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VERTEX ORDERING AND PARTITIONING PROBLEMS FOR RANDOM SPATIAL GRAPHS

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Given an ordering of the vertices of a finite graph, let the induced weight for an edge be the separation of its endpoints in the ordering. Layout problems involve choosing the ordering to minimize a cost functional such as the sum or maximum of the edge weights. We give growth rates for the costs of some of these problems on supercritical percolation processes and supercritical random geometric graphs, obtained by placing vertices randomly in the unit cube and joining them whenever at most some threshold distance apart.

1. Introduction. Several important optimization problems on graphs can be formulated as *layout problems*, where the aim is to order the vertices so that adjacent vertices are close together in the ordering. A (one-dimensional) layout of a finite input graph G is a bijection φ between its vertex set and a set of integers. Given a layout, the weight $\sigma(e)$ of an edge e is the difference between the integers associated with the two endpoints. A layout problem involves choosing φ so as to minimize some cost functional determined by the edge weights. For example, for the *minimum bandwidth* (MBW) problem [16, 32] the cost functional is $\max_e \sigma(e)$, while for the *minimum linear arrangement* (MLA) problem [18] the cost functional is $\sum_e \sigma(e)$. Moreover, the *minimum bisection* (MBIS) problem [14], of partitioning the vertices into two equal-sized sets so as to minimize the number of edges between them, can also be formulated as a layout problem.

Such problems have many applications. For example, MBIS and related problems are important in parallel processing [12, 13]. MBW is important for those computations on sparse symmetric matrices which are most rapidly carried out when all nonzero entries lie near the diagonal, and for minimizing delay of communication between adjacent nodes for routing problems. MLA has been used in brain cortex modeling [24], and there are applications of these problems in genomic sequencing and archeological dating. Last but not least, layout problems (and analogous problems for two-dimensional layouts) are important in Very Large Scale Integration (VLSI) problems of laying out the nodes of an integrated circuit on a board in an efficient manner [3, 30].

For the layout problems considered here, finding an optimal layout is known to be NP-hard for general graphs; see the references in [8]. Hence, computer

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scientists have been interested in looking for heuristics which can be performed rapidly and which offer good approximations in practice; a "heuristic," in this context, is a method for generating a layout that is hoped to be optimal or near-optimal. One way of testing a heuristic is to evaluate its performance on random instances, viewed as "typical" of the graphs that might arise in practice. Two classes of random instances have been widely used in the literature to enable comparisons of algorithms for layout and partitioning problems: *independent random graphs* and *random geometric graphs*.

In the case of independent random graphs, there are n vertices and each possible edge is included independently with probability p. Theoretical and empirical study of such random graphs has been extensive. For example, the approximation properties of sparse random graphs for different layout problems are considered in [11, 32] and partitioning algorithms for random graphs are studied in [5, 6]. However, for many problems independent random graphs fail to differentiate good from bad heuristics, in the sense that with high probability *all* orderings on such graphs have approximately the same cost [6, 11, 32].

In this paper we are concerned with random *geometric* graphs in which the n vertices correspond to points randomly distributed on the unit cube. Each of the possible edges appears if and only if the Euclidean distance between its two end-points is at most ρ. Graphs of this form are considered a relevant model for graphs that occur in practice, such as finite element graphs, VLSI circuits, and communication graphs [19, 20]. Many empirical studies of layout and partitioning problems have used random geometric graphs [2, 19, 20, 27, 28]. Typically, these have involved the experimental comparison of different heuristics for one or more of the layout problems under consideration, by trying them out on repeatedly simulated random geometric graphs.

The purpose of this paper and its companions [8, 9] is to provide a theoretical underpinning for these empirical studies, by establishing asymptotic growth rates for the optimal costs of layout problems on random geometric graphs, as n becomes large and ρ becomes small in a linked manner, so that the mean vertex degree tends to a limit (possibly infinity). It turns out that this limiting regime exhibits a phase transition with regard to these problems. Our results provide a benchmark by which to assess the performance on random geometric graphs of particular heuristics for these problems, for example those in [8, 19, 29]. There are some parallels between our results and those in the extensive literature on optimization problems such as the Traveling Salesman Problem (TSP) on random points [31, 34], but the methods used here are very different.

2. The main results. The layout problems considered here are formally defined as follows. Given a finite undirected graph $\mathcal{G} = (V, E)$ without self loops, a *layout* or *ordering* φ on $\mathcal{G}$ is a one-to-one function $\varphi\colon V \to \{1, 2, \dots, n\}$ with $n = |V|$ and $|\cdot|$ denoting cardinality. Given such a layout φ, for each edge $e = \{u, v\} \in E$ the associated weight is $\sigma(e, \varphi) = |\varphi(u) - \varphi(v)|$. For $v \in V$, define $L(v, \varphi) = \{u \in V\colon \varphi(u) \le \varphi(v)\}$ and $R(v, \varphi) = V \setminus L(v, \varphi)$. Then define the edge-boundary χ and interior vertex-boundary Δ of $L(v,\varphi)$ by

$$\chi(v,\varphi) = |\{\{u,w\} \in E\colon u \in L(v,\varphi) \text{ and } w \in R(v,\varphi)\}|,$$
$$\Delta(v,\varphi) = |\{u \in L(v,\varphi)\colon \exists w \in R(v,\varphi) \text{ with } \{u,w\} \in E\}|.$$

