Vertex labeling and routing in expanded Apollonian networks

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Abstract. We present a family of networks, expanded deterministic Apollonian networks, which are a generalization of the Apollonian networks and are simultaneously scale-free, small-world, and highly clustered. We introduce a labeling of their nodes that allows one to determine a shortest path routing between any two nodes of the network based only on the labels.

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1. Introduction

Recently, there has been a growing interest in the study of complex networks [1, 2, 3, 4], which can help to describe many social, biological, and communication systems, such as co-author networks [5], sexual networks [6], metabolic networks [7], protein networks in the cell [8], the Internet [9], and the World Wide Web [10]. Extensive observational studies show that many real-life networks have at least three important common statistical characteristics: the degree distribution exhibits a power law tail with an exponent taking a value between 2 and 3 (*scale-free*); two nodes having a common neighbor are far more likely to be linked to each other than are two nodes selected randomly (*highly clustered*); the expected number of links needed to go from one arbitrarily selected node to another one is low (*small-world property*).

These empirical findings have led to new kinds of network models [1, 3, 4, 2]. The research on these new models was started by the two seminal papers by Watts and Strogatz on small-world networks [11] and Barabási and Albert on scale-free networks [12]. A wide variety of network models and mechanisms, including initial attractiveness [13], nonlinear preferential attachment [14], aging and cost [15], competitive dynamics [16], edge rewiring [17] and removal [18], and duplication [19], which may represent processes realistically taking place in real-life systems, have been proposed.

Based on the classical Apollonian packing, Andrade *et al.* introduced Apollonian networks [20], which were simultaneously proposed by Doye and Massen in [21]. Apollonian networks belong to a class of deterministic networks studied earlier in [22, 23, 24, 25, 26], which have received much interest recently [27, 28, 29, 30, 31]. Two-dimensional Apollonian networks are simultaneously scale-free, small-world, Euclidean and space filling [20, 27]. They may provide valuable insight into real-life networks; moreover, they are maximal planar graphs, and this property is of particular interest for the layout of printed circuits and related problems [20, 27]. More recently, some interesting dynamical processes such as percolation, epidemic spread, synchronization, and random walks taking place on these networks, have been also investigated, see [27, 30, 32].

Networks are composed of nodes (vertices) and links (edges) and are very often studied using methods and results from graph theory, a branch of discrete mathematics. One active subject in graph theory is graph labeling [33]. This is due not only to its theoretical importance but also to the wide range of applications in different fields [34], such as X-rays, crystallography, coding theory, radar, astronomy, circuit design, and communication design.

In this paper, we present an extension of the general high-dimensional Apollonian networks [20, 21, 29] which includes the deterministic small-world network introduced in [35]. We give a vertex labeling that enables queries for the shortest path between any two vertices to be efficiently answered. Finding shortest paths in networks is a well studied and important problem with many applications [36]. Our labeling technique

may be useful in aspects such as network optimization and information dissemination, which are directly related to the problem of finding shortest paths between all pairs of vertices of the network.

2. Expanded Apollonian networks

In this section, we present a network model defined in an iterative way. The model, which we call an *expanded Apollonian network* (EAN), is an extension of the general high-dimensional Apollonian network [20, 21, 29] which includes the deterministic small-world network introduced in [35].

The networks, denoted by $A(d,t)$ after t iterations with $d \geq 1$ and $t \geq 0$, are constructed as follows. For $t = 0$, $A(d,0)$ is a complete graph K_{d+2} (also called a $(d+2)$-clique). For $t \geq 1$, $A(d,t)$ is obtained from $A(d,t-1)$. For each of the existing subgraphs of $A(d,t-1)$ that is isomorphic to a $(d+1)$-clique and created at step $t-1$, a new vertex is introduced and connected to all the vertices of this subgraph. Figure 1 shows the construction process for the particular case where $d = 2$. We note that EAN are a subfamily of the family of graphs known as k-trees. k-trees were introduced in the 1970s [37] and have been extensively studied since. They are defined as follows: (i) a k-clique K_k is a k-tree ; (ii) if G is a k-tree, then adding a vertex v to G and joining it to all the vertices of an existing k-clique in G yields a new graph G', which is also a k-tree. Hence, EAN are k-trees where $k = d + 1$, since it starts with a clique K_{d+2} (that is, the only possible $(d+1)$-tree with $d+2$ vertices), and since each time a vertex is added, it is joined to all the vertices of an existing $(d+1)$-clique. However, it can be seen that EAN are a very particular case of k-trees, since in their original definition, each step corresponds to adding only *one* vertex joined to *one* of the existing k-cliques; while in EAN, each step corresponds to adding *as many vertices* as new k-cliques were formed *in the previous step* with each vertex joined to one different new clique.

