VERTEX OPERATOR ALGEBRAS WITH TWO SIMPLE MODULES - THE MATHUR-MUKHI-SEN THEOREM REVISITED

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ABSTRACT. Let V be a strongly regular vertex operator algebra and let $\mathfrak{ch}_V$ be the space spanned by the characters of the irreducible V-modules. It is known that $\mathfrak{ch}_V$ is the space of solutions of a so-called *modular linear differential equation (MLDE)*. In this paper we obtain a near-classification of those V for which the corresponding MLDE is irreducible and monic of order 2. As a consequence we derive the complete classification when V has exactly two simple modules. It turns out that V is either one of four affine Kac-Moody algebras of level 1, or the Yang-Lee Virasoro model of central charge $-22/5$. Our proof establishes new connections between the characters of V and Gauss hypergeometric series, and puts the finishing touches to work of Mathur, Mukhi and Sen who first considered this problem forty years ago.

1. INTRODUCTION

In a remarkable paper that was ahead of its time [15], Mathur, Mukhi and Sen put forward the idea of classifying two-dimensional conformal field theories according to the differential equation satisfied by the characters of the simple modules (primary fields at vacuum). These differential equations, now call called MLDEs (*modular linear differential equations*), are polynomials in the so-called Serre derivation with coefficients which are modular forms. (Further details are given below.)

Mathur, Mukhi and Sen pushed through their ideas in the basic case when the MLDE has order two and is *monic* (leading coefficient 1) corresponding to some theories containing just two two primary fields. They achieved a classification result in this case, however some ambiguities remained and their methods are mathematically incomplete.

The purpose of the present paper is to revisit the classification of Mathur, Mukhi and Sen. Taking advantage of recent advances in the theory of MLDEs, in particular the connections with *Gauss hypergeometric series* [6], and also the theory of rational vertex operator algebras [12], we obtain a complete result that is mathematically rigorous. In particular, we give a new description of the characters of the modules of several familiar VOAs in terms of hypergeometric series. Our main results are stated below as Main Theorems 1 and 2. In the rest of the Introduction we give a more detailed discussion of our results and methods of proof.

The setting for our results is the theory of *rational vertex operator algebras* (VOAs), and in particular VOAs V that are *strongly regular*. Informally, this means

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that V is well-behaved. An overview of the theory can be found in [12]. However MLDEs are not treated in [12], and we discuss them here because they figure prominently in the present work.

The weight 2 "Eisenstein" series is

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n.$$

Here, and below, τ lies in the complex upper-half plane H and $q := e^{2\pi i\tau}$ The series $E_2(\tau)$ is holomorphic throughout H. Its main importance for us is its occurrence in the differential operators (sometimes called "Serre", or "Ramanujan", derivatives)

$$D_k := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{12}E_2(\tau) = q\frac{d}{dq} - \frac{k}{12}E_2(\tau) \quad (k\in\mathbf{Z}).$$

The operator D_k acts on the space $\mathcal{F}$ of holomorphic functions in H and in this regard it has a basic invariance property. To describe this, let $\Gamma := SL_2(\mathbf{Z})$ be the homogeneous modular group. For a given k, Γ acts on the right of $\mathcal{F}$ by the k^{th} *stroke operator*

$$f|_k\gamma(\tau) := (c\tau+d)^{-k}f(\gamma\tau) \quad \text{for} \quad f\in\mathcal{F} \quad \text{and} \quad \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma.$$

Then we have ([9, Chapter X, § 5])

$$D_k(f)|_{k+2}\gamma(\tau) = D_k(f|_k\gamma)(\tau). \tag{1}$$

An MLDE is a differential equation of the form

$$\left(P_0D_k^n + P_1D_k^{n-1} + \cdots + P_{n-1}D_k + P_n\right)f = 0 \tag{2}$$

Here, each P_j is a holomorphic modular form on Γ of weight $m-(k+2n-2j)\geq 0$ for some given integer m, and $D_k^n := D_{k+2n-2}\circ\cdots\circ D_{k+2}\circ D_k$ (for additional details, see [15] and [13]). For example, the simplest MLDE of order 2, having $n=2, k:=0, m:=4$, is

$$\left(D_0^2 + k_1E_4(\tau)\right)f = 0 \quad (k_1\in\mathbf{C}) \tag{3}$$

where $E_4(\tau)$ is the standard weight 4 Eisenstein series on Γ.

Eq. (2) may be rewritten as a (complicated) traditional linear differential equation involving the derivatives of f, but this will not be useful for us. The formulation (2) together with (1) makes it clear that the textitsolution space is invariant under the stroke action $|_m$ of Γ, and this representation of the modular group is the *monodromy* of the MLDE [7].

Suppose that V is a strongly regular VOA. Let $\mathfrak{ch}_V$ denote the $\mathbf{C}$-linear space spanned by the q-characters of the irreducible V-modules. An important theorem of Zhu ([18], [5]) states that $\mathfrak{ch}_V\subseteq\mathcal{F}$, moreover $\mathfrak{ch}_V$ is a Γ-submodule with respect to the zeroth stroke operator $|_0$. Furthermore, we may use the *modular Wronskian* ([13], [15]) together with Zhu's theorem to show that if $\dim\mathfrak{ch}_V = n$ then $\mathfrak{ch}_V$ is the solution space of some MLDE (2) of order n.

