_k / n\Omega_{\pm}| \rightarrow 0$  so that again  $\tan \delta \rightarrow 0$  (even when  $v_{||}/v_{\perp} \approx 1$ ).

Some care is necessary if there are cyclotron resonances with very small  $|v_{||}|$ , which happens when there are frequencies close to a cyclotron harmonic,  $\omega_k \sim n\Omega_{\pm}$ . This is the case for the low-density limit of the loss-cone instability of POST and ROSENBLUTH<sup>23</sup>. Then, neglecting terms of order  $\omega_k - n\Omega_{\pm}$ , the resonance characteristic equation reduces to

$$\frac{dv_{||}}{dv_{\perp}} = \frac{K_{||} v_{\perp}}{\omega_k} \quad (5.4)$$

and we have

$$\tan \delta = \frac{1}{\frac{v_{||}}{v_{\perp}} - \frac{K_{||} v_{\perp}}{\omega_k}} \quad (5.5)$$

Thus, when  $v_{\parallel} \approx 0$  and  $|k_{\parallel}/\omega_k|_{\max} v_{\perp} \gg 1$ ,  $\tan \delta \rightarrow 0$ . For  $v_{\perp}$  much less than  $|\omega_k/k_{\parallel}|_{\max}$  the scattering is primarily in the  $v_{\perp}$  variable, while for  $v_{\perp}$  much larger than  $|\omega_k/k_{\parallel}|_{\max}$ , it is primarily in  $v_{\parallel}$  and consequently, for these small  $v_{\parallel}$ , in pitch angle.

Summing up, the larger the particle energy relative to  $\frac{1}{2} M_{\perp} |\omega_k/k_{\parallel}|_{\max}^2$ , the more important is pitch angle scattering relative to energy scattering. In this region of velocity space, wave-particle scattering is approximately elastic. At smaller energies  $v_{\perp}^2 + v_{\parallel}^2 \lesssim |\omega_k/k_{\parallel}|_{\max}^2$ , diffusion makes particles adjust their energy and pitch angle equally rapidly. Here scattering is inelastic. Of course, the absolute diffusion rates depend strongly upon the characteristics of the spectrum excited.

For those spatially finite plasmas for which pitch angle scattering implies a loss of particles from the system, the maximum particle energy accessible to a two-dimensional turbulent acceleration process is roughly  $\frac{1}{2} M_{\perp} (\omega_k/k_{\parallel})_{\max}^2$  since higher energy cyclotron particles do not have time to gain appreciable energy before they are scattered from the system. Landau particles ordinarily only increase their energy by this amount at most. This theorem has immediate applications to laboratory mirror machines and the Earth's Van Allen Belts. It suggests that a correlation may exist between the  $|\omega_k/k_{\parallel}|_{\max}$  and the highest energy particles observed.

While a rigorous plateau seems difficult to achieve, a "quasi-plateau" characterized by weak diffusion may be possible in rather general conditions. The previous arguments lead to the following picture for a possible quasi-plateau. For small energies as defined above, the plateau distribution will be an almost constant function of either  $v_{\parallel}$  or of both  $v_{\perp}$  and  $v_{\parallel}$ , depending on whether only Landau, or cyclotron, or mixed Landau-cyclotron diffusion occurs. For larger energies, one expects an almost isotropic pitch angle distribution, accompanied by some flattening in the energy distribution. For the large  $B_0$  case, where the cyclotron particles have extremely large energies, the Landau particles approach among themselves a plateau of finite energy (cf. Fig. 1), leaving at best only a slow cyclotron

energy diffusion. Another possibility is where the initial growth rate is sufficiently strongly peaked as a function of  $\underline{k}$  that the maximum growth rate component dominates throughout the reconstruction of the distribution functions. Since the spectrum is narrow, characteristic crossing will not be a great problem in this case.

It should be emphasized that, if any, only quasi-plateaux are observable experimentally since ordinarily there are finite time scales associated with sinks and sources of particles and waves. In addition, when the diffusion and growth rates are sufficiently slow, the non-linear mode-mode couplings neglected here may give comparable effects.

