Vertex Normalordering as a Consequence of Nonsymmetric Bilinearforms in Clifford Algebras

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Abstract

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We consider Clifford algebras with nonsymmetric bilinear forms, which are isomorphic to the standard symmetric ones, but not equal. Observing, that the content of physical theories is dependent on the injection $\oplus^n \bigwedge \mathcal{V}^{(n)} \to CL(\mathcal{V}, Q)$ one has to transform to the standard construction. The injection is of course dependent on the antisymmetric part of the bilinear form. This process results in the appropriate vertex normalordering terms, which are now obtained from the theory itself and not added ad hoc via a regularization argument.

1   Introduction

Nonlinear spinor equations play an important role in several branches of physics. They appear in high energy, nuclear or solid state physics. The most recent examples are the Heisenberg, Nambu, Jona-Lasinio like models [1, 2] of elementary particle theory or nuclear physics. Even nonlinear sigma models bear an analogous structure [3]. In solid state physics the Hubbard model [4] is a widespread theoretical tool in describing phenomena from super conductivity up to spin chains and so on.

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The general structure of such models is of the form kinetic term equals a cubic interaction term with an arbitrary Vertex.

$$(\sum i\gamma^\mu\partial_\mu - m)_{II'}\Psi_{I'} \;=\; gV_{II'I''I'''}\Psi_{I'}\Psi_{I''}\Psi_{I'''}. \tag{1}$$

Here $\sum\gamma^\mu\partial_\mu$ is the Dirac–operator, with euclidean or lorentzian signature. The mass could be zero. With the multi index $I = \{K, \Lambda\}$ we represent the spinor and its adjoint by Λ and the other algebraic and spatio temporal indices by K. If we have fixed an adjoint spinor, the quadratic form of the Clifford algebra is already determined. A suitable quantization procedure also has to be applied.

There are several problems with these equations, which we want to consider now.

i) The equations are not renormalizable, because g will be in general a dimensional quantity.

ii) In order to define a unique quantization procedure one has to introduce an ordering, as say timeordering or on a space like hypersurface the (anti)symmetric one.

iii) In the language of diagrams, you have to consider only connected ones, by introducing a normalordering procedure.

For point one, there seems to be no principal problem in solid state physics, because there may be a physically motivated cut off at the Brillouin zone. In the case of particle physics, there are several ad hoc regularizations. An approach to these topics will be given elsewhere [5].

The second point is usually solved by using causality arguments, which introduce a natural ordering in the polynomials of the fields at hand. Therefore timeordered products are used in the covariant formulation, see for example [6]. If we would prefer the Hamilton formalism, the (anti)symmetric ordering could be used.

At this stage the definition process of the theory should stop, because including a somehow given Vacuum state[1] $|0>_{Ph}$, which gives the representation of the field operators $\prod_{Ph}(\Psi_I)$, all quantities are formally[2] defined.

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[1] One should be able to calculate this vacuum state in a nonlinear theory, by solving the dynamical problem at hand in a sort of self consistent problem.

[2] Formally, because one has to show uniqueness and existence of the defined objects too, which is a nontrivial problem.

Now the third point causes trouble, because the transformation to the connected components yields infinities. In Fock space[3] this transformation equals a Wick–Dyson normalordering of the vertex [8]. This process results in the desired connected amplitudes and so called contractions, which for bilinear and higher order terms yields singularities at least on the lightcone. In quantum theory these contractions are related to the finite ground state energies, which when field quantized become infinite. In this way, and by the convention that the vacuum has no nonzero quantum numbers a vertex regularization is also introduced. The field equations read now

$$(\sum i\gamma^\mu \partial_\mu - m)_{II'}\Psi_{I'} = gV_{II'I''I'''}{:}\Psi_{I'}\Psi_{I''}\Psi_{I'''}{:}. \tag{2}$$

With the physical propagator

$$P_{II'} := {}_{Ph}<0|\mathcal{T}(\Psi_I\Psi_{I'})|0>_{Ph} \tag{3}$$

the vertex term changes to

$${:}\Psi_{I'}\Psi_{I''}\Psi_{I'''}{:} = \Psi_{I'}\Psi_{I''}\Psi_{I'''} + P_{I'I''}\Psi_{I'''} - P_{I'I'''}\Psi_{I''} + P_{I''I'''}\Psi_{I'}. \tag{4}$$

But this procedure is nothing but a shift of the problem from one equation into the other. With this definition the timeordered equation becomes singular, and hence the whole theory is ill defined.