In this way, given a strongly regular V with space of characters $\mathfrak{ch}_V$, we obtain some important arithmetic/representation-theoretic data, namely the representation $\rho : \Gamma\to GL(\mathfrak{ch}_V)$, and an MLDE with monodromy ρ. This is related to, though rather different from, the usual S- and T- matrices of rational conformal field theories (RCFTs).

Mathur, Mukhi and Sen proposed to *classify* RCFTs according to the MLDE associated to them. They considered in detail the case where the MLDE is (3) and the monodromy ρ is *irreducible*. In particular $\dim \mathfrak{ch}_V = 2$. This means that either

(i) V has *exactly two* irreducible modules and they have linearly independent characters, or else
(ii) V has more than two irreducible modules and their characters are *not linearly independent*.

Inasmuch as an irreducible module and its dual have identical characters, the second possibility is commonplace.

In this paper we give a rigorous and full account of the classification of strongly regular VOAS in case (i). We prove that are exactly five isomorphism classes of such VOAs, under the assumption that the monodromy ρ is irreducible. Most of our analysis also applies to case (ii), although there are some cases (of central charges $c = -6, -8$ and -10) which remain open.

It is convenient to record here the assumptions and notation relating to V that we will be operating under:

– V is a strongly regular, simple vertex operator algebra of central charge c.
– $\mathfrak{ch}_V$ is the space of q-characters of the irreducible V-modules.
$(*)$ – V has an irreducible module M of conformal weight h and the q-characters $Z_V(\tau) := q^{-c/24}\sum_{n\geq 0}\dim V_n q^n$ and $Z_M(\tau) := q^{h-c/24}\sum_{n\geq 0}\dim M_{h+n}q^n$ span $\mathfrak{ch}_V$.
– $\mathfrak{ch}_V$ is the solution space of the order 2, monic MLDE (3) and the associated monodromy representation ρ is *irreducible*.

In order to describe our main results, we recall (cf [14]) that the *Gauss hypergeometric series* is the function

$$ {}_2F_1(a', b', c'; z) := 1 + \sum_{n\geq 0} \frac{(a')_n (b')_n}{(c')_n}\frac{z^n}{n!}, \tag{4}$$

where $(a')_n$ is the *Pochhammer symbol* (or rising factorial)

$$(a')_n := a'(a'+1)(a'+2)\cdots(a'+n-1).$$

The series ${}_2F_1(a', b', c'; z)$ converges for all $a', b', c' \in \mathbf{C}$ unless c' is a nonpositive integer. It is a solution of the *Gauss hypergeometric differential equation*

$$\frac{d^2f}{dz^2} + \frac{(c' - (a'+b'+1)z)}{z(1-z)}\frac{df}{dz} - \frac{a'b'}{z(1-z)}f = 0.$$

The following two Theorems are the main results of the paper:

Main Theorem 1. Suppose that V is a vertex operator algebra satisfying the assumptions $(*)$. If $c \geq 0$ then V is isomorphic to one of seven affine algebras of level 1:

$$L_{A_1}(1,0)), L_{A_2}(1,0), L_{G_2}(1,0), L_{F_4}(1,0), L_{D_4}(1,0), L_{E_6}(1,0), L_{E_7}(1,0).$$

If $c < 0$ then either V is isomorphic to the *Yang-Lee model*, i.e., the discrete series Virasoro algebra $Vir_{c_{2,5}}$ of central charge $-22/5$; or V is one of a series of (unkown) VOAs of central charge $c = -6, -8$ or -10.

In all cases both known and unknown, $Z_V(\tau)$ and $Z_M(\tau)$ are modular functions of weight 0 on a congruence subgroup of $SL_2(\mathbf{Z})$ and they may be described (up to an overall scalar) in terms of a pair of rational numbers (a, b) and the Gauss hypergeometric series as follows:

$$Z_V(\tau) = K^a \cdot {}_2F_1(a, a+1/3, 2a+5/6; K), \quad Z_M(\tau) = K^b \cdot {}_2F_1(b, b+1/3, 2b+5/6; K),$$

where K is the level 1 hauptmodul on Γ defined by

$$K := \frac{1728}{j} := \frac{E_4^3(\tau) - E_6^2(\tau)}{E_4^3(\tau)},$$

according to the cases in Table 1.

Type	a	b	c
A_1	$-1/24$	$5/24$	1
A_2	$-1/12$	$1/4$	2
G_2	$-7/6$	$17/60$	$14/5$
D_4	$-1/6$	$1/3$	4
F_4	$-13/60$	$23/60$	$26/5$
E_6	$-1/4$	$5/12$	6
E_7	$-7/24$	$11/24$	7
$Vir_{c_{2,5}}$	$11/60$	$-1/60$	$-22/5$
??	$1/4$	$-1/12$	-6
??	$1/3$	$-1/6$	-8
??	$5/12$	$-1/4$	-10

TABLE 1. Values of a, b, c

With a slightly stronger hypothesis the unknown cases of Main Theorem 1 do *not* exist:

Main Theorem 2. Suppose that V is a vertex operator algebra satisfying the assumptions $(*)$, and suppose further that (up to isomorphism) V and M are the *only* simple V-modules. Then V is isomorphic to one of the following five VOAs:

$$L_{A_1}(1,0)),\ L_{G_2}(1,0),\ L_{F_4}(1,0),\ L_{E_7}(1,0),\ \mathsf{Vir}_{c_{2,5}}.$$

