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Vertex Operators in Conformal Field Theory on $\mathbb{P}^1$ and Monodromy Representations of Braid Group

Dedicated to Professor Hirosi Toda on his 60th birthday

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§0. Introduction

The 2-dimensional conformal field theory was initiated by A.A. Belavin, A.N. Polyakov and A.B. Zamolodchikov [BPZ] and was developed by many physicists, e.g. [DF], [ZF] etc. In the paper [BPZ], the significance of the primary fields for this theory is pointed out. V.G. Knizhnik and A.B. Zamolodchikov [KZ] developed the theory with current algebra symmetry, proposed the notion of primary fields with gauge symmetry, and gave the differential equations of multipoint correlation functions.

Our aim in this paper is to give rigorous foundations to the work of [KZ], and to reformulate and develop the operator formalism in the conformal field theory on the complex projective line $\mathbb{P}^1$. The space $\mathscr{H}$ of operands is taken to be a sum $\mathscr{H}=\sum_{j=0}^{l/2}\mathscr{H}_j$ of the integrable highest weight modules $\mathscr{H}_j$ of the affine Lie algebra $\hat{\mathfrak{g}}=\mathfrak{sl}(2,\mathbb{C})\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ of type $A_1^{(1)}$. We fix the value ℓ (positive integer) of the central element c of $\hat{\mathfrak{g}}$ on the space $\mathscr{H}$. The Virasoro algebra $\mathscr{L}$ acts on each $\mathscr{H}_j$ through

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the Sugawara forms $L(m)$, $m \in \mathbb{Z}$. For each $X \in \mathfrak{sl}(2, \mathbb{C})$, the field operator $X(z)=\sum_{m\in Z} X(m)z^{-m-1}$ obeys the equations of motion:

$$[L(m), X(z)]=z^{m}\left(z\frac{d}{dz}+m+1\right)X(z).$$

The currents $X(z)$, $X \in \mathfrak{sl}(2, \mathbb{C})$ and the energy-momentum tensor $T(z)=\sum_{m\in Z} L(m)z^{-m-2}$ preserve each $\hat{\mathfrak{g}}$-module $\mathscr{H}_j$. Thus each space $\mathscr{H}_j$ may be considered as a free theory. In order to introduce operators describing the interactions in the theory, we define the vertex operators due to V.G. Knizhnik and A.B. Zamolodchikov [KZ].

The vertex operators play a central role in this paper. In Section 2, we show the existence and the uniqueness theorem of the vertex operators. In Section 3 we get the differential equations satisfied by N-point functions, which have only regular singularities. The properties of vertex operators are derived from these differential equations (called the fundamental equations). First, we get the convergence of compositions of vertex operators. The commutation relation of vertex operators is equivalently rephrased in terms of the connection matrix of the fundamental equations, and is calculated explicilty in a special case. The monodromies of the fundamental equations give rise to representations of the braid group B_N. We determine explicitly this monodromy representation in a more special case. In fact, it gives an irreducible representation of the Hecke algebra $H_N(q)$ of type A_{N-1}, where $q=\exp(2\pi\sqrt{-1}/(\ell+2))$. Here it is remarkable that the vacuum expectation values of the products of vertex operators provide canonical bases of these representation spaces and the commutation relations of vertex operators give a 'factorization' of the monodromy representations.

Fix a positive integer ℓ for the value of the central element c, and a half integer j with $0\leq 2j\leq \ell$, then there is a unique (up to isomorphisms) irreducible highest weight left $\hat{\mathfrak{g}}$-module $\mathscr{H}_j$ with a highest weight vector $u_j(j)$. The Lie algebra $\hat{\mathfrak{g}}$ has a decomposition $\hat{\mathfrak{g}}=\mathfrak{m}_+\oplus\mathfrak{g}\oplus\mathbb{C}c\oplus\mathfrak{m}_-$, where $\mathfrak{g}=\mathfrak{sl}(2, \mathbb{C})=\mathbb{C}F\oplus\mathbb{C}H\oplus\mathbb{C}E$ and $\mathfrak{m}_\pm=\mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]t^{\pm1}$ (see Section 1.1) The subspace $V_j=\{v \in \mathscr{H}_j;\ \mathfrak{m}_+v=0\}$ is an irreducible $\mathfrak{g}$-module of highest weight $2j$, i.e. of dimension $2j+1$.

We can define the corresponding irreducible highest weight right $\hat{\mathfrak{g}}$ (or $\mathfrak{g}$)-module $\mathscr{H}_j^\dagger$ (or $V_j^\dagger$) (and fix a highest weight vector $u_j^\dagger(j)$), and the nondegenerate bilinear pairing (called *vacuum expectation value*) $\langle\,|\,\rangle\colon \mathscr{H}_j^\dagger \times\mathscr{H}_j\to\mathbb{C}$ such that $\langle u_j^\dagger(j)\,|\,u_j(j)\rangle=1$ and $\langle va\,|\,w\rangle=\langle v\,|\,aw\rangle$ for any $v \in \mathscr{H}_j^\dagger$, $a \in \hat{\mathfrak{g}}$, $w \in \mathscr{H}_j$. Its restriction on $V_j^\dagger\times V_j$ is also nondegenerate.

Let $\mathscr{H}=\sum_{j=0}^{\ell/2}\mathscr{H}_j$ and $\mathscr{H}^\dagger=\sum_{j=0}^{\ell/2}\mathscr{H}_j^\dagger$. By an *operator* we mean a linear mapping $\Phi\colon \mathscr{H}\to\hat{\mathscr{H}}$, where $\hat{\mathscr{H}}$ is a completion of $\mathscr{H}$. Note that an operator Φ is characterized by a bilinear mapping $\hat{\Phi}: \mathscr{H}^\dagger \times \mathscr{H} \to \mathbb{C}$ defined by $\langle v|\hat{\Phi}|w\rangle = \langle v|\Phi(w)\rangle$ for any $v \in \mathscr{H}^\dagger$ and $w \in \mathscr{H}$. Two operators may not always be composable (see Section 2.1 for the definition of the composability).

For a positive half-integer j, a multi-valued, holomorphic, operator-valued function $\Phi(z)$ on the manifold $M_1 = \mathbb{C}^*$ is called a *vertex operator of spin j* if for any $u \in V_j$ and $z \in M_1$, $\Phi(z): V_j \otimes \mathscr{H} \to \hat{\mathscr{H}}$ satisfies the following:

(Gauge Condition) $\quad [X(m), \Phi(u; z)] = z^m \Phi(Xu; z) \qquad (X \in \mathfrak{g}, m \in \mathbb{Z})$;

(Equation of Motion) $\quad [L(m), \Phi(u; z)] = z^m \left\{ z\dfrac{d}{dz} + (m+1)\Delta_j \right\} \Phi(u; z)$

$$(m \in \mathbb{Z}),$$

for any $u \in V_j$ and $z \in M_1$, where the number $\Delta_j = (j^2+j)/(\ell+2)$ is called the *conformal dimension* of the vertex operator $\Phi(z)$ and $\Phi(u; z): \mathscr{H} \to \hat{\mathscr{H}}$ is an operator defined by $\Phi(u; z)(w) = \Phi(z)(u \otimes w)$ for $w \in \mathscr{H}$.

Remark (Proposition 2.4) that there are no vertex operators of spin j for $j > \ell/2$.

A triple $\mathrm{v} = \begin{pmatrix} j \\ j_2 j_1 \end{pmatrix}$ of nonnegative half integers j_2, j_1 and j is called a *vertex*. Put $\hat{\Delta}(\mathrm{v}) = \Delta_j + \Delta_{j_1} - \Delta_{j_2}$. Then the *Clebsch-Gordan condition*

$$|j_1 - j_2| \leq j \leq j_1 + j_2 \quad \text{and} \quad j_1 + j_2 + j \in \mathbb{Z}$$

for a vertex v is a condition for $\mathrm{Hom}_{\mathfrak{g}}(V_j \otimes V_{j_1}, V_{j_2}) \neq 0$. In this case $\mathrm{Hom}_{\mathfrak{g}}(V_j \otimes V_{j_1}, V_{j_2}) = \mathbb{C}$ and v is called a CG-vertex.

For a vertex $\mathrm{v} = \begin{pmatrix} j \\ j_2 j_1 \end{pmatrix}$ with $j_2, j_1 \leq \ell/2$, a vertex operator $\Phi(z)$ of spin j is called *of type* v, if $\Phi(u; z) = \Pi_{j_2} \Phi(u; z) \Pi_{j_1}$ for any $u \in V_j$, where Π_i is the projection of $\mathscr{H}$ (or $\hat{\mathscr{H}}$) onto $\mathscr{H}_i$ (or $\hat{\mathscr{H}}_i$ respectively). Then we get the condition for the existence of vertex operators:

Theorem 1 (Proposition 2.1 and Theorem 2.2).

i) *A vertex operator $\Phi(z)$ of type v is uniquely determined by the form (initial term) $\Phi_0 \in \mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^\dagger \otimes V_j \otimes V_{j_1}, \mathbb{C})$ defined by*

$$\Phi_0(v, u, w) = (z^{\hat{\Delta}(\mathrm{v})} \langle v|\Phi(u; z)|w\rangle)|_{z=0} \qquad (v \in V_{j_2}^\dagger, u \in V_j, w \in V_{j_1}).$$

ii) *There exists a nonzero vertex operator Φ of type $\mathrm{v} = \begin{pmatrix} j \\ j_2 j_1 \end{pmatrix}$ on $\mathscr{H}$, if and only if the vertex v is an ℓCG-vertex, that is, it satisfies the ℓ-constrained Clebsch-Gordan condition*:

$$|j_1 - j_2| \leq j \leq j_1 + j_2, \quad j_1 + j_2 + j \in \mathbb{Z} \quad \textit{and} \quad j_1 + j_2 + j \leq \ell.$$

Remark. i) The inequalities $j_1+j_2+j\leq\ell$ and $|j_1-j_2|\leq j\leq j_1+j_2$ imply the conditions $j_1, j_2, j\leq\ell/2$.

ii) Nonzero vertex operators of a fixed type v are unique up to a constant multiple. For each ℓCG-vertex $\mathrm{v}=\begin{pmatrix} j \\ j_2 j_1 \end{pmatrix}$, we choose and fix a nonzero element $\varphi_{\mathrm{v}}\in \mathrm{Hom}_{\mathfrak{g}}(V_j\otimes V_{j_1}, V_{j_2})=\mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^{\dagger}\otimes V_j\otimes V_{j_1}, \mathbb{C})\,(\cong\mathbb{C})$ and denote by $\Phi_{\mathrm{v}}(z)$ the associated vertex operator of type v with the initial term $\Phi_{\mathrm{v},0}=\varphi_{\mathrm{v}}$.

For each ℓCG-vertex $\mathrm{v}=\begin{pmatrix} j \\ j_2 j_1 \end{pmatrix}$, introduce the $\mathfrak{g}$-module $\mathscr{P}(\mathrm{v})$ defined by $\mathscr{P}(\mathrm{v})=\{\Phi_{\mathrm{v}}(u; z); u\in V_j\}$: $X\Phi_{\mathrm{v}}(u; z)=\Phi_{\mathrm{v}}(Xu; z)$ $(X\in\mathfrak{g})$.

We can show that any operators of the form $X(\zeta)$, $X\in\mathfrak{g}$, $T(\zeta)$ and vertex operators are composable. The composability of vertex operators is obtained by using the fact that the differential equations of N-point functions have only regular singular points.

Introduce the space $\mathcal{O}(\mathrm{v})$ of operators on $\mathscr{H}$ as the $\mathbb{C}$-vector space spanned by the set

$$\Big\{\frac{1}{(2\pi\sqrt{-1})^N}\int_{C_N}\cdots\int_{C_1} d\zeta_N\cdots d\zeta_1(\zeta_N-z)^{m_N}\cdots(\zeta_1-z)^{m_1}X_N(\zeta_N)\cdots$$
$$\cdots X_1(\zeta_1)\Phi(u; z);\ N\in\mathbb{Z}_{\geq 0},\ X_i\in\mathfrak{g},\ m_i\in\mathbb{Z}\ (1\leq i\leq N),\ u\in V_j\Big\},$$

where C_i's are contours around C_{i-1} such that 0 is outside C_N and z is inside C_1.

Introduce a $\hat{\mathfrak{g}}$-module structure and an $\mathscr{L}$-module structure in $\mathcal{O}(\mathrm{v})$ defined by

$$\hat{X}(m)A(z)=\frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^m X(\zeta)A(z)\in\mathcal{O}(\mathrm{v})$$

and

$$\hat{L}(m)A(z)=\frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^{m+1} T(\zeta)A(z)\in\mathcal{O}(\mathrm{v})$$

for $A(z)\in\mathcal{O}(\mathrm{v})$, $X\in\mathfrak{g}$, $m\in\mathbb{Z}$, and some contour C around z such that 0 is outside C.

Theorem 2 (Theorem 2.9). *For each ℓCG-vertex v, the $\mathfrak{g}$-module mapping Φ: $V_j\ni u\mapsto\Phi_{\mathrm{v}}(u; z)\in\mathscr{P}(\mathrm{v})$ is extended to the $\hat{\mathfrak{g}}$-isomorphism of $\mathscr{H}_j$ onto $\mathcal{O}(\mathrm{v})$.*

Here we summarize the relations satisfied by vertex operators:

Fundamental relations for vertex operators

Let $\Phi(z)$ be a vertex operator of spin j. Then

$$\hat{X}(m)\Phi(u;z)=0 \qquad (m\geq 1, X\in\mathfrak{g}, u\in V_j);$$
$$\hat{X}(0)\Phi(u;z)=[X(0),\Phi(u;z)]=\Phi(Xu;z) \qquad (X\in\mathfrak{g}, u\in V_j);$$
$$\hat{L}(m)\Phi(u;z)=0 \qquad (m\geq 1, u\in V_j);$$
$$\hat{L}(0)\Phi(u;z)=\Delta_j\Phi(u;z) \qquad (u\in V_j);$$
$$\hat{L}(-1)\Phi(u;z)=\frac{\partial}{\partial z}\Phi(u;z) \qquad (u\in V_j);$$
$$\hat{E}(-1)^{\ell-2j+1}\Phi(u_j(j);z)=0.$$

Remark that the last equation is derived from the structure of the irreducible $\hat{\mathfrak{g}}$-module $\mathscr{H}_j$ by using Theorem 2.

Now we call the vectors $|\mathrm{vac}\rangle=u_0(0)\in\mathscr{H}_0$ and $\langle\mathrm{vac}|=u_0^\dagger(0)\in\mathscr{H}_0^\dagger$ the *Virasoro vacuum*. They satisfies the equalities

$$X(m)|\mathrm{vac}\rangle=L(n)|\mathrm{vac}\rangle=0 \qquad (X\in\mathfrak{g}, m\geq 0, n\geq -1);$$
$$\langle\mathrm{vac}|X(m)=\langle\mathrm{vac}|L(n)=0 \qquad (X\in\mathfrak{g}, m\leq 0, n\leq 1).$$

For an N-ple $\mathbb{J}=(j_N,\cdots,j_1)$ of half integers with $0\leq 2j_i\leq\ell$, let $V^\vee(\mathbb{J})=V_{j_N}^\vee\otimes\cdots\otimes V_{j_1}^\vee$, and let $V_0^\vee(\mathbb{J})$ denote the invariant subspace of $V^\vee(\mathbb{J})$ under the diagonal $\mathfrak{g}$-action, where $V_j^\vee$ denotes the dual $\mathfrak{g}$-module of V_j. Let $\Phi_i(z_i)$ be a vertex operator of spin j_i $(1\leq i\leq N)$, then the vacuum expectation value of the composed operator

$$\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle=\langle\mathrm{vac}|\Phi_N(z_N)\cdots\Phi_1(z_1)|\mathrm{vac}\rangle$$

is considered as a $V^\vee(\mathbb{J})$-valued, formal Laurent series on $(z_N,\cdots,z_1)$ and is called an *N-point function* (of spin $\mathbb{J}$): If $\Phi_i(z_i)$ is of type $\mathbb{v}_i$ $(1\leq i\leq N)$,

$$\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle=\prod_{i=1}^N z_i^{-\Delta(\mathbb{v}_i)}\sum C_{m_N\cdots m_1}z_N^{-m_N}\cdots z_1^{-m_1},$$

where $C_{m_N\cdots m_1}\in V^\vee(\mathbb{J})$ and the sum is taken over integers $m_k\in\mathbb{Z}$ $(1\leq k\leq N)$ with $m_N\geq 0$ and $m_1\leq 0$.

Let π_i be the $\mathfrak{g}$-action on the i-th component of $V^\vee(\mathbb{J})$ and introduce the operator Ω_{ik} defined by

$$\Omega_{ik}=\frac{1}{2}\pi_i(H)\pi_k(H)+\pi_i(E)\pi_k(F)+\pi_i(F)\pi_k(E),$$

and $\Omega_i=\Omega_{ii}$ is the action of the Casimir element $\Omega=\frac{1}{2}HH+EF+FE$ on the i-th component of $V^{\vee}(\mathbb{J})$. Then $\Omega_{ik}=\frac{1}{2}\{(\pi_i+\pi_k)(\Omega)-\Omega_i-\Omega_k\}$ $(i\neq k)$ and $\Omega_i=2(j_i^2+j_i)$ id on $V^{\vee}(\mathbb{J})$.

Then we get a system of differential equations and a system of algebraic equations for N-point functions:

Theorem 3 (Theorem 3.1). *Let $\Phi_i(z_i)$ be a vertex operator of spin j_i $(1\leq i\leq N)$, then the N-point function $\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle$ satisfies the following equations:*

(I) *(projective invariance) For $m=-1, 0$ and 1,*

$$\sum_{i=1}^{N} z_i^m\left(z_i\frac{\partial}{\partial z_i}+(m+1)\Delta_{j_i}\right)\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle=0.$$

(II) *(gauge invariance) For any $X\in\mathfrak{g}$,*

$$\sum_{i=1}^{N}\pi_i(X)\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle=0.$$

(III) *For each $i=1,\cdots,N$,*

$$\left((\ell+2)\frac{\partial}{\partial z_i}-\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}\right)\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle=0.$$

(IV) *For each i $(1\leq i\leq N)$ and any $u_k\in V_{j_k}$ $(k\neq i)$,*

$$\sum_{\mathfrak{m}_i}\binom{L_i}{\mathfrak{m}_i}\sum_{k\neq i}(z_k-z_i)^{-m_k}\langle\Phi_N(E^{m_N}u_N;z_N)\cdots\Phi_i(u_{j_i}(j_i);z_i)\cdots\Phi_1(E^{m_1}u_1;z_1)\rangle =0,$$

where $\mathfrak{m}_i=(m_N,\cdots,\hat{m}_i,\cdots,m_1)\in(\mathbb{Z}_{\geq 0})^{N-1}$ with $\sum_{k\neq i}m_k=L_i=\ell-2j_i+ +1$ and $\binom{L_i}{\mathfrak{m}_i}$ is the multinomial coefficient.

Consider the systems E$(\mathbb{J})$ of differential equations and B$(\mathbb{J})$ of algebraic equations for $V_0^{\vee}(\mathbb{J})$-valued functions $\Phi(z_N,\cdots,z_1)$ on the manifold $X_N=\{z=(z_N,\cdots,z_1)\in\mathbb{C}^N;\ z_i\neq z_k\ (i\neq k)\}$:

$$\text{E}(\mathbb{J}):\quad\left((\ell+2)\frac{\partial}{\partial z_i}-\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}\right)\Phi(z_N,\cdots,z_1)=0\qquad(1\leq i\leq N),$$

$$\text{B}(\mathbb{J}):\ \sum_{\mathfrak{m}_i}\binom{L_i}{\mathfrak{m}_i}\prod_{k\neq i}(z_k-z_i)^{-m_k}\Phi(z_N,\cdots,z_1)(E^{m_N}u_N,\cdots,u_{j_i}(j_i),\cdots,E^{m_1}u_1) =0,$$

for each i $(1\leq i\leq N)$ and any $u_k\in V_{j_k}(k\neq i)$, where $\mathfrak{m}_i=(m_N,\cdots,\hat{m}_i,$

$\cdots, m_1) \in (\mathbb{Z}_{\geq 0})^{N-1}$ with $\sum_{k \neq i} m_k = L_i = \ell - 2j_i + 1$.

Introduce the set $\mathscr{P}_\ell(\mathbb{J})$ defined by

$$\mathscr{P}_\ell(\mathbb{J}) = \left\{ \mathbb{p} = (p_N, \cdots, p_1, p_0); p_i \in \frac{1}{2}\mathbb{Z}_{\geq 0} \quad \mathbb{v}_i = \begin{pmatrix} & j_i & \\ p_i & & p_{i-1} \end{pmatrix} \in (\mathrm{CG})_\ell, \right.$$

$$\left. p_N = p_0 = 0 \right\},$$

where $(\mathrm{CG})_\ell$ is the set of all ℓCG-vertices. For each $\mathbb{p} \in \mathscr{P}_\ell(\mathbb{J})$, the N-point function

$$\Phi_{\mathbb{p}}(z_N, \cdots, z_1) = \langle \Phi_{\mathbb{v}_N}(z_N) \cdots \Phi_{\mathbb{v}_1}(z_1) \rangle$$

of type $\mathbb{p}$ is a formal Laurent series solution of the joint system $\mathrm{E}(\mathbb{J})$ and $\mathrm{B}(\mathbb{J})$, moreover

Theorem 4 (Theorem 3.3).

i) *For any* $\mathbb{p} \in \mathscr{P}_\ell(\mathbb{J})$, *the Laurent series* $\Phi_{\mathbb{p}}(z_N, \cdots, z_1)$ *is absolutely convergent in the region* $\mathscr{R}_z = \{(z_N, \cdots, z_1) \in \mathbb{C}^N; |z_N| > \cdots > |z_1|\}$ *and is analytically continued to a multivalued holomorphic function on the manifold* X_N.

ii) $\{\Phi_{\mathbb{p}}(z_N, \cdots, z_1); \mathbb{p} \in \mathscr{P}_\ell(\mathbb{J})\}$ *gives a basis of the solution space of the joint system* $\mathrm{E}(\mathbb{J})$ *and* $\mathrm{B}(\mathbb{J})$.

As a corollary of Theorem 4, we get

Theorem 5 (Theorem 3.4). *Let* $\Phi_i(z_i)$ *be the vertex operator of spin* j_i *and* $u_i \in V_{j_i}$ $(1 \leq i \leq N)$. *Then the sequence* $\{\Phi_N(u_N; z_N), \cdots, \Phi_1(u_1; z_1)\}$ *is composable in the region* $\mathscr{R}_{z,0} = \{(z_N, \cdots, z_1) \in \mathbb{C}^N; |z_N| > \cdots > |z_1| > 0\}$ *and the composed operator* $\Phi_N(u_N; z_N) \cdots \Phi_1(u_1; z_1)$ *is analytically continued to a multivalued holomorphic function on the manifold* $M_N = \{(z_N, \cdots, z_1) \in X_N; z_i \neq 0\}$.

For ℓCG-vertices $\mathbb{v}_2 = \begin{pmatrix} & j_3 & \\ j_4 & & k \end{pmatrix}$ and $\mathbb{v}_1 = \begin{pmatrix} & j_2 & \\ k & & j_1 \end{pmatrix}$, the composed operator $\Phi_{\mathbb{v}_2}(w)\Phi_{\mathbb{v}_1}(z)$ of the vertex operators $\Phi_{\mathbb{v}_2}(w)$ and $\Phi_{\mathbb{v}_1}(z)$ is multi-valued holomorphic on the manifold M_2.

For a quadruple $\mathbb{J} = (j_4, j_3, j_2, j_1)$ of half integers with $0 \leq 2j_1 \leq \ell$, introduce the set $I_\ell(\mathbb{J})$ of *intermediate edges*, defined by

$$I_\ell(\mathbb{J}) = \left\{ k \in \frac{1}{2}\mathbb{Z}; 0 \leq 2k \leq \ell, \ \mathbb{v}_2(k) = \begin{pmatrix} & j_3 & \\ j_4 & & k \end{pmatrix} \in (\mathrm{CG})_\ell, \right.$$

$$\left. \mathbb{v}_1(k) = \begin{pmatrix} & j_2 & \\ k & & j_1 \end{pmatrix} \in (\mathrm{CG})_\ell \right\}.$$

Let $\bar{\mathbb{J}}=(j_4, j_2, j_3, j_1)$, then we get the g-isomorphism T: $V^\vee(\mathbb{J})\to V^\vee(\bar{\mathbb{J}})$ defined by

$$(T\varphi)(u_4\otimes u_2\otimes u_3\otimes u_1)=\varphi(u_4\otimes u_3\otimes u_2\otimes u_1)$$

for $\varphi \in V^\vee(\mathbb{J})$ and $u_4\otimes u_2\otimes u_3\otimes u_1 \in V(\bar{\mathbb{J}})$.

For an intermediate edge $\bar{k} \in I_\ell(\bar{\mathbb{J}})$, similarly define the ℓCG-vertices $\mathrm{v}_2(\bar{k})=\begin{pmatrix} & j_2 & \\ j_4 & & \bar{k}\end{pmatrix}$ and $\mathrm{v}_1(\bar{k})=\begin{pmatrix} & j_3 & \\ \bar{k} & & j_1\end{pmatrix}$ and consider the composed operator $\Phi_{\mathrm{v}_2(\bar{k})}(w)\Phi_{\mathrm{v}_1(\bar{k})}(z)$ of the vertex operators $\Phi_{\mathrm{v}_2(\bar{k})}(w)$ and $\Phi_{\mathrm{v}_1(\bar{k})}(z)$.

Assume that $I_\ell(\mathbb{J})\neq\emptyset$. For a point $(w, z) \in I_2=\{(z_2, z_1) \in \mathbb{R}^2;\ z_2>z_1>0\}$, let $\Phi_{\mathrm{v}_2(k)}(z)\Phi_{\mathrm{v}_1(k)}(w)$ denote the analytic continuation of the composition $\Phi_{\mathrm{v}_2(k)}(w)\Phi_{\mathrm{v}_1(k)}(z)$ of the vertex operators along the path $b(t)$, where the path $b(t)=(\eta(t), \zeta(t))$ from the point $(w, z) \in I_2$ to the point $(z, w) \in \bar{I}_2=\{(z_2, z_1) \in \mathbb{R}^2;\ z_1>z_2>0\}$ on the manifold M_2 is defined by

$$\eta(t)=\frac{w+z}{2}+e^{\pi\sqrt{-1}t}\frac{w-z}{2}, \qquad \zeta(t)=\frac{w+z}{2}-e^{\pi\sqrt{-1}t}\frac{w-z}{2} \quad (t \in [0, 1]).$$

Then

Proposition 6 (Proposition 4.2). i) *There exists a constant square matrix* $C(\mathbb{J})=(C_k^{\bar{k}}(\mathbb{J}))_{k\in I_\ell(\mathbb{J}),\bar{k}\in I_\ell(\bar{\mathbb{J}})}$ *such that for each intermediate edge* $k \in I_\ell(\mathbb{J})$,

$$T\Phi_{\mathrm{v}_2(k)}(z)\Phi_{\mathrm{v}_1(k)}(w)=\sum_{\bar{k}\in I_\ell(\bar{\mathbb{J}})}\Phi_{\mathrm{v}_2(\bar{k})}(w)\Phi_{\mathrm{v}_1(\bar{k})}(z)C_k^{\bar{k}}(\mathbb{J}).$$

ii) *Let* $\mathbb{J}=(t, j_3, j_2, j_1, s)$, *then the braid relation holds*:

$$\begin{aligned}&C(j_3, j_2, j_1, s)C(t, j_3, j_1, j_2)C(j_1, j_3, j_2, s)\\ &\quad =C(t, j_3, j_2, j_1)C(j_2, j_3, j_1, s)C(t, j_2, j_1, j_3).\end{aligned}$$

Now our fundamental problem is:

Fundamental Problem. Determine the matrix $C(\mathbb{J})=(C_k^{\bar{k}}(\mathbb{J}))$ for any quadruple $\mathbb{J}$ with $I_\ell(\mathbb{J})\neq\emptyset$.

In Section 4.2, we solve the fundamental problem for the case where $j_3=\frac{1}{2}$ in $\mathbb{J}$. For general j_3, we can solve it in principle by the fusion rule (see Section 5.4).

Now we take $j_2=j_3=\frac{1}{2}$. Then the conditions for the nontriviality, $V_0^\vee(\mathbb{J})\neq 0$, are divided into the following cases:

$$(\mathrm{D}2)_2 \quad \frac{\ell}{2}>j_1=j_4>0; \quad (\mathrm{D}2)_1 \quad \frac{\ell}{2}=j_1=j_4;$$

$(\mathrm{D1})_1$ $j_4=j_1+1$; $(\mathrm{D1})_2$ $j_1=j_4=0$; $(\mathrm{D1})_3$ $j_1=j_4+1$.

Proposition 7 (Proposition 4.8). *Let* $q=\exp(2\pi\sqrt{-1}/(\ell+2))$.

i) *For* $j\in\frac{1}{2}\mathbb{Z}$ *with* $0<2j<\ell$,

$$C\left(j,\frac{1}{2},\frac{1}{2},j\right)=q^{-3/4}\begin{pmatrix}\gamma_+^{-1} & \\ & \gamma_-^{-1}\end{pmatrix}\begin{bmatrix}\dfrac{-1}{[2j+1]} & \dfrac{\sqrt{q[2j][2j+2]}}{[2j+1]}\\ \dfrac{\sqrt{[2j][2j+2]}}{[2j+1]} & \dfrac{q^{2j+1}}{[2j+1]}\end{bmatrix}\begin{pmatrix}\gamma_+ & \\ & \gamma_-\end{pmatrix}$$

where $[\nu]$ *denotes the q-integer*

$$[\nu]=\frac{q^\nu-1}{q-1}\quad\text{and}\quad \gamma_\pm=\frac{\Gamma\left(\pm\dfrac{2j+1}{\ell+2}\right)}{\left(\Gamma\left(\pm\dfrac{2j+2}{\ell+2}\right)\Gamma\left(\dfrac{\pm 2j}{\ell+2}\right)\right)^{1/2}}.$$

ii) $C\left(\dfrac{\ell}{2},\dfrac{1}{2},\dfrac{1}{2},\dfrac{\ell}{2}\right)=C\left(0,\dfrac{1}{2},\dfrac{1}{2},0\right)=-q^{-3/4}$.

iii) $C\left(j+1,\dfrac{1}{2},\dfrac{1}{2},j\right)=C\left(j-1,\dfrac{1}{2},\dfrac{1}{2},j\right)=q^{1/4}$.

Let $N\geq 2$ and fix a half integer t (target edge) with $0\leq 2t\leq\ell$. Put $\mathbb{J}_t=(t,\frac{1}{2},\cdots,\frac{1}{2})$ and introduce the set

$$\mathscr{P}_\ell(N;t)=\left\{\mathbb{p}=(p_N,\cdots,p_1,p_0);\,p_N=t,\,p_0=0,\,p_i\in\frac{1}{2}\mathbb{Z},\,0\leq 2p_i\leq\ell,\right.$$
$$\left.|p_i-p_{i-1}|=\frac{1}{2}\quad(1\leq i\leq N)\right\}.$$

For each $\mathbb{p}\in\mathscr{P}_\ell(N;t)$, define the $V_0^\vee(\mathbb{J}_t)$-valued, multi-valued holomorphic function $\Psi_{\mathbb{p}}(z_N,\cdots,z_1)$ on X_N by

$$\Psi_{\mathbb{p}}(z_N,\cdots,z_1)(v,u_N,\cdots,u_1)=\langle\nu(v)|\Phi_{\mathrm{v}_N}(u_N;z_N)\cdots\Phi_{\mathrm{v}_1}(u_1;z_1)|\mathrm{vac}\rangle$$

for $v\in V_t$ and $u_i\in V_{1/2}$ $(1\leq i\leq N)$, where the vertex v_i is defined as $\mathrm{v}_i(\mathbb{p})=\begin{pmatrix} & 1/2 & \\ p_i & & p_{i-1}\end{pmatrix}$ $(1\leq i\leq N)$ and ν is the isomorphism $\nu\colon V_j\to V_j^\dagger$ defined in Section 2.3.

Then the function $\Psi_{\mathbb{p}}(z_N,\cdots,z_1)$ satisfies the systems $E(N;t)$ and $B(N;t)$ derived from the systems $E(\mathbb{J}_t)$ and $B(\mathbb{J}_t)$, where $\mathbb{J}_t=(t,\frac{1}{2},\cdots,\frac{1}{2})$ (see Section 5.2). Moreover we get that the solution space $W(N;t)$ of the systems $E(N;t)$ and $B(N;t)$ has a basis $\{\Psi_{\mathbb{p}}(z_N,\cdots,z_1);\,\mathbb{p}\in\mathscr{P}_\ell(N;t)\}$.

The braid group B_N acts on this space $W(N; t)$ as monodromies. The commutation relation of vertex operators gives a 'factorization' of this monodromy representation $(\pi_{N,t}, W(N; t))$. By the explicit formulae of the representation $\pi_{N,t}$ obtained from Proposition 7, we get

Theorem 8 (Theorem 5.2 and Proposition 5.3). *Let* $q=\exp\left(\frac{2\pi\sqrt{-1}}{\ell+2}\right)$.

i) *The monodromy representation* $q^{3/4}\pi_{N,t}$ *of the braid group* B_N *on the space* $W(N; t)$ *gives an irreducible and unitarizable representation of the group* B_N.

ii) *This representation factors through a representation of the Hecke algebra* $H_N(q)$ *of type* A_{N-1}.

iii) *Our representation* $(q^{3/4}\pi_{N,t}, W(N; t))$ *of the Hecke algebra* $H_N(q)$ *is equivalent to the representation* $(\pi_\lambda^{(2,\ell+2)}, V_\lambda^{(2,\ell+2)})$ *constructed by H. Wenzl* [W], *where* λ *is a Young diagram* $\lambda=[N/2+t, N/2-t]$.

Notations

$\mathfrak{g}=\mathfrak{sl}(2, \mathbb{C})=\mathbb{C}F\oplus\mathbb{C}H\oplus\mathbb{C}E$, where $F=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $E=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$\hat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]\oplus\mathbb{C}c$: the affine Lie algebra of type $A_1^{(1)}$

$\hat{\mathfrak{h}}=\mathbb{C}H(0)\oplus\mathbb{C}c$: the Cartan subalgebra of $\hat{\mathfrak{g}}$

$X(n)=X\otimes t^n$ for $X\in\mathfrak{g}$ and $n\in\mathbb{Z}$

$\mathfrak{m}_\pm=\mathfrak{g}\otimes t^\pm\mathbb{C}[t^\pm]$, $\mathfrak{n}_+=\mathfrak{m}_+\oplus\mathbb{C}E(0)$, $\mathfrak{n}_-=\mathfrak{m}_-\oplus\mathbb{C}F(0)$, $\mathfrak{p}_\pm=\mathfrak{m}_\pm\oplus\mathfrak{g}\oplus\mathbb{C}c$: subalgebras of $\hat{\mathfrak{g}}$

$\mathscr{L}=\sum_{n\in\mathbb{Z}}\mathbb{C}e_n+\mathbb{C}e_0'$: the Virasoro algebra

$\Omega=\frac{1}{2}H^2+EF+FE\in U(\mathfrak{g})$: the Casimir element of $\mathfrak{g}$

${:}X(m)Y(n){:}$: the normal ordered product for $X(m)$, $Y(n)\in\mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$

$X(z)=\sum_{n\in\mathbb{Z}}X(n)z^{-n-1}$ $(z\in\mathbb{C}^*, X\in\mathfrak{g})$: a current

$T(z)=\sum_{m\in\mathbb{Z}}L(m)z^{-m-2}$: the energy momentum tensor

ℓ: the central charge (we fix $\ell\in\mathbb{Z}_{>0}$ throughout the paper)

$\kappa=\ell+2$

V_j, $V_j^\dagger$: the irreducible left and right $\mathfrak{g}$-modules of spin j for $j\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ respectively

$V_j^\vee=\mathrm{Hom}(V_j, \mathbb{C})$: the dual (right) $\mathfrak{g}$-module of V_j

$\mathscr{H}_j=\mathscr{H}_j(\ell)$, $\mathscr{H}_j^\dagger=\mathscr{H}_j^\dagger(\ell)$: the integrable highest weight left and right $\hat{\mathfrak{g}}$-modules respectively

$\langle|\rangle$: $V_j^\dagger\times V_j\to\mathbb{C}$, $\mathscr{H}_j^\dagger\times\mathscr{H}_j\to\mathbb{C}$: the vacuum expectation values

$\mathscr{H}=\sum_{j=0}^{\ell/2}\mathscr{H}_j\subset\hat{\mathscr{H}}=\sum_{j=0}^{\ell/2}\hat{\mathscr{H}}_j$; $\mathscr{H}^\dagger=\sum_{j=0}^{\ell/2}\mathscr{H}_j^\dagger\subset\hat{\mathscr{H}}^\dagger=\sum_{j=0}^{\ell/2}\hat{\mathscr{H}}_j^\dagger$.