In this note we want to show, how an embedding of the theory in a Clifford algebra structure can overcome this problem. Therefore we consider nonsymmetric bilinearforms and the associated Clifford algebras. The transformation from such algebras to the symmetric ones is an isomorphism, but the linear space of antisymmetric p-vectors is moved. As they carry the physical information, this is therefore altered.

2   Clifford Algebras with nonsymmetric bilinearforms

The Clifford Algebra entered physics with Pauli and Dirac [9], who used it to linearize the D'Alambertian. So we should learn more about this procedure. Let Q be a nondegenerate quadratic form, $\mathcal{V}$ a vector space over R or C, then the Clifford map

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[3] The Fock space is only appropriate in free theories or in perturbation theory, which proves not useful in our case. A nonperturbative treatment is given in [7].

is an injection from $\mathcal{V}$ into $CL(\mathcal{V}, Q)$ with the property that every square of a vector element of the Clifford Algebra is a scalar.

$$\gamma \quad : \quad V \quad \rightarrow \quad CL(\mathcal{V}, Q), \quad e_i \mapsto \gamma_i$$
$$x^2 = x \cdot x = Q(x) \in (R, C) \tag{5}$$

With linearization we have on a generating set of $\mathcal{V}$

$$(e_i + e_j)(e_i + e_j) \;\; = \;\; e_i^2 + e_j^2 + e_i e_j + e_j e_i$$
$$e_i e_j + e_j e_i \;\; = \;\; Q(e_i + e_j) - Q(e_i) - Q(e_j) =: 2G(e_i, e_j) \quad \in (R, C). \tag{6}$$

It is evident from this calculation, that the bilinear form G is symmetric. The whole algebra is now generated from reduced products of one-vectors. Let N be the set of ordered partitions of n pieces, $|\alpha|$ the cardinality of such a subset, and include the empty set. We define $1_A = e_0$, then an algebra element read

$$A \;\; := \;\; \sum_{\alpha \in N} a_\alpha e_\alpha = A_0 + A_1 \ldots + A_n$$
$$e_\alpha \;\; := \;\; e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}, \quad i_1 < i_2 \ldots < i_r, \quad |\alpha| = r \in N. \tag{7}$$

The wedge product means antisymmetric multiplication as in the Grassmann case. Indeed as a linear space these two constructions are identical. Thereby the Clifford algebra has the direct sum decompositions

$$CL(V, Q) \;\; = \;\; CL_+ \oplus CL_- \quad \text{as algebra, and}$$
$$CL(V, Q) \;\; = \;\; \oplus^n \bigwedge \mathcal{V}^{(n)} \quad \text{as linear space.} \tag{8}$$

But the product intermingles the grades. Let $< \quad >_r$ be the projector to the homogeneous part of grade r, then one has

$$A_r B_s \;\; = \;\; < AB >_{|r-s|} + < AB >_{|r-s+2|} + \ldots + < AB >_{r+s}, \tag{9}$$

were in the Grassmann case $A_r B_s = AB_{r+s}$ results.

Physicists consider the anticommuting elements of grade r as eg. scalars, spinors (vectors) , spintensors, (tensors) and so on. That is, the physical content of the theory is coded explicitly in this structure.

Now let us see, in which way it is possible to introduce nonsymmetric bilinear forms. It is obvious, that we have to leave the above construction, in favor of a more general one. This can be done by introducing algebra derivatives as proposed by Chevaley and Riesz [10, 11].

First of all, we introduce two more algebraic constructions for further use. An involution[4] J of period two and the Reversion $\tilde{}$ by the rules

$$
\begin{aligned}
J \quad : \quad & J^2 := id_A \\
& J(XY) := J(X)J(Y) \\
& J(R,C) := (R,\bar{C}) \\
\tilde{} \quad : \quad & <\tilde{}>_{0+1} := id_{A_0+A_1} \\
& (XY)^{\tilde{}} := \tilde{Y}\tilde{X}.
\end{aligned} \tag{10}
$$