In our approach to the proofs of the Main Theorems, we first show that the q-characters of all irreducible modules are *modular functions on a congruence subgroup*. In the present situation we are able to prove this famous modular-invariance result based on recent advances in the theory of MLDEs [6]. We then closely consider the MLDE (2): we use a detailed knowledge of 2-dimensional congruence representations of Γ [11] to show that there are only 9 possibilities for the monodromy ρ. The description of the solutions of the MLDE in terms of Gauss hypergeometric series was given in [6], and this result is fundamental to our approach. We use it to show that there are only *finitely many* (14 in fact) possible values of the central charge c (and the *effective central charge* $\tilde{c}$) for a VOA satisfying the assumptions of the Main Theorems.[1] These are listed in Table 6. Our task is then to classify the VOAs according to this data. We use a number of classification results in the literature (summarized in Theorem 7) to show that of the fourteen possible sets of data, some *cannot* correspond to a VOA, while others characterize the VOAs uniquely.

[1]This finiteness result is somewhat surprising because, for example, there is no analogous result in dimension 3: there are *infinitely many* strongly regular VOAs with $\dim \mathfrak{ch}_V = 3$, and their c-values are unbounded.

All of these results are obtained in Section 2 under the assumptions of Main Theorem 1, in particular V may have more than two simple modules, although $\mathfrak{ch}_V$ is always assumed to have dimension 2. However there are three values of c which we cannot handle by these methods. To deal with these residual cases we must assume that V has exactly two irreducible modules. The reason for this is that we can then use our modular-invariance result to *explicitly identify* the q-characters as modular functions, and in particular we can write down the explicit S-matrix using known transformation laws for the modular functions in question. This is carried out in Section 3. In each case we obtain the curious contradiction that the S-matrix is *not symmetric*, thereby contradicting a basic fact of RCFT [8], and then Main Theorem 1 is a consequence.

2. Proof of Main Theorem 1

In this Section we discuss the proof of Main Theorem 1.

2.1. Modularity.

In this Subsection we do *not* need to assume that ρ is irreducible. We will prove

Theorem 1. *Let V be a vertex operator algebra satisfying the conditions $(*)$ and let $\rho:\Gamma\rightarrow GL(\mathfrak{ch}_V)$ be the representation of Γ furnished by the zeroth stroke action $|_0$. Then ρ is modular, i.e., $\ker\rho$ is a congruence subgroup of Γ. In particular, both $Z_V(\tau)$ and $Z_M(\tau)$ are modular functions of weight 0 on a congruence subgroup of $SL_2(\mathbf{Z})$.*

Proof. Let $M(\rho)$ denote the space of holomorphic vector-valued modular forms corresponding to ρ. This space is naturally $\mathbf{Z}$-graded by weight k:

$$M(\rho)=\bigoplus_{k\in\mathbf{Z}} M_k(\rho).$$

We assert that there is $F(\tau)\in M_k(\rho)$ for some integral weight k such that $F(\tau)$ has *bounded denominators*. Indeed, we may take $F(\tau):=\Delta(\tau)^kW(\tau)$ for some k, where

$$W(\tau):=\begin{pmatrix} Z_V(\tau)\\ Z_M(\tau)\end{pmatrix}$$

is the meromorphic vector-valued modular form defined by V. (There are no poles in H, but there may be poles at the cusps.) This assertion follows because $W(\tau)$ has integral Fourier coefficients, therefore the same is true for $F(\tau)$. And by choosing k large enough we can ensure that $F(\tau)$ is holomorphic at the cusps, hence it is a holomorphic vector-valued modular form.

Now we may apply Theorem 1.2 of [6], which says that if $M(\rho)$ contains a single nonzero vector-valued modular form with bounded denominators, then ρ is modular. The statement of the Theorem follows. □

Remark 2. There are exactly 54 equivalence classes of two-dimensional *irreducible* representations ρ that satisfy the conclusions of Theorem 1. They are explicitly listed in Tables 1–5 [11].

2.2. **The monic MLDE.** In this Subsection we take up consideration of the MLDE (3) which has $\mathfrak{ch}_V$ as its solution space. It will be convenient to deal with the *normalized* vector-valued modular form of weight 0

$$W_0(\tau) := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

whose components comprise a fundamental system of solutions of the MLDE. Thus

$$f_1(\tau){:=}Z_V(\tau){=}q^a{+}\cdots, \quad f_2(\tau){:=}(\dim M_h)^{-1}Z_M(\tau){=}q^b{+}\cdots,$$

where a, b are rational numbers satisfying $a{:=}{-}c/24$ and $b{:=}h{-}c/24$.

Lemma 3. *We have* $a{+}b{=}1/6$ *and* $ab{=}{-}k_1$.

Proof. We know [13] that $q{=}0$ is a regular singular point of (3), and that the corresponding indicial roots are a and b. The indicial equation is easily found [13] to be $x^2 - x/6 - k_1{=}0$, and the Lemma follows. Actually, our main need will be the formula $a + b{=}1/6$, which also follows immediately from the modular Wronskian argument, cf. [13], Theorems 3.7 and 4.3. □

Let us set

$$\rho(T){:=}\begin{pmatrix} e^{2\pi i m_1} & 0 \\ 0 & e^{2\pi i m_2} \end{pmatrix} \quad \text{with} \quad 0 \le m_j < 1.$$

We know by Theorem 1 that ρ has finite image, and in particular $\rho(T)$ has finite order. This implies that $m_1, m_2{\in}\mathbf{Q}$. Moreover $a{\equiv}m_1, b{\equiv}m_2 \pmod{\mathbf{Z}}$.