Let $n_v(t)$ and $n_e(t)$ denote the number of vertices and edges created at step t, respectively. According to the network construction, one can see that at a later step t' ($t' > 1$) the number of newly introduced vertices and edges is $n_v(t') = (d+2)(d+1)^{t'-1}$ and $n_e(t') = (d+2)(d+1)^{t'}$. From these results, we can easily compute the total number of vertices N_t and edges E_t at step t, which are $N_t = \frac{(d+2)[(d+1)^t+d-1]}{d}$ and $E_t = (d+2)(d+1)\frac{2(d+1)^t+d-2}{2d}$, respectively. So for large t, the average degree $\overline{k}_t = \frac{2E_t}{N_t}$ is approximately $2(d+1)$.

This general model includes existing models, as listed below.

Indeed, when $d = 1$, the network is the deterministic small-world network (DSWN) introduced in [35] and further generalized in [38]. DSWN is an exponential network, and its degree distribution $P(k)$ is an exponential of a power of degree k. For a vertex of degree k, the exact clustering coefficient is $\frac{2}{k}$. The average clustering coefficient of the DSWN approaches to the high constant value $\ln 2 \approx 0.6931$. The average path length of DSWN grows logarithmically with the number of network vertices [39].

When $d \geq 2$, the networks are exactly the same as the high-dimensional Apollonian
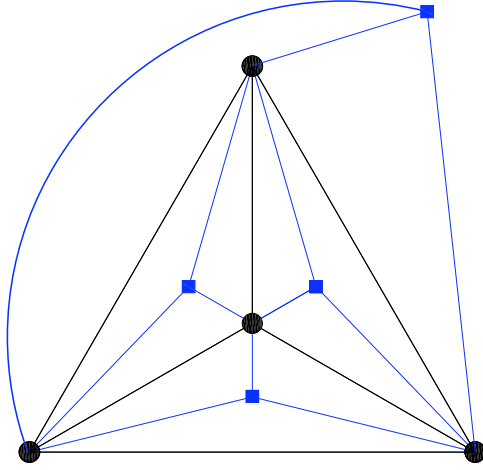
Figure 1. Illustration of a growing network in the case of $d = 2$, showing the first two steps of the process (round vertices correspond to step 0, while square vertices correspond to step 1.)

networks (HDAN) with d indicating the dimension [20, 21, 29, 31]. HDAN present the typical characteristics of real-life networks in nature and society, and their main topological properties are controlled by the dimension d. They have a power law degree distribution with exponent $\gamma = 1 + \frac{\ln(d+1)}{\ln d}$ belonging to the interval between 2 and 3 [20, 21, 29, 31]. For any individual vertex with degree k in HDAN, its clustering coefficient $C(k)$ is also (approximately) inversely proportional to its degree, as $C(k) = \frac{2d(k - \frac{d+1}{2})}{k(k-1)}$. The mean value C of the clustering coefficient of all vertices in HDAN is very large and is an increasing function of d. For instance, in the special cases where $d = 2$ and $d = 3$, C asymptotically reaches the values 0.8284 and 0.8852, respectively. The diameter of HDAN, defined as the longest shortest distance between any pair of vertices, increases logarithmically with the number of vertices, i.e. HDAN is small- world. So, the EAN model exhibits a transition from an exponential network ($d = 1$) to scale-free networks ($d \geq 2$).