$\mathbb{V}=\left\{\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix}; j, j_1, j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}\right\}$: the set of vertices

$\cup$

$\mathbb{V}_\ell=\left\{\mathrm{v} \in \mathbb{V};\ j_1,\ j_2 \leq \frac{\ell}{2}\right\}$

$(\mathrm{CG})=\{\mathrm{v} \in \mathbb{V};\ |j_1-j_2| \leq j \leq j_1+j_2,\ j_1+j_2+j \in \mathbb{Z}\}$: the set of all CG-vertices

$(\mathrm{CG})_\ell=\{\mathrm{v} \in (\mathrm{CG});\ j_1+j_2+j \leq \ell\}$: the set of all ℓCG-vertices

$\Delta_j=\dfrac{j^2+j}{\kappa}$: the conformal dimension of vertex operators of spin j

$\Delta(\mathrm{v})=\Delta_j$: the conformal dimension of a vertex v

$\hat{\Delta}(\mathrm{v})=\Delta_j+\Delta_{j_1}-\Delta_{j_2}$ for a vertex v

$\mathscr{V}(\mathrm{v})=\mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^{\dagger}\otimes V_j \otimes V_{j_1};\ \mathbb{C})$

$\varphi_{\mathrm{v}} \in \mathrm{Hom}_{\mathfrak{g}}(V_j \otimes V_{j_1},\ V_{j_2}) \cong \mathscr{V}(\mathrm{v})$: the nonzero element for each $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix}$ $\in (\mathrm{CG})$ fixed in Appendix I

$\Phi_{\mathrm{v}}(z)$: the vertex operator of type v whose initial term $\Phi_{\mathrm{v},0}$ is φ_{v} for each $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix} \in (\mathrm{CG})_\ell$ (considered as $V_j \otimes \mathscr{H}_{j_1} \to \hat{\mathscr{H}}_{j_2}$)

$\Phi(u; z)=\Phi(z)(u\otimes\cdot)=\sum_{n \in \mathbb{Z}} \Phi_n(u) z^{-n-\hat{\Delta}(\mathrm{v})}$: the homogeneous decomposition of a vertex operator $\Phi(z)$ of type v

Let $W=W_1\otimes\cdots\otimes W_N$ the tensor product of $\mathfrak{g}$-modules W_k, then

π_i: the $\mathfrak{g}$-action on the i-th component of W

$\Delta_{ik}=\pi_i+\pi_k$: the diagonal action on the i-th and k-th components of W

$\Omega_{ik}=\frac{1}{2}\pi_i(H)\pi_k(H)+\pi_i(E)\pi_k(F)+\pi_i(F)\pi_k(E)$

$\mathbb{J}=(j_N, \cdots, j_1)$: an N-ple of half-integers with $0 \leq 2j_i \leq \ell$

$V(\mathbb{J})=V_{j_N}\otimes\cdots\otimes V_{j_1}$, $V^\vee(\mathbb{J})=V_{j_N}^\vee\otimes\cdots\otimes V_{j_1}^\vee$

$V_0^\vee(\mathbb{J})$: the space of all $\mathfrak{g}$-invariant elements in $V^\vee(\mathbb{J})$

$\mathscr{P}(\mathbb{J})=\left\{\mathrm{p}=(p_N, \cdots, p_1, p_0);\ \mathrm{v}_i(\mathrm{p})=\begin{pmatrix} & j_i & \\ p_i & & p_{i-1}\end{pmatrix} \in (\mathrm{CG}),\ p_N=p_0=0\right\}$

$\mathscr{P}_\ell(\mathbb{J})=\{\mathrm{p}=(p_N, \cdots, p_1, p_0) \in \mathscr{P}(\mathbb{J});\ \mathrm{v}_i(\mathrm{p}) \in (\mathrm{CG})_\ell\}$

$\mathbb{J}=(j_4, j_3, j_2, j_1)$: a quadruple of half integers with $0 \leq 2j_i \leq \ell$

$$I(\mathbb{J})=\left\{k \in \frac{1}{2}\mathbb{Z};\ 0 \leq 2k \leq \ell,\ \mathrm{v}_2(k)=\begin{pmatrix} & j_3 & \\ j_4 & & k\end{pmatrix} \in (\mathrm{CG}),\ \mathrm{v}_1(k)=\begin{pmatrix} & j_2 & \\ k & & j_1\end{pmatrix} \in (\mathrm{CG})\right\}$$

$$I_\ell(\mathbb{J})=\left\{k \in \frac{1}{2}\mathbb{Z};\ 0 \leq 2k \leq \ell,\ \mathrm{v}_2(k) \in (\mathrm{CG})_\ell,\ \mathrm{v}_1(k) \in (\mathrm{CG})_\ell\right\}$$

$\mathrm{p}(k)=(\mathrm{v}_3, \mathrm{v}_2(k), \mathrm{v}_1(k), \mathrm{v}_0) \in \mathscr{P}_\ell(\mathbb{J})$ for $k \in I_\ell(\mathbb{J})$, where $\mathrm{v}_3=\begin{pmatrix} & j_4 & \\ 0 & & j_4\end{pmatrix}$ and $\mathrm{v}_0=\begin{pmatrix} & j_1 & \\ j_1 & & 0\end{pmatrix}$

$$\Delta_4(\mathbb{J})=\hat{\Delta}(\mathrm{v}_2)+\hat{\Delta}(\mathrm{v}_1)=\Delta_{j_1}+\Delta_{j_2}+\Delta_{j_3}-\Delta_{j_4}$$

$$J_\ell(\mathbb{J})=\left\{r\in\frac{1}{2}\mathbb{Z};\ 0\leq 2r\leq\ell,\ \mathrm{w}(r)=\begin{pmatrix} & r & \\ j_4 & & j_1\end{pmatrix}\in(\mathrm{CG})_\ell,\right.$$
$$\left.\overline{\mathrm{w}}(r)=\begin{pmatrix} & j_3 & \\ r & & j_2\end{pmatrix}\in(\mathrm{CG})_\ell\right\}$$

$$\varepsilon_0(\mathbb{J})=\frac{1}{\kappa}(j_4+j_3+j_2+j_1+1),\ \varepsilon_i(\mathbb{J})=\varepsilon_0(\mathbb{J})-\frac{1}{\kappa}(2j_i+1)\quad(i=1,\cdots,4)$$

$\mathbb{J}_t=\mathbb{J}_t(N)=\left(t,\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2}\right)$: an $(N+1)$-ple with $0\leq 2t\leq\ell$, $2t\in\mathbb{Z}$

$$\mathscr{P}_\ell(N;t)=\left\{\mathrm{p}=(p_N,\cdots,p_1,p_0);\ p_N=t,\ p_0=0,\ p_i\in\frac{1}{2}\mathbb{Z},\ 0\leq 2p_i\leq\ell,\right.$$
$$\left.|p_i-p_{i-1}|=\frac{1}{2}(1\leq i\leq N)\right\}.$$

$\mathbb{J}_{t,s}=\mathbb{J}_{t,s}(N)=\left(t,\frac{1}{2},\cdots,\frac{1}{2},s\right)$: an $(N+2)$-ple with $t,s\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ and $t,s\leq\frac{\ell}{2}$

$$\mathscr{P}_\ell(N;t,s)=\left\{\mathrm{p}=(p_N,\cdots,p_1,p_0);\ p_N=t,\ p_0=s,\ p_i\in\frac{1}{2}\mathbb{Z}_{\geq 0},\right.$$
$$\left.0\leq 2p_i\leq\ell,\ |p_i-p_{i-1}|=\frac{1}{2}\quad(1\leq i\leq N)\right\}$$

$X_N=\{(z_N,\cdots,z_1)\in\mathbb{C}^N;\ z_i\neq z_k\ (i\neq k)\}$
$\cup$
$M_N=\{(z_N,\cdots,z_1)\in(\mathbb{C}^*)^N;\ z_i\neq z_k\ (i\neq k)\}$
$\mathscr{R}_z=\{z=(z_N,\cdots,z_1)\in\mathbb{C}^N;\ |z_N|>\cdots>|z_1|\}\subset X_N$
$\cup$
$\mathscr{R}_{z,0}=\{(z_N,\cdots,z_1)\in\mathbb{C}^N;\ |z_N|>\cdots>|z_1|>0\}$
$\cup$
$I_N=\{(z_N,\cdots,z_1)\in\mathbb{R}^N;\ z_N>\cdots>z_1>0\}$
$\mathfrak{S}_N$: the N-th symmetric group
B_N: the braid group with N-strings of $\mathbb{C}$
$H_N(q)$: the Hecke algebra of type A_{N-1}
$\lambda=[f_1,\cdots,f_k]$: the Young diagram such that the number of nodes of the i-th row is f_i $(f_1\geq\cdots\geq f_k)$
$|\mathrm{vac}\rangle=u_0(0)$, $\langle\mathrm{vac}|=u_0^\dagger(0)$: the Virasoro vacuums
$\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle=\langle\mathrm{vac}|\Phi_N(z_N)\cdots\Phi_1(z_1)|\mathrm{vac}\rangle$: the N-point function of vertex operators $\{\Phi_N,\cdots,\Phi_1\}$
$\Gamma(z)$: the gamma function
$F(\alpha,\beta,\gamma;z)$: the Gauss' hypergeometric function

$[\nu]_q = \frac{q^\nu - 1}{q-1}$ $(q \neq 1)$, ν $(q=1)$: a q-integer $(\nu \in \mathbb{Z})$

$\binom{L}{\mathfrak{m}} = \frac{L!}{m_N! \cdots m_1!}$: the multinomial coefficient for $\mathfrak{m} = (m_N, \cdots, m_1)$ with $L = \sum m_k$

§ 1. Affine Lie Algebra of type $A_1^{(1)}$

In this section, we recall facts on the affine Lie algebra $\hat{\mathfrak{g}}$ of type $A_1^{(1)}$ (see V.G. Kac's book [Ka]).

1.1) Lie Algebra of type A_1 and its finite-dimensional modules

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ the Lie algebra of type A_1, that is, $\mathfrak{g}$ is a Lie algebra spanned by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The subspace $\mathfrak{h} = \mathbb{C}H$ is a Cartan subalgebra of $\mathfrak{g}$. Its dual $\mathfrak{h}^*$ is spanned by the element α, defined by $\alpha(H) = 2$. Put $\mathfrak{g}_\alpha = \mathbb{C}E$ and $\mathfrak{g}_{-\alpha} = \mathbb{C}F$, then $\mathfrak{g}$ has the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \mathfrak{g}_{-\alpha}.$$

Let $(\ ,\ ): \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form, defined by $(X, Y) = \mathrm{tr}\, XY$, where tr means the trace as 2×2-matrices. Then $(H, H) = 2$, $(E, F) = 1$ and $(H, E) = (H, F) = 0$.

The *Casimir element* Ω of $\mathfrak{g}$ is defined as

$$\Omega = \frac{1}{2} H^2 + EF + FE \in U(\mathfrak{g}).$$

Here we summarize the facts on finite dimensional modules of $\mathfrak{g}$:

Proposition 1.1. *Fix a half integer $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.*

I) i) *There exists a unique irreducible left $\mathfrak{g}$-module V_j (called* of spin *j) with highest weight $j\alpha$.*

ii) *V_j is of dimension $2j+1$ and has a basis $\{u_j(m); m = j, j-1, \cdots, 1-j, -j\}$ satisfying the relations*

$$(V_j) \qquad \begin{aligned} &Hu_j(m) = 2mu_j(m) && (-j \leq m \leq j); \\ &Eu_j(m) = \sqrt{(j+m+1)(j-m)}\, u_j(m+1) && (-j \leq m < j); \\ &Fu_j(m) = \sqrt{(j+m)(j-m+1)}\, u_j(m-1) && (-j < m \leq j). \end{aligned}$$

iii) *$Eu_j(j) = 0$, $F^n u_j(j) \neq 0$ $(0 \leq n \leq 2j)$ and $F^{2j+1} u_j(j) = 0$.*

iv) *$\Omega = 2(j^2 + j)$ on V_j.*

II) i) *There exists a unique irreducible right* $\mathfrak{g}$*-module* $V_j^\dagger$ *(called* of spin j*) with highest weight* $j\alpha$.

ii) $V_j^\dagger$ *is of dimension* $2j+1$ *and has a basis* $\{u_j^\dagger(m);\ m=j, j-1, \cdots, 1-j, -j\}$ *satisfying the relations*:

$$(V_j^\dagger)\qquad \begin{aligned} &u_j^\dagger(m)H=2mu_j^\dagger(m) && (-j\leq m\leq j);\\ &u_j^\dagger(m)E=\sqrt{(j+m)(j-m+1)}\,u_j^\dagger(m-1) && (-j<m\leq j);\\ &u_j^\dagger(m)F=\sqrt{(j+m+1)(j-m)}\,u_j^\dagger(m+1) && (-j\leq m<j).\end{aligned}$$

iii) $u_j^\dagger(j)F=0$, $u_j^\dagger(j)E^n\neq 0$ $(0\leq n\leq 2j)$ *and* $u_j^\dagger(j)E^{2j+1}=0$.

iv) $\Omega=2(j^2+j)$ *on* $V_j^\dagger$.

III) *There exists a unique bilinear form (called* vacuum expectation value)

$$\langle\ |\ \rangle\colon\ V_j^\dagger\times V_j\longrightarrow\mathbb{C}$$

such that 1) $\langle ua|v\rangle=\langle u|av\rangle$ *for any* $a\in\mathfrak{g}$, $\langle u|\in V_j^\dagger$ *and* $|v\rangle\in V_j$, *and* 2) $\langle u_j^\dagger(m)|u_j(m')\rangle=\delta_{m,m'}$. *Moreover this bilinear form is nondegenerate.*

1.2) The affine Lie algebra of type $A_1^{(1)}$

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra of type $A_1^{(1)}$, that is, $\hat{\mathfrak{g}}$ is defined by

$$\hat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$$

with the following commutation relations:

$$[X(m),Y(n)]=[X,Y](m+n)+(X,Y)m\delta_{m+n,0}c\qquad (X,Y\in\mathfrak{g},\ m,n\in\mathbb{Z}),$$

and

$$c\in\text{center of }\hat{\mathfrak{g}},$$

where $X(n)=X\otimes t^n$.

The Lie algebra $\mathfrak{g}$ is included in $\hat{\mathfrak{g}}$ by identifying X with $X(0)$. Introduce the subspace $\mathfrak{g}(n)=\mathfrak{g}\otimes t^n$ of $\hat{\mathfrak{g}}$ for any $n\in\mathbb{Z}$, and subalgebras $\mathfrak{m}_\pm=\sum_{n\geq 1}\mathfrak{g}(\pm n)$, then $\hat{\mathfrak{g}}$ is decomposed into

$$\hat{\mathfrak{g}}=\mathfrak{m}_+\oplus\mathfrak{g}\oplus\mathbb{C}c\oplus\mathfrak{m}_-.$$

The subspace $\hat{\mathfrak{h}}=\mathbb{C}H(0)\oplus\mathbb{C}c$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$. The dual $\hat{\mathfrak{h}}^*$ of $\hat{\mathfrak{h}}$ is identified with $\mathbb{C}^2\ni(\lambda,\mu)$, by the formulae:

$$(\lambda,\mu)(c)=\lambda\quad\text{and}\quad(\lambda,\mu)(H)=2\mu.$$

Now we summarize the facts about the integrable highest weight modules of the Lie algebra $\hat{\mathfrak{g}}$.

Proposition 1.2. *Irreducible integrable highest weight modules of* $\hat{\mathfrak{g}}$ *are parametrized by* $(\ell, j) \in \mathbb{Z}_{\geq 0} \oplus \frac{1}{2}\mathbb{Z}_{\geq 0}$ *with* $2j \leq \ell$. *Fix such* (ℓ, j).

i) *There exists a unique irreducible left* $\hat{\mathfrak{g}}$*-module* $\mathcal{H}_j(\ell)$ *with a nonzero vector* $|\ell, j\rangle$ (*called* vacuum) *such that*

$$\mathfrak{m}_+|\ell, j\rangle = E|\ell, j\rangle = 0, \quad c|\ell, j\rangle = \ell|\ell, j\rangle \quad \textit{and} \quad H|\ell, j\rangle = 2j|\ell, j\rangle.$$

ii) *There exists a unique irreducible right* $\hat{\mathfrak{g}}$*-module* $\mathcal{H}_j^\dagger(\ell)$ *with a non-zero vector* $\langle j, \ell|$ (*called* vacuum) *such that*

$$\langle j, \ell|\mathfrak{m}_- = \langle j, \ell|F = 0, \quad \langle j, \ell|c = \ell\langle j, \ell| \quad \textit{and} \quad \langle j, \ell|H = 2j\langle j, \ell|.$$

iii) *The subspaces* $\{|v\rangle \in \mathcal{H}_j(\ell);\ \mathfrak{m}_+|v\rangle = 0\}$ *of* $\mathcal{H}_j(\ell)$ *and* $\{\langle v| \in \mathcal{H}_j^\dagger(\ell);\ \langle v|\mathfrak{m}_- = 0\}$ *of* $\mathcal{H}_j^\dagger(\ell)$ *are* $\mathfrak{g}$*-stable and are isomorphic to the irreducible* $\mathfrak{g}$*-modules* V_j *and* $V_j^\dagger$ *respectively.*

The vacuums $|\ell, j\rangle$ *and* $\langle j, \ell|$ *can be identified with* $u_j(j)$ *and* $u_j^\dagger(j)$, *and* $\mathcal{H}_j(\ell)$ *and* $\mathcal{H}_j^\dagger(\ell)$ *are generated by* V_j *and* $V_j^\dagger$ *respectively.*

iv) *There exists a unique bilinear form* (*called* vacuum expectation value)

$$\langle\ |\ \rangle\colon \mathcal{H}_j^\dagger(\ell) \times \mathcal{H}_j(\ell) \longrightarrow \mathbb{C}$$

such that 1) $\langle j, \ell|\ell, j\rangle = 1$, *and* 2) $\langle ua|v\rangle = \langle u|av\rangle$ *for any* $a \in \hat{\mathfrak{g}}$, $\langle u| \in \mathcal{H}_j^\dagger(\ell)$ *and* $|v\rangle \in \mathcal{H}_j(\ell)$. *Moreover this bilinear form is non-degenerate, and its restriction on* $V_j^\dagger \times V_j$ *coincides with the vacuum expectation value as* $\mathfrak{g}$*-modules* (*Proposition* 1.1).

1.3) Segal-Sugawara form

In this paragraph, we give the actions on $\mathcal{H}_j(\ell)$ and $\mathcal{H}_j^\dagger(\ell)$ of another Lie algebra $\mathcal{L}$ called *Virasoro Algebra*, where $\mathcal{L} = \sum_{n \in \mathbb{Z}} \mathbb{C}e_n \oplus \mathbb{C}e_0'$ is the Lie algebra defined by the relations:

$$[e_m, e_n] = (m-n)e_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}e_0' \qquad (m, n \in \mathbb{Z});$$

$$[e_0', e_m] = 0.$$

Definition 1.3. Define the *normal ordered products* of elements of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ by

$$:X(m)Y(n): = \begin{cases} X(m)Y(n) & (m<n) \\ \dfrac{1}{2}\{X(m)Y(n) + Y(n)X(m)\} & (m=n) \\ Y(n)X(m) & (m>n). \end{cases}$$

Definition 1.4.

i) For each $X \in \mathfrak{g}$, we define the formal Laurent series

$$X(z)=\sum_{n\in \mathbf{Z}} X(n)z^{-n-1} \qquad (z \in \mathbb{C}^*).$$

ii) Energy-momentum tensor; Segal-Sugawara form ([Se] and [Su]) For $z \in \mathbb{C}^*$, define

$$\begin{aligned} T(z)&=\frac{1}{2(2+c)}\left\{\frac{1}{2}:H(z)H(z):+:E(z)F(z):+:F(z)E(z):\right\} \\ &=\sum_{m\in \mathbf{Z}} L(m)z^{-m-2}, \end{aligned}$$

that is,

$$\begin{aligned} L(m)=\frac{1}{2(2+c)}\sum_{k\in \mathbf{Z}}\Big\{\frac{1}{2}:H(-k)H(m+k):+:E(-k)F(m+k):+ \\ +:F(-k)E(m+k):\Big\}. \end{aligned}$$

Then we get

Proposition 1.5.

i) *For any* $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ *with* $2j\leq \ell$, *the operator* $L(m)$, $m \in \mathbb{Z}$, *and* $L'(0) =(3\ell/(2+\ell))\,\mathrm{id}$ *act on* $\mathscr{H}_j(\ell)$ *and* $\mathscr{H}_j^\dagger(\ell)$.

ii) *For any* $m, n \in \mathbb{Z}$,

$$[L(m),\ L(n)]=(m-n)L(m+n)+\frac{m^3-m}{12}\delta_{m+n,0}L'(0).$$

iii) *For each* $m \in \mathbb{Z}$ *and* $X \in \mathfrak{g}$,

$$\begin{aligned} &[L(m),\, X(z)]=z^m\Big(z\frac{d}{dz}+m+1\Big)X(z); \\ &[L(m),\, X(n)]=-nX(m+n) \qquad (n \in \mathbb{Z}). \end{aligned}$$

iv) *The modules* $\mathscr{H}_j(\ell)$ *and* $\mathscr{H}_j^\dagger(\ell)$ *have the eigenspace decompositions with respect to the operator* $L(0)$:

$$\mathscr{H}_j(\ell)=\sum_{d\geq 0}\mathscr{H}_{j,d}(\ell) \quad \textit{and} \quad \mathscr{H}_j^\dagger(\ell)=\sum_{d\geq 0}\mathscr{H}_{j,d}^\dagger(\ell).$$

where $\mathscr{H}_{j,d}(\ell)$ *and* $\mathscr{H}_{j,d}^\dagger(\ell)$ *are the eigenspaces of the eigenvalue* Δ_j+d, *and* $\Delta_j=(j^2+j)/(\ell+2)$. *In particular,* $\mathscr{H}_{j,0}(\ell)=V_j$ *and* $\mathscr{H}_{j,0}^\dagger(\ell)=V_j^\dagger$. *Moreover* $\dim \mathscr{H}_{j,d}(\ell)=\dim \mathscr{H}_{j,d}^\dagger(\ell)<\infty$.

v) *$\mathscr{H}^\dagger_{j,d}(\ell)\perp\mathscr{H}_{j,d'}(\ell)$ unless $d=d'$, and $\langle\,|\,\rangle$ is nondegenerate on $\mathscr{H}^\dagger_{j,d}(\ell)\times\mathscr{H}_{j,d}(\ell)$.*

vi) *For any $X\in\mathfrak{g}$, $m\in\mathbb{Z}$ and $d\geq 0$,*

$$X(m)\mathscr{H}_{j,d}(\ell),\quad L(m)\mathscr{H}_{j,d}(\ell)\subset\mathscr{H}_{j,d-m}(\ell)$$

and

$$\mathscr{H}^\dagger_{j,d}(\ell)X(m),\quad \mathscr{H}^\dagger_{j,d}(\ell)L(m)\subset\mathscr{H}^\dagger_{j,d+m}(\ell).$$

In the following of this paper, we fix an integer $\ell\geq 1$, put $\kappa=\ell+2$, and omit ℓ in the notations $\mathscr{H}_j(\ell)$, $\mathscr{H}_{j,d}(\ell)$ etc. (Note that $V_0=\mathscr{H}_0(0)=\mathbb{C}$.)

§ 2. Vertex Operators (Primary fields)

Throughout this paper we fix the value ℓ (a positive integer) of the central element c on the spaces $\mathscr{H}$ and $\mathscr{H}^\dagger$, and use the value $\kappa=\ell+2$ for convenience.

2.1) Field operators

Fix a half integer j with $0\leq 2j\leq\ell$. Introduce the product topology to the products $\hat{\mathscr{H}}_j=\prod_{d\geq 0}\mathscr{H}_{j,d}$ and $\hat{\mathscr{H}}^\dagger_j=\prod_{d\geq 0}\mathscr{H}^\dagger_{j,d}$, then the vacuum expectation $\langle\,|\,\rangle:\mathscr{H}^\dagger_j\times\mathscr{H}_j\to\mathbb{C}$ is uniquely extended to continuous bilinear pairings $\langle\,|\,\rangle:\mathscr{H}^\dagger_j\times\hat{\mathscr{H}}_j\to\mathbb{C}$ and $\hat{\mathscr{H}}^\dagger_j\times\mathscr{H}_j\to\mathbb{C}$, and there is a topological linear isomorphism $\hat{\mathscr{H}}^\dagger_j\cong\mathrm{Hom}_c(\mathscr{H}_j;\mathbb{C})$, where $\mathrm{Hom}_c(\mathscr{H}_j;\mathbb{C})$ is equipped with the weak topology. The actions of the Lie algebra $\hat{\mathfrak{g}}$ on $\mathscr{H}_j$ and $\mathscr{H}^\dagger_j$ can be extended to these completions.

Consider the direct sums of these modules:

$$\mathscr{H}=\sum_{j=0}^{\ell/2}\mathscr{H}_j\subset\hat{\mathscr{H}}=\sum_{j=0}^{\ell/2}\hat{\mathscr{H}}_j;\ \mathscr{H}^\dagger=\sum_{j=0}^{\ell/2}\mathscr{H}^\dagger_j\subset\hat{\mathscr{H}}^\dagger=\sum_{j=0}^{\ell/2}\hat{\mathscr{H}}^\dagger_j.$$

Denote by Π_j be the projection to the j-th component:

$$\Pi_j:\mathscr{H}\longrightarrow\mathscr{H}_j,\quad \hat{\mathscr{H}}\longrightarrow\hat{\mathscr{H}}_j;\quad \mathscr{H}^\dagger\longrightarrow\mathscr{H}^\dagger_j,\quad \hat{\mathscr{H}}^\dagger_j\longrightarrow\hat{\mathscr{H}}^\dagger_j,$$

then $\Pi_j\circ\Pi_k=\Pi_k\circ\Pi_j$ and Π_j commutes with the action of $\hat{\mathfrak{g}}$.

An *operator* A on $\mathscr{H}$ means a linear mapping $A:\mathscr{H}\longrightarrow\hat{\mathscr{H}}$, which is equivalent to give a bilinear map $\hat{A}:\mathscr{H}^\dagger\times\mathscr{H}\to\mathbb{C}$, and also to give a linear mapping $A^\dagger:\mathscr{H}^\dagger\to\hat{\mathscr{H}}^\dagger$ by the condition that for any $\langle v|\in\mathscr{H}^\dagger$ and $|w\rangle\in\mathscr{H}$,

$$\langle v|Aw\rangle=\langle v|\hat{A}|w\rangle=\langle vA|w\rangle.$$

In order to define compositions of operators, fix dual bases $\{|u_{d,1}\rangle,\cdots,|u_{d,m_d}\rangle\}$ of $\sum_{j=0}^{\ell/2}\mathscr{H}_{j,d}$ and $\{\langle v_{d,1}|,\cdots,\langle u_{d,m_d}|\}$ of $\sum_{j=0}^{\ell/2}\mathscr{H}^\dagger_{j,d}$

with respect to $\langle\,|\,\rangle$, where $m_d=\sum_{j=0}^{\ell/2}\dim \mathscr{H}_{j,d}=\sum_{j=0}^{\ell/2}\dim \mathscr{H}^{\dagger}_{j,d}$.

A sequence $\{A_N, \cdots, A_1\}$ of operators on $\mathscr{H}$ is called *composable*, if the series

$$\sum_{d_1,\cdots,d_{m-1}\geq 0}\left|\sum_{j_1=1}^{m_{d_1}}\cdots\sum_{j_{m-1}=1}^{m_{d_{m-1}}}\langle v|A_{n_m}|u_{d_{m-1},j_{m-1}}\rangle \langle u_{d_{m-1},j_{m-1}}|A_{n_{m-1}}|u_{d_{m-2},j_{m-2}}\rangle\cdots\langle u_{d_1,j_1}|A_{n_1}|w\rangle\right|$$

is convergent for any ordered subset $\{n_m, \cdots, n_1\}$ of $\{N, \cdots, 2, 1\}$ with $2\leq m\leq N$ and any vectors $\langle v|\in\mathscr{H}^{\dagger}$ and $|w\rangle\in\mathscr{H}$. Then the *composed operator* $A_N\cdots A_1$ is defined by the values

$$\langle v|A_N\cdots A_1|w\rangle=\sum_{d_1,\cdots,d_{N-1}\geq 0}\sum_{j_1=1}^{m_{d_1}}\cdots\sum_{j_{N-1}=1}^{m_{d_{N-1}}}\langle v|A_N|u_{d_{N-1},j_{N-1}}\rangle \langle u_{d_{N-1},j_{N-1}}|A_{N-1}|u_{d_{N-2},j_{N-2}}\rangle\cdots\langle u_{d_1,j_1}|A_1|w\rangle$$

for $\langle v|\in\mathscr{H}^{\dagger}$ and $|w\rangle\in\mathscr{H}$.

An operator-valued function $A(z)$: $\mathscr{H}\to\hat{\mathscr{H}}$ on a complex manifold M is called *holomorphic* with respect to the variable $z\in M$, if the function $\langle u|A(z)|v\rangle$ is holomorphic with respect to $z\in M$ for any $\langle u|\in\mathscr{H}^{\dagger}$ and $|v\rangle\in\mathscr{H}$.

Example. Operator-valued functions $X(z)$ ($X\in\mathfrak{g}$) and $T(z)$: $\mathscr{H}\to\hat{\mathscr{H}}$ are single-valued and holomorphic on $\mathbb{C}^*=\mathbb{P}^1\backslash\{0, \infty\}$.

Let $A_i(z_i)$ be an operator-valued function on $\mathscr{H}$ parametrized by a complex manifold M_i for each i with $1\leq i\leq N$, and assume the sequence $\{A_N(z_N), \cdots, A_1(z_1)\}$ is composable for any $(z_N, \cdots, z_1)\in M_N\times\cdots\times M_1$. Then the composed operator $A_N(z_N)\cdots A_1(z_1)$ is holomorphic on the complex manifold $M_N\times\cdots\times M_1$.

2.2) Vertex operators

Now we give the notion of vertex operators (or primary fields) which is introduced by V.G. Knizhnik and A.B. Zamolodchikov [KZ].

For a positive half integer j, a multi-valued, holomorphic, operator-valued function $\Phi(z)$ on the manifold $\mathbb{C}^*(=\mathbb{C}\backslash\{0\})$ is called a *vertex operator of spin j*, if

$$\Phi(z);\ V_j\otimes\mathscr{H}\longrightarrow\hat{\mathscr{H}}$$

satisfies the conditions:

(V2) $$[X(m), \Phi(u; z)]=z^m\Phi(Xu; z)$$

(V3) $$[L(m), \Phi(u; z)] = z^m \left\{ z\frac{d}{dz} + (m+1)\Delta_j \right\} \Phi(u; z)$$

for $X \in \mathfrak{g}$, $u \in V_j$, $m \in \mathbb{Z}$ and $z \in \mathbb{C}^*$, where the number $\Delta_j = (j^2+j)/\kappa$ is called the *conformal dimension* of the vertex operator $\Phi(z)$ and $\Phi(u; z)$: $\mathscr{H} \to \hat{\mathscr{H}}$ is the operator defined by

$$\Phi(u; z)(w) = \Phi(z)(u \otimes w) \qquad (u \in V_j,\ w \in \mathscr{H}).$$

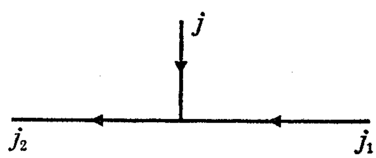
Remark. (V2) is the gauge condition for the field $\Phi(z)$ and (V3) means the equations of motion.

Introduce sets $\mathbb{V}$ and $\mathbb{V}_\ell$ defined by

$$\mathbb{V} = \left\{ \mathrm{v} = \begin{pmatrix} j \\ j_2\ j_1 \end{pmatrix}; j, j_1, j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0} \right\} \supset \mathbb{V}_\ell = \left\{ \mathrm{v} = \begin{pmatrix} j \\ j_2\ j_1 \end{pmatrix} \in \mathbb{V}; j_1, j_2 \leq \frac{\ell}{2} \right\}.$$

An element v of $\mathbb{V}$ is called a *vertex*. For a vertex $\mathrm{v} = \begin{pmatrix} j \\ j_2\ j_1 \end{pmatrix} \in \mathbb{V}$, we call j_1 an *incoming spin*, j_2 an *outgoing spin* and j an *outer spin*, and set $\Delta(\mathrm{v}) = \Delta_j$ $(=(j^2+j)/\kappa)$ and $\hat{\Delta}(\mathrm{v}) = \Delta_j + \Delta_{j_1} - \Delta_{j_2}$.

v:



For a vertex $\mathrm{v} = \begin{pmatrix} j \\ j_2\ j_1 \end{pmatrix} \in \mathbb{V}_\ell$, a vertex operator $\Phi(z)$ of spin j is called *of type* v, if $\Phi(u; z) = \Pi_{j_2}\Phi(u; z)\Pi_{j_1}$ for any $u \in V_j$.

Then we get the following (the proof will be given in Section 2.3):

Proposition 2.1.

i) *Any vertex operator* Φ *of type* v ($\in \mathbb{V}_\ell$) *has a Laurent series expansion*

$$\Phi(u; z) = \sum_{n \in \mathbb{Z}} \Phi_n(u) z^{-n-\hat{\Delta}(\mathrm{v})} \qquad (u \in V_j)$$

and $\Phi_n(u)$ *satisfies*

$$[L(0), \Phi_n(u)] = (\Delta_{j_2} - \Delta_{j_1} - n)\Phi_n(u) \qquad (n \in \mathbb{Z}),$$

that is,

$$\Phi_n(u): \mathscr{H}_{j_1,d} \longrightarrow \mathscr{H}_{j_2,d-n},\ \mathscr{H}^\dagger_{j_2,d} \longrightarrow \mathscr{H}^\dagger_{j_1,d+n} \qquad (n \in \mathbb{Z}).$$

ii) *Introduce a trilinear form* φ: $V^\dagger_{j_2} \otimes V_j \otimes V_{j_1} \to \mathbb{C}$ *defined by*

$$\varphi(v, u, w)=\langle v|\Phi_0(u)|w\rangle=\langle v|\Phi(u; z)|w\rangle z^{\Delta(\mathrm{v})}|_{z=0} \qquad (v \in V_{j_2}^{\dagger}, w \in V_{j_1}),$$

then φ *is* $\mathfrak{g}$*-invariant*:

$$\varphi(vX, u, w)=\varphi(v, Xu, w)+\varphi(v, u, Xw) \qquad (X \in \mathfrak{g}).$$

iii) *A vertex operator* Φ *of type* v *is uniquely determined by the form* $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^{\dagger}\otimes V_j \otimes V_{j_1}, \mathbb{C})$ *defined in* ii). *We call* φ *the* initial term *of the vertex operator* Φ *and sometimes denote* $\Phi=\Phi_{\varphi}$.

For each vertex $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix} \in \mathbb{V}$, introduce the space $\mathscr{V}(\mathrm{v})$ defined by

$$\mathscr{V}(\mathrm{v})=\mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^{\dagger}\otimes V_j\otimes V_{j_1}, \mathbb{C})\cong \mathrm{Hom}_{\mathfrak{g}}(V_j\otimes V_{j_1}, V_{j_2}).$$

It is well-known in the sl_2-theory that $\mathscr{V}(\mathrm{v})=\mathbb{C}$ or 0, and $\mathscr{V}(\mathrm{v})=\mathbb{C}$, if and only if v satisfies the *Clebsch-Gordan condition*:

$$|j_1-j_2|\leq j\leq j_1+j_2 \quad \text{and} \quad j_1+j_2+j \in \mathbb{Z}.$$

Call such vertex a CG-vertex and denote by (CG) the set of all CG vertices:

$$(\mathrm{CG})=\left\{\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix} \in \mathbb{V};\ |j_1-j_2|\leq j\leq j_1+j_2, j+j_1+j_2 \in \mathbb{Z}\right\}.$$

The following is the key lemma for the existence theorem of vertex operators:

Lemma 2.2. *For a vertex* $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix} \in (\mathrm{CG}) \cap \mathbb{V}_{\ell}$, *take a nonzero element* $\varphi \in \mathscr{V}(\mathrm{v})$. *Then the following conditions are equivalent.*

i) $j+j_1+j_2\leq \ell$.

ii) $\varphi(v, E^{\ell-2j_1+1}u, u_{j_1}(j))=0$ *for any* $v \in V_{j_2}^{\dagger}$ *and* $u \in V_j$.

iii) $\varphi(u_{j_2}^{\dagger}(j_2), F^{\ell-2j_2+1}u, w)=0$ *for any* $u \in V_j$ *and* $w \in V_{j_1}$.