Lemma 4. *We have* $m_1{+}m_2{=}7/6$. *There are just 9 possibilities for the (unordered) pair* $\{m_1,\, m_2\}$ *as follows:*

$\{m_1,\, m_2\}$	$\{5/6, 1/3\}$	$\{3/4, 5/12\}$	$\{11/12, 1/4\}$	$\{23/24, 5/24\}$
$\{17/24, 11/24\}$	$\{53/60, 17/60\}$	$\{47/60, 23/60\}$	$\{41/60, 29/60\}$	$\{59/60, 11/60\}$

TABLE 2. Values of m_1 and m_2

Proof. Since $a{+}b{\equiv}m_1{+}m_2 \pmod{\mathbf{Z}}$ and $0{\le}m_1{+}m_2{<}2$, after Lemma 3 the only possibilities are $m_1{+}m_2{=}1/6$ or $7/6$. On the other hand, by Remark 2 there are just 54 isomorphism classes of irreducible ρ (irreducibility of ρ is one of our hypotheses), and they are uniquely determined by the pair $\{m_1,\, m_2\}$. Indeed, Tables 1–5 in [11] list all 54 possibilities, and we observe from these Tables that the case $m_1{+}m_2{=}1/6$ never occurs and that there are just nine choices of ρ with $m_1{+}m_2{=}7/6$. The corresponding pairs $\{m_1,\, m_2\}$ are as listed, and the Lemma is proved. □

2.3. **Hypergeometric series.** In this Subsection we show that $Z_V(\tau)$ and $Z_M(\tau)$ are given by hypergeometric series evaluated at the level 1 hauptmodul K, as in the statement of Main Theorem 1. We follow the arguments of [6]. First rewrite (3) as follows:

$$\theta^2(f) - \tfrac{1}{6}E_2\theta(f) - k_1E_4f = 0, \tag{5}$$

where

$$\theta := q\frac{d}{dq}.$$

We then switch variables, from q to j. As computed in [6], we obtain

$$\frac{d^2 f}{dj^2} + \frac{7j - 4\cdot 1728}{6j(j-1728)}\frac{df}{dj} - \frac{k_1}{j(j-1728)}f = 0, \tag{6}$$

which is nothing but the Gauss normal form (cf. [14])

$$\frac{d^2 f}{dJ^2} + \frac{C-(A+B+1)J}{J(1-J)}\frac{df}{dJ} - \frac{AB}{J(1-J)}f = 0, \tag{7}$$

with

$$J = K^{-1},\ \ C = \frac{2}{3},\ \ A+B = \frac{1}{6},\ \ AB = -k_1 = ab.$$

Note that A, B satisfy the same equations as a, b (Lemma 3), so that we may, and shall, take $A=a, B=b$. The pair of fundamental solutions of (7)) at ∞ are then the hypergeometric series

$$f_1 := K^a \cdot {}_2F_1(a, a+1/3, 2a+5/6; K), \quad f_2 := K^b \cdot {}_2F_1(b, b+1/3, 2b+5/6; K). \tag{8}$$

2.4. **Bounds for c and m.** In this Subsection we show that there are *only a finite number of possibilities* for the central charge c of V and the integer m defined to be the dimension of the first nontrivial graded piece V_1 of V. To achieve this we will use the description (8) of $Z_V(\tau)$ and $Z_M(\tau)$ as a hypergeometric series.

We continue with previous notation, so that $a=-c/24$, $b=h-c/24$ and

$$f_1 = Z_V(\tau) = q^a + \cdots, \quad f_2 = Z_M(\tau) = q^b + \cdots$$

(up to an overall integral scalar). Using the hypergeometric description (8) and the explicit formula (4) we find that, up to an overall scalar,

$$f_1(\tau) \sim \left(12^3 q(1 - 744q + 356652q^2 + \cdots)\right)^a$$
$$\times \left\{1 + \frac{12^3 a(a+1/3)}{2a+5/6}q + \left(-\frac{12^3\cdot 744a(a+1/3)}{2a+5/6} + \frac{12^6 a(a+1)(a+1/3)(a+4/3)}{2(2a+5/6)(2a+11/6)}\right)q^2 + \cdots\right\}.$$

Remark 5. This series *does* converge. Indeed, it will converge as long as $2a+5/6$ is not a nonpositive integer, and this follows from Lemma 4.

To write the first factor as a q-expansion, we use Newton's binomial expansion

$$(1+X)^a = \sum_{k=0}^{\infty} \binom{a}{k} X^k$$

with $X := -744q + 356652q^2 + \cdots$ to obtain

$$\begin{aligned} f_1 &= q^a\left\{1 - 744aq + \left(356652a + \frac{744^2 a(a-1)}{2}\right)q^2 + \cdots\right\} \\ &\quad \times \left\{1 + \frac{12^3 a(a+1/3)}{2a+5/6}q + \left(-\frac{12^3\cdot 744a(a+1/3)}{2a+5/6} + \frac{12^6 a(a+1)(a+1/3)(a+4/3)}{2(2a+5/6)(2a+11/6)}\right)q^2 + \cdots\right\} \\ &= q^a\left\{1 + 24a\left(\frac{(6a+2)}{12a+5}72 - 31\right)q + \cdots\right\} \end{aligned}$$

The first nontrivial coefficient of f_1 is therefore

$$\dim V_1 =: m = -c\left(\frac{(8-c)}{(10-c)}36 - 31\right) = \frac{c(5c+22)}{10-c}.$$

This formula is known. It appears, for example, on P. 368 of [16], where it also arises from consideration of the MLDE (3), but instead of hypergeometric series Tuite and Van use special properties of the *exceptional* VOAs that they are studying.