3. Vertex labeling

A vertex labeling of a network is an assignment of labels to all the vertices in the network. In most applications, labels are nonnegative integers, though in general real numbers could be used [33]. In this section, we describe a way to label the vertices of $A(d, t)$, for any $d \geq 1$ and $t \geq 0$, such that a routing by shortest paths between any two vertices of $A(d, t)$ can be deduced from the labels. We note that a general result for shortest path routing in directed graphs with given treewidth is given in [40]. However, here we address the more specific case of the expanded Apollonian networks $A(d, t)$. In what follows, we will denote by $L(v)$ the label of vertex v, for any vertex v belonging to $A(d, t)$.
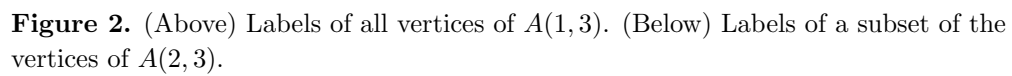
As noted in the previous section, EAN are a special case of k-trees, with $k = d+1$. There actually exist several results concerning labelings of k-trees that allow to route by shortest paths (see for instance [41, 42]). However, those labelings are too general for our purpose. In particular, they are constructed using the fact that, at each step, a vertex is added to a particular k-clique. However, as mentioned in the previous section, the notion of step for the usual k-trees is a totally different concept than the one we use for EAN, and as a consequence, their labeling is not of optimal length. In the following, we develop a new and original labeling, especially designed for EAN, which is of optimal length and allows to route by shortest paths.

The labeling idea, introduced in [43], is to assign to any vertex v created at step $t \geq 1$ a label of length t, in the form of a word of t digits, each digit being an integer between 1 and $d+2$ (the vertices obtained at step $t = 0$, i.e. the vertices of the *initial* $(d+2)$-clique $A(d, 0)$, are assigned a special label). More precisely, the labeling of each vertex v of $A(d, t)$ is done according to the following rules:

- Label the vertices of the initial $(d + 2)$-clique $A(d, 0)$ arbitrarily, with labels $1', 2', \ldots, (d+2)'$.

- At each step $t \geq 1$, when a new vertex v is added and joined to all vertices of a clique K_{d+1}:

  (i) If v is connected to $d + 1$ vertices of the initial $(d + 2)$-clique, then $L(v) = l$, where l' is the only vertex of the initial $(d + 2)$-clique that does not belong to this $(d + 1)$-clique.

  (ii) If not, then v is connected to $w_1, w_2, \ldots, w_{d+1}$, where at least one of the w_i's is not a vertex of the initial $(d + 2)$-clique. Thus, any such vertex has a label $L(w_i) = s_{1,i} s_{2,i} \ldots s_{k,i}$. W.l.o.g., let w_1 be the vertex not belonging to the initial $(d + 2)$-clique with the longest label. In that case, we give vertex v the label $L(v)$ defined as follows: $L(v) = \alpha \cdot L(w_1)$, where $1 \leq \alpha \leq d+2$ is the only integer not appearing alone as first digit in the labels of $w_1, w_2, \ldots, w_{d+1}$; In other words $\alpha = \{1, 2, \ldots, d, d + 1, d + 2\} \setminus \cup_{i=1}^{d+1} s_{1,i}$ (the fact that α is unique will be proved by Property 1 below).

Such a labeling is illustrated in Figure 2. In the upper part of this figure, we label the vertices of $A(d, t)$, for $d = 1$ and up to $t = 3$. We see that vertex u, created at step 1, has label $L(u) = 2$ because it is not connected to vertex $2'$ of the initial 3-clique (triangle). Vertex w is is not connected to any vertex of the initial 3-clique, its label is first composed of the only digit not appearing as first digit of its neighbors (in this case, 1), concatenated with the longest label of its neighbors (in this case, 23). Analogously, in the lower part of Figure 2 , where, for sake of clarity, only a part of $A(2, 3)$ is drawn, the vertices have been labeled. For the same reasons, we can see that vertex u has label $L(u) = 4$, while w has label $L(w) = 234$.

Thus, we see that for $t \geq 1$, a vertex v_t created at step t has a unique label, and that for a vertex v created at step $t \geq 1$, $L(v) = s_1 s_2 \ldots s_t$ is of length t, where each

Figure 2. (Above) Labels of all vertices of $A(1,3)$. (Below) Labels of a subset of the vertices of $A(2,3)$.

digit s_j satisfies $1 \leq s_j \leq d+2$; while the vertices created at step 0 have length 1 (these are the l', $1 \leq l \leq d+2$).

We note that since for step $t \geq 1$, the number of vertices that are added to the expanded Apollonian networks is equal to $(d+2)(d+1)^{t-1}$, the labeling we propose is optimal in the sense that each label $L(v_t)$ of a vertex created at step t is a $(d+2)$-ary word of length t. Globally, any vertex of $A(d,t)$ is assigned a label of length $O(\log_{d+2} t)$; since there are $N_t = (d+2)\frac{(d+1)^t+d-1}{d}$ vertices in $A(d,t)$, we can see that, overall, the labeling is optimal as well.