For the MLA problem, the cost $\mathrm{LA}(\varphi)$ of a layout φ is given by $\mathrm{LA}(\varphi) = \sum_{e\in E}\sigma(e,\varphi)$. An alternative formulation is $\mathrm{LA}(\varphi) = \sum_{v\in V}\chi(v,\varphi)$, which is equivalent because

$$\begin{aligned}\sum_{e\in E}\sigma(e,\varphi) &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=i}^{j-1}\mathbf{1}_{\{\{\varphi^{-1}(i),\varphi^{-1}(j)\}\in E\}}\\ &= \sum_{k=1}^{n-1}\sum_{i=1}^{k}\sum_{j=k+1}^{n}\mathbf{1}_{\{\{\varphi^{-1}(i),\varphi^{-1}(j)\}\in E\}} = \sum_{v\in V}\chi(v,\varphi).\end{aligned}$$

As well as MLA, MBW and MBIS, we study the problems of *minimum cut* (MCUT) (also known as the *isoperimetric problem* [17, 18, 22]), *minimum sum cut* (MSC) [7], and *minimum vertex separation* (MVS) [21]. In each of the six problems, given a graph $\mathscr{G}$, the object is to minimize some cost functional over the collection $\Phi(\mathscr{G})$ of all layouts on $\mathscr{G}$. The respective cost functionals for a given layout φ are denoted $\mathrm{LA}(\varphi)$, $\mathrm{BW}(\varphi)$, $\mathrm{BIS}(\varphi)$, $\mathrm{CUT}(\varphi)$, $\mathrm{SC}(\varphi)$, $\mathrm{VS}(\varphi)$, respectively, defined as follows:

$$\begin{aligned}
\mathrm{MLA}(\mathscr{G}) &= \min_{\varphi\in\Phi(\mathscr{G})}\mathrm{LA}(\varphi) \quad\text{with}\quad \mathrm{LA}(\varphi) = \sum_{e\in E}\sigma(e,\varphi) = \sum_{v\in V}\chi(v,\varphi),\\
\mathrm{MBW}(\mathscr{G}) &= \min_{\varphi\in\Phi(\mathscr{G})}\mathrm{BW}(\varphi) \quad\text{with}\quad \mathrm{BW}(\varphi) = \max_{e\in E}\sigma(e,\varphi),\\
\mathrm{MBIS}(\mathscr{G}) &= \min_{\varphi\in\Phi(\mathscr{G})}\mathrm{BIS}(\varphi) \quad\text{with}\quad \mathrm{BIS}(\varphi) = \chi(\varphi^{-1}(\lfloor n/2\rfloor),\varphi),\\
\mathrm{MCUT}(\mathscr{G}) &= \min_{\varphi\in\Phi(\mathscr{G})}\mathrm{CUT}(\varphi) \quad\text{with}\quad \mathrm{CUT}(\varphi) = \max_{v\in V}\chi(v,\varphi),\\
\mathrm{MSC}(\mathscr{G}) &= \min_{\varphi\in\Phi(\mathscr{G})}\mathrm{SC}(\varphi) \quad\text{with}\quad \mathrm{SC}(\varphi) = \sum_{v\in V}\Delta(v,\varphi),\\
\mathrm{MVS}(\mathscr{G}) &= \min_{\varphi\in\Phi(\mathscr{G})}\mathrm{VS}(\varphi) \quad\text{with}\quad \mathrm{VS}(\varphi) = \max_{v\in V}\Delta(v,\varphi).
\end{aligned}$$

Geometric graphs are defined as follows. Let $d \geq 2$ be an integer and let $\|\cdot\|$ be the Euclidean norm on $\mathbf{R}^d$. Given a set $\mathscr{X} \subset \mathbf{R}^d$, and given $\rho > 0$, let $\mathscr{G}(\mathscr{X};\rho)$ denote the graph with vertex set $\mathscr{X}$ and with $X, Y \in \mathscr{X}$ deemed adjacent if and only if $\|X - Y\| \leq \rho$ and $X \neq Y$.

Let $X_1, X_2, \ldots$ be independent and uniformly distributed on $[0,1]^d$, and let $\mathscr{X}_n$ be the point process $\{X_1, X_2, \ldots, X_n\}$. The *random geometric graphs* in this paper take the form $\mathscr{G}(\mathscr{X}_n;\rho_n)$, with $(\rho_n)_{n\geq 1}$ some chosen sequence of positive numbers tending to zero as $n \to \infty$. We shall assume $n\rho_n^d$ tends to a limit (possibly infinite). When this limit is finite, it will be denoted λ.