The previous display is equivalent to $5c^2 + (22+m)c - 10m = 0$ (note that $c{=}-24a{\neq}10$), so $c{=}\left(-(22{+}m) \pm \sqrt{m^2{+}484{+}244m}\right)/10$.

Because V is strongly regular then $c{\in}\mathbf{Q}$ (see [5]), so there is an integer s such that $s^2{=}m^2{+}244m{+}484{=}(m{+}122)^2{-}120^2$. Thus

$$s^2 + 120^2 = (m+122)^2 \tag{9}$$

and the solutions correspond to *Pythagorean triples* $(s, 120, m+122)$. (A triple of integers that may serve as lengths of sides of a (possibly degenerate) right triangle.)

There is an old and well-known algorithm (Euclid) that gives a parameterization of all Pythagorean triples. In our case there are only finitely many nonnegative integral pairs (s, m) that solve (9), and we may use Euclid's algorithm to readily find them all. They are set out in Table 3. We content ourselves by listing the resulting pairs. We also list the corresponding pairs of possible values of $c = -(m+22) \pm s)/10$ and $a = -c/24$, which we will need.

s	m	c	a
3599	3479	$-710,\ 49/5$	$355/12, -49/120$
896	782	$-170,\ 46/5$	$85/12,\ -23/60$
391	287	$-70,\ 41/5$	$35/12,\ -41/120$
209	119	$-35,\ 34/5$	$35/24,\ -17/60$
119	47	$-94/5,\ 5$	$47/60,\ -5/24$
64	14	$-10,\ 14/5$	$5/12,\ -7/60$
1798	1680	$-350,\ 48/5$	$175/12,\ -2/5$
442	336	$-80,\ 42/5$	$10/3,\ -7/20$
182	96	$-30,\ 32/5$	$5/4,\ -4/15$
22	0	$0,\ -22/5$	$0,\ 11/60$
1197	1081	$-230,\ 47/5$	$115/12, -47/120$
288	190	$-50,\ 38/5$	$25/12,\ -19/60$
27	1	$-5,\ 2/5$	$5/24,\ -1/60$
715	603	$-134,\ 9$	$67/12,\ -3/8$
35	3	$-6,\ 1$	$1/4,\ -1/24$
594	484	$-110,\ 44/5$	$55/12,\ -11/30$
126	52	$-20,\ 26/5$	$5/6,\ -13/60$
350	248	$-62,\ 8$	$31/12,\ -1/3$
50	8	$-8,\ 2$	$1/3,\ -1/12$
225	33	$-38,\ 7$	$19/12,\ -7/24$
160	78	$-26,\ 6$	$13/12,\ -1/4$
0	28	$-14,\ 4$	$7/12,\ -1/6$

TABLE 3. Values of s, m, c and a

We now compare the values of a in the fourth column of Table 3 with the values of m_j in Lemma 4. For we know that there is an index j such that $a{\equiv}m_j \pmod{\mathbf{Z}}$. A number of values of a do not survive this test, and those that do are listed in Table 4.

s	m	c	a
391	287	-70	$35/12$
209	119	-35	$35/24$
119	47	$-94/5$	$47/60$
64	14	$-10, 14/5$	$5/12, -7/60$
442	336	-80	$10/3$
182	96	-30	$5/4$
22	0	$-22/5$	$11/60$
288	190	$38/5$	$-19/60$
27	1	$-5, 2/5$	$5/24, -1/60$
35	3	$-6, 1$	$1/4, -1/24$
126	52	$26/5$	$-13/60$
50	8	$-8, 2$	$1/3, -1/12$
225	133	7	$-7/24$
160	78	6	$-1/4$
90	28	4	$-1/6$
126	52	-20	$5/6$

TABLE 4. Values of s, m, c and a

Next we record, for each a-value in Table 5 an initial segment of the q-expansion of $f_1{=}K^a \cdot {}_2F_1(a, a{+}1/3, 2a{+}5/6; K)$. These can be found in Table 5.

Thus the cases $a{=}35/12$, $35/24$, $47/60$, $5/4$, $10/3$ are eliminated because then f_1 has coefficients that are *not* integers. On the other hand, in the case $a{=}5/6$ we find that (up to an overall scalar) we have

$$f_2{\sim}q^{-2/3}(1{-}272q{-}34696q^2{-}1058368q^3{-}\ldots)$$

so that this possibility is eliminated on account of the negative coefficients. What remains is the list of possibilities in Table 6, where we also include the corresponding values of b and the *effective central charge* $\tilde{c}$. This invariant is discussed in Subsection 2.5, and calculated using Lemma 6. (Consideration of f_2 as in the case $a{=}5/6$ does not yield any useful information in these cases.)