Next, we give three properties about the above labeling. Property 1 ensures that our labeling is deterministic. Property 2 is a tool to prove Property 3, the latter being important to show that our routing protocol is valid and finds shortest paths.

Property 1 In $A(d,t)$, if $w_1, w_2, \ldots, w_{d+2}$ induce a $(d+2)$-clique, then every integer $1 \leq i \leq d+2$ appears exactly once as the first digit of the label of a w_j.

Proof. By induction on t. When $t = 0$, the property is true by construction. Suppose now that the property is true for $t' < t$, and let us then show it is true for t. A $(d+2)$-clique in $A(d,t)$ is composed of exactly one vertex v created at a given step t_1, and $d+1$ vertices $w_1, w_2, \ldots w_{d+1}$ created at steps strictly less than t_1. If $t_1 < t$, then the property is true by the induction hypothesis. If $t_1 = t$, we suppose that w_{d+2} is connected to a $(d+1)$-clique $\mathcal{C}$ composed of $w_1, w_2, \ldots, w_{d+1}$. It is clear that $\mathcal{C}$ did not exist at step $t-1$. In other words, one of the w_i's, say w_1, has been created at step $t-1$, based on $d+1$ vertices $w_2, w_3 \ldots w_{d+1}$ and x. By induction hypothesis, each integer $1 \leq i \leq d+2$ appears exactly once as first digit of the labels of $w_1, w_2, w_3, \ldots, w_{d+1}, x$. However, by construction, the first digit of $L(w_{d+2})$ is the first digit of $L(x)$. Thus we conclude that each integer $1 \leq i \leq d+2$ also appears exactly once as first digit of the labels of $w_1, w_2, w_3, \ldots, w_{d+1}, w_{d+2}$, and the result is proved by induction.

Property 2 Let v_t be a vertex of $A(d,t)$ created at step $t \geq 1$. Among the vertices $w_1, w_2, \ldots w_{d+1}$ forming the $(d+1)$-clique that generated v_t, let $w_1, w_2 \ldots w_k$, $k \leq d+1$, be the vertices that do not belong to the initial $(d+2)$-clique. Then $L(v_t)$ is a superstring of $L(w_i)$ for all $1 \leq i \leq k$.

Proof. By induction on t. When $t = 1$, any vertex v_1 created at step 1 is connected to vertices of the initial $(d+2)$-clique only. Thus the result is vacuously true. Now suppose the result is true for $1 \leq t' \leq t-1$, $t \geq 2$, and let us prove it is then true for t. For this, we consider a vertex v_t created at step t, and the $(d+1)$-clique $\mathcal{C}$ it is connected to. Suppose v_t is a neighbor of w_p which was created at step $t-1$. However, w_p was created itself thanks to a $(d+1)$-clique, say $\mathcal{C}'$, composed of vertices $x_1, x_2 \ldots x_{d+1}$. W.l.o.g., suppose that $k \leq d+1$ such vertices, $x_1, x_2 \ldots x_k$ do not belong to the initial $(d+2)$-clique. By induction hypothesis, $L(x_i) \subseteq L(w_p)$ for $1 \leq i \leq k$. Hence, in $\mathcal{C}$, w_p is the vertex not belonging to the initial $(d+2)$-clique that has the longest label. By construction of $L(v_t)$, we have that $L(w_p) \subseteq L(v_t)$; thus we also conclude that $L(x_i) \subseteq L(v_t)$ for $1 \leq i \leq k$. Thus $L(v_t)$ is a superstring of the labels of a vertex of $\mathcal{C}$ that does not belong to the initial $(d+2)$-clique, and the result is proved by induction.

Property 3 Let v_t be a vertex of $A(d,t)$ created at step $t \geq 1$. For $1 \leq i \leq d+2$, if

i does not appear in $L(v_t)$, then v_t is a neighbor of a vertex v' of the initial $(d+2)$-clique, such that $L(v') = i'$.

Proof. By induction on t. When $t = 1$, any vertex v_1 constructed at step 1 is assigned label i, where i' is the only vertex of the initial $(d+2)$-clique v_t is not connected to; thus, by construction, the property is satisfied.