A vertex $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix} \in \mathbb{V}_{\ell}$ is called an ℓCG-vertex, if it satisfies one of the conditions (called the *ℓ-constrained Clebsh-Gordan condition*) in Lemma 2.2 denoted by $(\mathrm{CG})_{\ell}$ the set of all ℓCG-vertices, *i.e.*

$$(\mathrm{CG})_{\ell}=\left\{\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix} \in (\mathrm{CG});\ j+j_1+j_2\leq \ell\right\}.$$

Remark 2.2′. i) The inequalities $|j_1-j_2|\leq j\leq j_1+j_2$ and $j+j_1+j_2 \leq \ell$ imply the inequalities $j, j_1, j_2\leq \ell/2$. In particular, outer spins of ℓCG-vertices are not greater than $\ell/2$.

ii) By the above remark and the proof of Lemma 2.2, one of the conditions of Lemma 2.2 is also equivalent to the condition:

$$\varphi(v, u_j(j), E^{\ell-2j+1}w)=0 \text{ for any } v \in V_{j_2}^\dagger \text{ and } w \in V_{j_1}.$$

Now we get the existence condition for vertex operators (the proof will be given in the paragraph 2.3):

Theorem 2.3. *There exists a nonzero vertex operator* Φ *of type* $\mathbb{v}=\begin{pmatrix} j \\ j_2\ j_1 \end{pmatrix} \in \mathbb{V}_\ell$ *on* $\mathscr{H}$, *if and only if the vertex* $\mathbb{v}$ *is an* ℓCG-*vertex.*

Moreover, nonzero vertex operators of a fixed type $\mathbb{v} \in (\mathrm{CG})_\ell$ *are unique up to a constant multiple.*

As a corollary, we get

Proposition 2.4. i) *For any* $j>\ell/2$, *there are no vertex operators of spin* j.

ii) *Let* $\Phi(z)$ *be a vertex operator of type* $\mathbb{v}=\begin{pmatrix} j \\ j_2\ j_1 \end{pmatrix} \in (\mathrm{CG})_\ell$. *Then as formal Laurent series,*

$$\Phi(u; z)=z^{L(0)-\Delta_j}\Phi(u; 1)z^{-L(0)} \qquad (u \in V_j).$$

Proof. ii) Let $\Phi(u; z)$ be a vertex operator of spin j. Then the condition (V3) for $m=0$ reads as

$$[L(0), \Phi(u; z)]=\left\{z\frac{d}{dz}+\Delta_j\right\}\Phi(u; z). \qquad \text{q.e.d.}$$

2.3) Proof of Proposition 2.1 and Theorem 2.3

We define the *parabolic subalgebras* $\mathfrak{p}_\pm$ of $\hat{\mathfrak{g}}$ as $\mathfrak{p}_\pm=\mathfrak{m}_\pm\oplus\mathfrak{z}\oplus\mathbb{C}c$, and the *Verma module* $\mathscr{M}_j$ as the $\hat{\mathfrak{g}}$-module $\mathscr{M}_j=U(\hat{\mathfrak{g}})\otimes_{\mathfrak{p}_+}V_j$ $(=U(\mathfrak{m}_-)V_j)$, where the $\mathfrak{g}$-module V_j is considered as a $\mathfrak{p}_+$-module by setting $\mathfrak{m}_+V_j=0$ and $c=\ell\,\mathrm{id}_{V_j}$. Then the irreducible $\hat{\mathfrak{g}}$-module $\mathscr{H}_j$ is obtained as the quotient of the Verma module $\mathscr{M}_j$ modulo the maximal proper submodule $\mathscr{J}_j$ (see V.G. Kac [Ka] (10.4.6)).

This $\hat{\mathfrak{g}}$-submodule $\mathscr{J}_j$ is also generated by the single vector $|J_j\rangle=E(-1)^{\ell-2j+1}u_j(j)$ and $\mathscr{J}_j=U(\mathfrak{p}_-)|J_j\rangle$. Moreover $\mathfrak{m}_+|J_j\rangle=E(0)|J_j\rangle=F(0)^{2\ell-2j+3}|J_j\rangle=0$, $H(0)|J_j\rangle=2(\ell-j+1)|J_j\rangle$, and $U(\mathfrak{g})|J_j\rangle$ is $\mathfrak{g}$-isomorphic to $V_{\ell-j+1}$. Denote by π_j the canonical projection $\pi_j\colon \mathscr{M}_j\to\mathscr{H}_j$.

The right $\hat{\mathfrak{g}}$-module $\mathscr{H}_j^\dagger$ is analogously obtained as $\mathscr{H}_j^\dagger=\mathscr{J}_j^\dagger\backslash\mathscr{M}_j^\dagger$, where $\mathscr{M}_j^\dagger$ is a right $\hat{\mathfrak{g}}$-module $\mathscr{M}_j^\dagger=V_j^\dagger\otimes_{\mathfrak{p}_-}U(\hat{\mathfrak{g}})$ (the $\mathfrak{g}$-module $V_j^\dagger$ is considered as a $\mathfrak{p}_-$-module by setting $V_j^\dagger\mathfrak{m}_-=0$ and $c=\ell\,\mathrm{id}_{V_j^\dagger}$), and its maximal proper $\hat{\mathfrak{g}}$-submodule $\mathscr{J}_j^\dagger$ is generated by a vector $\langle J_j|=u_j^\dagger(j)F(1)^{\ell-2j+1}$. Denote by $\pi_j^\dagger$ the canonical projection $\pi_j^\dagger\colon \mathscr{M}_j^\dagger\to\mathscr{H}_j^\dagger$.

The Verma modules $\mathcal{M}_j$ and $\mathcal{M}_j^\dagger$ have also eigenspace decompositions with respect to the operator $L(0)$:

$$\mathcal{M}_j=\sum_{d\geq 0}\mathcal{M}_{j,d} \quad \text{and} \quad \mathcal{M}_j^\dagger=\sum_{d\geq 0}\mathcal{M}_{j,d}^\dagger,$$

where the eigenvalue of $L(0)$ on $\mathcal{M}_{j,d}$ and $\mathcal{M}_{j,d}^\dagger$ is Δ_j+d.

In preparation of the proof, we introduce the filtrations in $\mathcal{M}_j$, $\mathcal{H}_j$, $\mathcal{M}_j^\dagger$ and $\mathcal{H}_j^\dagger$:

$$V_j=F_0\mathcal{H}_j=F_0\mathcal{M}_j\subset F_1\mathcal{M}_j\subset\cdots \quad \text{and} \quad V_j^\dagger=F_0\mathcal{H}_j^\dagger=F_0\mathcal{M}_j^\dagger\subset F_1\mathcal{M}_j^\dagger\subset\cdots$$

where $F_p\mathcal{M}_j$ and $F_p\mathcal{M}_j^\dagger$ are space spanned by the sets

$$\{Y_1(n_1)\cdots Y_q(n_q)|w\rangle;\,|w\rangle\in V_j,\ 0\leq q\leq p,\ Y_k(n_k)\in\hat{\mathfrak{g}}\ (1\leq k\leq q)\},$$

and

$$\{\langle v|X_q(m_q)\cdots X_1(m_1);\,\langle v|\in V_j^\dagger,\ 0\leq q\leq p,\ X_k(m_k)\in\hat{\mathfrak{g}}\ (1\leq k\leq q)\}$$

respectively, and

$$F_p\mathcal{H}_j=\pi_j(F_p\mathcal{M}_j) \quad \text{and} \quad F_p\mathcal{H}_j^\dagger=\pi_j^\dagger(F_p\mathcal{M}_j^\dagger).$$

Proof of Proposition 2.1.

i) Expand $\Phi(u;z)$ as a sum of homogeneous components:

$$\Phi(u;z)=\sum_{n\in Z}\Phi_n(u;z), \qquad \Phi_n(u;z)\colon \mathcal{H}_{j_1,d}\longrightarrow\mathcal{H}_{j_2,d-n}(d\geq 0),$$

then

$$[L(0),\Phi_n(u;z)]=(\Delta_{j_2}-\Delta_{j_1}-n)\Phi_n(u;z).$$

By (V3), we get

$$z\frac{d}{dz}\Phi_n(u;z)=-(\hat{\Delta}(\mathrm{v})+n)\Phi_n(u;z).$$

ii) The condition (V2) for $m=0$ implies

$$[X,\Phi(u,z)]=\Phi(Xu,z) \qquad (X\in\mathfrak{g},\ u\in V_j).$$

iii) Let Φ be a vertex operator of type v, and assume that $\varphi\in\mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^\dagger\otimes V_j\otimes V_{j_1},\mathbb{C})$ defined in ii) vanishes. We want to show $\Phi(z)=0$.

Now we show by the induction on $n=p+q$ that for any $u\in V_j$,

$$\langle v|\Phi(u;z)|w\rangle=0 \quad \text{for } \langle v|\in F_p\mathcal{H}_{j_2}^\dagger \quad \text{and} \quad |w\rangle\in F_q\mathcal{H}_{j_1}.$$

Assume that the assertion is valid for all $n\leq n_0$. It is sufficient to show

$$\langle v\,|\,X_p(m_p)\cdots X_1(m_1)\Phi(u;z)Y_q(-n_q)\cdots Y_1(-n_1)|\,w\rangle=0$$

for $p+q=n_0+1$, m_k, $n_k\geq 1$, $\langle v|\in V_{j_2}^{\dagger}$ and $|w\rangle\in V_{j_1}$.

We may assume that $p\geq 1$ (if $p=0$, we can take $q\geq 1$). Then

$$\begin{aligned}&\langle v\,|\,X_p(m_p)\cdots X_1(m_1)\Phi(u;z)Y_q(-n_q)\cdots Y_1(-n_1)|\,w\rangle\\&\quad=z^{m_1}\langle v\,|\,X_p(m_p)\cdots X_2(m_2)\Phi(X_1u;z)Y_q(-n_q)\cdots Y_1(-n_1)|\,w\rangle\\&\qquad+\langle v\,|\,X_p(m_p)\cdots X_2(m_2)\Phi(u;z)X_1(m_1)Y_q(-n_q)\cdots Y_1(-n_1)|\,w\rangle\\&\quad=0.\end{aligned}$$

q.e.d.

Proof of Theorem 2.3. Proposition 2.1 shows that a vertex operator $\Phi(z)$ of type $\mathbb{v}$ defines a form $\varphi\in\mathscr{V}(\mathbb{v})$ and is uniquely determined by φ. In particular, the existence of a vertex operator implies the Clebsh-Gordan condition for $\mathbb{v}$.

Let $\varphi\ (\neq 0)\in\mathscr{V}(\mathbb{v})=\mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^{\dagger}\otimes V_j\otimes V_{j_1};\mathbb{C})$. We want to construct a form $\tilde{\Phi}(z)\in\mathrm{Hom}(\mathscr{M}_{j_2}^{\dagger}\otimes V_j\otimes\mathscr{M}_{j_1};\mathbb{C})$ such that

(M1) $$\tilde{\Phi}(z)|_{V_{j_2}^{\dagger}\otimes V_j\otimes V_{j_1}}=z^{-\hat{\Delta}}\varphi \qquad (z\in\mathbb{C}^*),$$

(M2) $$\tilde{\Phi}(vX(m),u,w;z)-\tilde{\Phi}(v,u,X(m)w;z)=z^m\tilde{\Phi}(v,Xu,w;z) \qquad (m\in\mathbb{Z},\ X\in\mathfrak{g}),$$

and

(M3) $$\begin{aligned}&\tilde{\Phi}(vL(m),u,w;z)-\tilde{\Phi}(v,u,L(m)w;z)\\&\quad=z^m\left\{z\frac{d}{dz}+(m+1)\Delta_j\right\}\tilde{\Phi}(v,u,w;z)\qquad(m\in\mathbb{Z})\end{aligned}$$

for any $\langle v|\in\mathscr{M}_{j_2}^{\dagger}$, $u\in V_j$ and $|w\rangle\in\mathscr{M}_{j_1}$, where $\hat{\Delta}=\hat{\Delta}(\mathbb{v})$.
(We use the notation $\tilde{\Phi}(v,u,w;z)=\tilde{\Phi}(u;z)(v,w)=\tilde{\Phi}(z)(v,u,w)$.)

Step 0. (M1) defines $\tilde{\Phi}(z)$ on $V_{j_2}^{\dagger}\otimes V_j\otimes V_{j_1}$ satisfying (M2) for $m=0$.

Step 1. Define $\tilde{\Phi}(z)$ on $V_{j_2}^{\dagger}\otimes V_j\otimes F_q\mathscr{M}_{j_1}$ inductively as

$$\tilde{\Phi}(v,u,X(-m)w;z)=-z^{-m}\tilde{\Phi}(v,Xu,w;z)$$

for $m>0$, $X\in\mathfrak{g}$, $v\in V_{j_2}^{\dagger}$, $u\in V_j$, $w\in F_{q-1}\mathscr{M}_{j_1}$, then we get $\tilde{\Phi}(z)$ on $V_{j_2}^{\dagger}\otimes V_j\otimes\mathscr{M}_{j_1}$ satisfying (M1) and (M2) for $m\leq 0$.

Step 2. Define $\tilde{\Phi}(z)$ on $F_p\mathscr{M}_{j_2}^{\dagger}\otimes V_j\otimes\mathscr{M}_{j_1}$ inductively as

$$\tilde{\Phi}(vX(m),u,w;z)=z^m\tilde{\Phi}(v,Xu,w;z)+\tilde{\Phi}(v,u,X(m)w;z)$$

for $m>0$, $X\in\mathfrak{g}$, $v\in F_{p-1}\mathscr{M}_{j_2}^{\dagger}$, $u\in V_j$, $w\in\mathscr{M}_{j_1}$, then we get $\tilde{\Phi}(z)$ on $\mathscr{M}_{j_2}^{\dagger}\otimes V_j\otimes\mathscr{M}_{j_1}$. The well-definedness of $\tilde{\Phi}(z)$ and the condition (M2) can be verified again by the induction on p.

Step 3. Verify (M3) for $\tilde{\Phi}(z)$ defined in Step 2.

Let $v\otimes u\otimes w \in \mathcal{M}^{\dagger}_{j_2,d_2}\otimes V_j\otimes \mathcal{M}_{j_1,d_1}$, then $z^{\hat{\Delta}-d_2+d_1}\tilde{\Phi}(v, u, w; z)$ is proved to be constant by the construction of $\tilde{\Phi}(z)$. On the other hand,

$$\tilde{\Phi}(vL(0), u, w; z)-\tilde{\Phi}(v, u, L(0)w; z)=\{\Delta_{j_2}+d_2-\Delta_{j_1}-d_1\}\tilde{\Phi}(v, u, w; z)$$
$$=\left\{z\frac{d}{dz}+\Delta_j\right\}\tilde{\Phi}(v, u, w; z).$$

Thus we get (M3) for $m=0$.

Recall that $L(0)|_{V_j}=\Omega/2\kappa|_{V_j}=\Delta_j\,\mathrm{id}_{V_j}$, $L(0)|_{V_j^\dagger}=\Delta_j\,\mathrm{id}_{V_j^\dagger}$, and the expansion of $L(m)$:

$$L(m)=\frac{1}{2\kappa}\sum_{j\in Z}\left\{\frac{1}{2}{:}H(m-j)H(j){:}+{:}E(m-j)F(j){:}+{:}F(m-j)E(j){:}\right\}.$$

Then on each component $\mathcal{M}^{\dagger}_{j_2,d_2}\otimes V_j\otimes \mathcal{M}_{j_1,d_1}$ we can show (M3) for any $m\in\mathbb{Z}$ from (M3) for $m=0$ by case-by-case computations. We give here its proof in the case $m=2n+1>0$, $d_2\geq d_1$ (other cases are similarly obtained). In this case, $2\kappa L(m)=2\sum_{k\geq -n}\sum_{i=1}^{3}X^i(-k)X_i(m+k)$, where $X^1=2X_1=H$, $X^2=X_3=E$ and $X^3=X_2=F$.

Let $v\otimes u\otimes w \in \mathcal{M}^{\dagger}_{j_2,d_2}\otimes V_j\otimes \mathcal{M}_{j_1,d_1}$, then (M3) for $m=0$ reads as

$$2\kappa\left\{z\frac{d}{dz}+\Delta_j\right\}\tilde{\Phi}(v, u, w; z)=(2d_2+1)\tilde{\Phi}(v, \Omega u, w; z)$$
$$+2\sum_{k=1}^{d_1}z^{-k}\sum_{i=1}^{3}\tilde{\Phi}(v, X^iu, X_i(k)w; z)+2\sum_{k=0}^{d_2}z^{k}\sum_{i=1}^{3}\tilde{\Phi}(v, X^iu, X_i(-k)w; z).$$

And

$$2\kappa\{\tilde{\Phi}(vL(m), u, w; z)-\tilde{\Phi}(v, u, L(m)w; z)\}$$
$$=2\sum_{i=1}^{3}\sum_{k=-n}^{d_2}\{z^m\tilde{\Phi}(v, X^iX_iu, w; z)+z^{k+m}\tilde{\Phi}(v, X^iu, X_i(-k)w; z)\}$$
$$+2\sum_{i=1}^{3}\sum_{k=-n}^{d_1-m}z^{-k}\tilde{\Phi}(v, X^iu, X_i(m+k)w; z)$$
$$=2z^m\sum_{k=-n}^{d_2}\left\{\tilde{\Phi}(v, \Omega u, w; z)+\sum_{i=1}^{3}z^k\tilde{\Phi}(v, X^iu, X_i(-k)w; z)\right\}$$
$$+2z^m\sum_{i=1}^{3}\sum_{k=n+1}^{d_1}z^{-k}\tilde{\Phi}(v, X^iu, X_i(k)w; z).$$

Hence

$$2\kappa\left[\{\tilde{\Phi}(vL(m), u, w; z)-\tilde{\Phi}(v, u, L(m)w; z)\}-z^m\left\{z\frac{d}{dz}+\Delta_j\right\}\tilde{\Phi}(v, u, w; z)\right]$$
$$=(2n+1)z^m\tilde{\Phi}(v, \Omega u, w; z)$$
$$+2z^m\sum_{i=1}^{3}\left\{\sum_{k=-n}^{-1} z^k\tilde{\Phi}(v, X^i u, X_i(-k)w; z)-\sum_{k=1}^{n} z^{-k}\tilde{\Phi}(v, X^i u, X_i(k)w; z)\right\}$$
$$=mz^m\tilde{\Phi}(v, 2\kappa\Delta_j u, w; z),$$

thus we get (M3).

Step 4. Now we get $\tilde{\Phi}(z)\in \mathrm{Hom}(\mathcal{M}^\dagger_{j_2}\otimes V_j\otimes\mathcal{M}_{j_1}; \mathbb{C})$ satisfying (M1) $\sim$(M3). If $\tilde{\Phi}(z)$ factors to $\Phi(z)\in \mathrm{Hom}_{\mathfrak{g}}(\mathcal{H}^\dagger_{j_2}\otimes V_j\otimes\mathcal{H}_{j_1}; \mathbb{C})$, then the bilinear form $\Phi(u; z)$ $(u\in V_j)$ on $\mathcal{H}^\dagger_{j_2}\otimes\mathcal{H}_{j_1}$ defines an operator from $\mathcal{H}_{j_1}$ to $\mathcal{H}_{j_2}$ satisfying the conditions (V2) and (V3).

We must show that $\tilde{\Phi}(z)$ factors through $\mathrm{Hom}_{\mathfrak{g}}(\mathcal{H}^\dagger_{j_2}\otimes V_j\otimes\mathcal{H}_{j_1}; \mathbb{C})$, if and only if the vertex $\mathbb{v}$ is an ℓCG-vertex.

From the condition (M2), we get by the induction on p for $F_p\mathcal{M}^\dagger_{j_2}$ that $\tilde{\Phi}(u; z)$ factors through $\mathcal{M}^\dagger_{j_2}\otimes\mathcal{H}_{j_1}$, that is,

$$\tilde{\Phi}(v, u, \mathcal{J}_{j_1})=0 \qquad \text{for any } v\in\mathcal{M}^\dagger_{j_2} \text{ and } u\in V_j,$$

if and only if

$$\tilde{\Phi}(v, u, |J_{j_1}\rangle)=0 \qquad \text{for any } v\in V^\dagger_{j_2} \text{ and } u\in V_j.$$

In fact, $\mathcal{J}_{j_1}=U(\mathfrak{m}_-)U(\mathfrak{g})|J_{j_1}\rangle$ and $\mathfrak{m}_+|J_{j_1}\rangle=E|J_{j_1}\rangle=0$.

Since $|J_{j_1}\rangle=E(-1)^{\ell-2j_1+1}u_{j_1}(j_1)$, the last condition is equivalent to

$$\varphi(v, E^{\ell-2j_1+1}u, u_{j_1}(j_1))=0 \qquad \text{for any } v\in V^\dagger_{j_2} \text{ and } u\in V_j.$$

Similarly we get that $\tilde{\Phi}(u; z)$ factors through $\mathcal{H}^\dagger_{j_2}\otimes\mathcal{M}_{j_1}$, if and only if

$$\varphi(u^\dagger_{j_2}(j_2), F^{\ell-2j_2+1}u, w)=0 \qquad \text{for any } u\in V_j \text{ and } w\in V_{j_1}.$$

Step 5. Apply Lemma 2.2. q.e.d.

2.4) Normalization of vertex operators and Proof of Lemma 2.2.

The right $\mathfrak{g}$-module $V^\dagger_j$ can be identified with the dual (right) $\mathfrak{g}$-module $V^\vee_j=\mathrm{Hom}(V_j, \mathbb{C})$ through the vacuum expectation values:

$$v(u)=\langle v|u\rangle \qquad \text{for } v\in V^\dagger_j \text{ and } u\in V_j.$$

There exists an isomorphism ν: $V_j\to V^\dagger_j$ defined by $\nu(u_j(m))=(-1)^{j-m}\times u^\dagger_j(-m)$, then ν is an isomorphism over $(\mathfrak{g}, \nu)$:

$$\nu(X|v\rangle)=-\nu(|v\rangle)X \qquad (|v\rangle\in V_j, X\in\mathfrak{g}),$$

where $\nu: \mathfrak{g} \to \mathfrak{g}$ is the anti-automorphism defined by $\nu(X) = -X$. Moreover ν can be extended to the isomorphism $\nu: \mathscr{H}_j \to \mathscr{H}_j^\dagger$ such that $\nu(X(m)|v\rangle) = -\nu(|v\rangle)X(-m)$ ($|v\rangle \in \mathscr{H}_j$, $X \in \mathfrak{g}$, $m \in \mathbb{Z}$).

In Appendix I, we fix the element $\varphi_{\mathrm{v}} \in \mathscr{V}(\mathrm{v}) = \mathrm{Hom}_{\mathfrak{g}}(V_j \otimes V_{j_1}, V_{j_2})$ ($\cong \mathbb{C}$) for each CG-vertex $\mathrm{v} = \begin{pmatrix} & j & \\ j_2 & & j_1 \end{pmatrix}$. This notation will be used throughout this paper. And for each ℓCG-vertex v, denote by $\Phi_{\mathrm{v}}(z) = \Phi_{\varphi_{\mathrm{v}}}(z)$ the vertex operator of type v whose initial term is φ_{v}.

In a special case, we get

Proposition 2.5.

i) *Let j be an half-integer with $0 \leq 2j \leq \ell$ and put* $\mathrm{v} = \begin{pmatrix} & j & \\ j & & 0 \end{pmatrix}$. *Then* $\mathrm{v} \in (\mathrm{CG})_\ell$, $\hat{\Delta}(\mathrm{v}) = 0$, *and* $\varphi_{\mathrm{v}} = \mathrm{id}_{V_j} \in \mathscr{V}(\mathrm{v}) \cong \mathrm{Hom}(V_j, V_j)$. *Hence*

$$\lim_{z \searrow 0} \Phi_{\mathrm{v}}(w; z)|u_0(0)\rangle = |w\rangle \qquad (w \in V_j).$$

ii) *Let j be an half-integer with $0 \leq 2j \leq \ell$ and put* $\mathrm{v} = \begin{pmatrix} & j & \\ 0 & & j \end{pmatrix}$. *Then* $\mathrm{v} \in (\mathrm{CG})_\ell$, $\hat{\Delta}(\mathrm{v}) = 2\Delta_j$, *and* $\varphi_{\mathrm{v}} = \nu \in \mathscr{V}(\mathrm{v}) = \mathrm{Hom}(V_j, V_j^\dagger)$. *Hence*

$$\lim_{z \nearrow \infty} z^{2\Delta_j} \langle u_0^\dagger(0)|\Phi_{\mathrm{v}}(v; z) = \langle u_0^\dagger(0)|\varphi_{\mathrm{v}}(v) = \langle \nu(v)| \qquad (v \in V_j).$$

By the symmetry, it is sufficient to show the following for the proof of Lemma 2.2:

Lemma 2.2″. *For a vertex* $\mathrm{v} = \begin{pmatrix} & j & \\ j_2 & & j_1 \end{pmatrix} \in \mathbb{V}_\ell$, *assume* $\mathscr{V}(\mathrm{v}) \neq 0$ *and take its nonzero element φ. Then the following conditions are equivalent*:

(0) $$j_2 + j + j_1 \leq \ell.$$

(1) $\varphi(v, E^{\ell - 2j_1 + 1}u, u_{j_1}(j_1)) = 0$ *for any* $v \in V_{j_2}$ *and* $u \in V_j$.

Proof. Decompose the tensor product $V_{j_2} \otimes V_j$ into the sum of the irreducible components: $V_{j_2} \otimes V_j = \sum_k W_k$, where $W_k \cong V_k$ for $k \in \frac{1}{2}\mathbb{Z}$ with $|j - j_2| \leq k \leq j + j_2$ and $k + j + j_2 \in \mathbb{Z}$. By the assumption on φ, we may assume that $\varphi(W_{j_1} \otimes V_{j_1}) \neq 0$ and $\varphi(W_k \otimes V_{j_1}) = 0$ for $k \neq j_1$.

Since V_{j_1} is generated by the vector $u_{j_1}(j_1)$ and φ is invariant, there exists a vector $w \in W_{j_1, -j_1}$ such that $\varphi(w \otimes u_{j_1}(j_1)) \neq 0$ and $\varphi(W_{j_1, h} \otimes u_{j_1}(j_1)) = 0$ for any $h > -j_1$.

Put $L_1 = \ell - 2j_1 + 1$. Assume that $j_2 + j + j_1 \leq \ell$. Let $v \in V_{j_2, h_2}$ and $u \in V_{j, h}$. Since $h_2 + h + L_1 \geq 1 - j_1$, $\varphi(v, E^{L_1}u, u_{j_1}(j_1)) = 0$. Thus (0) implies (1).

Now express the vector w as $w=\sum_h a_h v_h \otimes u_h$, where $a_h \in \mathbb{C}$, $v_h \in V_{j_2,-h-j_1}$ and $u_h \in V_{j,h}$. Since $\mathfrak{n}_- w=0$, we get that $a_h \neq 0$ for $-j \leq h \leq j_2 - j_1$ by the induction on h. Hence $\varphi(v_{j_2-j_1}, u_{j_2-j_1}, u_{j_1}(j_1)) \neq 0$.

Assume that $j_2+j+j_1>\ell$. Then we get $j_2+j+j_1 \geq \ell+1$ and so

$$j_2-j_1-L_1 \geq j_1+j_2-\ell-1 \geq -j.$$

Hence the vector $u=F^{L_1}u_{j_2-j_1}$ does not vanish and $u_{j_2-j_1}=bE^{L_1}u$ for some nonzero constant b. Thus (1) implies (0). q.e.d.

2.5) Operator product expansions

The notion of operator product expansions in the 2-dimensional conformal field theory is due to A.A. Belavin et al. [BPZ].

Proposition 2.6.

i) *Ordered pairs* $\{X(\zeta), Y(z)\}$, $\{X(\zeta), T(z)\}$, $\{T(\zeta), X(z)\}$ *and* $\{T(\zeta), T(z)\}$ *of operators are composable for* $|\zeta|>|z|>0$ $(X, Y \in \mathfrak{g})$, *and their compositions* $X(\zeta)Y(z)$, $X(\zeta)T(z)$, $T(\zeta)X(z)$ *and* $T(\zeta)T(z)$ *are analytically continued to single-valued, operator-valued holomorphic functions on* $M_2=\{(\zeta, z) \in (\mathbb{C}^*)^2; \zeta \neq z\}$. *As operators on* $\mathscr{H}$, *the following identities hold*:

$$\text{(I)}\quad X(\zeta)Y(z)=\frac{\ell(X, Y)}{(\zeta-z)^2}\,\mathrm{id}+\frac{1}{\zeta-z}[X, Y](z)+R_{\mathrm{I}} \qquad (X, Y \in \mathfrak{g}).$$

$$\text{(II)}\quad T(\zeta)X(z)=\frac{1}{(\zeta-z)^2}X(z)+\frac{1}{\zeta-z}\frac{\partial}{\partial z}X(z)+R_{\mathrm{II}} \qquad (X \in \mathfrak{g}).$$

$$\text{(III)}\quad T(\zeta)T(z)=\frac{3\ell\,\mathrm{id}}{2\kappa(\zeta-z)^4}+\frac{2T(z)}{(\zeta-z)^2}+\frac{1}{\zeta-z}\frac{\partial}{\partial z}T(z)R_{\mathrm{III}}.$$

Here R_{I}, R_{II} *and* R_{III} *are regular at* $\zeta=z \in \mathbb{C}^*$.

Moreover

$$T(\zeta)T(z)=T(z)T(\zeta), \quad T(\zeta)X(z)=X(z)T(\zeta) \quad \textit{and} \quad X(\zeta)Y(z)=Y(z)X(\zeta).$$

ii) *Let* $\Phi(z)$ *be a vertex operator of spin* j *and* $u \in V_j$. *Ordered pairs* $\{X(\zeta), \Phi(u; z)\}$, $\{\Phi(u; \zeta), X(z)\}$, $\{T(\zeta), \Phi(u; z)\}$ *and* $\{\Phi(u; \zeta), T(z)\}$ *of operators are composable for* $|\zeta|>|z|>0$ $(X \in \mathfrak{g})$, *and their compositions* $X(\zeta)\Phi(u; z)$, $\Phi(u; \zeta)X(z)$, $T(\zeta)\Phi(u; z)$ *and* $\Phi(u; \zeta)T(z)$ *are analytically continued to multi-valued, operator-valued holomorphic functions on* M_2. *As operators on* $\mathscr{H}$, *the following identities hold*:

$$\text{(IV)}\quad X(\zeta)\Phi(u; z)=\frac{1}{\zeta-z}\Phi(Xu; z)+R_{\mathrm{IV}} \qquad (X \in \mathfrak{g}).$$

(V) $T(\zeta)\Phi(u;z)=\frac{\Delta_j}{(\zeta-z)^2}\Phi(u;z)+\frac{1}{\zeta-z}\frac{\partial}{\partial z}\Phi(u;z)+R_{\mathrm{V}}.$

Here R_{IV} and R_{V} are regular at $\zeta=z\in\mathbb{C}^*$.

Moreover $X(\zeta)\Phi(u;z)$ and $T(\zeta)\Phi(u;z)$ ($X\in\mathfrak{g}$) are single-valued and holomorphic function on $\zeta\in\mathbb{P}^1\backslash\{0, z, \infty\}$ for any fixed $z\in\mathbb{C}^*$, and

$$X(\zeta)\Phi(u;z)=\Phi(u;z)X(\zeta) \quad \textit{and} \quad T(\zeta)\Phi(u;z)=\Phi(u;z)T(\zeta).$$

Proof. All cases are obtained similarly, so we deal here with the case ii).

Let $\Phi(z)$ be a vertex operator of type $\mathbb{v}$. By Proposition 2.1 i), $\Phi(u;z)$ has the expansion

$$\Phi(u;z)=\sum_{n\in Z}z^{-n-\Delta}\Phi_n(u) \qquad (u\in V_j)$$

where $\Delta=\hat{\Delta}(\mathbb{v})=\Delta_j+\Delta_{j_1}-\Delta_{j_2}$. Then we get

$$[X(m), \Phi_n(u)]=[X(0), \Phi_{m+n}(u)]=\Phi_{m+n}(Xu) \qquad (X\in\mathfrak{g}, m, n\in\mathbb{Z})$$

and

$$[L(m), \Phi_n(u)]=\{(m+1)\Delta_j-m-n-\Delta\}\Phi_{m+n}(u) \qquad (m, n\in\mathbb{Z}).$$

Here we show (IV). For $|\zeta|>|z|>0$,

$$\begin{aligned}X(\zeta)\Phi(u,z)&=\sum_{m,n\in Z}\zeta^{-m-1}z^{-\Delta-n}X(m)\Phi_n(u)\\&=\sum_{k\in Z}\zeta^{-1}z^{-\Delta-k}\sum_{m\in Z}\left(\frac{z}{\zeta}\right)^m X(m)\Phi_{k-m}(u)\\&=\sum_{k\in Z}\zeta^{-1}z^{-\Delta-k}\sum_{m\geq 0}\left(\frac{z}{\zeta}\right)^m [X(m), \Phi_{k-m}(u)]+R_{\mathrm{IV}}\\&=\sum_{k\in Z}\zeta^{-1}z^{-\Delta-k}\sum_{m\geq 0}\left(\frac{z}{\zeta}\right)^m \Phi_k(Xu)+R_{\mathrm{IV}}\\&=\frac{\zeta^{-1}}{1-z/\zeta}\sum_{k\in Z}z^{-\Delta-k}\Phi_k(Xu)+R_{\mathrm{IV}}=\frac{1}{\zeta-z}\Phi(Xu;z)+R_{\mathrm{IV}},\end{aligned}$$

where

$$R_{\mathrm{IV}}=\sum_{k\in Z}\zeta^{-1}z^{-\Delta-k}\left\{\sum_{m>0}\left(\frac{\zeta}{z}\right)^m X(-m)\Phi_{k+m}(u)+\sum_{m\geq 0}\left(\frac{z}{\zeta}\right)^m\Phi_{k-m}(u)X(m)\right\}$$

is regular at $\zeta=z$.

For $|z|>|\zeta|>0$, we get

$$\Phi(u;z)X(\zeta)=\frac{1}{z-\zeta}[\Phi(u;z),X(0)]+R_{\rm IV}=\frac{-1}{z-\zeta}\Phi(Xu;z)+R_{\rm IV},$$

for the same Laurent series $R_{\rm IV}$. Hence for any $\langle u|\in\mathcal{H}^\dagger$, $|v\rangle\in\mathcal{H}$ and fixed $z\in\mathbb{C}^*$, the holomorphic function $\langle u|X(\zeta)\Phi(u;z)|v\rangle$ defined on $\{\zeta\in\mathbb{C};|\zeta|>|z|\}$ can be analytically continued to a (single-valued) holomorphic function on $\mathbb{P}^1\backslash\{0,\infty,z\}$ which coincides with the function $\langle u|\Phi(u;z)X(\zeta)|v\rangle$ on $\{\zeta;|z|>|\zeta|>0\}$. q.e.d.

Proposition 2.6 is generalized as follows:

Proposition 2.7. *Let $u\in V_j$ and $\Phi(z)=\Phi_{\rm v}(z)$ be the vertex operator of type* $\mathrm{v}=\begin{pmatrix} j \\ j_2\ j_1\end{pmatrix}\in(\mathrm{CG})_\ell$. *Let $A_N(z_N),\cdots,A_1(z_1)$ be operators of the form $T(z)$, $X(z)$ $(X\in\mathfrak{g})$ or $\Phi(u;z)$, and assume that there is a number i_0 such that $A_{i_0}(z_{i_0})=\Phi(u;z_{i_0})$ and $A_i(z_i)$ is not a vertex operator for $i\neq i_0$.*

Then $\{A_N(z_N),\cdots,A_1(z_1)\}$ is composable in the range $|z_N|>\cdots>|z_1|$, and the composed operator $A_N(z_N)\cdots A_1(z_1)$ is analytically continued to a multivalued and holomorphic function on $M_N=\{(z_N,\cdots,z_1)\in(\mathbb{C}^*)^N;z_i\neq z_j\ (i\neq j)\}$. If we fix $(z_N,\cdots,\hat{z}_j,\cdots,z_1)$ $(j\neq i_0)$, then this function is single-valued in $z_j\in\mathbb{P}^1\backslash\{\infty,z_N,\cdots,\hat{z}_j,\cdots,z_1,0\}$.