Now we may introduce the desired formulae

$$
\begin{aligned}
a \lrcorner B \quad &:= \quad \frac{1}{2}(aB - J(B)a); \quad a \in \mathcal{V}; \quad B \in A \\
a \dot{\wedge} B \quad &:= \quad \frac{1}{2}(aB + J(B)a),
\end{aligned} \tag{11}
$$

herewith we may decompose the Clifford product to

$$
aB \quad = \quad a \lrcorner B + a \dot{\wedge} B. \tag{12}
$$

The contraction $\lrcorner$ is a graded derivative of degree -1, as can be seen as follows (graded Leibnitz rule)

$$
\begin{aligned}
a \lrcorner (bc) \quad &:= \quad \frac{1}{2}(abc - J(bc)a) \\
&= \quad \frac{1}{2}(abc - J(b)ac + J(b)ac - J(b)J(c)a) \\
&= \quad (a \lrcorner b)c + J(b)(a \lrcorner c).
\end{aligned} \tag{13}
$$

With $bc = 1$ we have $a \lrcorner 1 = 2a \lrcorner 1$, so $a \lrcorner (R,C) = 0$, from which we could proof by induction the homogeneity of $a \lrcorner B_r$. Obviously the contraction is linear, that is

$$
(\alpha X + \beta Y) \lrcorner A \quad = \quad \alpha X \lrcorner A + \beta y \lrcorner A. \tag{14}
$$

These properties together state that $\lrcorner$ is an algebra derivation. One can easy proof the useful formulas [12]

$$
\begin{aligned}
(u \wedge v) \lrcorner X \quad &= \quad u \lrcorner (v \lrcorner (X)) \\
a \lrcorner (x_{i_1} \wedge \ldots \wedge x_{i_n}) \quad &= \quad \sum_{i=1}^{n}(-)^{i-1}(a \lrcorner x_i)(x_1 \wedge \ldots \wedge_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_n) \\
det(x_i \lrcorner x_j) \quad &= \quad (x_n \wedge \ldots \wedge x_1) \lrcorner (x_1 \wedge \ldots x_n) \\
&= \quad x_n \lrcorner (x_{n-1} \lrcorner \ldots (x_1 \lrcorner (x_1 \wedge \ldots \wedge x_n)) \ldots).
\end{aligned} \tag{15}
$$

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[4] This property is sometimes called conjugation.

Now the asymmetry of this result is obvious, and we may define an arbitrary non degenerate bilinear form B exactly as the contraction. In a not necessarily orthonormalized system of generating elements e_i we have

$$B = [B_{ij}] = [e_i \lrcorner e_j].$$ (16)

The injection, introduced by Chevalley, $\wedge \mathcal{V} \to CL(\mathcal{V}, Q)$, is of course known by physicists in the disguise of the Kähler–Atiyah isomorphism.[5]

Therefore we have identified the Clifford algebra as a subalgebra of $End(\oplus_n \wedge \mathcal{V}^n)$, the endomorphism algebra of the Grassmann algebra. A very explicit example will be given in the appendix, in a manner closely related to the work of Lounseto.

Clearly, if we had chosen J to be the common use involution on $\mathcal{V}$, that is $J(\mathcal{V}) = -\mathcal{V}$, we would reobtain the original formulas, with a symmetric bilinearform

$$G_{ij} = \frac{1}{2}(e_i e_j - J(e_j)e_i) = \frac{1}{2}(e_i e_j + e_j e_i).$$ (17)

Thus, if there exists a distinct involution of period two, we have the desired extension.

We are now able to construct a new generating system of the Clifford algebra, which is antisymmetric with respect to the reversion, by using the corresponding wedge product $\dot{\wedge}$.

$$\{e_0; e_{i_1}; e_{i_1}\dot{\wedge}e_{i_2}; e_{i_1}\dot{\wedge}e_{i_2}\dot{\wedge}e_{i_3}; \ldots\}, \quad \forall i_n; \quad i_1 < i_2 < \ldots$$ (18)

But now we have

$$e_{i_1}\dot{\wedge}e_{i_2} = e_{i_1}e_{i_2} - e_{i_1}\lrcorner e_{i_2} = e_{i_1}e_{i_2} - B_{i_1 i_2},$$ (19)

which is not antisymmetric with respect to the reversion as one can see as follows

$$
\begin{aligned}
(e_{i_1}\dot{\wedge}e_{i_2})^{\tilde{}} &= (e_{i_1}e_{i_2} - B_{i_1 i_2})^{\tilde{}} \\
&= e_{i_2}e_{i_1} - B_{i_2 i_1} + (B_{i_2 i_1} - B_{i_1 i_2}) \\
&= e_{i_2}\dot{\wedge}e_{i_1} + (B_{i_1 i_2}^T - B_{i_1 i_2}) \neq e_{i_2}\dot{\wedge}e_{i_1}
\end{aligned}
$$ (20)