The above arguments permit an understanding of how the transition occurs between the "infinite" magnetic field limit, where there is only one-dimensional Landau diffusion leading to plateau formation in  $g$  as shown in Fig. 1, and the regime  $\Omega_{\perp} \rightarrow 0$ , treated by BERNSTEIN and ENGELMANN<sup>13</sup> for electrostatic waves, where no plateau is possible. As  $B_0$  decreases, the cyclotron resonance regions move to smaller velocities. If the spectrum is broad enough, they may overlap with each other and with the  $n = 0$  Landau resonance region. Thus, crossing of characteristics appears in velocity space regions where there is an appreciable number of particles. Particles of finite energy then begin to leak towards infinite energies. As long as they are only weakly accessible to infinity, a quasi-plateau may be possible (cf. Fig. 2). A significant change occurs, however, when the magnetic field decreases enough to allow many different resonant regions to overlap (cf. Fig. 3), leading to crossing of characteristics at large angle. In the limit treated by BERNSTEIN and ENGELMANN the cyclotron resonances, of course, are no longer significant; instead  $\underline{k}_{\perp}$  enters into the resonance condition and a plateau requires

$$\underline{k} \cdot \frac{\partial g^+}{\partial \underline{v}} \delta(\omega_{\underline{k}} - \underline{k} \cdot \underline{v}) = 0 \quad (5.6)$$

for all  $\underline{v}$  and  $\underline{k}$  in the excited spectrum. Hence, for a sufficiently broad wave spectrum, the wave characteristics cross at rather large angles and the particles are fairly accessible to infinity, so that even a quasi-plateau becomes unlikely. An example of this type is visualized in Fig. 4.

#### ACKNOWLEDGMENTS

It is a pleasure to acknowledge many stimulating discussions with Professors R.Z. Sagdeev and R.N. Sudan. Also, Professors E.A. Frieman and M.N. Rosenbluth made useful comments. The authors would like to thank Professor Abdus Salam and the IAEA for the hospitality extended to them at the International Centre for Theoretical Physics, Trieste. One of them (C.F.K.) received financial support from the US National Science Foundation.



## FIGURE CAPTIONS

Fig. 1 Dependence of the distribution function  $g$  on  $v_{\parallel}$  in a plateau state for the "infinite" magnetic field limit: within the Landau resonance region,  $(\omega_{\underline{k}}/k_{\parallel})_{\min} < v_{\parallel} < (\omega_{\underline{k}}/k_{\parallel})_{\max}$ , associated with the spectrum of modes  $\underline{k}$  excited for large times,  $g(v_{\parallel})$  is flat, cyclotron resonances lie on the far tail of  $g(v_{\parallel})$ . It has been assumed that  $D_{zz}$  and  $\frac{\partial D_{zz}}{\partial v_{\parallel}}$  are continuous so that no discontinuity of  $g$  can occur.

Fig. 2 Schematic plot of the level lines of  $g(v_{\parallel}, v_{\perp})$  in the case of a quasi-plateau for strong magnetic field:  $g$  is independent of  $v_{\parallel}$  in the Landau resonance region  $(\omega_{\underline{k}}/k_{\parallel})_{\min} < v_{\parallel} < (\omega_{\underline{k}}/k_{\parallel})_{\max}$ , which does not overlap with cyclotron resonant regions

$((\omega_{\underline{k}} + n\Omega)/k_{\parallel})_{\min} < v_{\parallel} < ((\omega_{\underline{k}} + n\Omega)/k_{\parallel})_{\max}$  where  $g$  is isotropic. Only the lowest resonances  $n = \pm 1$  are shown explicitly. The level lines are approximately equal to many-wave diffusion characteristics within the resonant regions.

Fig. 3 Schematic representation of intersecting diffusion characteristics for moderate magnetic field such that overlapping between the Landau resonance region and the first cyclotron resonance region occurs. In the cyclotron resonance region two different families of characteristics associated with different  $k_{\perp}$  are shown.