2.5) Actions of $\hat{\mathfrak{g}}$ and $\mathscr{L}$ on vertex operators

For an ℓCG-vertex $\mathrm{v}=\begin{pmatrix} j \\ j_2\ j_1\end{pmatrix}$, introduce the $\mathfrak{g}$-module $\mathscr{P}(\mathrm{v})$ defined by

$$\mathscr{P}(\mathrm{v})=\{\Phi_{\rm v}(u;z);u\in V_j\}\quad\text{and}\quad X\Phi_{\rm v}(u;z)=\Phi_{\rm v}(Xu;z)\qquad(X\in\mathfrak{g}).$$

In this paragraph, we fix $\mathrm{v}\in(\mathrm{CG})_\ell$ and say $\Phi(z)=\Phi_{\rm v}(z)$.

Now introduce the space $\mathcal{O}(\mathrm{v})$ of operators on $\mathcal{H}$ as the $\mathbb{C}$-vector space spanned by the set

$$\left\{\frac{1}{(2\pi\sqrt{-1})^N}\int_{C_N}\cdots\int_{C_1}d\zeta_N\cdots d\zeta_1(\zeta_N-z)^{m_N}\cdots(\zeta_1-z)^{m_1}X_N(\zeta_N)\right.$$
$$\left.\cdots X_1(\zeta_1)\Phi(u;z);\,N\in\mathbb{Z}_{\geq0},\,X_i\in\mathfrak{g},\,m_i\in\mathbb{Z}\ (1\leq i\leq N),\,u\in V_j\right\},$$

where the contours C_i $(1\leq i\leq N)$ are taken as follows: the origin 0 is outside C_N, C_i is inside C_{i+1} and z is inside C_1.

Let $A(z)\in\mathcal{O}(\mathrm{v})$, $X\in\mathfrak{g}$ and $m\in\mathbb{Z}$, then define

$$\hat{X}(m)A(z)=\frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^m X(\zeta)A(z)\in\mathcal{O}(\mathrm{v})$$

for some contour C around z such that 0 is outside C. Then by Proposition 2.6,

Proposition 2.8. *Let* v *be an* ℓCG-*vertex.*

i) *The assignation* $X(m)\mapsto\hat{X}(m)$ *and* $c\mapsto\ell$ id *defines the* $\hat{\mathfrak{g}}$-*module structure on* $\mathcal{O}(\mathrm{v})$.

ii) *Let* $u\in V_j$, *then*

$$\hat{X}(m)\Phi_{\mathrm{v}}(u;z)=0 \qquad (m>0,\ X\in\mathfrak{g},\ u\in V_j);$$

$$\hat{X}(0)\Phi_{\mathrm{v}}(u;z)=[X(0),\Phi_{\mathrm{v}}(u;z)]=\Phi_{\mathrm{v}}(Xu;z) \qquad (X\in\mathfrak{g},\ u\in V_j).$$

iii) *The assignation* $V_j\ni u\mapsto\Phi_{\mathrm{v}}(u;z)$ *defines the* $\mathfrak{g}$-*isomorphism of* V_j *onto the space* $\mathscr{P}(\mathrm{v})$, *and it is extended to a surjective* $\hat{\mathfrak{g}}$-*module mapping* $\Phi=\Phi_{\mathrm{v}}\colon\mathcal{M}_j\to\mathcal{O}(\mathrm{v})$.

Define the action of the Virasoro algebra $\mathscr{L}$ on $\mathcal{O}(\mathrm{v})$ by

$$\hat{L}(m)A(z)=\frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^{m+1}T(\zeta)A(z) \qquad (m\in\mathbb{Z})$$

for some contour C around z such that 0 is outside C. Then by Proposition 2.6, we get

i) for any $u\in V_j$

$$\hat{L}(m)\Phi(u;z)=0 \qquad (m\geq 1);$$

$$\hat{L}(0)\Phi(u;z)=\Delta_j\Phi(u;z) \quad \text{and} \quad \hat{L}(-1)\Phi(u;z)=\frac{\partial}{\partial z}\Phi(u;z).$$

ii) the well-definedness of this $\mathscr{L}$-action: ($A(z)\in\mathcal{O}(\mathrm{v})$)

$$\hat{L}(m)\hat{L}(n)A(z)-\hat{L}(n)\hat{L}(m)A(z)$$
$$=(m-n)\hat{L}(m+n)A(z)+\frac{m^3-m}{12}c\delta_{m+n,0}A(z).$$

iii) the compatibility of $\hat{\mathfrak{g}}$-action and $\mathscr{L}$-action:

$$\hat{L}(m)\hat{X}(n)A(z)-\hat{X}(n)\hat{L}(m)A(z)=-n\hat{X}(m+n)A(z).$$

iv) this $\mathscr{L}$-action coincides with the one induced from the Sugawara form

$$\hat{L}(m)A(z)=\frac{1}{2\kappa}\sum_{k\in\mathbb{Z}}\left\{\frac{1}{2}{:}\hat{H}(-k)\hat{H}(m+k){:}+{:}\hat{E}(-k)\hat{F}(m+k){:}\right.$$
$$\left.+{:}\hat{F}(-k)\hat{E}(m+k){:}\right\}A(z).$$

Theorem 2.9 (Nuclear Democracy*⁾). *For each* ℓCG-*vertex* $\mathbb{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix}$, *the* $\hat{\mathfrak{g}}$-*mapping* Φ *gives the* $\hat{\mathfrak{g}}$-*isomorphism of* $\mathscr{H}_j$ *onto* $\mathcal{O}(\mathbb{v})$.

Note. The following fact is important for this theorem: The only one additional relation of $\mathscr{H}_j$ to the Verma module $\mathscr{M}_j$ is the equality $E(-1)^{\ell-2j+1}u_j(j)=0$.

Proof of Theorem 2.9. For each $\mathbb{v}\in(\mathrm{CG})_\ell$, set $\varphi=\varphi_{\mathbb{v}}$ and $\Phi(z)=\Phi_{\mathbb{v}}(z)$. Since the kernel of the projection of $\mathscr{M}_j$ onto $\mathscr{H}_j$ is generated by a vector $|J_j\rangle\in\mathscr{M}_j$ over $U(\hat{\mathfrak{g}})$, it is sufficient to show that $\Phi(|J_j\rangle;z)=0$.

Step 1. Recall that $|J_j\rangle=E(-1)^{\ell-2j+1}u_j(j)$, $\mathfrak{m}_+|J_j\rangle=0$, and $U(\mathfrak{g})|J_j\rangle=\sum_{k=0}^{2(\ell-j+1)}\mathbb{C}F(0)^k|J_j\rangle$, hence $\mathfrak{m}_+U(\mathfrak{g})|J_j\rangle=0$. Since Φ is $\hat{\mathfrak{g}}$-linear,

$$\hat{X}(m)\Psi(z)=0$$

for any $m>0$, $X\in\mathfrak{g}$ and $\Psi(z)\in U(\mathfrak{g})\Phi(|J_j\rangle;z)$.

Step 2. Let $\Psi(z)\in\mathcal{O}(\mathbb{v})$ such that $\hat{X}(m)\Psi(z)=0$ for any $X(m)\in\mathfrak{m}_+$, then

$$[X(0),\Psi(z)]=\hat{X}(0)\Psi(z)\quad\text{and}\quad[X(m),\Psi(z)]=z^m[X(0),\Psi(z)]\qquad(m\in\mathbb{Z}).$$

In fact, by Proposition 2.6, we get $X(\zeta)\Psi(z)=\Psi(z)X(\zeta)$, so

$$[X(m),\Psi(z)]=\frac{1}{2\pi\sqrt{-1}}\int_C d\zeta\,\zeta^m X(\zeta)\Psi(z)$$

for some contour C around z such that 0 is outside C, and by the assumption we get

$$X(\zeta)\Psi(z)=\frac{1}{\zeta-z}\hat{X}(0)\Psi(z)+\sum_{k\geq 0}\hat{X}(-k-1)\Psi(z)\,(\zeta-z)^k.$$

Step 3. Since $\mathbb{v}\in(\mathrm{CG})_\ell$, we get by Remark 2.2′ii),

$$\varphi(v,u_j(j),E^{\ell-2j+1}w)=0\qquad(v\in V_{j_2}^\dagger,\ w\in V_{j_1}).$$

By the induction on n, we get that for any $v\in V_{j_2}^\dagger$ and $w\in V_{j_1}$

*⁾ We owe the naming of Nuclear Democracy to Prof. T. Eguchi.

$$\langle v|E(\zeta_1)\cdots E(\zeta_n)\Phi(u_j(j);z)|w\rangle=\prod_{i=1}^{n}\zeta_i^{-1}z^{-\Delta(v)}\varphi(v,u_j(j),E^n w),$$

so

$$0=\langle v|(\hat{E}(-1)^{\ell-2j+1}\Phi(u_j(j);z)|w\rangle=\langle v|\Phi(|J_j\rangle;z)|w\rangle,$$

hence by Steps 1 and 2.

$$\langle v|\Psi(z)|w\rangle=0$$

for any $\Psi(z)\in U(\mathfrak{g})\Phi(|J_j\rangle;z)$.

Since $\mathscr{H}_{j_2}^{\dagger}=V_{j_2}^{\dagger}U(\mathfrak{m}_+)$ and $\mathscr{H}_{j_1}=U(\mathfrak{m}_-)V_{j_1}$, we get

$$\langle\mathscr{H}_{j_2}^{\dagger}|\Psi(z)|\mathscr{H}_{j_1}\rangle=0$$

for any $\Psi(z)\in U(\mathfrak{g})\Phi(|J_j\rangle;z)$, hence $U(\mathfrak{g})\Phi(|J_j\rangle;z)=0$. q.e.d.

Here we summarize the relations satisfied by vertex operators:

Fundamental relations for vertex operators

Let $\Phi(z)$ be a vertex operator of spin j. Then

$$\hat{X}(m)\Phi(u;z)=0 \qquad (m\geq 1,\ X\in\mathfrak{g},\ u\in V_j);$$
$$\hat{X}(0)\Phi(u;z)=[X(0),\Phi(u;z)]=\Phi(Xu;z) \qquad (X\in\mathfrak{g},\ u\in V_j);$$
$$\hat{L}(m)\Phi(u;z)=0 \qquad (m\geq 1,\ u\in V_j);$$
$$\hat{L}(0)\Phi(u;z)=\Delta_j\Phi(u;z) \qquad (u\in V_j);$$
$$\hat{L}(-1)\Phi(u;z)=\frac{\partial}{\partial z}\Phi(u;z) \qquad (u\in V_j);$$

and

$$\hat{E}(-1)^{\ell-2j+1}\Phi(u_j(j);z)=0.$$

§ 3. Differential Equations of N-point Functions and Composability of Vertex Operators

In this section, we will give the system of differential equations of N-point functions and show the composability of vertex operators.

3.1) N-point functions and their differential equations

The vacuums $u_0(0)$ and $u_0^{\dagger}(0)$ of $\mathscr{H}_0$ and $\mathscr{H}_0^{\dagger}$ are of special importance (and are called *Virasoro vacuums*): denote $|\mathrm{vac}\rangle=u_0(0)$ and $\langle\mathrm{vac}|=u_0^{\dagger}(0)$, then

$$\mathfrak{p}_+|\mathrm{vac}\rangle=0 \quad\text{and}\quad L(m)|\mathrm{vac}\rangle=0 \quad (m\geq -1);$$

$$\langle \mathrm{vac}|\mathfrak{p}_-=0 \quad \text{and} \quad \langle \mathrm{vac}|L(m)=0 \quad (m\leq 1).$$

For an operator A on $\mathscr{H}$, define its *vacuum expectation value* by

$$\langle A\rangle=\langle \mathrm{vac}|A|\mathrm{vac}\rangle.$$

Introduce the $\mathfrak{g}$-module $\mathscr{P}=\sum_{\mathbb{v}\in(\mathrm{CG})_\ell}\mathscr{P}(\mathbb{v})$, defined by the $\mathfrak{g}$-action

$$\hat{X}(0)\Phi(u;z)=\Phi(Xu;z) \qquad (X\in\mathfrak{g}).$$

Denote by Δ_{ik} $(1\leq i, k\leq N)$ the $\mathfrak{g}$-diagonal action on the i-th and k-th components of the N-th tensor product $\mathscr{P}^{\otimes N}$, that is, $\Delta_{ik}=\pi_i+\pi_k$, where π_i is the $\mathfrak{g}$-action on the i-th component of $\mathscr{P}^{\otimes N}$. Introduce the operator Ω_{ik} on $\mathscr{P}^{\otimes N}$ defined by

$$\Omega_{ik}=\frac{1}{2}\pi_i(H)\pi_k(H)+\pi_i(E)\pi_k(F)+\pi_i(F)\pi_k(E)$$

and denote $\Omega_i=\Omega_{ii}=\pi_i(\Omega)$, then

$$\Omega_{ik}=\frac{1}{2}\{\Delta_{ik}(\Omega)-\Omega_i-\Omega_k\}$$

and

$$[\Omega_{ik}, \Delta_{ik}(X)]=[\Omega_{ik}, \pi_j(X)]=0 \qquad (i\neq k,\ X\in\mathfrak{g},\ j\neq i, k).$$

For any half-integer j $(0\leq 2j\leq \ell)$, denote by $V_j^\vee$ the dual $\mathfrak{g}$-module of V_j. For any N-ple $\mathbb{J}=(j_N, \cdots, j_1)$ of half-integers with $0\leq 2j_i\leq \ell$, let $V^\vee(\mathbb{J})=V_{j_N}^\vee\otimes\cdots\otimes V_{j_1}^\vee$, and let $V_0^\vee(\mathbb{J})=(V_{j_N}^\vee\otimes\cdots\otimes V_{j_1}^\vee)^{\mathfrak{g}}$ the space of all $\mathfrak{g}$-invariant elements in $V^\vee(\mathbb{J})$. Then the operators Ω_{ik} act similarly on $V^\vee(\mathbb{J})$ and on $V_0^\vee(\mathbb{J})$.

Let $\Phi_i(z_i)$ be a vertex operator of spin j_i $(1\leq i\leq N)$, then the vacuum expectation value of the composed operator

$$\langle \Phi_N(z_N)\cdots\Phi_1(z_1)\rangle$$

is considered as a $V^\vee(\mathbb{J})$-valued, formal Laurent series on $(z_N, \cdots, z_1)$ and is called an *N-point function*: If $\Phi_i(z_i)$ is of type $\mathbb{v}_i$ $(1\leq i\leq N)$,

$$\langle \Phi_N(z_N)\cdots\Phi_1(z_1)\rangle=\prod_{i=1}^N z_i^{-\Delta(\mathbb{v}_i)}\sum_{m_N\geq 0}\cdots\sum_{m_k\in Z}\cdots\sum_{m_1\leq 0}C_{m_N\cdots m_1}z_N^{-m_N}\cdots z_1^{-m_1},$$

where

$$C_{m_N\cdots m_1}=\langle \mathrm{vac}|\Phi_{N,m_N}(\cdot)\Phi_{N-1,m_{N-1}}(\cdot)\cdots\Phi_{2,m_2}(\cdot)\Phi_{1,m_1}(\cdot)|\mathrm{vac}\rangle\in V^\vee(\mathbb{J}).$$

The aim of this section is to show that N-point functions are convergent in some region and analytically continued to a multivalued holomorphic function on M_N.

First we get a system of differential equations of N-point functions:

Theorem 3.1. *Let $\Phi_i(z_i)$ be a vertex operator of spin j_i $(1 \leq i \leq N)$, then the N-point function $\langle \Phi_N(z_N) \cdots \Phi_1(z_1) \rangle$ satisfies the following equations*:

(I) (projective invariance) *For $m=-1, 0$ and 1,*

$$\sum_{i=1}^{N} z_i^m \left(z_i \frac{\partial}{\partial z_i} + (m+1) \Delta_{j_i} \right) \langle \Phi_N(z_N) \cdots \Phi_1(z_1) \rangle = 0.$$

(II) (gauge invariance) *For any $X \in \mathfrak{g}$,*

$$\sum_{i=1}^{N} \pi_i(X) \langle \Phi_N(z_N) \cdots \Phi_1(z_1) \rangle = 0.$$

(III) *For each $i=1, \cdots, N$,*

$$\left(\kappa \frac{\partial}{\partial z_i} - \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{\Omega_{ik}}{z_i - z_k} \right) \langle \Phi_N(z_N) \cdots \Phi_1(z_1) \rangle = 0,$$

where $\kappa = \ell + 2$.

(IV) *For each $i=1, \cdots, N$,*

$$\langle \Phi_N(u_N; z_N) \cdots (\hat{E}(-1)^{\ell - 2j_i + 1} \Phi_i(u_{j_i}(j_i); z_i)) \cdots \Phi_1(u_1; z_1) \rangle = 0$$

for any $u_k \in V_{j_k}$ $(k \neq i)$.

Proof. These equations are obtained from the fundamental relations of vertex operators, the Sugawara form of $L(m)$ and the properties of the Virasoro vacuums. Here we give a brief proof of (III). First, note the identity:

$$\langle X(z) \Phi_N(z_N) \cdots \Phi_1(z_1) \rangle = \sum_{i=1}^{N} \frac{1}{z - z_i} \pi_i(X) \langle \Phi_N(z_N) \cdots \Phi_1(z_1) \rangle \qquad (X \in \mathfrak{g}).$$

Let $X^1 = 2X_1 = H$, $X^2 = X_3 = E$ and $X^3 = X_2 = F$, then the Casimir operator Ω is expressed as $\Omega = \sum_{k=1}^{3} X^k X_k$. By Proposition 2.6 and the relation $\hat{L}(-1)\Phi(z) = (\partial/\partial z)\Phi(z)$, we get

$$\kappa \frac{\partial}{\partial z_i} \Phi_i(u_i; z_i) = \lim_{z \searrow z_i} \left\{ \sum_{k=1}^{3} X^k(z) \Phi_i(X_k u_i; z_i) - \frac{1}{z - z_i} \Phi_i(\Omega u_i; z_i) \right\} \qquad (1 \leq i \leq N).$$

Hence for each i with $1\leq i\leq N$,

$$\sum_{k=1}^{3}\langle X^k(z)\Phi_N(z_N)\cdots(X_k\Phi_i)(z_i)\cdots\Phi_1(z_1)\rangle$$
$$=\sum_{k=1}^{3}\sum_{j=1}^{N}\frac{1}{z-z_j}\pi_j(X^k)\pi_i(X_k)\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle$$
$$=\left\{\frac{1}{z-z_i}\Omega_i+\sum_{j\neq i}\frac{1}{z-z_j}\Omega_{ij}\right\}\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle.$$

Thus we get the equation (III) by taking the limit $z\searrow z_i$.

Remark 3.2.

i) The equations (I) ~ (III) are obtained by V.G. Knizhnik and A.B. Zamolodchikov [KZ].

ii) The equations (II) mean that $\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle\in V_0^{\vee}(\mathbb{J})$.

iii) The equations (II) and (III) imply the equations (I). (Key is the property of the operators Ω_{ik}: $\sum_{k=1}^{N}\Omega_{ik}=0$ on $V_0^{\vee}(\mathbb{J})$.)

iv) The system (III) of differential equations is completely integrable. This complete integrability of (III) is reduced to the *infinitesimal pure braid relations* of Ω_{ik}:

$$[\Omega_{ik},\Omega_{mn}]=0 \qquad \text{(if } i, k, m, n \text{ are mutually disjoint);}$$

and

$$[\Omega_{im}, \Omega_{ik}+\Omega_{km}]=0 \qquad \text{(if } i, k, m \text{ are mutually disjoint).}$$

These infinitesimal pure braid relations were originally noted by K. Aomoto (see [A1] and [A2]). Moreover these pure braid relations are equivalent to the classical Yang-Baxter equations for $\mathfrak{sl}_2$ obtained by C.N. Yang [Y] and A.A. Belavin-V.G. Drinfel'd [BD].

v) N-point functions are translation invariant (Corollary of (I)):

$$\langle\Phi_N(z_N+z)\cdots\Phi_1(z_1+z)\rangle=\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle.$$

vi) The equations (IV) are equivalent to the algebraic equations: for each i $(1\leq i\leq N)$ and any $u_k\in V_{j_k}$ $(k\neq i)$, put $L_i=\ell-2j_i+1$.

$$\sum_{|\mathfrak{m}_i|=L_i}\binom{L_i}{\mathfrak{m}_i}\prod_{k\neq i}(z_k-z_i)^{-m_k}\langle\Phi_N(E^{m_N}u_N;z_N)\cdots\Phi_i(u_{j_i}(j_i);z_i)\cdots\Phi_1(E^{m_1}u_1;z_1)\rangle$$
$$=0,$$

where $\mathfrak{m}_i=(m_N,\cdots,\hat{m}_i,\cdots,m_1)\in(\mathbb{Z}_{\geq 0})^{N-1}$, $|\mathfrak{m}_i|=\sum_{k\neq i}m_k$ and $\binom{L_i}{\mathfrak{m}_i}$ is the multinomial coefficient.

3.2) Solutions of fundamental equation

Consider the systems $\mathrm{E}(\mathbb{J})$ of differential equations and $\mathrm{B}(\mathbb{J})$ of algebraic equations for $V_0^{\vee}(\mathbb{J})$-valued functions $\Phi(z_N, \cdots, z_1)$ on the manifold $X_N=\{(z_N, \cdots, z_1) \in \mathbb{C}^N;\ z_i \neq z_k\ (i \neq k)\} \supset M_N$:

$$E(\mathbb{J}):\qquad \left(\kappa\frac{\partial}{\partial z_i}-\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}\right)\Phi(z_N, \cdots, z_1)=0 \qquad (1\leq i\leq N)$$

and for each $i\ (1\leq i\leq N)$ and any $u_k \in V_{j_k}\ (k\neq i)$,

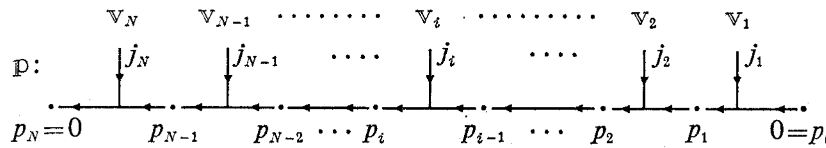
$$\mathrm{B}(\mathbb{J}):\sum_{|\mathrm{m}_i|=L_i}\binom{L_i}{\mathrm{m}_i}\prod_{k\neq i}(z_k-z_i)^{-m_k}\Phi(z_N, \cdots, z_1)(E^{m_N}u_N, \cdots, u_{j_i}(j_i), \cdots, E^{m_1}u)$$
$$=0,$$

where $\mathrm{m}_i=(m_N, \cdots, \hat{m}_i, \cdots, m_1) \in (\mathbb{Z}_{\geq 0})^{N-1}$, $|\mathrm{m}_i|=\sum_{k\neq i} m_k$ and $L_i=\ell-2j_i+1$.

By Remark 3.2, the system $E(\mathbb{J})$ is completely integrable.

Introduce the set $\mathscr{P}(\mathbb{J})$ defined by

$$\mathscr{P}(\mathbb{J})=\left\{\mathrm{p}=(p_N, \cdots, p_1, p_0);\ \mathrm{v}_i(\mathrm{p})=\begin{pmatrix} & j_i & \\ p_i & & p_{i-1}\end{pmatrix} \in (\mathrm{CG}),\ p_N=p_0=0\right\}.$$



For each $\mathrm{p} \in \mathscr{P}(\mathbb{J})$, define the vector φ_{p} of $V_0^{\vee}(\mathbb{J})$ from the (fixed) elements $\varphi_{\mathrm{v}_i} \in \mathrm{Hom}_{\mathfrak{g}}(V_{p_i}^{\dagger}\otimes V_{j_i}\otimes V_{p_{i-1}};\ \mathbb{C})\cong(V_{j_i}^{\vee}\otimes\mathrm{Hom}(V_{p_{i-1}}, V_{p_i}))^{\mathfrak{g}}$ $(1\leq i\leq N)$, as the trace of $\varphi_{\mathrm{v}_N}\otimes\cdots\otimes\varphi_{\mathrm{v}_1}$: for each $u_N\otimes\cdots\otimes u_1 \in V(\mathbb{J})$,

$$\varphi_{\mathrm{p}}(u_N, \cdots, u_1)=\langle \mathrm{vac}|\varphi_{\mathrm{v}_N}(u_N)\circ\varphi_{\mathrm{v}_{N-1}}(u_{N-1})\circ\cdots\circ\varphi_{\mathrm{v}_1}(u_1)|\mathrm{vac}\rangle.$$

Then the set $\{\varphi_{\mathrm{p}};\ \mathrm{p} \in \mathscr{P}(\mathbb{J})\}$ gives a basis of the space $V_0^{\vee}(\mathbb{J})$.

Introduce the operators $\Omega_m^{\vee}=\sum_{1\leq i\neq j\leq m}\Omega_{ij}$ on $V^{\vee}(\mathbb{J})$ for $m\ (2\leq m\leq N)$, then

$$\Omega_m^{\vee}=\hat{\Omega}_m-\sum_{i=1}^{m}\Omega_{ii},$$

where $\hat{\Omega}_m$ is the diagonal action of Ω on $V_{j_m}^{\vee}\otimes\cdots\otimes V_{j_1}^{\vee}$, and by the pure braid relations (Remark 3.2 iv), we get that $[\Omega_m^{\vee}, \Omega_n^{\vee}]=0$.

In the basis $\{\varphi_{\mathrm{p}};\ \mathrm{p} \in \mathscr{P}(\mathbb{J})\}$, these operators are diagonal:

$$\Omega_m^{\vee}\varphi_{\mathrm{p}}=2\kappa\Delta_m^{\vee}(\mathrm{p})\varphi_{\mathrm{p}} \qquad \left(\mathrm{p}=(p_N, \cdots, p_1, p_0);\ \mathrm{v}_i=\begin{pmatrix} j_i \\ p_i\, p_{i-1}\end{pmatrix}\right),$$

where

$$\Delta_m^{\vee}(\mathrm{p})=\Delta_{p_m}-\sum_{i=1}^{m}\Delta_{j_i}=-\sum_{i=1}^{m}\hat{\Delta}(\mathrm{v}_i) \qquad (2\leq m\leq N)$$

In fact, for each $i=2, \cdots, N$,

$$\hat{\Omega}_i\varphi_{\mathrm{p}}=2\kappa\Delta_{p_i}\varphi_{\mathrm{p}} \quad \text{and} \quad \Omega_{ii}\varphi_{\mathrm{p}}=2\kappa\Delta_{j_i}\varphi_{\mathrm{p}}.$$

Now introduce the subset $\mathscr{P}_\ell(\mathbb{J})$ of $\mathscr{P}(\mathbb{J})$ defined by

$$\mathscr{P}_\ell(\mathbb{J})=\left\{\mathrm{p}=(p_N, \cdots, p_1, p_0) \in \mathscr{P}(\mathbb{J});\ \mathrm{v}_i=\mathrm{v}_i(\mathrm{p})=\begin{pmatrix} j_i \\ p_i\, p_{i-1}\end{pmatrix} \in (\mathrm{CG})_\ell\right\},$$

then for each $\mathrm{p} \in \mathscr{P}_\ell(\mathbb{J})$, the N-point function

$$\Phi_{\mathrm{p}}(z_N, \cdots, z_1)=\langle\Phi_{\mathrm{v}_N}(z_N)\cdots\Phi_{\mathrm{v}_1}(z_1)\rangle$$

of type p is a formal Laurent series solution of the system $E(\mathbb{J})$ and $B(\mathbb{J})$ by Theorem 3.1, where its Laurent series expansion is given as

$$\begin{aligned}\Phi_{\mathrm{p}}(z_N, \cdots, z_1)&=\prod_{i=1}^{N} z_i^{-\hat{\Delta}(\mathrm{v}_i)} \sum_{m_N\geq 0}\cdots\sum_{m_i\in Z}\cdots\sum_{m_1\leq 0} C_{m_N\cdots m_1}z_N^{-m_N}\cdots z_1^{-m_1}\\ &=\prod_{i=1}^{N} z_i^{-\Delta_{j_i}}\left\langle z_N^{L(0)}\Phi_{\mathrm{v}_N}(1)\left(\frac{z_{N-1}}{z_N}\right)^{L(0)}\Phi_{\mathrm{v}_{N-1}}(1)\cdots\left(\frac{z_1}{z_2}\right)^{L(0)}\Phi_{\mathrm{v}_1}(1)z_1^{-L(0)}\right\rangle\end{aligned}$$

where

$$C_{m_N\cdots m_1}=\langle\mathrm{vac}|\Phi_{\mathrm{v}_N,m_N}(\cdot)\Phi_{\mathrm{v}_{N-1},m_{N-1}}(\cdot)\cdots\Phi_{\mathrm{v}_2,m_2}(\cdot)\Phi_{\mathrm{v}_1,m_1}(\cdot)|\mathrm{vac}\rangle \in V^{\vee}(\mathbb{J}).$$

Moreover

Theorem 3.3. *Consider the region $\mathscr{R}_z$ in the manifold X_N, defined by*

$$\mathscr{R}_z=\{\mathrm{z}=(z_N, \cdots, z_1) \in \mathbb{C}^N;\ |z_N|>\cdots>|z_1|\}.$$

Then

i) *for any $\mathrm{p} \in \mathscr{P}_\ell(\mathbb{J})$, the Laurent series $\Phi_{\mathrm{p}}(z_N, \cdots, z_1)$ is absolutely convergent in the region $\mathscr{R}_z$, and is analytically continued to a multivalued holomorphic function on X_N.*

ii) *$\{\Phi_{\mathrm{p}}(z_N, \cdots, z_1);\ \mathrm{p} \in \mathscr{P}_\ell(\mathbb{J})\}$ is linearly independent and gives a basis of the solution space of the joint system* $\mathrm{E}(\mathbb{J})$ *and* $\mathrm{B}(\mathbb{J})$.

Proof. The system $\mathrm{E}(\mathbb{J})$ of differential equations is equivalent to the Pfaffian system:

$$\mathrm{P}(\mathbb{J}): \qquad \kappa d\Phi(\mathbb{z}) - \sum_{k<i} \frac{d(z_i - z_k)}{z_i - z_k} \Omega_{ik} \Phi(\mathbb{z}) = 0.$$

Now we change coordinates $\mathbb{z}$ to $\mathbb{w}$ by

$$w_N = z_N;\ w_i = z_i/z_{i+1}\ (1 \leq i \leq N-1).$$

Then the region $\mathscr{R}_z$ transforms bijectively onto the region

$$\mathscr{R}_{w,0} = \{\mathbb{w} = (w_N, \cdots, w_1) \in \mathbb{C}^N;\ w_N \neq 0,\ 1 > |w_i| > 0\ (2 \leq i \leq N-1),\ 1 > |w_1|\},$$

where the inverse transformation is given as

$$z_i = w_N \cdots w_i\ (1 \leq i \leq N).$$

And introduce the region $\mathscr{R}_w = \{\mathbb{w} \in \mathbb{C}^N;\ 1 > |w_i|\ (1 \leq i \leq N-1)\} \supset \mathscr{R}_{w,0}$.

The system $\mathrm{P}(\mathbb{J})$ is written in the coordinates $\mathbb{w}$ as

$$\tilde{\mathrm{P}}(\mathbb{J}): \kappa \sum_{i=1}^{N} \frac{\partial \tilde{\Phi}}{\partial w_i} dw_i = \sum_{m=2}^{N} \frac{1}{w_m} dw_m \sum_{k<i\leq m} \Omega_{ik} \tilde{\Phi} - \sum_{m=1}^{N-1} dw_m \sum_{k\leq m<i} \frac{w_{i-1} \cdots \hat{w}_m \cdots w_k}{1 - w_{i-1} \cdots w_k} \Omega_{ik} \tilde{\Phi},$$

where $\tilde{\Phi}(\mathbb{w}) = \Phi(\mathbb{z})$.

Hence by using the operators $\Omega_m^{\vee}$, the system $\mathrm{E}(\mathbb{J})$ turns to be

$$\tilde{\mathrm{E}}(\mathbb{J}): \qquad \left\{2\kappa \frac{\partial}{\partial w_m} - \frac{\Omega_m^{\vee}}{w_m} + A_m(\mathbb{w})\right\} \tilde{\Phi}(\mathbb{w}) = 0 \qquad (2 \leq m \leq N),$$

$$\left\{2\kappa \frac{\partial}{\partial w_1} + A_1(\mathbb{w})\right\} \tilde{\Phi}(\mathbb{w}) = 0$$

where

$$A_m(\mathbb{w}) = \sum_{k\leq m<i} \frac{w_{i-1} \cdots \hat{w}_m \cdots w_k}{1 - w_{i-1} \cdots w_k} \Omega_{ik} \qquad (2 \leq m \leq N-1),$$

$$A_1(\mathbb{w}) = \frac{1}{1 - w_1} \Omega_{21} + \sum_{i\geq 3} \frac{w_{i-1} \cdots w_2}{1 - w_{i-1} \cdots w_1} \Omega_{i1} \quad \text{and} \quad A_N(\mathbb{w}) = 0.$$

Since $A_m(w)$'s are holomorphic in the region $\mathscr{R}_w$, the system $\tilde{\mathrm{E}}(\mathbb{J})$ is with regular singularities along the divisors $D_i = \{w_i = 0\}$ for $i = 2, \cdots, N$. The basis $\{\varphi_{\mathbb{p}};\ \mathbb{p} \in \mathscr{P}(\mathbb{J})\}$ of $V_0^{\vee}(\mathbb{J})$ diagonalizes the principal parts of the system $\mathrm{E}(\mathbb{J})$ with the exponents $\{\Delta_i^{\vee}(\mathbb{p});\ 1 \leq i \leq N\}$ corresponding to $\varphi_{\mathbb{p}}$.

The formal Laurent series solution $\tilde{\Phi}_{\mathbb{p}}(w_N, \cdots, w_1) = \Phi_{\mathbb{p}}(z_N, \cdots, z_1)$, $\mathbb{p} \in \mathscr{P}_{\ell}(\mathbb{J})$, of the system $\tilde{\mathrm{E}}(\mathbb{J})$ is written as

$$\tilde{\Phi}_{\mathbb{p}}(w_N, \cdots, w_1) = \prod_{i=2}^{N} w_i^{\Delta_i^{\vee}(\mathbb{p})} S_{\mathbb{p}}(w_N, \cdots, w_1),$$

where

$$S_{\mathbb{p}}(w_N, \cdots, w_1)$$
$$=\prod_{i=1}^{N} w_i^{-\Delta_{p_i}}\langle w_N^{L(0)}\Phi_{\mathrm{v}_N}(1) w_{N-1}^{L(0)}\Phi_{\mathrm{v}_{N-1}}(1)\cdots\Phi_{\mathrm{v}_2}(1) w_1^{L(0)}\Phi_{\mathrm{v}_1}(1)\rangle$$

is a formal power series in $\mathbb{w}$, since $w_i^{L(0)}=w_i^{\Delta_{p_i}+d}$ id on $\mathscr{H}_{p_i,d}$.

By the theory of partial differential equations with regular singular points (see e.g. [CL] Chap. 3 and [Kn] Appendix B), the function $\tilde{\Phi}_{\mathbb{p}}(\mathbb{w})$ is a solution of the system $\tilde{\mathrm{E}}(\mathbb{J})$ of differential equation in the region $\mathscr{R}_{w,0}$ for each $\mathbb{p}\in\mathscr{P}_\ell(\mathbb{J})$. Hence the formal power series $S_{\mathbb{p}}(\mathbb{w})$ gives a holomorphic function in $\mathscr{R}_w$, and so the function $\tilde{\Phi}_{\mathbb{p}}(w_N, \cdots, w_1)$ is holomorphic in the region $\mathscr{R}_{w,0}$. Thus the N-point function $\Phi_{\mathbb{p}}(z_N, \cdots, z_1)$ is holomorphic in $\mathscr{R}_z$ for any $\mathbb{p}\in\mathscr{P}_\ell(\mathbb{J})$.

ii) By the remark before the statement of the theorem, for each $\mathbb{p}\in\mathscr{P}_\ell(\mathbb{J})$

$$S_{\mathbb{p}}(0, \cdots, 0)=\langle \mathrm{vac}|\Phi_{\mathrm{v}_N,0}(\cdot)\Phi_{\mathrm{v}_{N-1},0}(\cdot)\cdots\Phi_{\mathrm{v}_2,0}(\cdot)\Phi_{\mathrm{v}_1,0}(\cdot)|\mathrm{vac}\rangle$$
$$=\langle \mathrm{vac}|\varphi_{\mathrm{v}_N}\varphi_{\mathrm{v}_{N-1}}\cdots\varphi_{\mathrm{v}_2}\varphi_{\mathrm{v}_1}|\mathrm{vac}\rangle=\langle \mathrm{vac}|\varphi_{\mathbb{p}}|\mathrm{vac}\rangle\in V_0^\vee(\mathbb{J}).$$

This implies the linear independence of $\{\Phi_{\mathbb{p}}(z_N, \cdots, z_1)\colon \mathbb{p}\in\mathscr{P}_\ell(\mathbb{J})\}$.