Here T means matrix transposition. To avoid such a situation, and for establishing the reversion as the (hermitean) transpose of the matrix representation, we are

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[5] See [12] for an account on that, and for a review on the historical development.

forced to choose the bi– and multivectors in a definite way. With $i_1 < i_2 < \ldots$ we set

$$
\begin{aligned}
e_{i_1 i_2} &:= \frac{1}{2}(e_{i_1}\dot{\wedge}e_{i_2} - e_{i_2}\dot{\wedge}e_{i_1}) \\
&= \frac{1}{2}(e_{i_1}e_{i_2} - B_{i_1 i_2} - e_{i_2}e_{i_1} + B_{i_2 i_1}) \\
&= \frac{1}{2}(e_{i_1}e_{i_2} - e_{i_2}e_{i_1}) - \frac{1}{2}(B_{i_1 i_2} + B_{i_2 i_1}) \\
&= e_{i_1} \wedge e_{i_2} - F_{i_1 i_2},
\end{aligned}
\tag{21}
$$

were B is now splitted into symmetric and antisymmetric parts $B = G_S + F_A$, with respect to the usual matrix transpose. We obtain the following rules, utilizing now the reversion and the *standard* involution.

$$
\begin{aligned}
\frac{1}{2}(e_{i_1}e_{i_2} + (e_{i_1}e_{i_2})\tilde{}) &= \frac{1}{2}(e_{i_1}e_{i_2} + e_{i_2}e_{i_1}) = G_{i_1 i_2} \\
\frac{1}{2}(e_{i_1}e_{i_2} - (e_{i_1}e_{i_2})\tilde{}) &= \frac{1}{2}(e_{i_1}e_{i_2} - e_{i_2}e_{i_1}) =: e_{i_1 i_2} \\
&= \frac{1}{2}(e_{i_1} \lrcorner e_{i_2} + e_{i_1} \wedge e_{i_2} - e_{i_2} \lrcorner e_{i_1} - e_{i_2} \wedge e_{i_1}) \\
&= e_{i_1} \wedge e_{i_2} + F_{i_1 i_2} \\
e_{i_1 i_2}\tilde{} &= e_{i_2 i_1} = -e_{i_1 i_2} \\
J(e_{i_1 i_2}) &= e_{i_1 i_2}.
\end{aligned}
\tag{22}
$$

A third order term will be given as

$$
\begin{aligned}
e_{i_1 i_2 i_3} &= \frac{1}{2}(e_{i_1}e_{i_2 i_3} + e_{i_2 i_3}e_{i_1}) \\
&= e_{i_1} \wedge e_{i_2} \wedge e_{i_3} + F_{i_1 i_2}e_{i_3} + F_{i_2 i_3}e_{i_1} - F_{i_1 i_3}e_{i_2}.
\end{aligned}
\tag{23}
$$

If we would like to have the transposition to act trivial on the matrix representation of the vector elements, we have to introduce a dual set of generating elements.

We finish this section, by recalling the main consequences of the analysis done, with respect to the application in the next section.

If there is a nonsymmetric part in the contraction, then the usual multivectors are not the desired algebraic elements. The nondiagonal part of the contraction leads to a refined treatment of the algebraic properties. The antisymmetric parts are incorporated in the multivectorial structure, where as the symmetric part should be handled with dual sets of generating elements[6].

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[6] The matrix transpose is only in this special situation equivalent to the reversion.

By looking at this constructions, we are forced to introduce a new kind of multi-vectors. As a Clifford algebra, the two constructions prove to be isomorphic, at least in the nondegenerate case. But the linear subspaces $\oplus^n \bigwedge \mathcal{V}^{(n)}$ and $\oplus^n \text{span}\{e_{i_1...i_n}\}$ are quite different represented.

3   Application to the nonlinear spinor field model

Now we want to have a look at the vertex term of the nonlinear spinor field theory. This should correctly be done in the functional space formulation, which exhibits the structure quite more clearly [7]. For brevity and simplicity, we will give our arguments direct on the level of the field operators.