Fig. 4 The resonance region and diffusion characteristics for vanishing external magnetic field (cf.ref.13). The horizontal straight lines correspond to diffusion characteristics associated with  $k_{\perp} = 0$ ; they intersect at large angles with a family of characteristics corresponding to fairly large  $\omega_{\underline{k}}/k_{\perp}$ .

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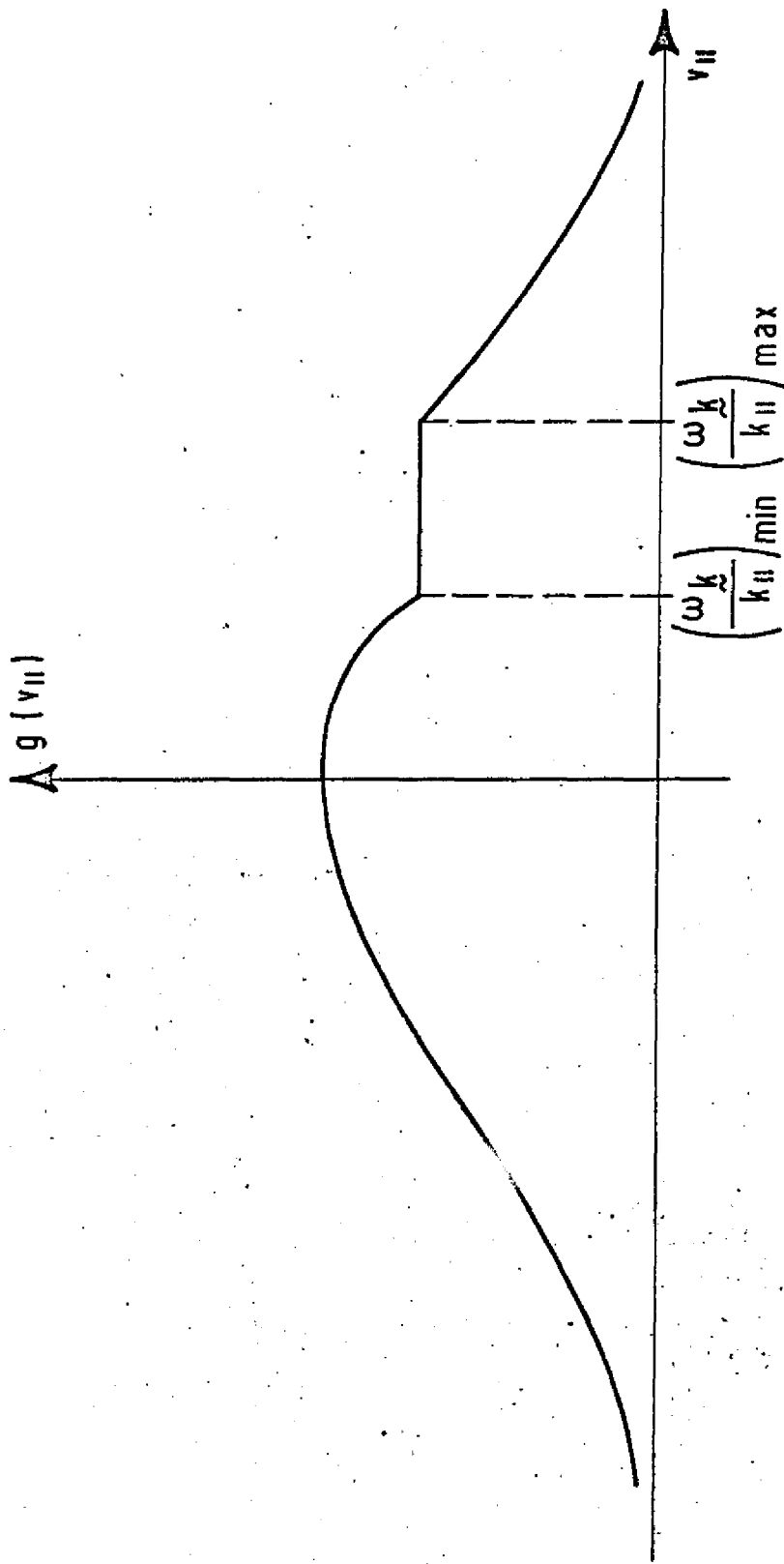