Finally we want to show that the dimension of the solution space of the joint system $\tilde{\mathrm{E}}(\mathbb{J})$ and $\tilde{\mathrm{B}}(\mathbb{J})$ is not greater than $\sharp\mathscr{P}_\ell(\mathbb{J})$, where $\tilde{\mathrm{B}}(\mathbb{J})$ is the system $\mathrm{B}(\mathbb{J})$ written in the coordinates $\mathbb{w}$: for each i with $1\le i\le N$, let $L=\ell-2j_i+1$,

$$\tilde{\mathrm{B}}_i(\mathbb{J})\colon\ \sum_{K=0}^{L}\binom{L}{K}\sum_{\substack{|\mathbb{m}'|=L-K\\|\mathbb{m}''|=K}}\binom{L-K}{\mathbb{m}'}\binom{K}{\mathbb{m}''}\prod_{k=i}^{N} w_k^{-K-m_{i+1}-\cdots-m_k}(1+O(w))$$
$$\times\tilde{\Phi}(w_N, \cdots, w_1)(E^{m_N}u_N, \cdots, u_{j_i}(j_i), \cdots, E^{m_1}u_1)=0,$$

where $\mathbb{m}'=(m_N, \cdots, m_{i+1})\in(\mathbb{Z}_{\ge0})^{N-1}$, $\mathbb{m}''=(m_{i-1}, \cdots, m_1)\in(\mathbb{Z}_{\ge0})^{i-1}$ and $O(\mathbb{w})$ is a convergent power series in $\mathscr{R}_w$ and $O(\mathbb{0})=0$.

For each $\mathbb{p}\in\mathscr{P}(\mathbb{J})$, take a solution

$$\Psi_{\mathbb{p}}(w_N, \cdots, w_1)=\prod_{i=2}^{N} w_i^{\Delta_i^\vee(\mathbb{p})}\,T_{\mathbb{p}}(w_N, \cdots, w_1)\qquad(\mathbb{p}\in\mathscr{P}(\mathbb{J}))$$

of the system $\tilde{\mathrm{E}}(\mathbb{J})$, where $T_{\mathbb{p}}(\mathbb{w})$ is a convergent power series in $\mathscr{R}_w$ with the constant term $T_{\mathbb{p}}(0)=\varphi_{\mathbb{p}}$. Apply $\tilde{\mathrm{B}}_i(\mathbb{J})$ to $\Psi_{\mathbb{p}}(\mathbb{w})$ for $i\ge2$, then its leading term must vanish, and the term is obtained by taking $K=L$, since

$$\sum_{k=i}^{N}(K+m_{i+1}+\cdots+m_k)=(N-i+1)K+\sum_{k=i+1}^{N}(k-i)m_k$$

and

$$K+\sum_{k=i+1}^{N} m_k = L.$$

Hence

$$0=\sum_{|\mathrm{m}''|=L}\binom{L}{\mathrm{m}''}T_{\mathrm{p}}(0)(u_N, \cdots, u_{i+1}, u_{j_i}(j_i), E^{m_{i-1}}u_{i-1}, \cdots, E^{m_1}u_1)$$
$$=\varphi_{\mathrm{p}}(u_N, \cdots, u_{i+1}, u_{j_i}(j_i), E^L(u_{i-1}\otimes\cdots\otimes u_1)).$$

By Remark 2.2′ ii), we get that $\ell \geq j_i+p_i+p_{i-1}$, that is, $\mathrm{v}_i \in (\mathrm{CG})_\ell$ for $i\geq 2$. Hence $\mathbb{p} \in \mathscr{P}_\ell(\mathbb{J})$, since $\mathrm{v}_1 \in (\mathrm{CG})_\ell$ automatically.

Introduce a partial order $\prec$ in the set $\mathscr{P}(\mathbb{J})$ defined by

$$\mathbb{p}\prec\mathbb{p}', \qquad \text{if } (\Delta_N(\mathbb{p}')-\Delta_N(\mathbb{p}), \cdots, \Delta_2(\mathbb{p}')-\Delta_2(\mathbb{p})) \in (\mathbb{Z}_{\geq 0})^{N-1}.$$

Let $\Psi(\mathrm{w})$ be a solution of the systems $\tilde{\mathrm{E}}(\mathbb{J})$ and $\tilde{\mathrm{B}}(\mathbb{J})$, and express it as $\Psi(\mathrm{w})=\sum_{\mathbb{p}\in\mathscr{P}_\Psi} c_{\mathbb{p}}\Psi_{\mathbb{p}}(\mathrm{w})$, where $\mathscr{P}_\Psi=\{\mathbb{p} \in \mathscr{P}(\mathbb{J});\ c_{\mathbb{p}}\neq 0\}$. Apply $\tilde{\mathrm{B}}_i(\mathbb{J})$ to $\Psi(\mathrm{w})$, then by the linear independence of solutions of $\tilde{\mathrm{E}}(\mathbb{J})$ with different exponents modulo $\mathbb{Z}^N$, the leading term for $\Psi_{\mathbb{p}}(\mathrm{w})$ must vanish for any minimal $\mathbb{p}$ in $\mathscr{P}_\Psi$. Hence any minimal $\mathbb{p} \in \mathscr{P}_\Psi$ belongs to $\mathscr{P}_\ell(\mathbb{J})$. Since $\tilde{\Phi}_{\mathbb{p}}(\mathrm{w})$ satisfies $\tilde{\mathrm{B}}(\mathbb{J})$ for any $\mathbb{p} \in \mathscr{P}_\ell(\mathbb{J})$, $\Psi(\mathrm{w})$ must belong to the space spanned by $\{\tilde{\Phi}_{\mathbb{p}}(\mathrm{w});\ \mathbb{p} \in \mathscr{P}_\ell(\mathbb{J})\}$. q.e.d.

3.3) Composability of vertex operators

As a corollary of Theorem 3.3, we get the following

Theorem 3.4. *Let $\Phi_i(z_i)$ be a vertex operator of spin j_i and $u_i \in V_{j_i}$ $(1\leq i\leq N)$. Then the sequence $\{\Phi_N(u_N; z_N), \cdots, \Phi_1(u_1; z_1)\}$ is composable in the region $\mathscr{R}_{z,0}=\{(z_N, \cdots, z_1) \in \mathbb{C}^N;\ |z_N|>\cdots>|z_1|>0\}$ and the composed operator $\Phi_N(u_N; z_N)\cdots\Phi_1(u_1; z_1)$ is analytically continued to a multi-valued holomorphic function on M_N.*

Proof. We may assume that $\Phi_i(u_i; z_i)=\Phi_{\mathrm{v}_i}(u_i; z_i)$ for some vertex $\mathrm{v}_i=\begin{pmatrix} & j_i & \\ p_i & & p_{i-1}\end{pmatrix} \in (\mathrm{CG})_\ell$. Put $\mathbb{J}=(p_N, j_N, \cdots, j_1, p_0)$, then $\mathbb{p}=(0, p_N, \cdots, p_1, p_0, 0) \in \mathscr{P}_\ell(\mathbb{J})$.

For the vertices $\mathrm{v}_{N+1}=\mathrm{v}_{N+1}(\mathbb{p})=\begin{pmatrix} & p_N & \\ 0 & & p_N\end{pmatrix}$ and $\mathrm{v}_0=\mathrm{v}_0(\mathbb{p})=\begin{pmatrix} & p_0 & \\ p_0 & & 0\end{pmatrix}$, we get by Proposition 2.5,

$$\lim_{z\searrow 0}\Phi_{\mathrm{v}_0}(w; z)|\mathrm{vac}\rangle=|w\rangle \qquad (w \in V_{p_0});$$

and

$$\lim_{z\nearrow\infty} z^{2\Delta_{p_N}}\langle \mathrm{vac}|\Phi_{\mathrm{v}_{N+1}}(v;z)=\langle \nu(v)| \qquad (v\in V_{p_{N+1}}).$$

The $(N+2)$-point function $\langle \Phi_{\mathrm{v}_{N+1}}(v_N;z_{N+1})\,\Phi_{\mathrm{v}_N}(u_N;z_N)\cdots\Phi_{\mathrm{v}_1}(u_1;z_1)\,\Phi_{\mathrm{v}_0}(w;z_0)\rangle$ is holomorphic in $\mathscr{R}_z^{N+2}=\{(z_{N+1},\cdots,z_0)\in\mathbb{C}^{N+2};\,|z_{N+1}|>\cdots>|z_0|\}$, so it is an absolutely convergent Laurent series in the region $\mathscr{R}_z^{N+2}$. Hence

$$\begin{aligned}&\langle \nu(v)|\Phi_{\mathrm{v}_N}(u_N;z_N)\cdots\Phi_{\mathrm{v}_1}(u_1;z_1)|w\rangle\\ &\quad=\lim_{z_0\searrow 0}\lim_{z_{N+1}\nearrow\infty} z_{N+1}^{2\Delta_{p_N}}\langle \Phi_{\mathrm{v}_{N+1}}(v;z_{N+1})\Phi_{\mathrm{v}_N}(u_N;z)\cdots\Phi_{\mathrm{v}_1}(u_1;z_1)\Phi_{\mathrm{v}_0}(w;z_0)\rangle\end{aligned}$$

is absolutely convergent at any point $(z_N,\cdots,z_1)\in\mathscr{R}_{z,0}$ for any $v\in V_{p_{N+1}}$, $u_i\in V_{j_i}$ $(1\leq i\leq N)$ and $w\in V_{p_0}$.

For general $v\in\mathscr{H}^{\dagger}_{p_{N+1}}$ and $w\in\mathscr{H}_{p_0}$, we may put

$$v=\langle \nu(v_0)|\,Y_q(m_q)\cdots Y_1(m_1)\quad\text{and}\quad w=X_1(-n_1)\cdots X_r(-n_r)|w_0\rangle$$

for some $v_0\in V_{p_{N+1}}$, $w_0\in V_{p_0}$, $Y_i, X_i\in\mathfrak{g}$, $m_i, n_i\geq 0$.

Then it is sufficient for the convergence of the function $\langle v|\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)|w\rangle$ to note

$$\lim_{z\searrow 0}\hat{X}_1(-n_1)\cdots\hat{X}_r(-n_r)\Phi_{\mathrm{v}_0}(w_0;z)|\mathrm{vac}\rangle=|w\rangle,$$

and

$$\lim_{z\nearrow\infty} z^{2\Delta_{p_N}}\langle\mathrm{vac}|(\hat{Y}_q(m_q)\cdots\hat{Y}_1(m_1)\Phi_{\mathrm{v}_{N+1}}(v_0;z))=\langle v|. \qquad \text{q.e.d.}$$

Remark 3.5. If we take the value ℓ of the central element c of $\hat{\mathfrak{g}}$ as $\ell\notin\mathbb{Q}$, then we can construct an analogous theory without the ℓ-constraint condition. In this case, the Verma module $\mathscr{M}_j$ (defined as in the top of Section 2.3) is irreducible for any nonnegative half integer j, and the space $\mathscr{H}$ is taken as $\mathscr{H}=\sum\mathscr{M}_j$, where j runs over $\frac{1}{2}\mathbb{Z}_{\geq 0}$. Then there exists a vertex operator on $\mathscr{H}$ of type $\mathrm{v}\in\mathbb{V}$, if and only if $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix}\in(\mathrm{CG})$. In this case, $\mathcal{O}(\mathrm{v})\cong\mathscr{M}_j$, so the last equation $\hat{E}(-1)^{\ell-2j+1}\Phi(u_j(j);z)=0$ is eliminated among the the fundamental equations for vertex operators.

§ 4. Commutation Relations of Vertex Operators

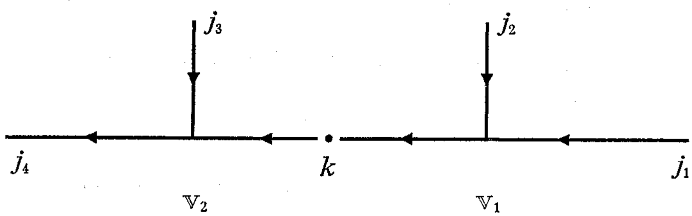
4.1) Formulation of the problem

For a quadruple $\mathbb{J}=(j_4,j_3,j_2,j_1)$ of half integers with $0\leq 2j_i\leq l$, introduce the set $I_\ell(\mathbb{J})$ of *intermediate edges*, defined by

$$I_\ell(\mathbb{J})=\left\{k \in \frac{1}{2}\mathbb{Z};\ 0\leq 2k\leq \ell,\ \mathbb{v}_2(k)=\begin{pmatrix} & j_3 & \\ j_4 & & k\end{pmatrix} \in (\mathrm{CG})_\ell,\right.$$
$$\left.\mathbb{v}_1(k)=\begin{pmatrix} & j_2 & \\ k & & j_1\end{pmatrix} \in (\mathrm{CG})_\ell\right\}.$$

For each $k \in I_\ell(\mathbb{J})$, put $\mathbb{p}(k)=(\mathbb{v}_3, \mathbb{v}_2(k), \mathbb{v}_1(k), \mathbb{v}_0) \in \mathscr{P}_\ell(\mathbb{J})$, where $\mathbb{v}_3=\begin{pmatrix} & j_4 & \\ 0 & & j_4\end{pmatrix}$ and $\mathbb{v}_0=\begin{pmatrix} & j_1 & \\ j_1 & & 0\end{pmatrix}$. And put $\Delta_4(\mathbb{J})=\hat{\Delta}(\mathbb{v}_2)+\hat{\Delta}(\mathbb{v}_1)=\Delta_{j_1}+\Delta_{j_2}+\Delta_{j_3}-\Delta_{j_4}$ (independent of k).

$\mathbb{p}(k)$:



Assume $I_\ell(\mathbb{J})\neq\emptyset$, then we get two vertex operators $\Phi_{\mathbb{v}_2(k)}(w)$ and $\Phi_{\mathbb{v}_1(k)}(z)$. By Theorem 3.4, they are composable in the region $\mathscr{R}_2=\{(w,z) \in \mathbb{C}^2;\ |w|>|z|>0\}$ and the composed operator $\Phi_k(w,z)=\Phi_{\mathbb{v}_2}(w)\Phi_{\mathbb{v}_1}(z)$ is analytically continued to a multi-valued holomorphic and operator-valued function on $M_2=\{(w,z) \in (\mathbb{C}^*)^2;\ w\neq z\}$. Introduce a $V_0^\vee(\mathbb{J})$-valued holomorphic function $\Psi_k(w,z)$ on M_2 defined by

$$\Psi_k(w,z)(u_4\otimes u_3\otimes u_2\otimes u_1)=\langle\nu(u_4)\,|\,\Phi_{\mathbb{v}_2}(u_3;w)\Phi_{\mathbb{v}_1}(u_2;z)\,|\,u_1\rangle \qquad (u_i \in V_{j_i}).$$

In the region $\mathscr{R}_2$, this function has a convergent Laurent expansion:

$$\Psi_k(w,z)(u_4\otimes u_3\otimes u_2\otimes u_1)$$
$$=z^{-\Delta_4(\mathbb{J})}\sum_{n\geq 0}\left(\frac{z}{w}\right)^{n-\hat{\Delta}(\mathbb{v}_2)}\langle\nu(u_4)|\Phi_{\mathbb{v}_2,n}(u_3)\Phi_{\mathbb{v}_1,-n}(u_2)|u_1\rangle,$$

with the initial term $\langle\nu(u_4)|\varphi_{\mathbb{v}_2}(u_3)\varphi_{\mathbb{v}_1}(u_2)|u_1\rangle$ for any $u_i \in V_{j_i}$.

By Propositions 2.1, 5 and Theorems 2.3, 3.3, we get

Proposition 4.1. *Assume $I_\ell(\mathbb{J})\neq\emptyset$. Then for each $k \in I_\ell(\mathbb{J})$,*

i) *the operator $\Phi_k(w,z)$ on $\mathscr{H}$ is uniquely determined by the $V_0^\vee(\mathbb{J})$-valued function $\Psi_k(w,z)$.*

ii) *The function $\Psi_k(w,z)$ satisfies the joint system $E'(\mathbb{J})$ and $B'(\mathbb{J})$ of equations*:

$$E'(\mathbb{J}):\ \left\{\kappa\frac{\partial}{\partial w}-\frac{\Omega_{13}}{w}-\frac{\Omega_{23}}{w-z}\right\}\Psi_k(w,z)=\left\{\kappa\frac{\partial}{\partial z}-\frac{\Omega_{12}}{z}-\frac{\Omega_{23}}{z-w}\right\}\Psi_k(w,z)=0.$$

and

$$B'(\mathbb{J}):\ \sum_{m=0}^{L_1}\binom{L_1}{m}w^{-m}z^{m-L_1}\Psi_k(w,z)(u_4,E^m u_3,E^{L_1-m}u_2,u_{j_1}(j_1))=0,$$

$$\sum_{m=0}^{L_2}\binom{L_2}{m}(w-z)^{-m}(-z)^{m-L_2}\Psi_k(w,z)(u_4,E^m u_3,u_{j_2}(j_2),E^{L_2-m}u_1)=0,$$

$$\sum_{m=0}^{L_3}\binom{L_3}{m}(z-w)^{-m}(-w)^{m-L_3}\Psi_k(w,z)(u_4,u_{j_3}(j_3),E^m u_2,E^{L_3-m}u_1)=0,$$

$$\sum_{|\mathfrak{m}|=L_4}\binom{L_4}{\mathfrak{m}}\Psi_k(w,z)(u_{j_4}(j_4),E^{m_3}u_3,E^{m_2}u_2,E^{m_1}u_1)=0,$$

where $L_i=\ell-2j_i+1\ (1\le i\le 4)$ *and* $\mathfrak{m}=(m_3,m_2,m_1)\in(\mathbb{Z}_{\ge 0})^3$.

iii) *The family* $\{\Psi_k(w,z);\ k\in I_\ell(\mathbb{J})\}$ *gives a basis of the solution space of the systems* $E'(\mathbb{J})$ *and* $B'(\mathbb{J})$.

Now assign a new quadruple $\bar{\mathbb{J}}=(j_4,j_2,j_3,j_1)$ to the quadruple $\mathbb{J}=(j_4,j_3,j_2,j_1)$ of half integers with $0\le 2j_i\le\ell$, then we get the $\mathfrak{g}$-isomorphism $T\colon V^\vee(\mathbb{J})\to V^\vee(\bar{\mathbb{J}})$ defined by

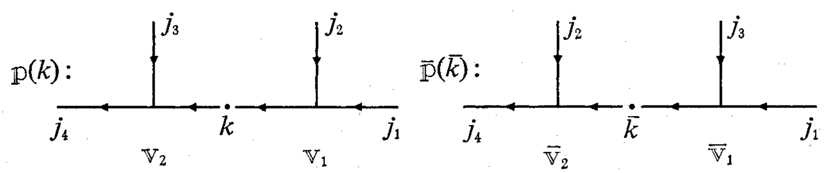
$$(T\varphi)(u_4\otimes u_2\otimes u_3\otimes u_1)=\varphi(u_4\otimes u_3\otimes u_2\otimes u_1)$$

for $\varphi\in V^\vee(\mathbb{J})$ and $u_4\otimes u_2\otimes u_3\otimes u_1\in V(\bar{\mathbb{J}})$. Since $T(V_0^\vee(\mathbb{J}))=V_0^\vee(\bar{\mathbb{J}})$, we get $\dim V_0^\vee(\mathbb{J})=\dim V_0^\vee(\bar{\mathbb{J}})$. Note $\Delta_4(\mathbb{J})=\Delta_4(\bar{\mathbb{J}})$ and $\sharp I_\ell(\mathbb{J})=\sharp I_\ell(\bar{\mathbb{J}})$.

For an intermediate edge $\bar{k}\in I_\ell(\bar{\mathbb{J}})$, similarly define the vertices $\bar{\mathrm{v}}_2(\bar{k})=\begin{pmatrix} & j_2 & \\ j_4 & & \bar{k}\end{pmatrix}$, $\bar{\mathrm{v}}_1(\bar{k})=\begin{pmatrix} & j_3 & \\ \bar{k} & & j_1\end{pmatrix}\in(\mathrm{CG})_\ell$, the composed operator $\bar{\Phi}_{\bar{k}}(w,z)$ of the vertex operators $\Phi_{\bar{\mathrm{v}}_2(\bar{k})}(w)$ and $\Phi_{\bar{\mathrm{v}}_1(\bar{k})}(z)$, and the $V_0^\vee(\bar{\mathbb{J}})$-valued holomorphic function $\bar{\Psi}_{\bar{k}}(w,z)$ on M_2. In the region $\mathscr{R}_2$, this function $\bar{\Psi}_{\bar{k}}(w,z)$ also has a convergent Laurent expansion:

$$\begin{aligned}&\bar{\Psi}_{\bar{k}}(w,z)(u_4\otimes u_2\otimes u_3\otimes u_1)\\&=z^{-\Delta_4(\mathbb{J})}\sum_{n\ge 0}\left(\frac{z}{w}\right)^{n-\Delta(\bar{\mathrm{v}}_2)}\langle\nu(u_4)|\Phi_{\bar{\mathrm{v}}_2(\bar{k}),n}(u_2)\Phi_{\bar{\mathrm{v}}_1(\bar{k}),-n}(u_3)|u_1\rangle,\end{aligned}$$

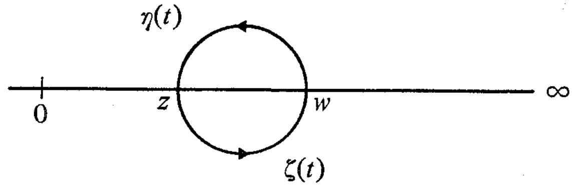
with the initial term $\langle\nu(u_4)|\varphi_{\bar{\mathrm{v}}_2(\bar{k})}(u_2)\varphi_{\bar{\mathrm{v}}_1(\bar{k})}(u_3)|u_1\rangle$ for any $u_i\in V_{j_i}$.



Now introduce the path $b(t)=(\eta(t),\zeta(t))$ from a point (w,z) in the set $I_2=\{(w,z)\in\mathbb{R}^2;\ w>z>0\}$ to the point (z,w) in the set $\bar{I}_2=\{(z,w)\in\mathbb{R}^2;$

$w>z>0\}$ on the manifold M_2, defined by

$$\eta(t)=\frac{w+z}{2}+e^{\pi\sqrt{-1}t}\frac{w-z}{2},\quad \zeta(t)=\frac{w+z}{2}-e^{\pi\sqrt{-1}t}\frac{w-z}{2}\quad (t\in[0,1]).$$



Denote by $\Psi_k(z, w)$ the analytic continuation of the convergent Laurent series $\Psi_k(w, z)$ in the region $\mathscr{R}_2$ along the path $b(t)$ and consider $\Psi_k(z, w)$ near $\tilde{I}_2$, then the $V_0^{\vee}(\mathbb{J})$-valued function $T\Psi_k(z, w)$ satisfies the equations $E'(\mathbb{J})$ and $B'(\mathbb{J})$, so it is expressed as a linear combination:

$$T\Psi_k(z, w)=\sum_{\bar{k}\in I_\ell(\mathbb{J})}\Psi_{\bar{k}}(w, z)C_k^{\bar{k}}(\mathbb{J}),$$

where $C(\mathbb{J})=(C_k^{\bar{k}}(\mathbb{J}))_{k\in I_\ell(\mathbb{J}),\bar{k}\in I_\ell(\mathbb{J})}$ is a square matrix.

Hence by Proposition 4.1,

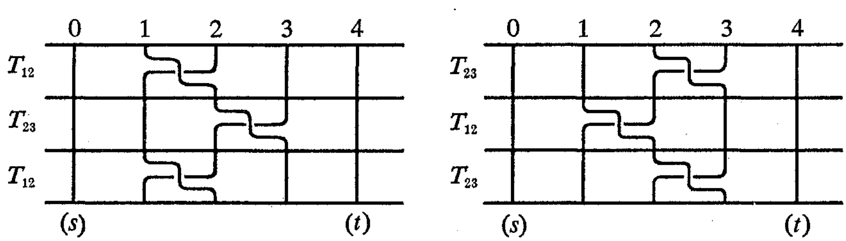
Proposition 4.2. i) *Let* $\mathbb{J}=(j_4, j_3, j_2, j_1)$ *with* $I_\ell(\mathbb{J})\neq\emptyset$. *Then for each intermediate edge* $k\in I_\ell(\mathbb{J})$ *and* $(w, z)\in I_2$,

$$T\Phi_{\mathrm{v}_2(k)}(z)\Phi_{\mathrm{v}_1(k)}(w)=\sum_{\bar{k}\in I_\ell(\mathbb{J})}\Phi_{\mathrm{v}_2(\bar{k})}(w)\Phi_{\mathrm{v}_1(\bar{k})}(z)C_k^{\bar{k}}(\mathbb{J}),$$

where the operator in the left hand side is considered as the analytic continuation of the composition of the vertex operators $\Phi_{\mathrm{v}_2}(w)$ *and* $\Phi_{\mathrm{v}_1}(z)$ *along the path* $b(t)$ *in the manifold* X_N.

ii) *Let* $\mathbb{J}=(t, j_3, j_2, j_1, s)$, *then the braid relation holds*:

$$\begin{aligned}&C(j_3, j_2, j_1, s)C(t, j_3, j_1, j_2)C(j_1, j_3, j_2, s)\\&\quad=C(t, j_3, j_2, j_1)C(j_2, j_3, j_1, s)C(t, j_2, j_1, j_3).\end{aligned}$$



Now our fundamental problem is:

Fundamental Problem. Determine the matrix $C(\mathbb{J})=(C_k^{\bar{k}}(\mathbb{J}))$ for any quadruple $\mathbb{J}$ with $I_\ell(\mathbb{J})\neq\varnothing$.

4.2) Reduced Equation

Take an intermediate edge $k\in I_\ell(\mathbb{J})$ and introduce a variable $\zeta=z/w$, then the $V_0^\vee(\mathbb{J})$-valued function $z^{\Delta_4(\mathbb{J})}\Psi_k(w,\zeta w)$ is independent of w, since by Theorem 3.1, I,

$$\left\{w\frac{\partial}{\partial w}+z\frac{\partial}{\partial z}-\Delta_4(\mathbb{J})\right\}\Psi_k(w,z)=0.$$

So we abbreviate $z^{\Delta_4(\mathbb{J})}\Psi_k(w,\zeta w)$ to $\Psi_k(\zeta)$, then the $V_0^\vee(\mathbb{J})$-valued function $\Psi_k(\zeta)$ (called reduced 4-point function) has a convergent Laurent expansion

$$\Psi_k(\zeta)(u_4\otimes u_3\otimes u_2\otimes u_1)=\zeta^{-\hat{\Delta}(\mathrm{v}_2(k))}\sum_{n\geq 0}\langle\nu(u_4)|\Phi_{\mathrm{v}_2,n}(u_3)\Phi_{\mathrm{v}_1,-n}(u_2)|u_1\rangle\zeta^n$$

in $\zeta\in\mathbb{C}^*$ with the initial term $\langle\nu(u_4)|\varphi_{\mathrm{v}_2}(u_3)\varphi_{\mathrm{v}_1}(u_2)|u_1\rangle$ for $u_i\in V_{j_i}$. Then by Proposition 4.1,

Proposition 4.3 (Reduced equation). *The $V_0^\vee(\mathbb{J})$-valued function $\Psi_k(\zeta)$ satisfies the joint system* RE($\mathbb{J}$) *and* RB($\mathbb{J}$) *of equations*:

$$\mathrm{RE}(\mathbb{J}):\qquad \left(\kappa\frac{d}{d\zeta}-\frac{\Omega_{12}+\kappa\Delta_4(\mathbb{J})}{\zeta}-\frac{\Omega_{23}}{\zeta-1}\right)\Psi_k(\zeta)=0$$

and

$$\mathrm{RB}(\mathbb{J}):\qquad \sum_{m=0}^{L_1}\binom{L_1}{m}\zeta^m\Psi_k(\zeta)(u_4,E^m u_3,E^{L_1-m}u_2,u_{j_1}(j_1))=0,$$

$$\sum_{m=0}^{L_2}\binom{L_2}{m}\left(\frac{\zeta}{\zeta-1}\right)^m\Psi_k(\zeta)(u_4,E^m u_3,u_{j_2}(j_2),E^{L_2-m}u_1)=0,$$

$$\sum_{m=0}^{L_3}\binom{L_3}{m}\left(\frac{1}{1-\zeta}\right)^m\Psi_k(\zeta)(u_4,u_{j_3}(j_3),E^m u_2,E^{L_3-m}u_1)=0,$$

$$\sum_{|\mathbb{m}|=L_4}\binom{L_4}{\mathbb{m}}\Psi_k(\zeta)(u_{j_4}(j_4),E^{m_3}u_3,E^{m_2}u_2,E^{m_1}u_1)=0,$$

where $L_i=\ell-2j_i+1$ $(1\leq i\leq 4)$ *and* $\mathbb{m}=(m_3,m_2,m_1)\in(\mathbb{Z}_{\geq 0})^3$.

Proof. The system $E'(\mathbb{J})$ of equations turns to a single differential equation RE($\mathbb{J}$), since $\Omega_{12}+\Omega_{13}+\Omega_{23}=-\kappa\Delta_4(\mathbb{J})$. q.e.d.

In the following, we want to solve the fundamental problem for the case where $j_3=\frac{1}{2}$ in $\mathbb{J}$. For this aim, we investigate first the reduced equation RE$(\mathbb{J})$ in detail for each quadruple $\mathbb{J}=(j_4, \frac{1}{2}, j_2, j_1)$ with $V_0^{\vee}(\mathbb{J})\neq 0$ and thereafter take the equation RB$(\mathbb{J})$ into account. For the investigation of the reduced equation RE$(\mathbb{J})$, introduce the set $I(\mathbb{J})$ defined by

$$I(\mathbb{J})=\left\{k\in\frac{1}{2}\mathbb{Z}_{\geq 0};\ \mathrm{v}_2(k)=\begin{pmatrix} & j_3 & \\ j_4 & & k\end{pmatrix}\in(\mathrm{CG}),\ \mathrm{v}_1(k)=\begin{pmatrix} & j_2 & \\ k & & j_1\end{pmatrix}=(\mathrm{CG})\right\}.$$

First note that $\sharp I(\mathbb{J})=\dim V_0^{\vee}(\mathbb{J})\leq 2$. And $\dim V_0^{\vee}(\mathbb{J})=2$ if and only if

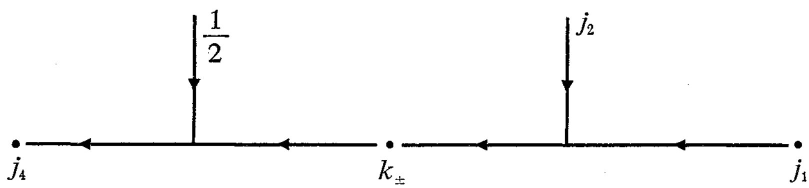
(D2) $\quad |j_1-j_2|\leq j_4-\dfrac{1}{2},\ j_4+\dfrac{1}{2}\leq j_1+j_2$ and $j_1+j_2+\dfrac{1}{2}+j_4\in\mathbb{Z}$.

In this case, $I(\mathbb{J})=\{k_\pm=j_4\pm\frac{1}{2}\}$.

The case (D2) is divided into three cases $(\mathrm{D2})_i$ such that $\sharp I_\ell(\mathbb{J})=i$ $(i=0, 1, 2)$. Introduce the number $\varepsilon_0(\mathbb{J})=(j_1+j_2+j_4+\frac{3}{2})/\kappa$, then

$(\mathrm{D2})_2$ $\quad \varepsilon_0<1$; then $j_1, j_2, j_4\leq\dfrac{l-1}{2}$ and $I_\ell(\mathbb{J})=\left\{k_\pm=j_4\pm\dfrac{1}{2}\right\}$,

$(\mathrm{D2})_1$ $\quad \varepsilon_0=1$; $I_\ell(\mathbb{J})=\left\{k_-=j_4-\dfrac{1}{2}\right\}$, $\qquad (\mathrm{D2})_0$ $\quad \varepsilon_0>1$; $I_\ell(\mathbb{J})=\varnothing$.



Moreover $\dim V_0^{\vee}(\mathbb{J})=1$, if and only if either of the following conditions (D1) holds:

$(\mathrm{D1})_1$ $\quad j_1=j_4+\dfrac{1}{2}+j_2$; $\quad (\mathrm{D1})_2$ $\quad j_2=j_4+\dfrac{1}{2}+j_1$; $\quad (\mathrm{D1})_3$ $\quad j_4=\dfrac{1}{2}+j_2+j_1$.

And $I(\mathbb{J})=\{j_4+\frac{1}{2}\}$ for the case $(\mathrm{D1})_{1,2}$ and $I(\mathbb{J})=\{j_4-\frac{1}{2}\}$ for the case $(\mathrm{D1})_3$. Note that one of the conditions $(\mathrm{D1})_i$ implies $\sharp I_\ell(\mathbb{J})=1$.

Denote by (D0) the case where $V_0^{\vee}(\mathbb{J})=0$, i.e. $I(\mathbb{J})=\varnothing$.