2.5. **The effective central charge $\tilde{c}$.** In this Subsection we will show that some additional cases listed in Table 6 *cannot* correspond to strongly regular VOAs. To do this we make use of the *effective central charge* $\tilde{c}$ of V defined as follows:

$$\tilde{c}{:=}c{-}24h_{\min}$$

where $h_{\min}$ is defined to be the *minimum* of 0 and h. (Recall that h is the conformal weight of the irreducible V-module M.) By (1.3) in [2], a strongly regular VOA necessarily satisfies $\tilde{c}{>}0$. In the present situation we have

Lemma 6. *Exactly one of $-24a$, $-24b$ is positive, and this is equal to $\tilde{c}$.*

Proof. First observe from Table 3 that exactly one of a, b is negative. If $h{\geq}0$ then $h_{\min}{=}0$ and then $\tilde{c}{=}c{=}{-}24a$. On the other hand, if $h{<}0$ then $h_{\min}{=}h$ and furthermore $\tilde{c}{=}c{-}24h{=}c{-}24(b{+}c/24){=}{-}24b$. The Lemma follows. □

In the following omnibus Theorem we collect some further results, gleaned from [2], [3] and [12], having to do with the effective central charge $\tilde{c}$ in an arbitrary strongly regular VOA.

TABLE 5. q-expansion of f_1

a	f_1
$35/12$	$q^{35/12}(1+287q+\frac{847903}{23}q^2+\cdots)$
$35/24$	$q^{35/24}(1+119q+\frac{113358}{19}q^2+\cdots)$
$47/60$	$q^{47/60}(1+47q+\frac{15369}{17}q^2+\cdots)$
$5/12$	$q^{5/12}(1+14q+92q^2+456q^3+1848q^4+6580q^5+21141q^6+62806q^6+174777q^7+\cdots)$
$-7/60$	$q^{-7/60}(1+14q+42q^2+140q^3+350q^4+840q^5+1827q^6+3858q^7+7637q^8+\cdots)$
$10/3$	$q^{10/3}\left(1+336q+\frac{868136}{17}q^2)+\frac{1541266112}{323}q^3)+\frac{5323642484}{17}q^4+\frac{264979509920}{17}q^5+\cdots\right)$
$5/4$	$q^{5/4}\left(1+96q+\frac{49869}{13}q^2+\cdots\right)$
$5/6$	$q^{-6/5}(1+1292q+701246q^2+207599288q^3+36592296829q^4+3988939885028q^5+\cdots)$
$11/60$	$q^{11/60}(1+q^2+q^3+q^4+q^5+2q^6+2q^7+3q^8+\cdots)$
$-19/60$	$q^{-19/60}(1+190q+2831q^2+22306q^3+129276q^4+611724q^5+2511667q^6+\cdots)$
$5/24$	$q^{5/24}(1+q+3q^2+4q^3+7q^4+10q^5+17q^6+23q^7+35q^8+\cdots)$
$-1/60$	$q^{-1/60}(1+q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+\cdots)$
$1/4$	$q^{1/4}(1+3q+9q^2+19q^3+42q^4+81q^5+155q^6+276q^6+486q^7+\cdots)$
$-1/24$	$q^{-1/24}(1+3q+4q^2+7q^3+13q^4+19q^5+29q^6+43q^7+62q^8+\cdots)$
$-13/60$	$q^{-13/60}(1+52q+377q^3q^2+1976q^3+7852q^4+27404q^5+84981q^6+243230q^7+\cdots)$
$1/3$	$q^{1/3}(1+8q+36q^2+128q^3+394q^4+1088q^5+2776q^6+6656q^7+15155q^8+\cdots\cdots)$
$-1/12$	$q^{-1/12}(1+8q+17q^2+46q^3+98q^4+198q^5+371q^6+692q^7+1205q^8+\cdots)$
$-7/24$	$q^{-7/24}(1+133q+1673q^2+11914q^3+63252q^4+278313q^5+1070006q^6+\cdots$
$-1/4$	$q^{-1/4}(1+78q+729q^2+4382q^3+19917q^4+77274q^5+264664q^6+827388q^7+\cdots)$
$-1/6$	$q^{-1/6}(1+28q+134q^2+568q^3+1809q^4+5316q^5+13990q^6+34696q^7+\cdots)$

TABLE 6. Residual possibilities

s	m	c	a	b	$\tilde{c}$
64	14	$-10, 14/5$	$5/12, -7/60$	$-1/4, 17/60$	$6,\ 14/5$
22	0	$-22/5$	$11/60$	$-1/60$	$2/5$
288	190	$38/5$	$-19/60$	$29/60$	$38/5$
27	1	$-5, 2/5$	$5/24,\ -1/60$	$-1/24,\ 11/60$	$1,\ 2/5$
35	3	$-6,\ 1$	$1/4,\ -1/24$	$-1/12,\ 5/24$	$2,\ 1$
126	52	$26/5$	$-13/60$	$23/60$	$26/5$
50	8	$-8,\ 2$	$1/3, -1/12$	$-1/6, 1/4$	$4,\ 2$
225	133	7	$-7/24$	$11/24$	7
160	78	6	$-1/4$	$5/12$	6
90	28	4	$-1/6$	$1/3$	4

Theorem 7. *Assume that $V = \mathbf{C}\mathbf{1}\oplus V_1\oplus\cdots$ is a strongly regular VOA of central charge c. Let ℓ be the Lie rank of the Lie algebra V_1 (dimension of a maximal abelian subalgebra of V_1). Then the following hold:*
(a) *The Lie algebra V_1 is reductive.*
(b) $\ell \leq \tilde{c}$.
(c) *If $c{=}\ell{=}\tilde{c}$ then V is isomorphic to a lattice theory V_Λ for some even, positive-definite lattice Λ of rank ℓ.*
(d) *Suppose that $\tilde{c}{<}\ell{+}1$. Then $c{=}\ell{+}c_{p,q}$ where $c_{p,q}$ is a central charge in the Virasoro discrete series.*
(e) *If $L{\subseteq}V_1$ is a Levi factor of V_1 then the subVOA $U{:=}\langle L\rangle{\subseteq}V$ generated by L is isomorphic to a tensor product of affine algebras $L(\mathfrak{g}_i, k_i)$ at positive integral levels k_i. (U and V may have different conformal vectors.)*
(f) *If $\tilde{c}{=}2/5$ then V is isomorphic to the Virasoro (Yang-Lee) model with $c{=}{-}22/5$.*