Figure 1

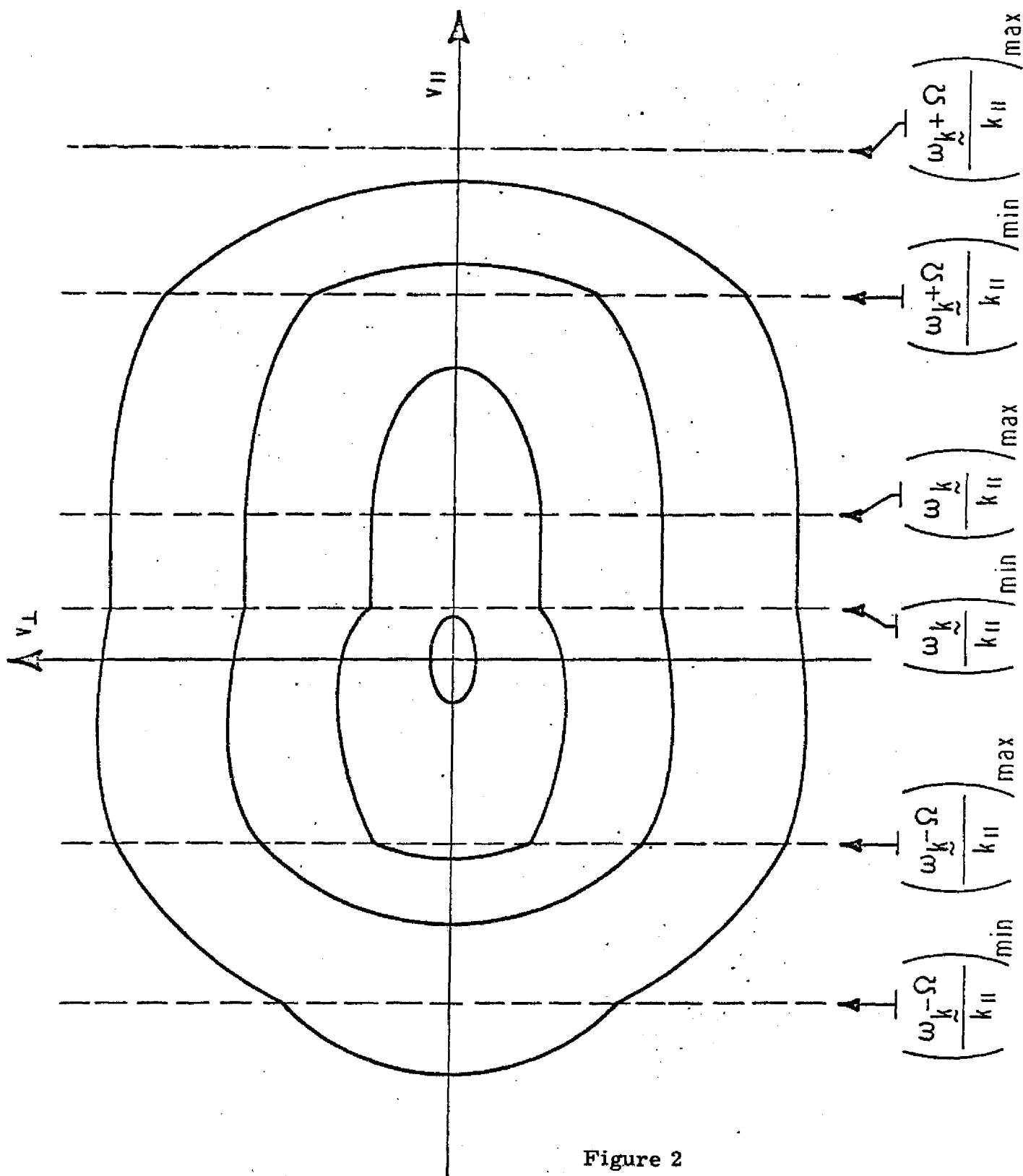


Figure 2