Now we suppose that the property is true for $1 \leq t' \leq t-1$, $t \geq 2$, and we will show it then holds for t as well. As for the previous property, we consider a vertex v_t created at step t, and the $(d+1)$-clique $\mathcal{C}$ it is connected to.

Suppose v_t is connected to a vertex w_{t-1} that was created at step $t-1$. However, w_{t-1} was created itself thanks to a $(d+1)$-clique $\mathcal{C}'$ composed of vertices $x_1, x_2 \ldots x_{d+1}$. Among those $d+1$ vertices, only one, say x_p, does not belong to $\mathcal{C}$. W.l.o.g., suppose that $k \leq d+1$ such vertices, $x_1, x_2 \ldots x_k$ do not belong to the initial $(d+2)$-clique. Now suppose that i does not appear in $L(v_t)$; then i appears as the first digit of one of the $L(x_j)$'s, $j \in [1, p-1] \cup [p+1, d+1]$, or of $L(w_{t-1})$ (by Property 1). However, $L(x_j) \subseteq L(w_{t-1}) \subseteq L(v_t)$ for $1 \leq j \leq k$ (by Property 2). Thus, neither w_{t-1} nor any vertex among the x_j's, $1 \leq j \leq k$, contains the digit i in its label. Hence, only a vertex y from the initial $(d+2)$-clique can have i in its label, and thus $L(y) = i'$. Hence it suffices to show that v_t and y are neighbors to prove the property. The only case for which this would not happen is when $y = x_p$; we will show that this is not possible. Indeed, by construction of the labels, the first digit of $L(v_t)$ is the only integer not appearing as first digit of the labels of the vertices of $\mathcal{C}$, that is $w_{t-1}, x_1, x_2 \ldots x_{p-1}, x_{p+1} \ldots x_{d+1}$. However, the fact that we suppose $y = x_p$ means that no vertex of $\mathcal{C}$ contains i in its label. Thus this would mean that the first digit of $L(v_t)$ is i, a contradiction. Thus, v_t is connected to y with $L(y) = i'$, and the induction is proved.

4. Routing by shortest path

Now we describe the routing protocol between any two vertices u and v of $A(d, t)$, with labels respectively equal to $L(u)$ and $L(v)$. We note that since $A(d, 0)$ is isomorphic to the complete graph K_{d+2}, we can assume $t \geq 1$. The routing protocol is unusual here in the sense that the routing is done both from u and v, until they reach a common vertex. Hence, the routing strategy will be used simultaneously from u and from v. In order to find a shortest path between any two vertices u and v, the routing protocol is as follows. First we compute the longest common suffix $LCS(L(u), L(v))$ of $L(u)$ and $L(v)$. We distinguish two cases:

(i) If $LCS(L(u), L(v)) = \emptyset$:

   (a) Simultaneously from u and v (say, from u): let $u = u_0$ and go from u_i to u_{i+1}, $i \geq 0$ where u_{i+1} is the neighbor of u_i with shortest label.
   (b) Stop when u_k is a neighbor of the initial $(d+2)$-clique.
      Let $\bar{L}(u_k)$ (resp. $\bar{L}(v_{k'})$) be the integers not present in $L(u_k)$ (resp. $L(v_{k'})$), and let $S = \bar{L}(u_k) \cap \bar{L}(v_{k'})$.

1. If $S \neq \emptyset$, pick any $l \in S$, and close the path by taking the edge from u_k to l', and the edge from l' to $v_{k'}$.

2. If $S = \emptyset$, route from u_k to any neighbor l'_1 (belonging to the initial $(d+2)$-clique) of u_k, and do similarly from $v_{k'}$ to a neighbor l'_2 (belonging to the initial $(d+2)$-clique) of $v_{k'}$. Then, take the edge from l'_1 to l'_2 and thus close the path from u to v.

(ii) If $LCS(L(u), L(v)) \neq \emptyset$, then let us call *the least common clique* of u and v, or $LCC(u,v)$, the $(d+2)$-clique composed of the vertex with label $LCS(L(u), L(v))$ and the $d+1$ vertices forming the $(d+1)$-clique that generated the vertex of label $LCS(L(u), L(v))$. We simultaneously route from u and v to (respectively) u_k and $v_{k'}$, going each time to the neighbor with $LCS(L(u), L(v))$ as label suffix, and having the shortest label. Similarly as above, we stop at u_k (resp. $v_{k'}$), where u_k (resp. $v_{k'}$) is the first of the u_i's (resp. of the v_j's) to be a neighbor of $LCC(u,v)$. Then there are two subcases, depending on $Q = L(u_k) \cap L(v_{k'})$.