Now consider the equation

RE$(\mathbb{J})$: $$\left(\kappa\frac{d}{d\zeta}-\frac{\Omega_{12}+\kappa\Delta_4(\mathbb{J})}{\zeta}-\frac{\Omega_{23}}{\zeta-1}\right)\Psi(\zeta)=0$$

for $V_0^\vee(\mathbb{J})$-valued functions $\Psi(\zeta)$ on $\zeta \in \mathbb{C}^*$. The coordinate change $\zeta \mapsto \eta = 1/\zeta$ makes the equation RE($\mathbb{J}$) into

$$\text{RE}(\mathbb{J})_\infty: \qquad \left(\kappa\frac{d}{d\eta} - \frac{\Omega_{13}}{\eta} - \frac{\Omega_{23}}{\eta-1}\right)\Psi\left(\frac{1}{\eta}\right) = 0.$$

Case (D2). First we get three bases $\{U_\pm^{(0)}\}$, $\{U_\pm^{(1)}\}$ and $\{U_\pm^{(\infty)}\}$ of $V_0^\vee(\mathbb{J})$ such that they diagonalize the operators Ω_{12}, Ω_{23} and Ω_{13} respectively (see Appendix I):

$$\Omega_{12}U_\pm^{(0)} = \kappa(\gamma_\pm^{(0)} - \Delta_4(\mathbb{J}))U_\pm^{(0)}, \quad \Omega_{23}U_\pm^{(1)} = \kappa\gamma_\pm^{(1)}U_\pm^{(1)}, \quad \Omega_{13}U_\pm^{(\infty)} = \kappa\gamma_\pm^{(\infty)}U_\pm^{(\infty)},$$

and

$$U_\pm^{(0)}(u_4, u_3, u_2, u_1) = \frac{1}{\sqrt{2j_4+1}}\langle \nu(u_4)|\varphi_{v_2(k_\pm)}(u_3)\varphi_{v_1(k_\pm)}(u_2)|u_1\rangle$$

for $u_4 \otimes u_3 \otimes u_2 \otimes u_1 \in V(\mathbb{J})$, where

$$\gamma_+^{(0)} = \frac{2j_4+3}{2\kappa}, \quad \gamma_-^{(0)} = \frac{2j_4-1}{-2\kappa}, \quad \gamma_+^{(1)} = \frac{j_2}{\kappa}, \quad \gamma_-^{(1)} = \frac{j_2+1}{-\kappa},$$

$$\gamma_+^{(\infty)} = \frac{j_1}{\kappa}, \quad \gamma_-^{(\infty)} = \frac{j_1+1}{-\kappa}.$$

Introduce the differences $\gamma^{(i)} = \gamma_+^{(i)} - \gamma_-^{(i)}$ ($i = 0, 1, \infty$), then $0 < \gamma^{(i)} < 1$, in particular, they are not integers:

$$\gamma^{(0)} = \frac{2j_4+1}{\kappa}, \quad \gamma^{(1)} = \frac{2j_2+1}{\kappa} \quad \text{and} \quad \gamma^{(\infty)} = \frac{2j_1+1}{\kappa} \qquad (\kappa = \ell + 2).$$

The transformation matrices $S^{(i,k)}$ between the bases $\{U_\pm^{(i)}\}$ and $\{U_\pm^{(k)}\}$ are given as

$$(U_+^{(k)}, U_-^{(k)}) = (U_+^{(i)}, U_-^{(i)})S^{(i,k)},$$

where

$$S^{(0,1)} = S^{(1,0)} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, \qquad S^{(0,\infty)} = S^{(\infty,0)} = \begin{pmatrix} -A'' & B'' \\ B'' & A'' \end{pmatrix} \in O(2)\setminus SO(2),$$

$$S^{(\infty,1)} = {}^tS^{(1,\infty)} = \begin{pmatrix} A' & -B' \\ B' & A' \end{pmatrix} \in SO(2),$$

and the constants $A \sim B''$ are given as

$$A = \left(\frac{\varepsilon_2\varepsilon_4}{\gamma^{(1)}\gamma^{(0)}}\right)^{1/2}, \quad A' = \left(\frac{\varepsilon_1\varepsilon_2}{\gamma^{(\infty)}\gamma^{(1)}}\right)^{1/2}, \quad A'' = \left(\frac{\varepsilon_1\varepsilon_4}{\gamma^{(\infty)}\gamma^{(0)}}\right)^{1/2},$$

and

$$B=\left(\frac{\varepsilon_0\varepsilon_1}{\gamma^{(1)}\gamma^{(0)}}\right)^{1/2},\quad B'=\left(\frac{\varepsilon_0\varepsilon_4}{\gamma^{(\infty)}\gamma^{(1)}}\right)^{1/2},\quad B''=\left(\frac{\varepsilon_0\varepsilon_2}{\gamma^{(\infty)}\gamma^{(0)}}\right)^{1/2},$$

where

$$\varepsilon_0=\varepsilon_0(\mathbb{J})\quad\text{and}\quad \varepsilon_i=\frac{1}{\kappa}\left\{j_1+j_2+j_4+\frac{1}{2}-2j_i\right\}\quad (i=1, 2, 4).$$

Now we get the fundamental solutions of the linear differential equation RE($\mathbb{J}$) with regular singular points at $\zeta=0$, 1 and ∞ by means of the Gauss' hypergeometric function $F(\alpha, \beta, \gamma; \zeta)$ (see Appendix II for the proof):

Proposition 4.4. *Introduce the constants* $\alpha=\varepsilon_0$, $\beta=\varepsilon_1$, $\beta^{(\infty)}=\varepsilon_4$ *and let* $\Psi_{\pm}^{(i)}(\zeta)$ *be the fundamental solutions of the equation* RE($\mathbb{J}$) *normalized at* $\zeta=i$ $(i=0, 1, \infty)$:

$$(\Psi_{+}^{(i)}(\zeta), \Psi_{-}^{(i)}(\zeta))=(U_{+}^{(i)}, U_{-}^{(i)})\begin{pmatrix}\varphi_{++}^{(i)}(\zeta) & \varphi_{-+}^{(i)}(\zeta)\\ \varphi_{+-}^{(i)}(\zeta) & \varphi_{--}^{(i)}(\zeta)\end{pmatrix}.$$

Then

(i) $\varphi_{++}^{(0)}(\zeta)=\zeta^{\gamma_{+}^{(0)}}(1-\zeta)^{\gamma_{+}^{(1)}}F(\alpha, \beta, \gamma^{(0)}; \zeta);$

$\varphi_{+-}^{(0)}(\zeta)=c_{+}^{(0)}\zeta^{1+\gamma_{+}^{(0)}}(1-\zeta)^{\gamma_{+}^{(1)}}F(\alpha+1, \beta+1, 2+\gamma^{(0)}; \zeta);$

$\varphi_{-+}^{(0)}(\zeta)=c_{-}^{(0)}\zeta^{1+\gamma_{-}^{(0)}}(1-\zeta)^{\gamma_{-}^{(1)}}F(-\alpha+1, -\beta+1, 2-\gamma^{(0)}; \zeta);$

$\varphi_{--}^{(0)}(\zeta)=\zeta^{\gamma_{-}^{(0)}}(1-\zeta)^{\gamma_{-}^{(1)}}F(-\alpha, -\beta, -\gamma^{(0)}; \zeta).$

(ii) $\varphi_{++}^{(1)}(\zeta)=\zeta^{\gamma_{+}^{(0)}}(1-\zeta)^{\gamma_{+}^{(1)}}F(\alpha, \beta, \gamma^{(1)}; 1-\zeta);$

$\varphi_{+-}^{(1)}(\zeta)=c_{+}^{(1)}\zeta^{\gamma_{+}^{(0)}}(1-\zeta)^{1+\gamma_{+}^{(1)}}F(\alpha+1, \beta+1, 2+\gamma^{(1)}; 1-\zeta);$

$\varphi_{-+}^{(1)}(\zeta)=c_{-}^{(1)}\zeta^{\gamma_{-}^{(0)}}(1-\zeta)^{1+\gamma_{-}^{(1)}}F(-\alpha+1, -\beta+1, 2-\gamma^{(1)}; 1-\zeta);$

$\varphi_{--}^{(1)}(\zeta)=\zeta^{\gamma_{-}^{(0)}}(1-\zeta)^{\gamma_{-}^{(1)}}F(-\alpha, -\beta, -\gamma^{(1)}; 1-\zeta).$

(iii) $\varphi_{++}^{(\infty)}(\zeta)=\zeta^{-\gamma_{+}^{(\infty)}}\left(1-\frac{1}{\zeta}\right)^{\gamma_{+}^{(1)}}F\left(\alpha, \beta^{(\infty)}, \gamma^{(\infty)}; \frac{1}{\zeta}\right);$

$\varphi_{+-}^{(\infty)}(\zeta)=c_{+}^{(\infty)}\zeta^{-1-\gamma_{+}^{(\infty)}}\left(1-\frac{1}{\zeta}\right)^{\gamma_{+}^{(1)}}F\left(\alpha+1, \beta^{(\infty)}+1, 2+\gamma^{(\infty)}; \frac{1}{\zeta}\right);$

$\varphi_{-+}^{(\infty)}(\zeta)=c_{-}^{(\infty)}\zeta^{-1-\gamma_{-}^{(\infty)}}\left(1-\frac{1}{\zeta}\right)^{\gamma_{-}^{(1)}}$

$\times F\left(-\alpha+1, -\beta^{(\infty)}+1, 2-\gamma^{(\infty)}; \frac{1}{\zeta}\right);$

$\varphi_{--}^{(\infty)}(\zeta)=\zeta^{-\gamma_{-}^{(\infty)}}\left(1-\frac{1}{\zeta}\right)^{\gamma_{-}^{(1)}}F\left(-\alpha, -\beta^{(\infty)}, -\gamma^{(\infty)}; \frac{1}{\zeta}\right),$

where

$$c_{\pm}^{(i)} = -\frac{\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_4}}{\gamma^{(i)}(1\pm\gamma^{(i)})} \qquad (i=0,\ 1,\ \infty).$$

Note. The reduced 4-point function $\Psi_{\kappa_\pm}(\zeta)$ is the solution of RE$(\mathbb{J})$ with exponent $\gamma_{\pm}^{(0)}$ at $\zeta=0$, so by the normalization of $U_{\pm}^{(0)}$, $\Psi_{\kappa_\pm}(\zeta)=\sqrt{2j_4+1}\,\Psi_{\pm}^{(0)}(\zeta)$.

Case (D1). Since $\dim V_0^{\curlyvee}(\mathbb{J})=1$, the choice of basis vectors of $V_0^{\curlyvee}(\mathbb{J})$ is not of importance. But from the compatibility with the case (D2), we choose basis vectors $\{U^{(i)};\ i=0,1,\infty\}$ of $V_0^{\curlyvee}(\mathbb{J})$ such that

$$U^{(0)}=U^{(1)}=U^{(\infty)} \quad \text{for } (\mathrm{D1})_{1,3};\quad U^{(0)}=U^{(1)}=-U^{(\infty)} \quad \text{for } (\mathrm{D1})_2.$$

The exponents $\gamma^{(0)}$, $\gamma^{(1)}$ and $\gamma^{(\infty)}$ of the equation RE$(\mathbb{J})$ at $\zeta=0,\ 1,\ \infty$ are given as

$(\mathrm{D1})_1$ $\quad \gamma^{(0)}=\dfrac{3+2j_4}{2\kappa},\quad \gamma^{(1)}=\dfrac{j_2}{\kappa},\quad \gamma^{(\infty)}=-\dfrac{j_1+1}{\kappa},$

$(\mathrm{D1})_2$ $\quad \gamma^{(0)}=\dfrac{3+2j_4}{2\kappa},\quad \gamma^{(1)}=-\dfrac{1+j_2}{\kappa},\quad \gamma^{(\infty)}=\dfrac{j_1}{\kappa},$

$(\mathrm{D1})_3$ $\quad \gamma^{(0)}=\dfrac{1-2j_4}{2\kappa},\quad \gamma^{(1)}=\dfrac{j_2}{\kappa},\quad \gamma^{(\infty)}=\dfrac{j_1}{\kappa}.$

Then we get

Proposition 4.4′. *The fundamental solution* $\Psi^{(i)}(\zeta)=U^{(i)}\varphi^{(i)}(\zeta)$ *of the equation* RE$(\mathbb{J})$ *normalized at* $\zeta=i$ $(i=0,1,\infty)$ *is given as*

$(\mathrm{D1})_{1,3}$ $\quad \varphi^{(0)}(\zeta)=\varphi^{(1)}(\zeta)=\zeta^{\gamma^{(0)}}(1-\zeta)^{\gamma^{(1)}},\quad \varphi^{(\infty)}(\zeta)=q^{-j_2/2}\varphi^{(0)}(\zeta),$

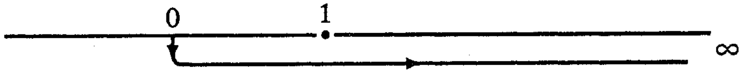
and

$(\mathrm{D1})_2$ $\quad \varphi^{(0)}(\zeta)=\varphi^{(1)}(\zeta)=\zeta^{\gamma^{(0)}}(1-\zeta)^{\gamma^{(1)}},\quad \varphi^{(\infty)}(\zeta)=q^{(j_2+1)/2}\varphi^{(0)}(\zeta),$

where the exponents $\gamma^{(i)}$ *are corresponding ones and* $q=\exp(2\pi\sqrt{-1}/\kappa)$.

4.3) Connection matrices for $\mathbb{J}=(j_4,\frac{1}{2},j_2,j_1)$

The path $b(t)$ from a point $(w,z)\in I_2$ to $(z,w)\in \bar{I}_2$ on M_2 introduced in Section 4.1 corresponds a path from the point $\zeta=z/w$ in the set $J_1=\{\zeta\in\mathbb{R};\ 1>\zeta>0\}$ to the point $1/\zeta$ in the set $\bar{J}_1=\{\zeta\in\mathbb{R};\ \zeta>1\}$ on the manifold C^*. If z tends to zero, then the corresponding path tends to the path $\bar{b}(t)$ from 0 to the infinity figured below:



Now take an intermediate edge k for a quadruple $\mathbb{J}=(j_4, \frac{1}{2}, j_2, j_1)$ with $I(\mathbb{J})\neq\emptyset$. We want to know the analytic continuation of the reduced 4-point function $\Psi_k(\zeta)$ along the path $\bar{b}(t)$.

For the case (D1), we get easily the connection matrix (scalar) $K(\mathbb{J})$ of the fundamental solution $\Psi^{(0)}(\zeta)$ at $\zeta=0$ to $\Psi^{(\infty)}(\zeta)$ at $\zeta=\infty$ of the equation RE$(\mathbb{J})$: $S^{(0,\infty)}\varphi^{(0)}(\zeta)=\varphi^{(\infty)}(\zeta)K(\mathbb{J})$ as follows:

$$(\mathrm{D1})_{1,3}\quad K(\mathbb{J})=q^{j_2/2};\quad (\mathrm{D1})_2\quad K(\mathbb{J})=-q^{-(1+j_2)/2}\quad \left(q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)\right).$$

Now we deal with the case (D2). By the formulae for connection matrices of the hypergeometric functions, we get the connection matrix $K(\mathbb{J})=\begin{pmatrix}K_+^+ & K_-^+\\ K_+^- & K_-^-\end{pmatrix}$ of the fundamental solutions $(\Psi_+^{(0)}, \Psi_-^{(0)})$ at $\zeta=0$ to $(\Psi_+^{(\infty)}, \Psi_-^{(\infty)})$ at $\zeta=\infty$ of the equation RE$(\mathbb{J})$:

$$(\Psi_+^{(0)}, \Psi_-^{(0)})=(\Psi_+^{(\infty)}, \Psi_-^{(\infty)})\begin{pmatrix}K_+^+ & K_-^+\\ K_+^- & K_-^-\end{pmatrix},$$

that is,

$$S^{(\infty,0)}\begin{pmatrix}\varphi_{++}^{(0)}(\zeta) & \varphi_{-+}^{(0)}(\zeta)\\ \varphi_{+-}^{(0)}(\zeta) & \varphi_{--}^{(0)}(\zeta)\end{pmatrix}=\begin{pmatrix}\varphi_{++}^{(\infty)}(\zeta) & \varphi_{-+}^{(\infty)}(\zeta)\\ \varphi_{+-}^{(\infty)}(\zeta) & \varphi_{--}^{(\infty)}(\zeta)\end{pmatrix}\begin{pmatrix}K_+^+ & K_-^+\\ K_+^- & K_-^-\end{pmatrix}.$$

(see Appendix II for more details):

Proposition 4.5.

$$K_+^+=-q^{-(j_1+j_4+3/2)/2}\left(\frac{\gamma^{(0)}\gamma^{(\infty)}}{\varepsilon_1\varepsilon_4}\right)^{1/2}\frac{\Gamma(\gamma^{(0)})\Gamma(-\gamma^{(\infty)})}{\Gamma(\varepsilon_1)\Gamma(-\varepsilon_4)},$$

$$K_-^+=q^{(j_4-j_1-1/2)/2}\left(\frac{\gamma^{(0)}\gamma^{(\infty)}}{\varepsilon_0\varepsilon_2}\right)^{1/2}\frac{\Gamma(-\gamma^{(0)})\Gamma(-\gamma^{(\infty)})}{\Gamma(-\varepsilon_0)\Gamma(-\varepsilon_2)},$$

$$K_+^-=q^{(j_1-j_4-1/2)/2}\left(\frac{\gamma^{(0)}\gamma^{(\infty)}}{\varepsilon_0\varepsilon_2}\right)^{1/2}\frac{\Gamma(\gamma^{(0)})\Gamma(\gamma^{(\infty)})}{\Gamma(\varepsilon_0)\Gamma(\varepsilon_2)},$$

$$K_-^-=q^{(j_1+j_4+1/2)/2}\left(\frac{\gamma^{(0)}\gamma^{(\infty)}}{\varepsilon_1\varepsilon_4}\right)^{1/2}\frac{\Gamma(-\gamma^{(0)})\Gamma(\gamma^{(\infty)})}{\Gamma(-\varepsilon_1)\Gamma(\varepsilon_4)},$$

where we denote $q=\exp(2\pi\sqrt{-1}/\kappa)$.

The conditions $(\mathrm{D2})_i$ $(i=2, 1, 0)$ and (D1) for $\mathbb{J}$ are equivalent to $(\mathrm{D2})_i$ and (D1) for $\bar{\mathbb{J}}$ respectively. Intermediate edges for $\bar{\mathbb{J}}$ must be $\bar{k}=$

$j_1 \pm \frac{1}{2}$ under the condition $(\mathrm{D2})_2$, and the intermediate edge for $\bar{\mathbb{J}}$ must be

$$(\mathrm{D2})_1, (\mathrm{D1})_1 \quad \bar{k}=j_1-\frac{1}{2}; \qquad (\mathrm{D1})_{2,3} \quad \bar{k}=j_1+\frac{1}{2}.$$

From the three bases $\{U_\pm^{(i)}=U_\pm^{(i)}(\mathbb{J});\ i=0, 1, \infty\}$ of $V_0^\vee(\mathbb{J})$, put

$$\bar{U}_\pm^{(i)}=\bar{U}_\pm^{(i)}(\bar{\mathbb{J}})=TU_\pm^{(1/i)} \quad (i=0, \infty) \quad \text{and} \quad \bar{U}_\pm^{(1)}=\pm TU_\pm^{(1)},$$

then they are three bases of $V_0^\vee(\bar{\mathbb{J}})$ such that

$$\Omega_{12}\bar{U}_\pm^{(0)}=\kappa\bar{\gamma}_\pm^{(0)}\bar{U}_\pm^{(0)}, \quad \Omega_{23}\bar{U}_\pm^{(1)}=\kappa\bar{\gamma}_\pm^{(1)}\bar{U}_\pm^{(1)}, \quad \Omega_{13}\bar{U}_\pm^{(\infty)}=\kappa(\bar{\gamma}_\pm^{(\infty)}-\Delta_4(\mathbb{J}))\bar{U}_\pm^{(\infty)},$$
$$\bar{\gamma}_\pm^{(0)}=\gamma_\pm^{(\infty)}, \qquad \bar{\gamma}_\pm^{(1)}=\gamma_\pm^{(1)}, \qquad \bar{\gamma}_\pm^{(\infty)}=\gamma_\pm^{(0)},$$

and

$$\bar{U}_\pm^{(0)}(u_4, u_2, u_3, u_1)=\frac{1}{\sqrt{2j_4+1}}\langle\nu(u_4)|\varphi_{v_2(\bar{k}_\pm)}(u_2)\varphi_{v_1(\bar{k}_\pm)}(u_3)|u_1\rangle$$

for $u_4\otimes u_3\otimes u_2\otimes u_1 \in V^\vee(\mathbb{J})$.

By Proposition 4.1, the composition $\Phi_k(w, z)$ of vertex operators is determined by the $V_0^\vee(\mathbb{J})$-valued function $\Psi_k(w, z)$ which is written as $\Psi_k(w, z)=z^{-\Delta_4(\mathbb{J})}\Psi_k(z/w)$ by the reduced 4-point function $\Psi_k(\zeta)$. And the composed operator $\bar{\Phi}_{\bar{k}}(z, w)$ is also determined by the $V_0^\vee(\bar{\mathbb{J}})$-valued function $\bar{\Psi}_{\bar{k}}(z, w)=w^{-\Delta_4(\mathbb{J})}\bar{\Psi}_{\bar{k}}(w/z)$.

The functions $\Psi_k(\zeta)$ and $\bar{\Psi}_{\bar{k}}(\eta)$ satisfy the differential equations RE $(\mathbb{J})$ and RE $(\bar{\mathbb{J}})$ with the initial conditions:

$$\zeta^{\hat{\Delta}(v_2(k))}\Psi_k(\zeta)(u_4, u_3, u_2, u_1)|_{\zeta=0}=\langle\nu(u_4)|\varphi_{v_2}(u_3)\varphi_{v_1}(u_2)|u_1\rangle$$

and

$$\eta^{\hat{\Delta}(v_2(\bar{k}))}\bar{\Psi}_{\bar{k}}(\eta)(u_4, u_2, u_3, u_1)|_{\eta=0}=\langle\nu(u_4)|\varphi_{v_2}(u_2)\varphi_{v_1}(u_3)|u_1\rangle.$$

By the relations among the exponents $\{\gamma_\pm^{(i)}\}$ and $\{\bar{\gamma}_\pm^{(i)}\}$ of the equations RE $(\mathbb{J})$ and RE $(\bar{\mathbb{J}})$, we get

$$T\Psi_\pm^{(0)}(\zeta)=\zeta^{\Delta_4(\mathbb{J})}\bar{\Psi}_\pm^{(\infty)}\left(\frac{1}{\zeta}\right) \quad \text{and} \quad T\Psi_\pm^{(\infty)}(\zeta)=\zeta^{\Delta_4(\mathbb{J})}\bar{\Psi}_\pm^{(0)}\left(\frac{1}{\zeta}\right),$$

where $\bar{\Psi}_\pm^{(i)}(\eta)$ denotes the fundamental solutions of the equation RE $(\bar{\mathbb{J}})$ similarly obtained as in Proposition 4.4.

By the note after Proposition 4.4, we get

$$\Psi_{k_\pm}(\zeta)=\sqrt{2j_4+1}\,\Psi_\pm^{(0)}(\zeta) \quad \text{and} \quad \bar{\Psi}_{\bar{k}_\pm}(\eta)=\sqrt{2j_4+1}\,\bar{\Psi}_\pm^{(0)}(\eta).$$

Hence by Propositions 4.1, 5, we get

Proposition 4.6. *Let* $\mathbb{J}=(j_4, \frac{1}{2}, j_2, j_1)$ *with* $I(\mathbb{J})\neq\emptyset$. *Then*

(D1) $C(\mathbb{J})=C_k^{\bar{k}}(\mathbb{J})=K(\mathbb{J})$, *where* $k\in I(\mathbb{J})$ *and* $\bar{k}\in I(\bar{\mathbb{J}})$.

$(\mathrm{D2})_1$ $C(\mathbb{J})=C_{k_-}^{\bar{k}_-}(\mathbb{J})=K_-^-(\mathbb{J})$, *where* $\bar{k}_-=j_1-\dfrac{1}{2}$ *and* $k_-=j_4-\dfrac{1}{2}$.

$(\mathrm{D2})_2$ $C(\mathbb{J})=(C_k^{\bar{k}}(\mathbb{J}))_{k\in I(J),\bar{k}\in I(\bar{J})}=K(\mathbb{J})$ *as* 2×2*-matrices.*

Remark. In the case $(\mathrm{D2})_2$, all entries of the matrix $C(\mathbb{J})=K(\mathbb{J})$ do not vanish. In the case $(\mathrm{D2})_1$, $\varepsilon_0=1$ implies $K_-^+(\mathbb{J})=0$, hence the matrix $K(\mathbb{J})$ is of the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.

4.4. Case $\mathbb{J}=(j_4, \frac{1}{2}, \frac{1}{2}, j_1)$

As a special case, we take $j_2=j_3=\frac{1}{2}$, then the conditions (D2) and (D1) read as

$(\mathrm{D2})_2$ $\dfrac{\ell}{2}>j_1=j_4>0$; $(\mathrm{D2})_1$ $\dfrac{\ell}{2}=j_1=j_4$;

and

$(\mathrm{D1})_1$ $j_4=j_1+1$; $(\mathrm{D1})_2$ $j_1=j_4=0$; $(\mathrm{D1})_3$ $j_1=j_4+1$.

Under the assumption (D2), the constants $\gamma_\pm^{(i)}$, ε_i and the matrix $K(\mathbb{J})$ turns to be the following (here $j=j_1=j_4$):

$$\gamma_+^{(0)}=\frac{2j+3}{2\kappa},\quad \gamma_-^{(0)}=\frac{2j-1}{-2\kappa},\quad \gamma_+^{(1)}=\frac{1}{2\kappa},\quad \gamma_-^{(1)}=\frac{3}{-2\kappa},$$

$$\gamma_+^{(\infty)}=\frac{j}{\kappa},\quad \gamma_-^{(\infty)}=\frac{j+1}{-\kappa};\quad \gamma^{(0)}=\gamma^{(\infty)}=\frac{2j+1}{\kappa},\quad \gamma^{(1)}=\frac{2}{\kappa};$$

$$\varepsilon_0=\frac{2j+2}{\kappa},\quad \varepsilon_1=\varepsilon_4=\frac{1}{\kappa},\quad \varepsilon_2=\frac{2j}{\kappa};$$

$$K_+^+=-(2j+1)q^{-j-3/4}\frac{\Gamma\left(\frac{2j+1}{\kappa}\right)\Gamma\left(\frac{2j+1}{-\kappa}\right)}{\Gamma\left(\frac{1}{\kappa}\right)\Gamma\left(\frac{-1}{\kappa}\right)},$$

$$K_-^+=\frac{2j+1}{2\sqrt{j(j+1)}}q^{-1/4}\frac{\Gamma\left(\frac{2j+1}{-\kappa}\right)^2}{\Gamma\left(\frac{2j+2}{-\kappa}\right)\Gamma\left(\frac{2j}{-\kappa}\right)},$$

$$K_{+}^{-}=\frac{2j+1}{2\sqrt{j(j+1)}}q^{-1/4}\frac{\Gamma\left(\frac{2j+1}{\kappa}\right)^{2}}{\Gamma\left(\frac{2j+2}{\kappa}\right)\Gamma\left(\frac{2j}{\kappa}\right)},$$

and

$$K_{-}^{-}=(2j+1)q^{j+1/4}\frac{\Gamma\left(\frac{2j+1}{\kappa}\right)\Gamma\left(\frac{2j+1}{-\kappa}\right)}{\Gamma\left(\frac{1}{\kappa}\right)\Gamma\left(\frac{-1}{\kappa}\right)}.$$

Now recall the notion of *q-integers* for $q \in \mathbb{C}^*$: for each integer $\nu \in \mathbb{Z}$, introduce the q-integer $[\nu]=[\nu]_q$ defined by

$$[\nu]_q=\begin{cases}\dfrac{q^{\nu}-1}{q-1} & (q\neq 1)\\ \nu & (q=1)\,.\end{cases}$$

Then

Lemma 4.7.

i) $[0]_q=0,\quad [1]_q=1$ *and* $[2]_q=1+q$.

ii) $[-\nu]_q=-q^{-\nu}[\nu]_q$ *and* $[\nu]_{1/q}=q^{1-\nu}[\nu]_q$ $(\nu \in \mathbb{Z})$.

iii) $[\nu]_q=0$, *if and only if* $q^{\nu}=1$. $\left([\kappa]_q=0 \text{ if } q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right).\right)$

iv) $\lim_{q\to 1}[\nu]_q=\nu$ *for any* $\nu \in \mathbb{Z}$.

Then in the case $(D2)_2$, the matrix $K(\mathbb{J})$ can be symmetrized by means of q-integers:

Proposition 4.8. *For* $j \in \frac{1}{2}\mathbb{Z}$ *with* $0<2j<\ell$,

$$K\left(j, \frac{1}{2}, \frac{1}{2}, j\right)$$

$$=q^{-3/4}\begin{pmatrix}\gamma_{+}^{-1} & \\ & \gamma_{-}^{-1}\end{pmatrix}\begin{bmatrix}\dfrac{-1}{[2j+1]} & \dfrac{\sqrt{q[2j][2j+2]}}{[2j+1]}\\ \dfrac{\sqrt{q[2j][2j+2]}}{[2j+1]} & \dfrac{q^{2j+1}}{[2j+1]}\end{bmatrix}\begin{pmatrix}\gamma_{+} & \\ & \gamma_{-}\end{pmatrix}$$

where

$$\gamma_{\pm}=\frac{\Gamma\left(\frac{2j+1}{\pm\kappa}\right)}{\left(\Gamma\left(\frac{2j+2}{\pm\kappa}\right)\Gamma\left(\frac{2j}{\pm\kappa}\right)\right)^{1/2}},\quad q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right) \text{ and } [\nu]=[\nu]_q.$$

We can get the connection matrix (=scalar) $K(\mathbb{J})$ in the cases $(\mathrm{D2})_1$ and (D1):

Proposition 4.8′. *Let* $q=\exp(2\pi\sqrt{-1}/\kappa)$.

i) $K\left(\frac{\ell}{2}, \frac{1}{2}, \frac{1}{2}, \frac{\ell}{2}\right)=q^{\ell+1/4}/[\ell+1]_q=-q^{-3/4}$.

ii) $K\left(0, \frac{1}{2}, \frac{1}{2}, 0\right)=-q^{-3/4}$.

iii) $K\left(j+1, \frac{1}{2}, \frac{1}{2}, j\right)=K\left(j-1, \frac{1}{2}, \frac{1}{2}, j\right)=q^{1/4}$.

Remark. These values are also obtained from the calculations in the case $(\mathrm{D2})_2$. $K^-_+(0, \frac{1}{2}, \frac{1}{2}, 0)=0$ and $K^+_+(0, \frac{1}{2}, \frac{1}{2}, 0)=-q^{-3/4}$. For $\mathbb{J}_\pm=(j\pm 1, \frac{1}{2}, \frac{1}{2}, j)$, $K^+_+(\mathbb{J}_\pm)=K^-_-(\mathbb{J}_\pm)=0$ and $K^+_-(\mathbb{J}_+)=K^-_+(\mathbb{J}_-)=q^{1/4}$.

§ 5. Monodromy Representations of Braid Groups

In this section, we construct representations of braid groups on the spaces of multi-correlation functions, and show that they give the same representations of Hecke algebras constructred by H. Wenzl.

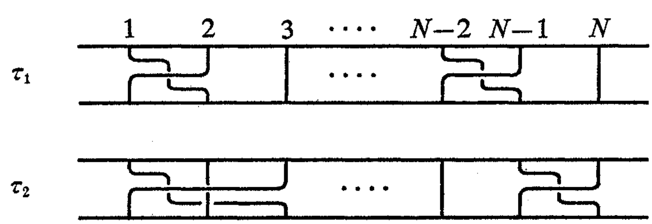
5.1) Braid groups and Hecke algebras

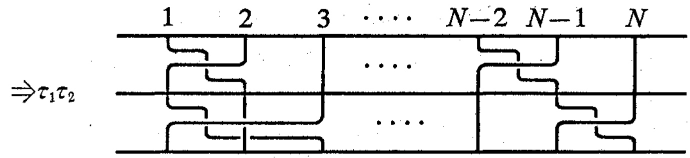
Recall our X_N is a complex manifold defined by

$$X_N=\{(z_N, z_{N-1}, \cdots, z_1) \in \mathbb{C}^N;\ z_i \neq z_j\ (i\neq j)\}.$$

The N-th symmetric group $\mathfrak{S}_N$ acts on the manifold X_N as $(z_N, \cdots, z_1)\sigma=(z_{(N)\sigma}, \cdots, z_{(1)\sigma})$ $(\sigma \in \mathfrak{S}_N)$, then we get a covering space $\pi_N\colon X_N \to \bar{X}_N=X_N/\mathfrak{S}$. Let $\tilde{\pi}_N\colon \tilde{X}_N \to X_N$ be a universal covering manifold of X_N, then $\bar{\pi}_N=\tilde{\pi}_N \circ \pi_N\colon \tilde{X}_N \to \bar{X}_N$ is also a universal covering of $\bar{X}_N$.

Now recall the braid groups according to J. S. Birman [Bi]. The fundamental group $\pi_1(\bar{X}_N, \bar{p}_N)$ of the manifold $\bar{X}_N$ is called the *braid group with N strings of the manifold* $\mathbb{C}$, that is, the *classical braid group of Artin*, and is denoted by B_N, where we take the base point as $\bar{p}_N=\pi_N(p_N)$. The composition of τ_1 and τ_2 in the group B_N is figured as



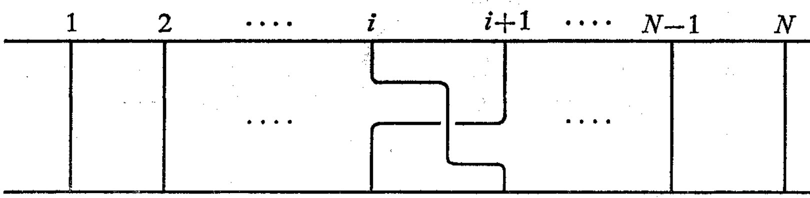


The fundamental group $\pi_1(X_N, p_N)$ of the manifold X_N is called the *pure braid group with N strings of the manifold* $\mathbb{C}$, and is denoted by P_N, where p_N is a base point of X_N, e. g. $p_N=(N, N-1, \cdots, 1)$. Then the group P_N is the kernel of the natural homomorphism ρ of B_N onto $\mathfrak{S}_N$.

It is well-known that the group B_N has a system $\{b_i;\ 1\le i\le N-1\}$ of generators with the fundamental relations

(BR) $b_i b_j = b_j b_i$ $(|i-j|\ge 2)$ and $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ $(1\le i\le N-2)$.

where b_i is figured as a geometric braid by



The subgroup P_N has a system $\{a_{ij};\ 1\le i<j\le N\}$ of generators, defined by

$$a_{ij}=b_{j-1}b_{j-2}\cdots b_{i+1}b_i^2 b_{i+1}^{-1}\cdots b_{j-1}^{-1}.$$

Introduce a subset I_N of the manifold X_N defined by

$$I_N=\{(z_N, \cdots, z_1)\in \mathbb{R}^N;\ z_N>z_{N-1}>\cdots>z_1>0\}.$$

Specify a base point $\tilde{p}_N$ of the manifold $\tilde{X}_N$ such that $\tilde{\pi}_N(\tilde{p}_N)=p_N$, then there is a subset $\tilde{I}_N$ of $\tilde{X}_N$ such that $\tilde{p}_N\in\tilde{I}_N$ and I_N is homeomorphic to $\tilde{I}_N$.

For a finite dimensional $\mathfrak{S}_N$-module W, denote by $\mathcal{O}(\tilde{X}_N; W)$ the space of all W-valued holomorphic functions on $\tilde{X}_N$. The values of $\varphi\in\mathcal{O}(\tilde{X}_N; W)$ on the whole $\tilde{X}_N$ are determined by the values of φ in $\tilde{I}_N$, which we call the *principal branch* of the multi-valued function φ on X_N. For a point $(z_N, \cdots, z_1)\in I_N$, sometimes we write $\varphi(z_N, \cdots, z_1)=\varphi(\tilde{p})$, where $\tilde{p}\in\tilde{I}_N$ such that $\tilde{\pi}_N(\tilde{p})=(z_N, \cdots, z_1)$.

The action of the braid group B_N on the space $\mathcal{O}(\tilde{X}_N; W)$ is defined as follows: Let $\tau\in B_N=\pi_1(\overline{X}_N)$. For each φ of $\mathcal{O}(X_N; W)$ and $\tilde{p}\in\tilde{X}_N$, put

$$(\tau\varphi)(\tilde{p})=\rho(\tau)\cdot\varphi(\tilde{p}\cdot\tau) \qquad (\tilde{p}\in\tilde{X}_N),$$

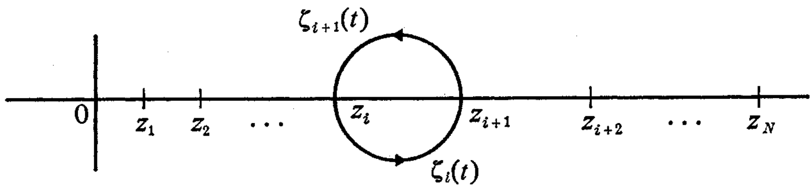
where the group B_N acts on each fiber $\tilde{\pi}_N^{-1}(\tilde{\pi}_N(\tilde{p}))$ as the covering transformation of $\tilde{X}_N\to\bar{X}_N$.

We will give more explicitly the principal branch of $\tau\varphi$ for a generator $\tau=b_i$ $(1\leq i\leq N-1)$. For each $\tilde{p}\in\tilde{I}_N$, let $\bar{p}=\tilde{\pi}_N(\tilde{p})\in\bar{I}_N$ and $(z_N,\cdots,z_1)=\tilde{\pi}_N(\tilde{p})\in I_N$,

$$(b_i\varphi)(\tilde{p})=(i,i+1)\cdot\varphi(\tilde{p}\cdot b_i)$$

where $(i,i+1)$ denotes the transposition, and $\varphi(\tilde{p}\cdot b_i)$ is nothing but the analytic continuation of the principal branch $\varphi(z_N,\cdots,z_1)$ along the path $(\zeta_N(t),\cdots,\zeta_1(t))$ in X_N $(t\in[0,1])$: $\zeta_k(t)=z_k$ $(k\neq i,i+1)$,

$$\zeta_i(t)=\frac{z_i+z_{i+1}}{2}-e^{\pi t\sqrt{-1}}\frac{z_{i+1}-z_i}{2};\quad \zeta_{i+1}(t)=\frac{z_i+z_{i+1}}{2}+e^{\pi t\sqrt{-1}}\frac{z_{i+1}-z_i}{2}$$



Related to braid groups, the notion of Hecke algebras is important (see e.g. D. Kazhdan-G. Lusztig [KL] and V. H. R. Jones [Jo]).

Let $N\geqq 2$ and $q\in\mathbb{C}^*$. Then the *Hecke algebra* $H_N(q)$ *of type* A_{N-1} is defined as the associative complex algebra with generators $1, T_1,\cdots, T_{N-1}$ with the defining relations:

(H1) $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$ for $i=1,2,\cdots,N-2$.

(H2) $T_iT_j=T_jT_i$ for $|i-j|\geq 2$.

(H3) $(T_i-q)(T_i+1)=0$, that is, $T_i^2=(q-1)T_i+q$.

Note that (H1) and (H2) are nothing but the braid relations (BR), hence there is a natural epimorphism of the group algebra $\mathbb{C}[B_N]$ onto $H_N(q)$. For $q=1$, the Hecke algebra $H_N(1)$ is isomorphic to the group algebra $\mathbb{C}\mathfrak{S}_N$ of the N-th symmetric group $\mathfrak{S}_N$, by sending T_i to the transposition $(i,i+1)$. If q is not a root of unity, it is known by H. Wenzl [W] that there exists an isomorphisms of $H_q(N)$ with the group ring $\mathbb{C}[\mathfrak{S}_N]$ as algebras.