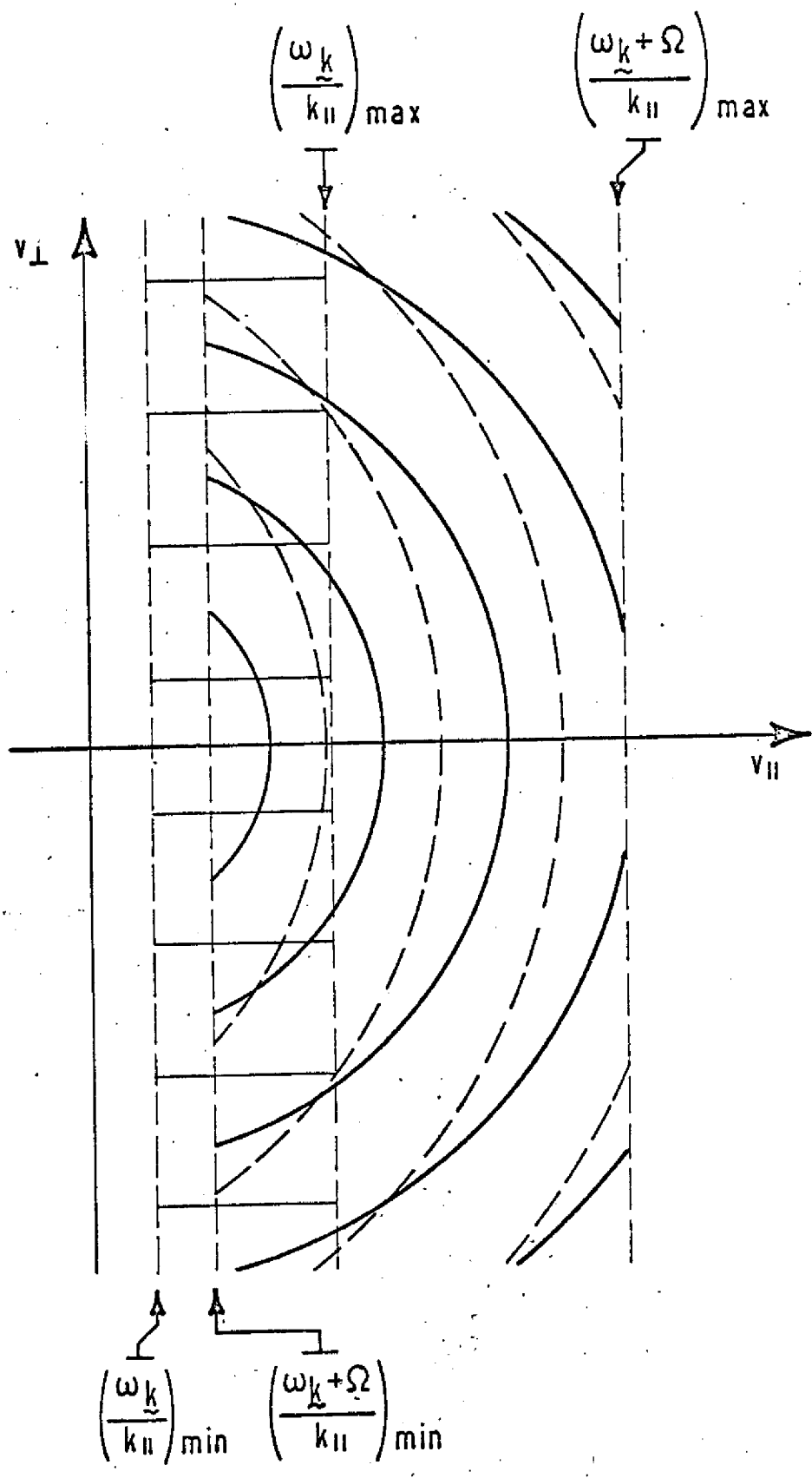


Figure 3