(a) If $Q \neq \emptyset$, close the path by going to any vertex w with label l, $l \in Q$.

(b) If $Q = \emptyset$, then route from u_k (resp. $v_{k'}$) to any neighbor w_1 (resp. w_2) in $LCC(u,v)$, and close the path by taking the edge (w_1, w_2), which exists since both vertices w_1 and w_2 belong to the same clique $LCC(u,v)$.

Proposition 1 The above mentioned routing algorithm is valid, and produces a shortest path.

Proof. Let us first give the main ideas for the validity of the above routing protocol. Take any two vertices u and v. By construction of $L(u)$ and $L(v)$, the longest common suffix $LCS(L(u), L(v))$ indicates to which $(d+2)$-clique u and v have to go. We can consider this as a way for u and v to reach their least common ancestor in the graph of cliques induced by the construction of $A(d,t)$, or the "*least common clique*". In Case (1), this least common clique is the initial $(d+2)$-clique ; thus, u and v have to get back to it. In Case (2), the shortest path does not go through the initial $(d+2)$-clique, and the least common clique of u and v, say $LCC(u,v)$, is indicated by the longest common suffix $LCS(L(u), L(v))$. In other words, the length of $LCS(L(u), L(v))$ indicates the depth of $LCC(u,v)$ in the graph of cliques induced by the construction of $A(d,t)$. In that case, the routing is similar as in Case (1), except that the initial $(d+2)$-clique has to be replaced by the clique $LCC(u,v)$. Hence, the idea is to adopt the same kind of routing, considering only neighbors that also have $LCS(L(u), L(v))$ as suffix in their labels.

When this least common ancestor is determined, one can see, still by construction, that the shortest route to reach this clique (either from u or v) is to go to the neighbor which has smallest label, since the length of the label indicates at which step the vertex was created. Indeed, the earlier the neighbor w was created, the smaller the distance from w to the least common clique is.

After we have reached, from u (resp. from v), a vertex u_k (resp. $v_{k'}$) that is a neighbor of the least common clique, the last thing we need to know is whether u_k and

$v_{k'}$ are neighbors. Thanks to Property 3, we know that looking at $L(u_k)$ and $L(v_{k'})$ is sufficient to answer this question. More precisely:

- In Case (1)(b)-i, u_k and $v_{k'}$ share a neighbor in the initial $(d+2)$-clique (by Property 3). All those common neighbors have label l', where $l \in S$. Hence, if we pick any $l \in S$, then there exists an edge between u_k and l', as well as an edge between l' and $v_{k'}$.

- In Case (1)(b)-ii, u_k and $v_{k'}$ do not share a neighbor in the initial $(d+2)$-clique. Hence, taking a route from u_k (resp. $v_{k'}$) to any neighbor l'_1 (resp. l'_2) belonging to the initial $(d+2)$-clique, we can finally take the edge from l'_1 to l'_2 (which are neighbors, since they both belong to the initial $(d+2)$-clique) in order to close the path from u to v.

- In Case (ii)(a), u_k and $v_{k'}$ share a neighbor in $LCC(u,v)$. Hence we can close the path by going to any vertex w with label l, $l \in Q$, since w is a neighbor of both u_k and $v_{k'}$.

- In Case (ii)(b), u_k and $v_{k'}$ do not share a neighbor in $LCC(u,v)$. Hence we route from u_k (resp. $v_{k'}$) to any neighbor w_1 (resp. w_2) in $LCC(u,v)$, and we close the path by taking the edge (w_1, w_2). This edge exists since both vertices w_1 and w_2 belong to the same clique $LCC(u,v)$.

5. Conclusion

We have proposed an expanded deterministic Apollonian network model, which represents a transition for degree distribution between exponential and power law distributions. Our model successfully reproduces some remarkable characteristics in many natural and man-made networks. We have also introduced a vertex labeling for these networks. The length of the label is optimal. Using the vertex labels, it is possible to find in an efficient way a shortest path between any pair of vertices. Nowadays, efficient handling and delivery in communication networks (e.g. the Internet) has become one important practical issue, and it is directly related to the problem of finding shortest paths between any two vertices. Our results, therefore, can be useful when describing new communication protocols for complex communication systems.

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