Proof. Parts (a), (b) and (c) correspond to Theorems 1.1, 1.2 and 1.3 respectively of [2]. Part (d) is an immediate consequence of Theorem 7 of [12]. Part (e) follows from Theorem 1.1 of [3] and Theorem 3 of [12], while (f) is a restatement of Corollary 9 of [12]. □

ℓ	1	2	3	4	5	6	7	8	9	10
A_ℓ	3	8	15	24	35	48	63	80	99	120
B_ℓ	3	10	21	36	55	78	105	136	171	210
C_ℓ	3	10	21	36	55	78	105	136	171	210
D_ℓ	3	6	15	28	45	66	91	120	153	190
F_4	52									
G_2	14									
E_6	78									
E_7	133									
E_8	248									

TABLE 7. Dimensions of simple Lie algebras

2.6. **Proof of Main Theorem 1.** In this Subsection we complete the proof of Main Theorem 1. This involves a more detailed consideration of the possibilities listed in Table 6 based on the results of Theorem 7. The list of low-dimensional simple Lie algebras in Table 7 is also useful. First we deal with the 8 known cases.

Case $c{=}1$**.** From Table 6 we have $m{=}3, c{=}\tilde{c}{=}1$. By Theorem 7(a) and (b) V_1 is a reductive Lie algebra of dimension $m{=}3$ and Lie rank $\ell{\leq}1$. Thus we must have $V_1 \cong \mathfrak{sl}_2$, so that $\ell{=}1$. Now Theorem 7(c) applies and establishes that $V{\cong}V_{A_1}{\cong}L(A_1, 1)$.

Case $c{=}2$**.** This is similar to the previous Case. We have $m{=}8$ and $c{=}\tilde{c}{=}2$, so V_1 is a reductive Lie algebra of dimension 8 and $\ell{\leq}2$. The only possibility is $V_1{\cong}\mathfrak{sl}_3$, and we can conclude with Theorem 7(c) once more that $V{\cong}V_{A_2}{\cong}L(A_2, 1)$.

Case $c{=}4$**.** Here, V_1 is a reductive Lie algebra of dimension 28 and Lie rank $\ell{\leq}\tilde{c}{=}c{=}4$. By the Cartan-Killing classification of semisimple Lie algebras one checks that the only possibility is either $V_1{\cong}\mathfrak{so}_8$ or $V_1{\cong}G_2 \oplus G_2$, and by Theorem 7(c) we obtain $V{\cong}L_{D_4}{\cong}L(D_4, 1)$.

Case $c{=}6$**.** Here, V_1 is a reductive Lie algebra of dimension 78 and Lie rank $\ell{\le}\tilde{c}{=}c{=}6$. By the Cartan-Killing classification of semisimple Lie algebras the possibilities are $V_1{\cong}\mathfrak{e}_6, \mathfrak{sp}_{12}$ or $\mathfrak{so}_{13}$. In each case we have $\tilde{c}{=}c{=}\ell$ and by Theorem 7(c) we obtain $V \cong L_{E_6} \cong L(E_6, 1)$.

Case $c{=}7$**.** Here, V_1 is a reductive Lie algebra of dimension 133 and Lie rank $\ell{\le}\tilde{c} = c = 7$. By the Cartan-Killing classification of semisimple Lie algebras the only possibility is $V_1{\cong}\mathfrak{e}_7$ and by Theorem 7 (c) we obtain $V{\cong}L_{E_7}{\cong}L(E_7, 1)$.

This deals with the cases of affine algebras with simply-laced root systems. In other cases the argument is a bit more complicated:

Case $c{=}14/5$**.** Here, V_1 is a reductive Lie algebra of dimension 14 and Lie rank $\ell{\le}\tilde{c} = c = 14/5$. Thus $\ell{\le}2$ and by the Cartan-Killing classification of semisimple Lie algebras the only possibility is $V_1{\cong}\mathfrak{g}_2$. Now from Tables 5 and 6, the character $Z_V(\tau)$ is uniquely determined from the hypotheses of the main Theorem together with the numerical restrictions $c{=}14/5$ and $\dim V_1{=}14$. Because the affine algebra $L(G_2, 1)$ also satisfies these conditions then it follows $f_1{=}Z_V(\tau){=}q^{-7/60} + \cdots$ as given in Table 5 is exactly the graded character of $L(G_2, 1)$.

On the other hand, if $U{:=}\langle V_1\rangle$ is as in the statement of Theorem 7(e) then that result shows that $U{\cong}L(G_2, k)$ for some positive integer k. It follows from the last paragraph that the graded character of $L(G_2, k)$ is *majorized* by that of $L(G_2, 1)$ in the following sense: *every* coefficient in the graded character of $L(G_2, k)$ is *no greater* than the corresponding coefficient in the graded character of $L(G_2, 1)$.