Assume that $[2]_q\neq 0$, that is, $q\neq -1$. Then we can give another system $\{1, e,\cdots, e_{N-1}\}$ of generators of $H_N(q)$ consisting of idempotents:

$$e_i=\frac{q-T_i}{[2]_q},\quad \text{i.e. } T_i=q-[2]_q e_i \quad (i=1,\cdots,N-1).$$

Then the defining relations (H1)~(H3) translate to

(H1)′ $e_i e_{i+1} e_i-\frac{q}{[2]_q^2}e_i=e_{i+1}e_i e_{i+1}-\frac{q}{[2]_q^2}e_{i+1}$ for $i=1,2,\cdots,N-2$.

(H2)′ $e_i e_j=e_j e_i$ for $|i-j|\geq 2$.

(H3)′ $e_i^2=e_i$ for $i=1,2,\cdots,N-1$.

5.2) Monodromy representations

Let $N\geq 2$, $\kappa=\ell+2$, $q=\exp(2\pi\sqrt{-1}/\kappa)$ and fix a half integer t with $0\leq 2t\leq \ell$ which we call a *target edge*. Introduce an $(N+1)$-ple $\mathbb{J}_t=(t,\frac{1}{2},\cdots,\frac{1}{2})$, and consider the systems $E(N;t)$ and $B(N;t)$ of equations for $V_0^\vee(\mathbb{J}_t)$-valued functions on the manifold X_N:

$$E(N;t):\quad \left(\kappa\frac{\partial}{\partial z_i}-\sum_{\substack{k=1\\k\neq i}}^N\frac{\Omega_{ik}}{z_i-z_k}\right)\Psi(z_N,\cdots,z_1)=0\quad (1\leq i\leq N)$$

and for any $u_k\in V_{j_k}$ $(j_{N+1}=t, j_i=\frac{1}{2}\ (1\leq i\leq N))$,

$$B(N;t):\quad \sum_{\mathfrak{m}_i}\binom{L_i}{\mathfrak{m}_i}\prod_{\substack{k=1\\k\neq i}}^N(z_k-z_i)^{-m_k}$$
$$\times\Psi(\mathbb{z})(u_{N+1},E^{m_N}u_N,\cdots,u_{j_i}(j_i),\cdots,E^{m_1}u_1)=0$$

for $1\leq i\leq N$, and

$$\sum_{\mathfrak{m}_{N+1}}\binom{L_{N+1}}{\mathfrak{m}_{N+1}}\Psi(\mathbb{z})(u_{j_{N+1}}(j_{N+1}),E^{m_N}u_N,\cdots,E^{m_1}u_1)=0$$

where $\mathfrak{m}_i=(m_N,\cdots,\hat{m}_i,\cdots,m_1)\in(\mathbb{Z}_{\geq 0})^{N-1}$ $(1\leq i\leq N)$ and $\mathfrak{m}_{N+1}=(m_N,\cdots,m_1)\in(\mathbb{Z}_{\geq 0})^N$ with $|\mathfrak{m}_i|=L_i=\ell-2j_i+1$ $(1\leq i\leq N+1)$.

Let $W(N;t)$ be the solution space of the joint system $E(N;t)$ and $B(N;t)$. Then by Theorem 3.3, the space $W(N;t)$ has a basis $\{\Psi_{\mathfrak{p}}(z_N,\cdots,z_1);\ \mathfrak{p}\in\mathscr{P}_\ell(N;t)\}$ defined as follows: Let

$$\mathscr{P}_\ell(N;t)=\Big\{\mathfrak{p}=(p_N,\cdots,p_1,p_0);\ p_N=t,\ p_0=0,\ p_i\in\frac{1}{2}\mathbb{Z},\ 0\leq 2p_i\leq\ell,$$
$$|p_i-p_{i-1}|=\frac{1}{2}\ (1\leq i\leq N)\Big\}.$$

For each $\mathfrak{p}\in\mathscr{P}_\ell(N;t)$, define the $V_0^\vee(\mathbb{J}_t)$-valued, multi-valued holomorphic function $\Psi_{\mathfrak{p}}(z_N,\cdots,z_1)$ on X_N by

$$\Psi_{\mathrm{p}}(z_N, \cdots, z_1)(v, u_N, \cdots, u_1) = \langle \nu(v) | \Phi_{\mathrm{v}_N}(u_N; z_N) \cdots \Phi_{\mathrm{v}_1}(u_1; z_1) | \mathrm{vac} \rangle$$

for $v \in V_t$ and $u_i \in V_{1/2}$ $(1 \leq i \leq N)$, where the vertex $\mathrm{v}_i = \mathrm{v}_i(\mathrm{p})$ is defined as $\mathrm{v}_i = \begin{pmatrix} \frac{1}{2} \\ p_i p_{i-1} \end{pmatrix}$ $(1 \leq i \leq N)$.

The braid group B_N acts on this space $W(N; t)$ as monodromies. The commutation relations of vertex operators give a factorization of this monodromy representation $(\pi_{N,t}, W(N; t))$. The $\mathfrak{S}_N$-module structure of the space $V_0^{\vee}(\mathbb{J})$ is defined by

$$(\sigma\varphi)(u_N, \cdots, u_1) = \varphi(u_{(N)\sigma}, \cdots, u_{(1)\sigma}) \qquad (\varphi \in V_0^{\vee}(\mathbb{J}),\ \sigma \in \mathfrak{S}_N),$$

and the B_N-module structure on the space of $V_0^{\vee}(\mathbb{J})$-valued functions on X_N is defined in Section 5.1. By Propositions 4.8, 4.8′ and 5.1, we will give this representation $\pi = \pi_{N,t}$ explicitly.

For each i $(1 \leq i \leq N-1)$, the action $\pi(b_i)$ of the generator b_i of the group B_N on the space $W(N; t)$ is given as follows.

At first, divide the set $\mathscr{P}_\ell(N; t)$ into the four parts: Let $\mathrm{p} = (p_N, p_{N-1}, \cdots, p_i, \cdots, p_1, p_0) \in \mathscr{P}_\ell(N; t)$, $p_N = t$, $p_0 = 0$.

$$\mathrm{p} \in \mathscr{P}_i^a(N; t) \longleftrightarrow p_{i+1} = p_{i-1} = 0.$$

$$\mathrm{p} \in \mathscr{P}_i^b(N; t) \longleftrightarrow |p_{i+1} - p_{i-1}| = 1.$$

$$\mathrm{p} \in \mathscr{P}_i^c(N; t) \longleftrightarrow \frac{\ell}{2} > p_{i+1} = p_{i-1} > 0.$$

$$\mathrm{p} \in \mathscr{P}_i^d(N; t) \longleftrightarrow p_{i+1} = p_{i-1} = \frac{\ell}{2}.$$

Then the operation $\pi(b_i)$ is given on the basis vectors $\{\Psi_{\mathrm{p}};\ \mathrm{p} \in \mathscr{P}_\ell(N; t)\}$ as:

a, d) If $\mathrm{p} \in \mathscr{P}_i^a(N; t)$ or $\mathrm{p} \in \mathscr{P}_i^d(N; t)$, $\quad \pi(b_i)\Psi_{\mathrm{p}} = -q^{-3/4}\Psi_{\mathrm{p}}$.

b) If $\mathrm{p} \in \mathscr{P}_i^b(N; t)$, $\quad \pi(b_i)\Psi_{\mathrm{p}} = q^{1/4}\Psi_{\mathrm{p}}$.

c) If $\mathrm{p} \in \mathscr{P}_i^c(N; t)$, there is only one $\mathrm{p}' \in \mathscr{P}_i^c(N; t)$ such that $p_k = p'_k$ for any $k \neq i$ and $|p_i - p'_i| = 1$. We define the action $\pi(b_i)$ for which $\mathbb{C}\Psi_{\mathrm{p}} + \mathbb{C}\Psi_{\mathrm{p}'}$ is invariant. We modify the notations as $\mathrm{p}_\pm = (t, p_{N-1}, \cdots, p_{i+1}, p_i^\pm, p_{i-1}, \cdots, p_1)$, where $p_i^+ = \max(p_i, p'_i)$ and $p_i^- = \min(p_i, p'_i)$. Then the action $\pi(b_i)$ on $\mathbb{C}\Psi_{\mathrm{p}+} + \mathbb{C}\Psi_{\mathrm{p}-}$ is given as $\pi(b_i) = K(p, \frac{1}{2}, \frac{1}{2}, p)$, where $0 < p = p_{i+1} = p_{i-1} < \ell/2$:

$$\pi(b_i)|_{\mathbb{C}\Psi_{\mathrm{p}+} + \mathbb{C}\Psi_{\mathrm{p}-}}$$

$$= q^{-3/4} \begin{pmatrix} \gamma_+^{-1} & \\ & \gamma_-^{-1} \end{pmatrix} \begin{bmatrix} \dfrac{-1}{[2p+1]} & \dfrac{\sqrt{p[2p][2p+2]}}{[2p+1]} \\ \dfrac{\sqrt{q[2p][2p+2]}}{[2p+1]} & \dfrac{q^{2p+1}}{[2p+1]} \end{bmatrix} \begin{pmatrix} \gamma_+ & \\ & \gamma_- \end{pmatrix}$$

where

$$\gamma_{\pm}=\frac{\Gamma\left(\frac{2p+1}{\pm\kappa}\right)}{\left(\Gamma\left(\frac{2p+2}{\pm\kappa}\right)\Gamma\left(\frac{2p}{\pm\kappa}\right)\right)^{1/2}}.$$

In each case, $\{q, -1\}$ are only possible eigenvalues of the operators $q^{3/4}\pi(b_i)$. Thus the actions $q^{3/4}\pi(b_i)$ on the space $W(N; t)$ satisfy the relation (H3) of the Hecke algebra $H_N(q)$.

Theorem 5.2. *The monodromy representation $q^{3/4}\pi_{N,t}$ of the braid group B_N on the space $W(N; t)$ gives a representation of the Hecke algebra $H_N(q)$, where $q=\exp(2\pi\sqrt{-1}/\kappa)$.*

Remark. It is remarkable that our representations are obtained for the Hecke algebra $H_N(q)$ with a root q of unity, since the algebra $H_N(q)$ is not semi-simple for a root q of unity (cf. V. F. R. Jones [Jo]).

5.3) Wenzl's representations of Hecke algebra

H. Wenzl [W] constructed irreducible representations (π_λ, V_λ) of Hecke algebras $H_N(q)$ for any q not being roots of unity, parametrized by the set Λ_N of all Young diagrams on N nodes. If $q=\exp(2\pi\sqrt{-1}/\kappa)$ with $\kappa(=\ell+2)\geq 4$ (i.e. $\ell\geq 2$), he also constructed irreducible representations $(\pi_\lambda^{(k,\kappa)}, V_\lambda^{(k,\kappa)})$ of $H_N(q)$ parametrized by the set $\Lambda^{(k,\kappa)}$ of all (k, κ)-diagrams on N nodes. Note that the representations $\pi_\lambda^{(k,\kappa)}$ are unitarizable as representations of the group B_N.

In this paragraph, we show that our representation $(\pi_{N,t}, W(N; t))$ of the Hecke algebra $H_N(q)$ $(q=\exp(2\pi\sqrt{-1}/\kappa))$ is equivalent to the representation $(\pi_\lambda^{(2,\kappa)}, V_\lambda^{(2,\kappa)})$.

Let Λ_N^2 be the set of all Young diagrams λ on N nodes with depth (λ) ≤ 2. For each $\lambda\in\Lambda_N^2$, $d(\lambda)$ denotes the difference of the number of the first row of λ and the one of the second. Introduce the set $\Lambda_N^{(2,\kappa)}$ of all $(2, \kappa)$-diagrams on N nodes, defined by $\Lambda_N^{(2,\kappa)}=\{\lambda\in\Lambda_N^2;\ d(\lambda)\leq\kappa-2\ (=\ell)\}$. Any $\lambda\in\Lambda_N^{(2,\kappa)}$ is written as $[N/2+t, N/2-t]$ for some half-integer $t\geq 0$.

We shall write $\mu<\lambda$, if the Young diagram μ can be obtained by taking away appropriate nodes of λ. For each $\lambda\in\Lambda_N^{(2,\kappa)}$, let

$$\mathscr{P}_\ell(\lambda)=\{p=(\lambda_{(N)},\cdots,\lambda_{(1)});\ \lambda_{(i)}\in\Lambda_i^{(2,\kappa)},\ \lambda_{(i)}<\lambda_{(i+1)},\ \lambda_{(N)}=\lambda\}.$$

H. Wenzl defines an irreducible representation $(\pi_\lambda^{(2,\kappa)}, V_\lambda^{(2,\kappa)})$ of the algebra $H_N(q)$ for each $\lambda\in\Lambda_N^{(2,\kappa)}$, where $V_\lambda^{(2,\kappa)}$ has the form $\bigoplus_{p\in\mathscr{P}_\ell(\lambda)}\mathbf{C}\vec{v}_p$. This gives a unitary representation of the group B_N.

Note that for each N, the number $d(\lambda)$ determines the Young diagram

$\lambda \in \Lambda_N^2$ uniquely. For each $p=(\lambda_{(N)}, \cdots, \lambda_{(1)}) \in \mathscr{P}_\ell(\lambda)$ with $\lambda \in \Lambda_N^{(2,\kappa)}$, let $K(p)=(t, \frac{1}{2}d(\lambda_{(N-1)}), \cdots. \frac{1}{2}d(\lambda_{(1)}), 0) \in \mathscr{P}_\ell(N; t)$ with $t=\frac{1}{2}d(\lambda)$. Then the mapping K gives a bijection of $\mathscr{P}_\ell(\lambda)$ with $\mathscr{P}_\ell(N; \frac{1}{2}d(\lambda))$.

For each $p=(\lambda_{(N)}, \cdots, \lambda_{(1)}) \in \mathscr{P}_\ell(\lambda)$, introduce the numbers $\gamma_i(p)$ $(1\leq i\leq N-1)$ defined by

$$\gamma_i(p)=1, \quad \text{if } d(\lambda_{(i+1)})=d(\lambda_{(i-1)})=0 \quad \text{or} \quad |d(\lambda_{(i+1)})-d(\lambda_{(i-1)})|=2,$$

$$\gamma_i(p)=\frac{\Gamma\left(\frac{d+1}{\kappa}\right)}{\left(\Gamma\left(\frac{d+2}{\kappa}\right)\Gamma\left(\frac{d}{\kappa}\right)\right)^{1/2}}, \quad \text{if } d=d(\lambda_{(i-1)})=d(\lambda_{(i+1)})=d(\lambda_{(i)})-1,$$

and

$$\gamma_i(p)=\frac{\Gamma\left(\frac{d+1}{-\kappa}\right)}{\left(\Gamma\left(\frac{d+2}{-\kappa}\right)\Gamma\left(-\frac{d}{\kappa}\right)\right)^{1/2}}, \quad \text{if } d=d(\lambda_{(i-1)})=d(\lambda_{(i+1)})=d(\lambda_{(i)})+1.$$

Define the mapping K: $V_\lambda^{(2,\kappa)}\to W(N; \frac{1}{2}d(\lambda))$ by

$$K(\vec{v}_p)=\prod_{i=1}^{N-1}\gamma_i(p)\Psi_{K(p)} \qquad \text{for } p \in \mathscr{P}_\ell(\lambda).$$

(note $\gamma_1(p)=1$ for any $p \in \mathscr{P}_\ell(\lambda)$.)

Then the mapping K intertwines Wenzl's representations $(\pi_\lambda^{(2,\kappa)}, V_\lambda^{(2,\kappa)})$ and our $(\pi_{N,t}, W(N; t))$:

Proposition 5.3. *For each $\lambda \in \Lambda_N^{(2,\kappa)}$, set $t=\frac{1}{2}d(\lambda)$. Then*

$$K\pi_\lambda=q^{3/4}\pi_{N,t}K.$$

Note 1. If we construct the theory for $\ell \notin \mathbf{Q}$ as in Remark 3.5, we get the monodromy representations of the Hecke algebra $H_N(q)$, $q=\exp(2\pi\sqrt{-1}/(\ell+2))$, which are isomorphic to the representations (π_λ, V_λ) parametrized by $\lambda \in \Lambda_N^{(2)}$.

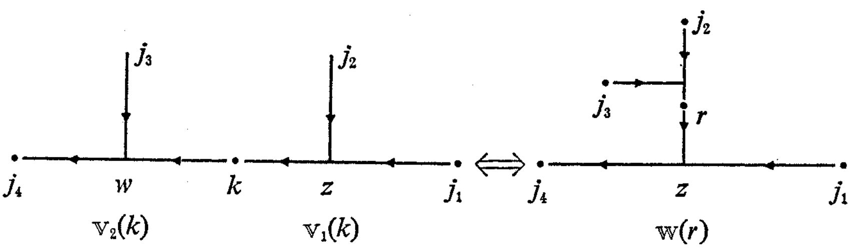
Note 2. By means of $A_1^{(1)}$-modules, we obtained here the representations $(\pi_\lambda^{(2,\kappa)}, V_\lambda^{(2,\kappa)})$ of the algebra $H_N(q)$ parametrized by the $(2, \kappa)$-diagrams. For general $k>2$, Wenzl's representations $(\pi_\lambda^{(k,\kappa)}, V_\lambda^{(k,\kappa)})$ are obtained by means of integrable highest weight modules of affine Lie algebras of type $A_n^{(1)}$ $(n>1)$. We will discuss them in our succeeding paper.

5.4) Fusion rule

For a quadruple $\mathbb{J}=(j_4, j_3, j_2, j_1)$, introduce the set $J_\ell(\mathbb{J})$ defined by

$$J_\ell(\mathbb{J})=\Big\{r\in\frac{1}{2}\mathbb{Z};\quad 0\leq 2r\leq \ell,\quad \mathrm{w}(r)=\begin{pmatrix} r \\ j_4 j_1\end{pmatrix}\in(\mathrm{CG})_\ell,$$
$$\overline{\mathrm{w}}(r)=\begin{pmatrix} j_3 \\ r\ j_2\end{pmatrix}\in(\mathrm{CG})_\ell\Big\},$$

and consider the fusion of vertex operators $\Phi_{\mathrm{v}_2(k)}(w)$ and $\Phi_{\mathrm{v}_1(k)}(z)$ for $k\in I_\ell(\mathbb{J})$ to $\Phi_{\mathrm{w}(r)}(z)$ (the first term of the short range expansion of the product $\Phi_{\mathrm{v}_2(k)}(w)\Phi_{\mathrm{v}_1(k)}(z)$):



Now we restrict ourselves to the case $j_3=\frac{1}{2}$. Assume that $V_0^\vee(\mathbb{J})\neq\emptyset$, then in cases listed in Section 4.2. we get

(D2)$_2$ $\quad I_\ell(\mathbb{J})=\Big\{k_\pm=j_4\pm\frac{1}{2}\Big\},\qquad J_\ell(\mathbb{J})=\Big\{r_\pm=j_2\pm\frac{1}{2}\Big\},$

(D2)$_1$ $\quad I_\ell(\mathbb{J})=\Big\{j_4-\frac{1}{2}\Big\},\qquad J_\ell(\mathbb{J})=\Big\{j_2-\frac{1}{2}\Big\},$

(D1)$_1$ $\quad I_\ell(\mathbb{J})=\Big\{j_4+\frac{1}{2}\Big\},\qquad J_\ell(\mathbb{J})=\Big\{j_2+\frac{1}{2}\Big\},$

(D1)$_2$ $\quad I_\ell(\mathbb{J})=\Big\{j_4+\frac{1}{2}\Big\},\qquad J_\ell(\mathbb{J})=\Big\{j_2-\frac{1}{2}\Big\},$

(D1)$_3$ $\quad I_\ell(\mathbb{J})=\Big\{j_4-\frac{1}{2}\Big\},\qquad J_\ell(\mathbb{J})=\Big\{j_2+\frac{1}{2}\Big\}.$

Here we discuss the case (D2), since other cases are much simpler. In this case, fix notations

$$\mathrm{v}_2(k_\pm)=\begin{pmatrix} \frac{1}{2} \\ j_4\, k_\pm\end{pmatrix},\quad \mathrm{v}_1(k_\pm)=\begin{pmatrix} j_2 \\ k_\pm\ j_1\end{pmatrix},\quad \mathrm{w}(r_\pm)=\begin{pmatrix} r_\pm \\ j_4\ j_1\end{pmatrix},\quad \overline{\mathrm{w}}(r_\pm)=\begin{pmatrix} \frac{1}{2} \\ r_\pm\, j_2\end{pmatrix}$$

and note the relations:

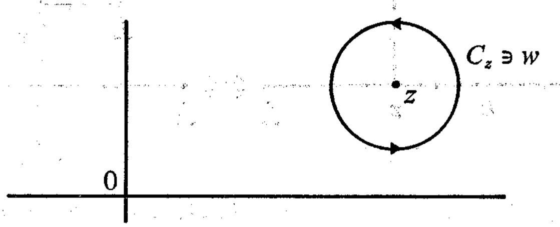
$$\gamma_\pm^{(0)}=\hat{\Delta}(\mathrm{v}_2(k_\pm))=\Delta_4(\mathbb{J})-\hat{\Delta}(\mathrm{v}_1(k_\pm))\quad\text{and}\quad \gamma_\pm^{(1)}=\Delta_4(\mathbb{J})-\hat{\Delta}(\mathrm{w}(r_\pm)).$$

Then

Proposition 5.4. *For a quadruple* $\mathbb{J}=(j_4, \frac{1}{2}, j_2, j_1)$ *in the case* $(\mathrm{D2})_2$ *and for each* $k \in I_\ell(\mathbb{J})$,

$$\frac{1}{2\pi\sqrt{-1}}\int_{C_z}(w-z)^{-\gamma_{\pm}^{(1)}-1}\Phi_{v_3(k)}(u_3;w)\Phi_{v_2(k)}(u_2;z)dw$$
$$=F_k^{\pm}(\mathbb{J})\Phi_{w(r_\pm)}(\varphi_{w(r_\pm)}(u_3,u_2);z) \qquad (u_3 \in V_{1/2}, u_2 \in V_{j_2}),$$

where C_z *is a contour around* z *such that* 0 *is outside* C_z, *and the coefficients* $F_\pm^{\pm}=F_{k\pm}^{\pm}$ *are given in Proposition* A. 2.



Proof. The composition $\Phi_k(w, z)$ of vertex operators $\Phi_{v_3(k)}(w)$ and $\Phi_{v_2(k)}(z)$ is determined by the $V_0^{\vee}(\mathbb{J})$-valued function $\Psi_k(w, z)$ on M_2 defined in Section 4.1 for each $k \in I_\ell(\mathbb{J})$. By Propositions 4.3 and 4.4, we get the expansion of $\Psi_k(w, z)$ near $w=z$ as

$$\Psi_k(w,z)=(w-z)^{\gamma_+^{(1)}}\{\sqrt{2j_4+1}\,F_k^+U_+^{(1)}+O(w-z)\}$$
$$+(w-z)^{\gamma_-^{(1)}}\{\sqrt{2j_4+1}\,F_k^-U_-^{(1)}+O(w-z)\}$$

where $O(w-z)$ is holomorphic near $w=z$ and vanishes on $\{w=z\}$.

Now introduce the operator $\Xi_k^{\pm}(u_3, u_2; z)$ of $\mathscr{H}_{j_1}$ to $\hat{\mathscr{H}}_{j_4}$ defined by the integral

$$\Xi_k^{\pm}(u_3,u_2;z)=\frac{1}{2\pi\sqrt{-1}}\int_{C_z}(w-z)^{-\gamma_{\pm}^{(1)}-1}\Phi_{v_3(k)}(u_3;w)\Phi_{v_2(k)}(u_2;z)dw.$$

And define an operator $\Xi_k^{\pm}(z)\,(v)=\Xi_k^{\pm}\,(v;z)\colon \mathscr{H}_{j_1}\to\hat{\mathscr{H}}_{j_4}$ parametrized by $V_{r_\pm}$ as follows: For any vector $v \in V_{r_\pm}$ is written as a linear combination $v=\sum_i c_i\varphi_{w(r_\pm)}(u_3^i, u_2^i)$ for some $u_3^i\otimes u_2^i \in V_{j_3}\otimes V_{j_2}$. Then put $\Xi_k^{\pm}(v;z)=\sum_i c_i\Xi_k^{\pm}(u_3^i, u_2^i; z)$, then $\Xi_k^{\pm}(z)$ is independent of the expression of v, and is a vertex operator of type $w(r_\pm)$, that is, of spin $r_\pm$ (note $-\Delta_{r_\pm}=\gamma_{\pm}^{(1)}-\Delta_{j_3}-\Delta_{j_2}$). In fact, for $X \in \mathfrak{g}$ and $m \in \mathbb{Z}$,

$$[X(m), \Xi_k^{\pm}(u_3,u_2;z)]=z^m[\Xi_k^{\pm}(Xu_3,u_2;z)+\Xi_k^{\pm}(u_3,Xu_2,z)]$$

and

$$[L(m), \Xi_k^{\pm}(u_3, u_2, z)]$$

$$=\frac{1}{2\pi\sqrt{-1}}\int_{C_z}(w-z)^{-r_{\pm}^{(1)}-1}\Big\{w^{m+1}\frac{\partial}{\partial w}+z^{m+1}\frac{\partial}{\partial z}$$

$$+(m+1)(\Delta_{j_3}w^m+\Delta_{j_2}z^m)\Big\}\Phi_{v_3(k)}(u_3; w)\Phi_{v_2(k)}(u_2; z)dw$$

$$=z^m\Big\{z\frac{\partial}{\partial z}+(m+1)(\Delta_{j_3}+\Delta_{j_2})\Big\}\Xi_k^{\pm}(u_3, u_2; z)$$

$$-\frac{1}{2\pi\sqrt{-1}}\int_{C_z}\frac{\partial}{\partial w}\{(w-z)^{-r_{\pm}^{(1)}-1}w^{m+1}\}\Phi_{v_3(k)}(u_3; w)\Phi_{v_2(k)}(u_2; z)dw$$

$$-\frac{z^{m+1}}{2\pi\sqrt{-1}}\int_{C_z}\frac{\partial}{\partial z}\{(w-z)^{-r_{\pm}^{(1)}-1}\}\Phi_{v_3(k)}(u_3; w)\Phi_{v_2(k)}(u_2; z)dw$$

$$=z^m\Big\{z\frac{\partial}{\partial z}+(m+1)(\Delta_{j_3}+\Delta_{j_2}-1)\Big\}\Xi_k^{\pm}(u_3, u_2; z)$$

$$+\frac{r_{\pm}^{(1)}+1}{2\pi\sqrt{-1}}\int_{C_z}(w^{m+1}-z^{m+1})(w-z)^{-r_{\pm}^{(1)}-2}\Phi_{v_3(k)}(u_3; w)\Phi_{v_2(k)}(u_2; z)dw$$

$$=z^m\Big\{z\frac{\partial}{\partial z}+(m+1)\Delta_{r_{\pm}}\Big\}\Xi_k^{\pm}(u_3, u_2; z).$$

Thus $\Xi_k^{\pm}(z)$ is a vertex operator of type $\mathbb{w}(r_{\pm})$, so it is a constant multiple of $\Phi_{\mathbb{w}(r_{\pm})}(z)$.

Hence we get the proposition, by computing the initial term of $\Phi_{\mathbb{w}(r_{\pm})}(\varphi_{\mathbb{w}(r_{\pm})}(u_3, u_2); z)$:

$$\langle\nu(u_4)|\varphi_{\mathbb{w}(r_{\pm})}(\varphi_{\mathbb{w}(r_{\pm})}(u_3, u_2), u_1)\rangle=\sqrt{2j_4+1}\ U_{\pm}^{(1)}(u_4, u_3, u_2, u_1). \qquad \text{q.e.d.}$$

Let $N\geq 2$. Fix an N-ple $\mathbb{J}=(\frac{1}{2}, \cdots, \frac{1}{2})$ and half integers t (*target edge*) and s (*source edge*) with $0\leq 2t, 2s\leq \ell$, and put $\mathbb{J}_{t,s}=(t, \frac{1}{2}, \cdots, \frac{1}{2}, s)$.

Consider the systems $\mathrm{E}_{t,s}(\mathbb{J})$ and $\mathrm{B}_{t,s}(\mathbb{J})$ of equations for $V_0^{\vee}(\mathbb{J}_{t,s})$-valued functions $\Psi(\mathbb{z})$ on the manifold M_N:

$$\mathrm{E}_{t,s}(\mathbb{J}): \qquad \Big(\kappa\frac{\partial}{\partial z_i}-\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}-\frac{\Omega_{i0}}{z_i}\Big)\Psi(\mathbb{z})=0 \qquad (1\leq i\leq N)$$

and for any $u_k\in V_{j_k}$ $(j_N=t, j_i=\frac{1}{2}\ (1\leq i\leq N-1), j_0=s)$,

$$\mathrm{B}_{t,s}(\mathbb{J}):\ \sum_{\mathbb{m}_0}\binom{L_0}{\mathbb{m}_0}\prod_{k=1}^{N}z_k^{-m_k}\Psi(\mathbb{z})(E^{m_{N+1}}u_{N+1}, E^{m_N}u_N, \cdots, E^{m_1}u_1, u_s(s))=0,$$

$$\sum_{\mathbb{m}_i}\binom{L_i}{\mathbb{m}_i}\prod_{\substack{k=1\\k=i}}^{N}(z_k-z_i)^{-m_k}\Psi(\mathbb{z})(u_{N+1}, E^{m_N}u_N, \cdots, u_{j_i}(j_i), \cdots, E^{m_0}u_0)=0$$

for $1 \leq i \leq N$ $(z_0=0)$, and

$$\sum_{\mathbb{m}_{N+1}} \binom{L_{N+1}}{\mathbb{m}_{N+1}} \Psi(\mathbb{z})(u_{j_{N+1}}(j_{N+1}), E^{m_N}u_N, \cdots, E^{m_0}u_0)=0,$$

where $\mathbb{m}_i=(m_N, \cdots, \hat{m}_i, \cdots, m_0) \in (\mathbb{Z}_{\geq 0})^N$ $(0 \leq i \leq N)$ and $\mathbb{m}_{N+1}=(m_N, \cdots, m_0) \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\mathbb{m}_i|=L_i=\ell-2j_i+1$ $(0 \leq i \leq N+1)$.

Let $W(N; t, s)$ be the solution space of the joint system $\mathrm{E}_{t,s}(\mathbb{J})$ and $\mathrm{B}_{t,s}(\mathbb{J})$. Then Theorem 3.3 implies that the space $W(N; t, s)$ has a basis $\{\Psi_{\mathbb{p}}(z_N, \cdots, z_1); \mathbb{p} \in \mathscr{P}_\ell(N; t, s)\}$ defined as follows: Let

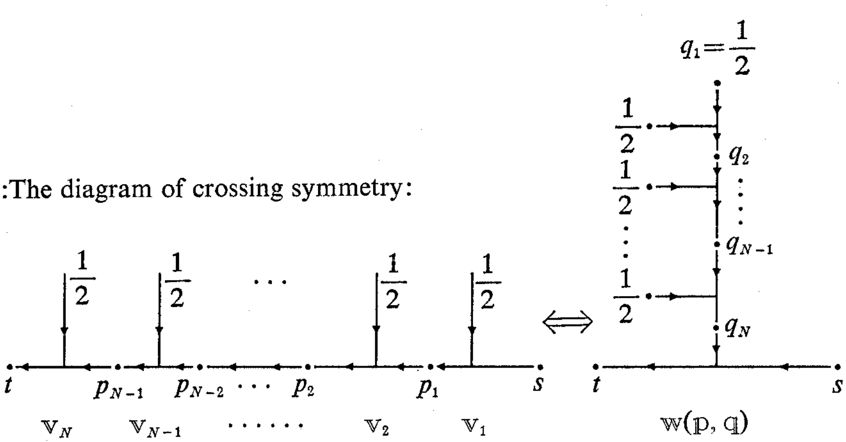
$$\mathscr{P}_\ell(N; t, s)=\Big\{\mathbb{p}=(p_N, \cdots, p_1, p_0); p_N=t, p_0=s, p_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}, 2p_i \leq \ell,$$
$$|p_i-p_{i-1}|=\frac{1}{2} \ (1 \leq i \leq N)\Big\}.$$

For each $\mathbb{p} \in \mathscr{P}_\ell(N; t, s)$, define the $V_0^\vee(\mathbb{J}_{t,s})$-valued, multi-valued holomorphic function $\Psi_{\mathbb{p}}(z_N, \cdots, z_1)$ on M_N by

$$\langle \nu(v)| \Psi_{\mathbb{p}}(z_N, \cdots, z_1)(u_N, \cdots, u_1)|w\rangle = \langle \nu(v)| \Phi_{\mathbb{v}_N}(u_N; z_N) \cdots \Phi_{\mathbb{v}_1}(u_1; z_1)|w\rangle$$

for $v \in V_t$, $u_i \in V_{1/2}(1 \leq i \leq N)$ and $w \in V_s$, where $\mathbb{v}_i(\mathbb{p})=\begin{pmatrix} & \frac{1}{2} & \\ p_i & & p_{i-1}\end{pmatrix}$ $(1 \leq i \leq N)$.

:The diagram of crossing symmetry:



Now introduce the set $\mathscr{Q}_\ell(N)$ defined by

$$\mathscr{Q}_\ell(N)=\Big\{\mathbb{q}=(q_N, \cdots, q_1); \ q_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}, \ q_1=\frac{1}{2}, \ 2q_i \leq \ell,$$

$$\left.|q_i-q_{i-1}|=\frac{1}{2}\ (2\leq i\leq N)\right\}$$

For each $\mathfrak{p}\in\mathscr{P}_\ell(N;t,s)$, $\mathfrak{q}\in\mathscr{Q}_\ell(N)$ and i $(2\leq i\leq N)$, define the quadruples $\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})=(p_i,\frac{1}{2},q_{i-1},s)$, these quadruples $\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})$ satisfy one of the conditions $(\mathrm{D2})_{1,2}$, $(\mathrm{D1})_{1,2,3}$ and (D0). Moreover, define numbers $\gamma_i(\mathfrak{p},\mathfrak{q})$ and $F_i(\mathfrak{p},\mathfrak{q})$ as follows: if $\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})$ satisfies the condition (D1), let

$$\gamma_i(\mathfrak{p},\mathfrak{q})=\gamma^{(1)}(\mathbb{Q}_i(\mathfrak{p},\mathfrak{q}))\quad\text{and}\quad F_i(\mathfrak{p},\mathfrak{q})=1.$$

If $\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})$ satisfies the condition $(\mathrm{D2})_1$, let

$$\gamma_i(\mathfrak{p},\mathfrak{q})=\gamma^{(1)}_{-}(\mathbb{Q}_i(\mathfrak{p},\mathfrak{q}))\quad\text{and}\quad F_i(\mathfrak{p},\mathfrak{q})=F^{-}_{-}(\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})).$$

Assume that $\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})$ satisfies the condition $(\mathrm{D2})_2$. If $p_{i-1}-p_i=\pm\frac{1}{2}$, then put $k=\pm$, and if $q_i-q_{i-1}=\pm\frac{1}{2}$, then put $\bar{k}=\pm$. Let

$$\gamma_i(\mathfrak{p},\mathfrak{q})=\gamma^{(1)}_{\bar{k}}(\mathbb{Q}_i(\mathfrak{p},\mathfrak{q}))\quad\text{and}\quad F_i(\mathfrak{p},\mathfrak{q})=F^{\bar{k}}_{k}(\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})).$$

If $\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})$ satisfies the condition (D0), let $F_i(\mathfrak{p},\mathfrak{q})=0$.

Then we get

Proposition 5.5. *For each* $\mathfrak{p}\in\mathscr{P}_\ell(N;t,s)$ *and* $\mathfrak{q}\in\mathscr{Q}_\ell(N;f)$ *such that* $V_0(\mathbb{Q}_i(\mathfrak{p},\mathfrak{q}))\neq 0$, *i.e.* $\mathbb{Q}_i(\mathfrak{p},\mathfrak{q})\in(\mathrm{D2})_{1,2}\cup(\mathrm{D1})_{1,2,3}$,

$$\begin{aligned}(2\pi\sqrt{-1})^{1-N}\int_{C_N}\cdots\int_{C_2}\sum_{i=2}^{N}(z_i-z)^{-\gamma_i(Q_i(\mathfrak{p},\mathfrak{q}))-1}&\\ \Phi_{\mathrm{v}_N(\mathfrak{p})}(u_N;z_N)\cdots\Phi_{\mathrm{v}_1(\mathfrak{p})}(u_1;z)dz_N\cdots dz_2&\\ =\prod_{i=2}^{N}F_i(\mathbb{Q}_i(\mathfrak{p},\mathfrak{q}))\Phi_{\mathrm{w}}(\varphi_{\mathfrak{q}}(u_N,\cdots,u_2,u_1);z)&\end{aligned}$$

for each $u_i\in V_{j_i}$, *where* C_i*'s are contours around* C_{i-1} $(3\leq i\leq N)$ *such that* 0 *is outside* C_N *and* C_2 *is around* z. *The vertices* w *and* w_i $(2\leq i\leq N)$ *are defined as*

$$\mathrm{w}=\mathrm{w}(\mathfrak{p},\mathfrak{q})=\begin{pmatrix} & f & \\ t & & s\end{pmatrix},\quad f=q_N,\quad \mathrm{w}_i=\mathrm{w}_i(\mathfrak{q})=\begin{pmatrix} & \frac{1}{2} & \\ q_i & & q_{i-1}\end{pmatrix},$$

and $\varphi_{\mathfrak{q}}(u_N,\cdots,u_2,u_1)\in V_f$ *is defined by*

$$\varphi_{\mathfrak{q}}(u_N,\cdots,u_2,u_1)=\varphi_{\mathrm{w}_N}(u_N,\varphi_{\mathrm{w}_{N-1}}(u_{N-1},\cdots,\varphi_{\mathrm{w}_2}(u_2,u_1)\cdots)).$$

Appendix I. Bases of Tensor Products of $\mathfrak{sl}$-modules

Here we use notation on the Lie algebra $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$ and its modules given in Section 1.