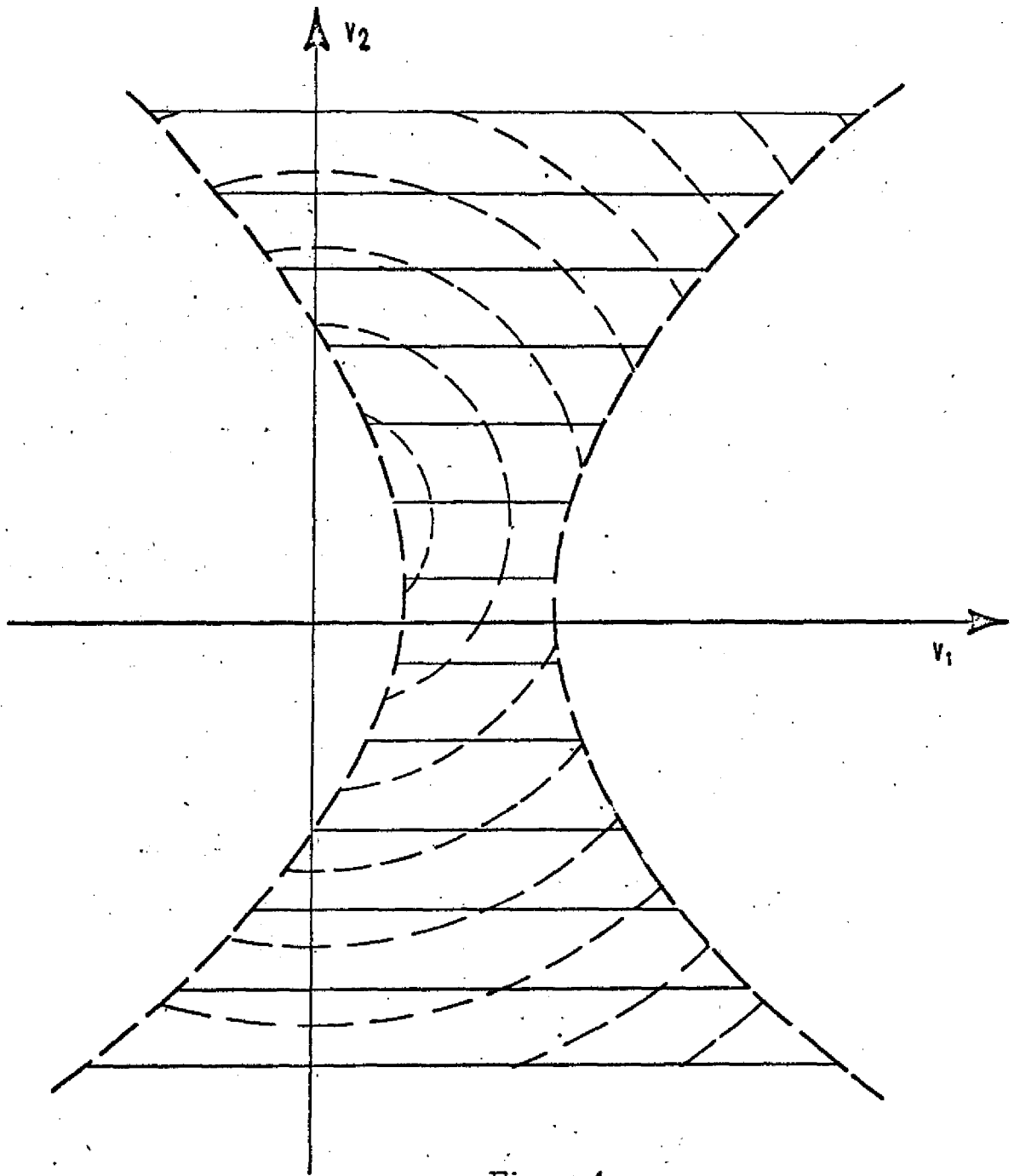
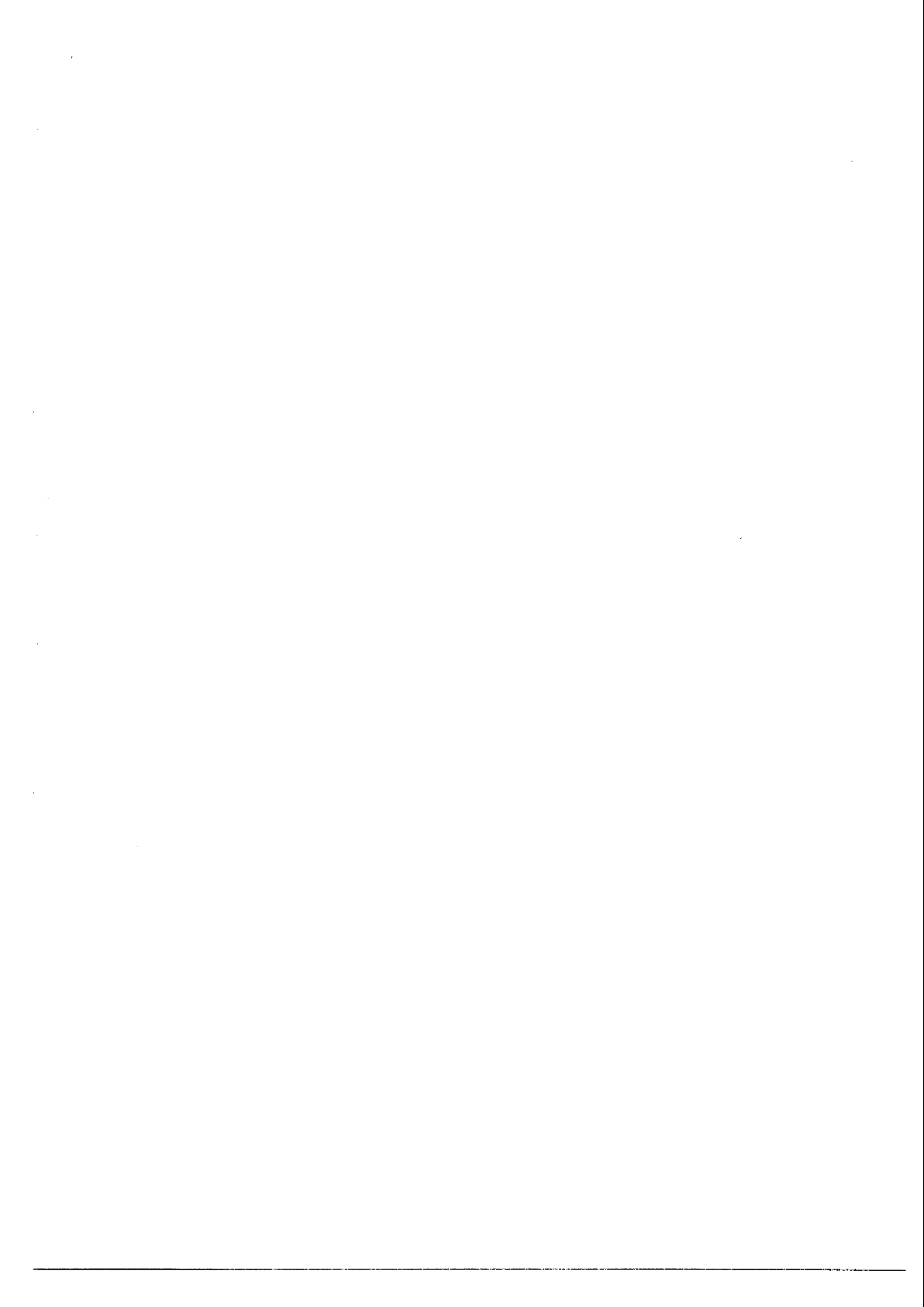


Figure 4







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