The quantization of fermionic fields is in effect the introduction of a Clifford algebra, or CAR algebra as in this context usually named [6, 13].

$$\{\Psi_K^\dagger, \Psi_{K'}\}_+ \;\; = \;\; \delta_{KK'} \tag{24}$$

With our indexing $\Psi_I = \Psi_{K\Lambda} = \{\Psi_{K1}^\dagger; \Psi_{K2}\}$ we have[7]

$$\{\Psi_I, \Psi_{I'}\} \;\; = \;\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\Lambda\Lambda'} \delta_{KK'} \tag{25}$$

This relation can be rewritten in the form

$$\Psi_I \lrcorner \Psi_{I'} + \Psi_{I'} \lrcorner \Psi_I \;\; = \;\; 2G_{II'} = \delta_{II'} \tag{26}$$

which now can be extended to an arbitrary bilinear form B. We obtain in this way the antisymmetric part, exhibiting a new term

$$[\Psi_I, \Psi_{I'}] \;\; = \;\; 2F_{II'} + 2\Psi_I \wedge \Psi_{I'}. \tag{27}$$

From the Clifford algebraic point of view, this corresponds to the scalar and bivector part, if we use the usual wedge product.

This entity is in fact related with the propagator of the theory.

$$
\begin{aligned}
P_{II'} \;\; &= \;\; {}_{Ph}<0|T(\Psi_I \Psi_{I'})|0>_{Ph} \\
&= \;\; {}_{Ph}<0|\theta(t_I - t_{I'})\Psi_I \Psi_{I'} - \theta(t_{I'} - t_I)\Psi_{I'}\Psi_I|0>_{Ph}.
\end{aligned}
\tag{28}
$$

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[7] This is of course a special basis, we may call it a Witt basis [13]. If the hermitean conjugation is the usual one, then the connection to Fock space is very close [14]. Therefore we will expect to have a (in this formulation) invisible antisymmetric part. So it is essential to have non Fock–states.

For equal times we have

$$P^t_{II'} = \,_{Ph}<0|\Psi_I\Psi_{I'} - \Psi_{I'}\Psi_I|0>_{Ph,t=t'}$$
$$= \,_{Ph}<0|2F_{II'} + 2\Psi_I \wedge \Psi_{I'}|0>_{Ph,t=t'} . \tag{29}$$

Now the $F_{II'}$ are 'scalars', that is in field theory a distribution valued function, and act *not* as operators.

Looking in this way at the vertex term, we have antisymmetric products, and are free to choose the appropriate one, which absorbs the additional terms, resulting in the normalordering procedure. Of course, this should be done in such a way that the transposition and reversion behave in the right way, but here we will not bother about that[8].

By comparing [7]

$$:\Psi_{I'}\Psi_{I''}\Psi_{I'''}: = \Psi_{I'}\Psi_{I''}\Psi_{I'''} - P_{I'I''}\Psi_{I'''} + P_{I'I'''}\Psi_{I''} - P_{I''I'''}\Psi_{I'}$$
$$e_{i_1} \wedge e_{i_2} \wedge e_{i_3} = e_{i_1 i_2 i_3} - F_{i_1 i_2}e_{i_3} + F_{i_1 i_3}e_{i_2} - F_{i_2 i_3}e_{i_1}, \tag{30}$$

it is shown, that if we choose $P_{II'}$ as the antisymmetric part of the contraction, then we are forced to introduce the normalordering terms in the field equation from the beginning. This is, because we want the usual conjugation and the multivectorial construction to hold in the algebraic and matrix case.

For the field equation this yields

$$(\sum i\gamma^\mu \partial_\mu - m)_{II'}\Psi_{I'} = gV_{II'I''I'''}\Psi_{I'} \wedge \Psi_{I''} \wedge \Psi_{I'''}, \tag{31}$$

or

$$(\sum i\gamma^\mu \partial_\mu - m)_{II'}\Psi_{I'} + gV_{II'I''I'''}\{P_{I'I''}\Psi_{I'''} - P_{I'I'''}\Psi_{I''} + P_{I''I'''}\Psi_{I'}\} =$$
$$gV_{II'I''I'''}\Psi_{I'I''I'''}. \tag{32}$$

Omitting now the interaction term (RHS of 32) we are left with a still singular equation, but now the singularity is only the dynamical one. As proposed in the introduction, the dynamical singularities may also be treated in an algebraic manner, which will be shown elsewhere.