Since the vacuum expectation value on $V_j^\dagger \times V_j$ is nondegenerate, we can identify the dual right $\mathfrak{g}$-module $V_j^\vee$ of V_j with $V_j^\dagger$. The basis $\{\varphi_j(m);\ m=j, j-1, \cdots, -j\}$ of $V_j^\dagger$ dual to the basis $\{u_j(m)\}$ is identified with $\{u_j^\dagger(m)\}$ by $\varphi_j(m)(u_j(m'))=\langle u_j^\dagger(m)|u_j(m')\rangle=\delta_{m,m'}$.

The isomorphism ν: $V_j \to V_j^\dagger$ is defined by $\nu(u_j(j))=u_j^\dagger(-j)$ and $\nu(X|v\rangle)=-\nu(|v\rangle)X$ ($|v\rangle \in V_j$, $X \in \mathfrak{g}$). Then

$$\nu(u_j(m))=(-1)^{j-m}u_j^\dagger(-m)=(-1)^{j-m}\varphi_j(-m).$$

Introduce the $\mathbb{C}$-bilinear forms (,) on V_j, $V_j^\dagger$ and $V_j^\vee$ for which the bases $\{u_j(m)\}$, $\{u_j^\dagger(m)\}$ and $\{\varphi_j(m)\}$ are orthonormal, then E and F are mutually adjoint with each other and H is self-adjoint in all cases.

Here we refer to the famous textbook [LL] of L.D. Landau and E.M. Lifshitz.

Now for each vertex $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix} \in (CG)$, we choose and fix the element φ_{v} of $\mathrm{Hom}_{\mathfrak{g}}(V_j \otimes V_{j_1}, V_{j_2})=(V_{j_2}\otimes V_j^\vee \otimes V_{j_1}^\vee)^{\mathfrak{g}}$ as

$$\varphi_{\mathrm{v}}=\sum_{m_1+m=m_2} C_{m_1 m}^{j_2 m_2} u_{j_2}(m_2)\otimes\varphi_j(m)\otimes\varphi_{j_1}(m_1),$$

where the *Clebsch-Gordan coefficients* $C_{m_1 m_2}^{j_3 m_3}$ are real numbers and expressed by the well-known *Wigner's 3j-symbols* $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}$ as

$$C_{m_1 m_2}^{j_3 m_3}=(-1)^{j_1-j_2+m_3}\sqrt{2j_3+1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3\end{pmatrix}.$$

Wigner's $3j$-symbols are defined for half integers $j_i \geq 0$ with $j_3+j_2+j_1 \in \mathbb{Z}$, $|j_2-j_1| \leq j_3 \leq j_2+j_1$, $j_i-m_i \in \mathbb{Z}$, and satisfy the following:

i) If $|j_i|<m_i$ for some i, or $m_3+m_2+m_1 \neq 0$, then $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}=0$.

ii) $$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3\end{pmatrix}=\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2\end{pmatrix}=\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3\end{pmatrix}$$
$$=(-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix},$$

iii) $$\begin{pmatrix} j & j & 0 \\ m & -m & 0\end{pmatrix}=(-1)^{j-m}(2j+1)^{-1/2}.$$

In particular, if $j_1=0$, then $j_2=j$ and

$$\varphi_{\left(\begin{smallmatrix} & j & \\ j & & 0\end{smallmatrix}\right)}=\sum_{m=-j}^{j} C_{0\ m}^{j\ m} u_j(m)\otimes\varphi_j(m)\otimes\varphi_0(0)=\sum_{m=-j}^{j} u_j(m)\otimes\varphi_j(m)=\mathrm{id}_{V_j}.$$

If $j_2=0$, then $j_1=j$ and

$$\varphi_{\binom{j}{0\ j}}=\sum_{m=-j}^{j} C_{m\ -m}^{0\ \ 0} u_0(0)\otimes\varphi_j(-m)\otimes\varphi_j(m)=\sum_{m=-j}^{j}(-1)^{j-m}\varphi_j(-m)\otimes\varphi_j(m)$$

is identified with the isomorphism ν: $V_j\to V_j^{\vee}$ which is given by $\nu(u_j(m))=(-1)^{j-m}\varphi_j(-m)$.

For a quadruple $\mathbb{J}=(j_4, j_3, j_2, j_1)$ of half integers $j_i\geq 0$, there are three orthonomal bases $\{U_{j_{12}}^{(0)}; j_{12}\in I(\mathbb{J})\}$, $\{U_{j_{23}}^{(1)}; j_{23}\in I(j_4, j_1, j_3, j_2)\}$ and $\{U_{j_{13}}^{(\infty)}; j_{13}\in I(j_4, j_2, j_3, j_1)\}$ of $V_0^{\vee}(\mathbb{J})$ defined by

$$U_{j_{12}}^{(0)}=\frac{1}{\sqrt{2j_4+1}}\sum_{\substack{m_1+m_2=m_{12}\\ m_3+m_{12}=m_4}}(-1)^{j_4-m_4}C_{m_{12}m_3}^{j_4m_4}C_{m_1m_2}^{j_{12}m_{12}}$$
$$\varphi_{j_4}(-m_4)\otimes\varphi_{j_3}(m_3)\otimes\varphi_{j_2}(m_2)\otimes\varphi_{j_1}(m_1),$$

$$U_{j_{23}}^{(1)}=\frac{1}{\sqrt{2j_4+1}}\sum_{\substack{m_2+m_3=m_{23}\\ m_3+m_{23}=m_4}}(-1)^{j_4-m_4}C_{m_1m_{23}}^{j_4m_4}C_{m_2m_3}^{j_{23}m_{23}}$$
$$\varphi_{j_4}(-m_4)\otimes\varphi_{j_3}(m_3)\otimes\varphi_{j_2}(m_2)\otimes\varphi_{j_1}(m_1),$$

and

$$U_{j_{13}}^{(\infty)}=\frac{1}{\sqrt{2j_4+1}}\sum_{\substack{m_1+m_3=m_{13}\\ m_2+m_{13}=m_4}}(-1)^{j_4-m_4}C_{m_{13}m_2}^{j_4m_4}C_{m_1m_3}^{j_{13}m_{13}}$$
$$\varphi_{j_4}(-m_4)\otimes\varphi_{j_3}(m_3)\otimes\varphi_{j_2}(m_2)\otimes\varphi_{j_1}(m_1),$$

then the operator $\Omega_{12}=\frac{1}{2}[\Delta_{12}(\Omega)-\Omega_{11}-\Omega_{22}]$ is diagonalized by this basis $\{U_{j_{12}}^{(0)}; j_{12}\in I(\mathbb{J})\}$ as

$$\Omega_{12}U_{j_{12}}^{(0)}=\kappa(\Delta_{j_{12}}-\Delta_{j_2}-\Delta_{j_1})U_{j_{12}}^{(0)}.$$

The operators Ω_{23} and Ω_{13} are also diagonalized by these bases $\{U_{j_{23}}^{(1)}\}$ and $\{U_{j_{13}}^{(\infty)}\}$ respectively as

$$\Omega_{23}U_{j_{23}}^{(1)}=\kappa(\Delta_{j_{23}}-\Delta_{j_2}-\Delta_{j_3})U_{j_{23}}^{(1)}\quad\text{and}\quad\Omega_{13}U_{j_{13}}^{(\infty)}=\kappa(\Delta_{j_{13}}-\Delta_{j_1}-\Delta_{j_3})U_{j_{13}}^{(\infty)}.$$

Moreover the basis vectors $U_{j_{12}}^{(0)}$ are expressed by the fixed φ_{v}'s as

$$U_{j_{12}}^{(0)}(u_4, u_3, u_2, u_1)=\frac{1}{\sqrt{2j_4+1}}\langle\nu(u_4)|\varphi_{\mathrm{v}_2(j_{12})}(u_3)\varphi_{\mathrm{v}_1(j_{12})}(u_2)|u_1\rangle$$

for any $u_i\in V_{j_i}$.

The transformation matrices $S^{(1,0)}=(S_{j_{12}}^{j_{23}})$, $S^{(\infty,0)}=(S_{j_{12}}^{j_{13}})$, $S^{(\infty,1)}=(S_{j_{23}}^{j_{13}})$ between three bases of $V_0^{\vee}(\mathbb{J})$ defined by

$$U_{j_{12}}^{(0)}=\sum_{j_{23}}U_{j_{23}}^{(1)}S_{j_{12}}^{j_{23}},\quad U_{j_{12}}^{(0)}=\sum_{j_{13}}U_{j_{13}}^{(\infty)}S_{j_{12}}^{j_{13}},\quad U_{j_{23}}^{(1)}=\sum_{j_{13}}U_{j_{13}}^{(\infty)}S_{j_{23}}^{j_{13}}$$

are real orthogonal matrices and are given as

$$S_{j_{12}}^{j_{23}}=(-1)^{j_1+j_2+j_3+j_4}\sqrt{(2j_{12}+1)(2j_{23}+1)}\begin{Bmatrix}j_1 j_2 j_{12}\\ j_3 j_4 j_{23}\end{Bmatrix},$$

$$S_{j_{12}}^{j_{13}}=(-1)^{2j_1-j_2+j_3+j_{13}-j_{12}}\sqrt{(2j_{12}+1)(2j_{13}+1)}\begin{Bmatrix}j_2 j_1 j_{12}\\ j_3 j_4 j_{13}\end{Bmatrix},$$

and

$$S_{j_{23}}^{j_{13}}=(-1)^{-j_1+2j_2+2j_3-j_4+j_{23}}\sqrt{(2j_{23}+1)(2j_{13}+1)}\begin{Bmatrix}j_2 j_3 j_{23}\\ j_1 j_4 j_{13}\end{Bmatrix},$$

where $\begin{Bmatrix}j_1 j_2 j_3\\ j_4 j_5 j_6\end{Bmatrix}$ is the *6j-symbol* or *Racah coefficient* which is defined by 3*j*-symbols as

$$\begin{Bmatrix}j_1 j_2 j_3\\ j_4 j_5 j_6\end{Bmatrix}=\sum_{\mathrm{m}}(-1)^{\Sigma_i(j_i-m_i)}\begin{pmatrix}j_1 & j_2 & j_3\\ -m_1 & -m_2 & -m_3\end{pmatrix}\begin{pmatrix}j_1 & j_5 & j_6\\ m_1 & -m_5 & m_6\end{pmatrix}$$
$$\times\begin{pmatrix}j_4 & j_2 & j_6\\ m_4 & m_2 & -m_6\end{pmatrix}\begin{pmatrix}j_4 & j_5 & j_3\\ -m_4 & m_5 & -m_3\end{pmatrix}.$$

In the case (D2) for a quadruple $\mathbb{J}=(j_4, \frac{1}{2}, j_2, j_1)$, then $I(\mathbb{J})=\{j_4\pm\frac{1}{2}\}$, $I(j_4, j_1, j_2, \frac{1}{2})=\{j_2\pm\frac{1}{2}\}$ and $I(j_4, j_2, \frac{1}{2}, j_1)=\{j_1\pm\frac{1}{2}\}$. Denote

$$U_{\pm}^{(0)}=U_{j_4\pm1/2}^{(0)},\quad U_{\pm}^{(1)}=U_{j_2\pm1/2}^{(1)}\quad\text{and}\quad U_{\pm}^{(\infty)}=U_{j_1\pm1/2}^{(\infty)},$$

then we get easily the formulae in Section 4.2, by using some values of 6*j*-symbols (see [LL] Section 108):

$$\begin{Bmatrix}0 & b & b\\ a & c & c\end{Bmatrix}=\begin{Bmatrix}a & b & c\\ 0 & c & b\end{Bmatrix}=\frac{(-1)^{a+b+c}}{\sqrt{(2b+1)(2c+1)}}.$$

Let $s=a+b+c+\frac{1}{2}$, then

$$\begin{Bmatrix}c & \frac{1}{2} & c+\frac{1}{2}\\ b & a & b+\frac{1}{2}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{2} & c & c+\frac{1}{2}\\ a & b & b+\frac{1}{2}\end{Bmatrix}=(-1)^s\left(\frac{(s-2b)(s-2c)}{(2b+1)(2b+2)(2c+1)(2c+2)}\right)^{1/2},$$

$$\begin{Bmatrix}c & \frac{1}{2} & c+\frac{1}{2}\\ b & a & b-\frac{1}{2}\end{Bmatrix}=\begin{Bmatrix}b & \frac{1}{2} & b-\frac{1}{2}\\ c & a & c+\frac{1}{2}\end{Bmatrix}=(-1)^s\left(\frac{(s+1)(s-2a)}{2b(2b+1)(2c+1)(2c+2)}\right)^{1/2},$$

and

$$\begin{Bmatrix}c & \frac{1}{2} & c-\frac{1}{2}\\ b & a & b-\frac{1}{2}\end{Bmatrix}=-(-1)^s\left(\frac{(s-2b)(s-2c)}{2b(2b+1)2c(2c+1)}\right)^{1/2}.$$

Appendix II. Connection Matrices of Reduced Equation

A.II.1) Solutions of reduced equation

For a quadruple $\mathbb{J}=(j_4, \frac{1}{2}, j_2, j_1)$ of half integers, we will give fundamental solution of the reduced equation

$$\mathrm{RE}(\mathbb{J}): \qquad \left(\kappa\frac{d}{d\zeta}-\frac{\Omega_{12}+\kappa\Delta_4(\mathbb{J})}{\zeta}-\frac{\Omega_{23}}{\zeta-1}\right)\Psi(\zeta)=0$$

for $V_0^{\vee}(\mathbb{J})$-valued functions $\Psi(\zeta)$ on $\zeta\in\mathbb{C}^*$. The coordinate change $\zeta\mapsto\eta=1/\zeta$ makes the equation $\mathrm{RE}(\mathbb{J})$ into

$$\mathrm{RE}(\mathbb{J})_\infty: \qquad \left(\kappa\frac{d}{d\eta}-\frac{\Omega_{13}}{\eta}-\frac{\Omega_{23}}{\eta-1}\right)\Psi\left(\frac{1}{\eta}\right)=0.$$

In this section we deal only with the case (D2) and prove Proposition 4.4, since the case (D1) is much simpler.

Write a solution $\Psi(\zeta)$ as

$$\Psi(\zeta)=(U_+^{(i)}, U_-^{(i)})\Psi^{(i)}(\zeta)=(U_+^{(i)}, U_-^{(i)})\begin{pmatrix}\varphi_+^{(i)}(\zeta)\\ \varphi_-^{(i)}(\zeta)\end{pmatrix} \qquad (i=0, 1)$$
$$=(U_+^{(\infty)}, U_-^{(\infty)})\Psi^{(\infty)}(\zeta^{-1})=(U_+^{(\infty)}, U_-^{(\infty)})\begin{pmatrix}\varphi_+^{(\infty)}(\zeta^{-1})\\ \varphi_-^{(\infty)}(\zeta^{-1})\end{pmatrix}$$

where $\{U_\pm^{(i)}; i=0, 1, \infty\}$ are three bases $\{U_\pm^{(i)}; i=0, 1, \infty\}$ of $V_0^{\vee}(\mathbb{J})$ such that

$$\Omega_{12}U_\pm^{(0)}=\kappa(\gamma_\pm^{(0)}-\Delta_4(\mathbb{J}))U_\pm^{(0)}, \quad \Omega_{23}U_\pm^{(1)}=\kappa\gamma_\pm^{(1)}U_\pm^{(1)}, \quad \Omega_{13}U_\pm^{(\infty)}=\kappa\gamma_\pm^{(\infty)}U_\pm^{(\infty)},$$

and the exponents $\gamma_\pm^{(i)}$ are given in Section 4.2. The differences $\gamma^{(i)}=\gamma_+^{(i)}-\gamma_-^{(i)}$ of exponents are given as

$$\gamma^{(0)}=\frac{2j_4+1}{\kappa}, \quad \gamma^{(1)}=\frac{2j_2+1}{\kappa} \quad \text{and} \quad \gamma^{(\infty)}=\frac{2j_1+1}{\kappa} \quad (\kappa=\ell+2).$$

Since the transformation matrices $S^{(i,k)}$ between the bases $\{U_\pm^{(i)}\}$ and $\{U_\pm^{(k)}\}$ are given in Section 4.2, we get the matrix forms of the equations $\mathrm{RE}(\mathbb{J})$ and $\mathrm{RE}(\mathbb{J})_\infty$:

$$\mathrm{RE}(\mathbb{J})_i: \quad \frac{d}{d\zeta}\Psi^{(i)}(\zeta)=A^i(\zeta)\Psi^{(i)}(\zeta)=\begin{pmatrix}a_{++}^{(i)}(\zeta) & a_{-+}^{(i)}(\zeta)\\ a_{+-}^{(i)}(\zeta) & a_{--}^{(i)}(\zeta)\end{pmatrix}\Psi^{(i)}(\zeta) \qquad (i=0, 1)$$

and

$$\mathrm{RE}(\mathbb{J})_\infty: \quad \frac{d}{d\zeta}\Psi^{(\infty)}(\eta)=A^\infty(\eta)\Psi^{(\infty)}(\zeta)=\begin{pmatrix} a^{(\infty)}_{++}(\eta) & a^{(\infty)}_{-+}(\eta) \\ a^{(\infty)}_{+-}(\eta) & a^{(\infty)}_{--}(\eta)\end{pmatrix}\Psi^{(\infty)}(\eta),$$

where the coefficient matrices A^i are given as

$$a^0_{++}(\zeta)=\frac{\gamma^{(0)}_+}{\zeta}+\frac{\gamma^{(1)}_+-a^0}{\zeta-1}, \qquad a^0_{+-}(\zeta)=a^0_{-+}(\zeta)=\frac{b^0}{\zeta-1},$$

$$a^0_{--}(\zeta)=\frac{\gamma^{(0)}_-}{\zeta}+\frac{\gamma^{(1)}_-+a^0}{\zeta-1}, \qquad a^1_{++}(\zeta)=\frac{\gamma^{(0)}_+-a^1}{\zeta}+\frac{\gamma^{(1)}_+}{\zeta-1},$$

$$a^1_{+-}(\zeta)=a^1_{-+}(\zeta)=\frac{b^1}{\zeta}, \qquad a^1_{--}(\zeta)=\frac{\gamma^{(0)}_-+a^1}{\zeta}+\frac{\gamma^{(1)}_-}{\zeta-1},$$

$$a^\infty_{++}(\eta)=\frac{\gamma^{(\infty)}_+}{\eta}+\frac{\gamma^{(1)}_+-a^\infty}{\eta-1}, \qquad a^\infty_{+-}(\eta)=a^\infty_{-+}(\eta)=\frac{b^\infty}{\eta-1},$$

$$a^\infty_{--}(\eta)=\frac{\gamma^{(\infty)}_-}{\eta}+\frac{\gamma^{(1)}_-+a^\infty}{\eta-1},$$

and

$$a^0=\frac{\varepsilon_0\varepsilon_1}{\gamma^{(0)}}, \qquad a^1=\frac{\varepsilon_0\varepsilon_1}{\gamma^{(1)}}, \qquad a^\infty=\frac{\varepsilon_0\varepsilon_4}{\gamma^{(\infty)}},$$

$$b^0=\frac{\sqrt{\varepsilon}}{\gamma^{(0)}}, \qquad b^1=\frac{\sqrt{\varepsilon}}{\gamma^{(1)}}, \qquad b^\infty=\frac{\sqrt{\varepsilon}}{\gamma^{(\infty)}} \qquad (\varepsilon=\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_4).$$

Now look at the function $\varphi^{(0)}_+(\zeta)$. The equation $\mathrm{RE}(\mathbb{J})_0$ turns into the equation for $\varphi^{(0)}_+(\zeta)$:

$$\frac{d^2}{d\zeta^2}\varphi^{(0)}_+(\zeta)=\left(\frac{\gamma^{(0)}_++\gamma^{(0)}_-}{\zeta}+\frac{\gamma^{(1)}_++\gamma^{(1)}_--1}{\zeta-1}\right)\frac{d}{d\zeta}\varphi^{(0)}_+(\zeta)$$
$$-\left\{\frac{\gamma^{(0)}_+(1+\gamma^{(0)}_-)}{\zeta^2}+\frac{\gamma^{(0)}_+(\gamma^{(1)}_-+a^0-1)+\gamma^{(0)}_-(\gamma^{(1)}_+-a^0)}{\zeta(\zeta-1)}+\frac{\gamma^{(1)}_+\gamma^{(1)}_-}{(\zeta-1)^2}\right\}\varphi^{(0)}_+(\zeta),$$

which is a second-order equation of Fuchsian type.

Now recall that a second-order equation of Fuchsian type is of the form

$$\frac{d^2\varphi}{d\zeta^2}(\zeta)=\left(\frac{\lambda+\lambda'-1}{\zeta}+\frac{\mu+\mu'-1}{\zeta-1}\right)\frac{d\varphi}{d\zeta}(\zeta)$$
$$-\left\{\frac{\lambda\lambda'}{\zeta^2}+\frac{\nu\nu'-\lambda\lambda'-\mu\mu'}{\zeta(\zeta-1)}+\frac{\mu\mu'}{(\zeta-1)^2}\right\}\varphi(\zeta),$$

where $\lambda, \lambda'; \mu, \mu'; \nu, \nu'$ are exponents at $\zeta=0; 1; \infty$ respectively. The solution space of this equation is denoted by the Riemann P-function

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ \lambda & \mu & \nu & z \\ \lambda' & \mu' & \nu' & \end{Bmatrix}.$$

The equations $\mathrm{RE}(\mathbb{J})_i$ for other functions are also reduced to Fuchsian equations of similar forms. Then we get

Proposition A.1.

$$\varphi_+^{(0)}(\zeta) \in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma_+^{(0)} & \gamma_+^{(1)} & \gamma_+^{(\infty)} & \zeta \\ 1+\gamma_-^{(0)} & \gamma_-^{(1)} & \gamma_-^{(\infty)} & \end{Bmatrix}, \quad \varphi_-^{(0)}(\zeta) \in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma_-^{(0)} & \gamma_-^{(1)} & \gamma_-^{(\infty)} & \zeta \\ 1+\gamma_+^{(0)} & \gamma_+^{(1)} & \gamma_+^{(\infty)} & \end{Bmatrix},$$

$$\varphi_+^{(1)}(\zeta) \in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma_+^{(0)} & \gamma_+^{(1)} & \gamma_+^{(\infty)} & \zeta \\ \gamma_-^{(0)} & 1+\gamma_-^{(1)} & \gamma_-^{(\infty)} & \end{Bmatrix}, \quad \varphi_-^{(1)}(\zeta) \in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma_-^{(0)} & \gamma_-^{(1)} & \gamma_-^{(\infty)} & \zeta \\ \gamma_+^{(0)} & 1+\gamma_+^{(1)} & \gamma_+^{(\infty)} & \end{Bmatrix},$$

and

$$\varphi_+^{(\infty)}(\eta) \in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma_+^{(0)} & \gamma_+^{(1)} & \gamma_+^{(\infty)} & \eta \\ \gamma_-^{(0)} & \gamma_-^{(1)} & 1+\gamma_-^{(\infty)} & \end{Bmatrix}, \quad \varphi_-^{(\infty)}(\eta) \in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma_-^{(0)} & \gamma_-^{(1)} & \gamma_-^{(\infty)} & \eta \\ \gamma_+^{(0)} & \gamma_+^{(1)} & 1+\gamma_+^{(\infty)} & \end{Bmatrix}.$$

Before we give the proof of Proposition 4.4 we recall the facts on the hypergeometric function $F(\alpha, \beta, \gamma; \zeta)$ (see e.g. [E]):

$$F(\alpha, \beta, \gamma; \zeta) = 1 + \sum_{n=1} \frac{\alpha_{(n)}\beta_{(n)}}{n!\,\gamma_{(n)}}\zeta^n = 1 + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{\zeta^n}{n!},$$

where

$$\alpha_{(n)} = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

i) If $\gamma \notin \mathbb{Z}_{\leq 0}$, $F(\alpha, \beta, \gamma; \zeta)$ is a solution of the Gaussian equation:

$$\zeta(1-\zeta)\varphi''(\zeta) + \{\gamma - (\alpha+\beta+1)\zeta\}\varphi'(\zeta) - \alpha\beta\varphi(\zeta) = 0,$$

that is,

$$F(\alpha, \beta, \gamma; \zeta) \in P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & \zeta \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix}.$$

ii) $F(\alpha, \beta, \gamma; \zeta)$ and $\zeta^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; \zeta)$ give a basis of the solution space $P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & \zeta \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix}$ of the Gaussian equation.

iii) $F(\alpha, \beta, \gamma; \zeta)=(1-\zeta)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma; \zeta)$.

iv) $(d/d\zeta)F(\alpha, \beta, \gamma; \zeta)=(\alpha\beta/\gamma)F(\alpha+1, \beta+1, \gamma+1; \zeta)$.

v) $$F(\alpha, \beta, \gamma; \zeta)=(1-\zeta)F(\alpha+1, \beta+1, \gamma+1; \zeta) + \frac{(\alpha-\gamma)(\beta-\gamma)}{\gamma(\gamma+1)}\zeta F(\alpha+1, \beta+1, \gamma+2; \zeta).$$

vi) $$(\gamma+1)F(\alpha, \beta, \gamma; \zeta) =\{(\gamma+1)-(\alpha+\beta+1-(\alpha\beta/\gamma))\zeta\}F(\alpha+1, \beta+1, \gamma+2; \zeta) + \frac{(\alpha+1)(\beta+1)}{\gamma+2}\zeta(1-\zeta)F(\alpha+2, \beta+2, \gamma+3; \zeta).$$

vii) $$(1-\zeta)F(\alpha, \beta, \gamma; \zeta) =F(\alpha-1, \beta-1, \gamma; \zeta)+\frac{\alpha+\beta-\gamma-1}{\gamma}\zeta F(\alpha, \beta, \gamma+1; \zeta).$$

viii) $$F(\alpha, \beta, \gamma+1; \zeta)=\frac{\alpha\beta}{(\alpha-\gamma)(\beta-\gamma)}(1-\zeta)F(\alpha+1, \beta+1, \gamma+1;\zeta) + \frac{\gamma(\gamma-\alpha-\beta)}{(\alpha-\gamma)(\beta-\gamma)}F(\alpha, \beta, \gamma; \zeta).$$

ix) $$\frac{\alpha(\gamma-\beta)}{\beta(\gamma-\alpha)}F(\alpha+1, \beta, \gamma+1; \zeta) =F(\alpha, \beta+1, \gamma+1; \zeta)+\frac{\gamma(\alpha-\beta)}{\beta(\gamma-\alpha)}F(\alpha, \beta, \gamma; \zeta).$$

x) $$F(\alpha, \beta+1, \gamma+1; \zeta) =F(\alpha+1, \beta, \gamma+1\ \zeta)+\frac{\alpha-\beta}{1+\gamma}\zeta F(1+\alpha, 1+\beta, 2+\gamma; \zeta).$$

xi) $$F(\alpha, \beta, \gamma; \zeta) =\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-\zeta)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-\zeta) + \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-\zeta).$$

xii) $$F(\alpha, \beta, \gamma; \zeta) =\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-\zeta)^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, 1/\zeta) + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-\zeta)^{-\beta}(1-(1/\zeta))^{\gamma-\alpha-\beta} \times F(1-\alpha, \gamma-\alpha, \beta-\alpha+1; 1/\zeta).$$

Proof of Proposition 4.4. Similarly as Theorem 3.3, we get the functions $\varphi^{(0)}_{\pm\pm}(\zeta)$ in a neighbourhood of $\zeta=0$ such that

$$\Psi^{(0)}_{+}(\zeta)=(U^{(0)}_{+}, U^{(0)}_{-})\begin{pmatrix}\varphi^{(0)}_{++}(\zeta)\\ \varphi^{(0)}_{+-}(\zeta)\end{pmatrix};\qquad \Psi^{(0)}_{-}(\zeta)=(U^{(0)}_{+}, U^{(0)}_{-})\begin{pmatrix}\varphi^{(0)}_{-+}(\zeta)\\ \varphi^{(0)}_{--}(\zeta)\end{pmatrix}$$

such that $\varphi^{(0)}_{\pm}(\zeta)$ have the expansion with respect to ζ as

$$\varphi^{(0)}_{++}(\zeta)=\zeta^{\gamma^{(0)}_{+}}(1+\cdots),\qquad \varphi^{(0)}_{+-}(\zeta)=\zeta^{\gamma^{(0)}_{+}}(c\zeta+\cdots),$$

and

$$\varphi^{(0)}_{-+}(\zeta)=\zeta^{\gamma^{(0)}_{-}}(d\zeta+\cdots),\quad \varphi^{(0)}_{--}(\zeta)=\zeta^{\gamma^{(0)}_{-}}(1+\cdots),$$

where c and d are some constants.

Then by Proposition A.1,

$$\varphi^{(0)}_{++}(\zeta),\ \varphi^{(0)}_{-+}(\zeta)\in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma^{(0)}_{+} & \gamma^{(1)}_{+} & \gamma^{(\infty)}_{+} & \zeta \\ 1+\gamma^{(0)}_{-} & \gamma^{(1)}_{-} & \gamma^{(\infty)}_{-} & \end{Bmatrix},$$

and

$$\varphi^{(0)}_{+-}(\zeta),\ \varphi^{(0)}_{--}(\zeta)\in P\begin{Bmatrix} 0 & 1 & \infty & \\ \gamma^{(0)}_{-} & \gamma^{(1)}_{+} & \gamma^{(\infty)}_{+} & \zeta \\ 1+\gamma^{(0)}_{+} & \gamma^{(1)}_{-} & \gamma^{(\infty)}_{-} & \end{Bmatrix}.$$

Hence

$$\varphi^{(0)}_{++}(\zeta)\in \zeta^{\gamma^{(0)}_{+}}(1-\zeta)^{\gamma^{(1)}_{+}}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & \zeta \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix},$$

$$\varphi^{(0)}_{+-}(\zeta)\in \zeta^{1+\gamma^{(0)}_{+}}(1-\zeta)^{\gamma^{(1)}_{+}}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha+1 & \zeta \\ -1-\gamma & \gamma-\alpha-\beta & \beta+1 & \end{Bmatrix},$$

$$\varphi^{(0)}_{-+}(\zeta)\in \zeta^{1+\gamma^{(0)}_{+}}(1-\zeta)^{\gamma^{(1)}_{-}}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & 1-\alpha & \zeta \\ \gamma-1 & \alpha+\beta-\gamma & 1-\beta & \end{Bmatrix},$$

and

$$\varphi^{(0)}_{--}(\zeta)\in \zeta^{\gamma^{(0)}_{-}}(1-\zeta)^{\gamma^{(1)}_{-}}P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & -\alpha & \zeta \\ 1+\gamma & \alpha+\beta-\gamma & -\beta & \end{Bmatrix},$$

where α, β and $\gamma=\gamma^{(0)}$ are given in the proposition. Then by the formulae iv)$\sim$vi) above, we get the statement (i) of Proposition 4.4. Other statements of Proposition 4.4 are similarly obtained.

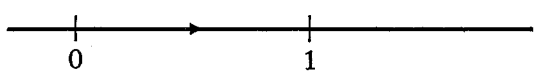
A.II.2) Connection matrices of reduced equation

We must prove Proposition 4.5 on the connection matrix of the fundamental solutions of the reduced equation RE($\mathbb{J}$) along the path from 0 to ∞ figured in Section 4.3. Fortunately the formula xii) of the hypergeometric function gives its connection matrix along the same path. And we may take $(-\zeta)^{\lambda}=\exp(-\lambda\pi\sqrt{-1})\zeta^{\lambda}$ by the choice of the path. Then it is sufficient for the proof of Proposition 4.5 to note the following relations among constants in Section 4.2:

$$-c_{\pm}^{(i)}\gamma^{(i)}(1\pm\gamma^{(i)})=\sqrt{\varepsilon}=\beta\beta^{(\infty)}\frac{B''}{A''}=\alpha(\gamma^{(\infty)}-\beta^{(\infty)})\frac{A''}{B''}$$
$$=\alpha\beta\frac{A}{B}=(\gamma-\beta)(\alpha-\gamma)\frac{B}{A}$$

for $i=0,\ 1,\ \infty$.

Similarly we get the connection matrix of the fundamental solutions of the reduced equation RE($\mathbb{J}$) along the path from 0 to 1 figured below. The formula xi) of the hypergeometric function also gives its connection matrix along the same path.



Then by relations above, we get in the case (D2):

Proposition A.2. *Denote by* $F(\mathbb{J})=\begin{pmatrix}F_+^+ & F_-^+\\ F_+^- & F_-^-\end{pmatrix}$ *the connection matrix of the fundamental solutions* $(\Psi_+^{(0)},\ \Psi_-^{(0)})$ *at* $\zeta=0$ *to* $(\Psi_+^{(1)},\ \Psi_-^{(1)})$ *at* $\zeta=1$ *of the equation* RE($\mathbb{J}$):

$$(\Psi_+^{(0)},\ \Psi_-^{(0)})=(\Psi_+^{(1)},\ \Psi_-^{(1)})\begin{pmatrix}F_+^+ & F_-^+\\ F_+^- & F_-^-\end{pmatrix}$$

that is,

$$S^{(0,1)}\begin{pmatrix}\varphi_{++}^{(0)}(\zeta) & \varphi_{-+}^{(0)}(\zeta)\\ \varphi_{+-}^{(0)}(\zeta) & \varphi_{--}^{(0)}(\zeta)\end{pmatrix}=\begin{pmatrix}\varphi_{++}^{(1)}(\zeta) & \varphi_{-+}^{(1)}(\zeta)\\ \varphi_{+-}^{(1)}(\zeta) & \varphi_{--}^{(1)}(\zeta)\end{pmatrix}\begin{pmatrix}F_+^+ & F_-^+\\ F_+^- & F_-^-\end{pmatrix}.$$

Then

$$F_+^+=\left(\frac{\gamma^{(0)}\gamma^{(1)}}{\varepsilon_2\varepsilon_4}\right)^{1/2}\frac{\Gamma(\gamma^{(0)})\Gamma(-\gamma^{(1)})}{\Gamma(\varepsilon_2)\Gamma(-\varepsilon_4)},$$

$$F_-^+=\left(\frac{\gamma^{(0)}\gamma^{(1)}}{\varepsilon_0\varepsilon_1}\right)^{1/2}\frac{\Gamma(-\gamma^{(0)})\Gamma(-\gamma^{(1)})}{\Gamma(-\varepsilon_0)\Gamma(-\varepsilon_4)},$$

$$F_+^-=\left(\frac{\gamma^{(0)}\gamma^{(1)}}{\varepsilon_0\varepsilon_1}\right)^{1/2}\frac{\Gamma(\gamma^{(0)})\Gamma(\gamma^{(1)})}{\Gamma(\varepsilon_0)\Gamma(\varepsilon_1)},$$

$$F_-^-=\left(\frac{\gamma^{(0)}\gamma^{(1)}}{\varepsilon_2\varepsilon_4}\right)^{1/2}\frac{\Gamma(-\gamma^{(0)})\Gamma(\gamma^{(1)})}{\Gamma(-\varepsilon_2)\Gamma(\varepsilon_4)},$$

Remark. Since $\varepsilon_0=1$ in the case $(\mathrm{D2})_1$, $F_-^\pm(\mathbb{J})=0$ and $F(\mathbb{J})=F_-^-(\mathbb{J})=[(2j_2+1)(\kappa-1-2j_2)/(2j_4+1)(\kappa-1-2j_4)]^{1/2}$. By Proposition 4.4′, it is obvious that $F(\mathbb{J})=1$ in the case (D1). In the case (D1), the matrix $F(\mathbb{J})$ in the proposition A.2 is written as $F(\mathbb{J})=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ in the case $(\mathrm{D1})_1$ and $F(\mathbb{J})=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ in the case $(\mathrm{D1})_{2,3}$.

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