Now $L(G_2, k)$ is constructed as a graded quotient of the universal VOA $M(G_2, k)$ associated with the Lie algebra G_2, and the (unique) maximal submodule of $M(G_2, k)$ is generated by $e_\theta(-1)^{k+1}\mathbf{1}$, where e_θ is the longest root (cf. [10, Chapter 6.6]). Because the graded dimension of $L(G_2, k)$ is *majorized* by that of $L(G_2, 1)$ in the sense of the previous paragraph, it follows that $k{=}1$.

Case $c{=}26/5$**.** Here, V_1 is a reductive Lie algebra of dimension 52 and Lie rank $\ell{\le}\tilde{c}{=}c{=}26/5$. Thus $\ell{\le}5$ and by the Cartan-Killing classification of semisimple Lie algebras the only possibility is $V_1{\cong}\mathfrak{f}_4$. The rest of the argument proving that $V{\cong}L(F_4, 1)$ is completely parallel to that of the previous case, except that of course we replace G_2 with F_4.

The remaining entry in Table 6 corresponding to a known VOA is the following:

Case $c{=}{-}22/5$**.** In this Case we have $\tilde{c}{=}2/5$ from Table 4, therefore by Theorem 7 (f), V is the Virasoro VOA in the discrete series with $c{=}{-}22/5$. Alternatively, we have $\dim V_1{=}0$ from Table 4, whence the identification of V follows from the characterization of the same Virasoro algebra given in [1].

Next we show by arguments similar to those already used that the cases with $c{=}2/5,\ 38/5,\ -5$ *do not* correspond to strongly regular VOAS.

Case $c{=}2/5$**.** Here, Table 6 informs us that $\tilde{c}{=}2/5$ and $\dim V_1{=}1$. It follows that $V{=}\mathbf{C}$ and therefore $\ell{=}1{>}\tilde{c}$, contradicting Theorem 7(b). Alternatively, we may apply Theorem 7 (e) to see that V is the Yang-Lee model with $c{=}{-}22/5$, a contradiction.

Case $c{=}38/5$**.** From Table 6 and various parts of Theorem 7, we find that V_1 is a reductive Lie algebra of dimension 190 and Lie rank $\ell{\le}7$. But there is *no* such Lie algebra, as we can see using Table 7. So this Case cannot occur.

Case $c{=}{-}5$**.** From Table 6, $\tilde{c}{=}1$ and $\dim V_1{=}1$, so $\ell{=}1$. So Theorem 7 (d) applies, and tells us that $c{=}1{+}c_{p,q}$. This is impossible because $c{-}1{=}{-}6$ is *never* equal to any $c_{p,q}$.

3. Proof of Main Theorem 2

In this Section we give the proof of Main Theorem 2. Essentially, we must handle the three remaining cases, where $c = -6,\ -8$ and -10. We will show that they cannot occur under the assumption that V and M are the only simple V-modules. The methods employed in Section 2.6 are less effective when dealing with these cases. Instead we will use the modularity Theorem 1 coupled with the fact [8] that the S-matrix is *symmetric*.

Case $c = -6$**.** We will need the explicit identification of f_1 and f_2 as modular functions of level 12. (The level is the least common of the denominators of $a = 1/4$ and $b = -1/12$.) In fact, we have

$$f_1(\tau){=}\Delta_3(\tau)/\eta(\tau)^2,\quad f_2(\tau){=}I_3(\tau)/\eta(\tau)^2,$$

where

$$\Delta_3(\tau){:=}\eta(3\tau)^3/\eta(\tau),\qquad I_3(\tau){:=}1+6\sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{d}{3}\right)q^n,$$

and $\left(\frac{d}{3}\right)$ is the Legendre symbol. (We use the provisional notation Δ_3 and I_3 as there is no standard way to denote the corresponding modular forms.) This can be checked in various ways: (a) show that the indicated modular forms solve the MLDE (3); (b) check that $\Delta_3(\tau)$ and $I_3(\tau)$ are holomorphic modular forms of weight 1 and level 12 and that the first few terms of their q-expansions agree with those of $\eta(\tau)^2 f_1(\tau)$ and $\eta(\tau)^2 f_2(\tau)$ respectively.

Using standard transformation laws, we find that

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\Big|_0 S = -\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -\frac{1}{3} \\ 6 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Thus the S-matrix for V is visibly *not* symmetric, and therefore V cannot exist.

Case 2 $c{=}{-}8$**.** We proceed as in Case 1. We find that

$$f_1 = \frac{\eta(2\tau)^8}{\eta(\tau)^8},\quad f_2 = \frac{2E_2(2\tau)^2 - E_2(\tau)}{\eta(\tau)^4}$$

and

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\Big|_0 S = \frac{1}{2}\begin{pmatrix} -1 & \frac{1}{8} \\ 24 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

We see that the S-matrix is not symmetric.

Case $c{=}{-}10$. Proceed as in Cases 1 and 2. We find that

$$f_1 = \frac{I_3(\tau)^3\Delta_3(\tau)^3}{\eta(q)^6},\quad f_2 = \frac{I_3(\tau)^3 + 54\Delta_3(\tau)^3}{\eta(q)^6}$$

and

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\Big|_0 S = \tfrac{1}{\sqrt{3}}\begin{pmatrix} -1 & -\frac{1}{27} \\ 54 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Once again, the S-matrix is not symmetric.

This completes the proof that the three cases where $c{=}{-}6,\ {-}8,\ {-}10$ *cannot* occur. Now our Main Theorem 2 follows from Main Theorem 1.

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