The Clifford algebraic point of view should of course be taken from the very beginning.

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[8] See the remarks in the appendix.

4 Conclusion

With help of some results obtained by studying Clifford algebras with non symmetric bilinear forms, we are able to understand the process of normalordering in a new and deeper way. Without this sort of tool, it seems hardly to be possible to recognize the algebraic difference between T and N ordered transition matrix elements. In fact they belong to quite different multivector constructions. In ordinary treatments the vertex normalordering is done ad hoc, simply motivated by obtaining an afterwards finite theory.

At least in the computation of composites one has to expect the appearance of nonsymmetric parts of the bilinear forms. This stems from the antisymmetric constructions of the composite, in which case the effective bilinearform should have such a part.

The next step is the observation, that the usually obtained divergencies are related to the dynamical ones. Therefore it is obvious, that they are irrelevant to the not yet well defined theory, because they evaporate when the theory is regularized. A posteriori the 'dot' procedure is thus justified. But the important thing is, that we have, even in this case, to choose an other timeordered equation. The construction itself gives the hint, that we should start from the very beginning with Clifford methods. Thereby Clifford analysis, or monogenetic function theory, should us provide a finite theory, from first principles.

Acknowledgement

Appendix

In the Appendix an example is given, in the spirit of Lounesto[12]. Because all used quantities can only be constructed explicitly in very low dimensional cases, we use the Pauli algebra. It is well known and the smallest Clifford algebra over the reals which exhibits a three–vector quantity.

The bilinear form is decomposed into symmetric and antisymmetric parts, using matrix transposition. We have the linear independent not normalized, not orthogo-

nal set $\{e_1, e_2, e_3\}$ spanning $\mathcal{V}$. The algebra is generated by

$$\{e_0; e_1, e_2, e_3; e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1; e_1 \wedge e_2 \wedge e_3\}. \tag{33}$$

In this basis the bilinear form is

$$
\begin{aligned}
B &= [B_{ij}] = [e_i \lrcorner e_j] = [g_{ij}] + [f_{ij}] \\
[g_{ij}]^T &= [g_{ij}] \\
[f_{ij}]^T &= -[f_{ij}].
\end{aligned}
\tag{34}
$$

Next we search for a matrix representation. This can be done [12] by Clifford multiplying from the right an algebra element with all elements of the algebraic basis and expanding the result in homogeneous parts. Those are written as columns of the matrices. Matrix multiplication corresponds to the Clifford product. Of course we have

$$[1] = [\delta_{ij}], \tag{35}$$

and we calculate as an example $[e_1]$

$$
\begin{aligned}
e_1 1 &= e_1 \\
e_1 e_1 &= g_{11} \\
e_1 e_2 &= e_1 \lrcorner e_2 + e_1 \wedge e_2 = g_{12} + f_{12} + e_1 \wedge e_2 \\
e_1 e_3 &= g_{13} + f_{13} + e_1 \wedge e_3 \\
e_1 (e_1 \wedge e_2) &= e_1 (e_1 e_2 - e_1 \lrcorner e_2) = g_{11} e_2 - (g_{12} + f_{12}) e_1 \\
e_1 (e_2 \wedge e_3) &= (g_{12} + f_{12}) e_3 - (g_{13} + f_{13}) e_2 + e_1 \wedge e_2 \wedge e_3 \\
e_1 (e_3 \wedge e_1) &= -g_{11} e_3 + (g_{13} + f_{13}) e_1 \\
e_1 (e_1 \wedge e_2 \wedge e_3) &= g_{11} e_2 \wedge e_3 + (g_{12} + f_{12}) e_3 \wedge e_1 + (g_{13} + f_{13}) e_1 \wedge e_2 \quad (36)
\end{aligned}
$$

The same can be done for the other elements, which yields

$$
[e_1] = \begin{bmatrix}
0 & g_{11} & g_{12}+f_{12} & g_{13}+f_{13} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -g_{12}-f_{12} & 0 & g_{13}+f_{13} & 0 \\
0 & 0 & 0 & 0 & g_{11} & -g_{13}-f_{13} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_{12}+f_{12} & -g_{11} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & g_{13}+f_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{11} \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & g_{12}+f_{12} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

$$[e_2] \;=\; \begin{bmatrix} 0 & g_{21}+f_{21} & g_{22} & g_{23}+f_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_{22} & 0 & g_{23}+f_{23} & 0 \\ 1 & 0 & 0 & 0 & g_{21}-f_{12} & -f_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{22} & -g_{12}+f_{12} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & g_{23}+f_{23} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & g_{12}-f_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[e_3] \;=\; \begin{bmatrix} 0 & g_{13}-f_{13} & g_{23}-f_{23} & g_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g_{23}+f_{23} & 0 & g_{33} & 0 \\ 0 & 0 & 0 & 0 & g_{13}-f_{13} & -g_{33} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & g_{23}-f_{23} & -g_{13}+f_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{33} \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & g_{13}-f_{13} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & g_{23}-f_{23} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

This 8×8 dimensional representation of the Pauli algebra is not reducible to a real 4 $\times$ 4 or complex 2×2 one. The matrix transposition is not the reversion, because the $[e_i]$ are not symmetric matrices. Also the trace is not an algebraic invariant object, because there are elements with non vanishing trace beside $[\delta^i_j]$, which means, that the trace is not a projection on to the homogenouse part of degree zero. So the matrix trace is not a linear form *in* the algebra, because there are algebra elements except $[\delta_{ij}]$ which are not traceless. The trace is clearly a linear form on the matrix representation, but into the field (R, C) and not in the image of the field in the algebra.

The volume element has nearly the bilinearform as entries in the vector–vector block. The element e_{123} reads

$$[e_{123}] \;=\; [e_1 e_2 e_3 - f_{12} e_3 + f_{13} e_2 - f_{23} e_1], \tag{37}$$

which yields a matrix not easy to display. The entries are linear, quadratic and cubic functions of the f_{ij} and g_{ij} parameters.

The same procedure can be done with the reordered basis, belonging to the dotted wedge product or with respect to the basis

$$\{e_1, e_2, e_3; e_{123}; e_0; e_{12}, e_{23}, e_{31}\}. \tag{38}$$

ordered in odd and even elements. The vector elements read

$$[e_1] \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -g_{12} & 0 & g_{31} \\ 0 & 0 & 0 & 0 & 0 & g_{11} & -g_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_{12} & -g_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ g_{11} & g_{12} & g_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & g_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & g_{12} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[e_2] \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -g_{22} & 0 & g_{23} \\ 0 & 0 & 0 & 0 & 1 & g_{12} & -g_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_{22} & -g_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ g_{12} & g_{22} & g_{23} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & g_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & g_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{22} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[e_3] \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -g_{23} & 0 & g_{33} \\ 0 & 0 & 0 & 0 & 0 & g_{13} & -g_{33} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & g_{23} & -g_{13} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ g_{13} & g_{23} & g_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{33} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & g_{13} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -g_{23} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which yields a much more convenient and symmetric form. If the bilinear form is in the symmetric part diagonal, then this representation becomes symmetric with respect to the trace. In this case trace and reversion are identical. The antisymmetric part has been absorbed fully in the multivectorial construction.

A full satisfactory representation could be obtained by using a dual set of generating algebra elements, to the above ones. Therefore let the Volume element be

$$E \quad := \quad e_{123} = -E^{\sim}$$

$$\begin{aligned} E^{-1} &= fracE\tilde{}E\tilde{}E \\ E\tilde{}E &= detG = |G|. \end{aligned} \tag{39}$$

Then we may construct

$$e^i := (-)^{i+1}e_{1...i-1i+1...n}E^{-1} \tag{40}$$

which is a generalization to nonsymmetric bilinearforms of the detailed results in [15].

Now the representation with the algebra basis X^i yields via $[e_i X^J]$ symmetric matrices, even if the symmetric part of B is nontrivial.

This form is the most distinguished and symmetric one. Transposition and conjugation are the usual operations, but for arbitrary B the representation is still of dimension 8×8.

In this light, we have to change the 'quantization' process, to use this dual elements. Therefore we should write

$$\Psi^I \Psi_{I\prime} + \Psi_{I\prime} \Psi^I = \delta^i_{I\prime}. \tag{41}$$

But now the dual set depends on the possibly varying volume element of the algebra, and makes the definition of 'creation' and 'destruction' operators position dependent. We may hope to get a better understanding of quantization on curved space in this way.

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