For an infinite-volume analogue, let $\mathscr{P}_\lambda$ denote a homogeneous Poisson process on $\mathbf{R}^d$ of intensity λ, and set $\mathscr{P}_{\lambda,0} := \mathscr{P}_\lambda \cup \{0\}$. The *continuum percolation probability* $\tilde{\theta}(\lambda)$ is the probability that the added point at the origin lies in an infinite component of $\mathscr{G}(\mathscr{P}_{\lambda,0}; 1)$. Then $\tilde{\theta}(\lambda)$ is nondecreasing in λ. Set $\lambda_c = \inf\{\lambda > 0\colon \tilde{\theta}(\lambda) > 0\}$. It is well known [15, 23] that $\lambda_c \in (0, \infty)$.

The significance of continuum percolation in the present context is as follows. Suppose that $\lim_{n\to\infty} n\rho_n^d = \lambda$. Then, for n large, after appropriate scaling and centering at a randomly chosen point of $\mathscr{X}_n$, the graph $\mathscr{G}(\mathscr{X}_n; \rho_n)$ looks locally like $\mathscr{G}(\mathscr{P}_{\lambda,0}; 1)$. If $\lambda < \lambda_c$, then all components of $\mathscr{G}(\mathscr{X}_n; \rho_n)$ are likely to have at most $O(\log n)$ points, while if $\lambda > \lambda_c$, there is likely to be a unique "big" component containing a nonvanishing proportion of the points of $\mathscr{X}_n$; in fact, the proportion of points in the big component will be approximately $\tilde{\theta}(\lambda)$. This dichotomy (phase transition) between $\lambda < \lambda_c$ and $\lambda > \lambda_c$ was demonstrated in [25, 26]. As it turns out, the same dichotomy occurs in describing the growth rates of the optimal cost functionals for layout problems.

We shall show that probabilities tend to zero rapidly. Given sequences (x_n) and (α_n) of positive numbers with $\lim_{n\to\infty} \alpha_n = \infty$, we shall say the sequence (x_n) *decays exponentially in* α_n if

$$\limsup_{n\to\infty} (\log(x_n)/\alpha_n) < 0.$$

We first give upper bounds on the optimal cost, holding with high probability, for each of the six problems. These upper bounds are rather crude in the sense that they are established by simply looking at the lexicographic ordering (the "projection algorithm" or "projection heuristic" [8]) with points of $\mathscr{X}_n$ ordered by their first co-ordinate.

THEOREM 2.1. *Suppose* $\lim_{n\to\infty} n\rho_n^d = \lambda \in (0, \infty]$. *Then there exists a constant* K *such that, except on an event of probability decaying exponentially in* $n\rho_n$,

$$\mathrm{MBW}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le Kn\rho_n, \tag{2.1}$$

$$\mathrm{MVS}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le Kn\rho_n, \tag{2.2}$$

$$\mathrm{MSC}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le Kn^2\rho_n, \tag{2.3}$$

and except on an event of probability decaying exponentially in $\rho_n^{(1-d)/2} \cdot |\log \rho_n|^{-2}$,

$$\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le Kn^3\rho_n^{d+1}, \tag{2.4}$$

$$\mathrm{MCUT}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le Kn^2\rho_n^{d+1}, \tag{2.5}$$

$$\mathrm{MBIS}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le Kn^2\rho_n^{d+1}. \tag{2.6}$$

At the start of Section 3, we shall explain informally why these bounds arise naturally from the lexicographic ordering.

The subcritical case $\lambda < \lambda_c$, and also the case $\tilde{\theta}(\lambda) < \frac{1}{2}$ for MBIS, is considered in [9], a preliminary version of which is in [10]. In this case, the upper bounds given by Theorem 2.1 grow at a different rate from the actual optimal cost; for example, $\mathrm{MBIS}(\mathscr{G}(\mathscr{X}_n; \rho_n)) = 0$ with high probability if $\tilde{\theta}(\lambda) < \frac{1}{2}$. In the supercritical case $\lambda > \lambda_c$, however, they are more relevant. We now give our main result, which establishes *lower* bounds on the cost functionals, of the same order of magnitude as the upper bounds in Theorem 2.1, valid for $\lambda > \lambda_c$ (or for $\tilde{\theta}(\lambda) > \frac{1}{2}$ in the case of MBIS).

THEOREM 2.2. *Suppose* $\lim_{n\to\infty} n\rho_n^d = \lambda \in (\lambda_c, \infty]$. *Then there exists a constant* $\eta > 0$ *such that, except on an event of probability decaying exponentially in* ρ_n^{1-d},

$$\mathrm{MBW}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \geq \eta n \rho_n, \tag{2.7}$$

$$\mathrm{MVS}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \geq \eta n \rho_n, \tag{2.8}$$

$$\mathrm{MSC}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \geq \eta n^2 \rho_n, \tag{2.9}$$

$$\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \geq \eta n^3 \rho_n^{d+1}, \tag{2.10}$$

$$\mathrm{MCUT}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \geq \eta n^2 \rho_n^{d+1}. \tag{2.11}$$

If also $\tilde{\theta}(\lambda) > \frac{1}{2}$ *or* $\lambda = \infty$, *then there exists a constant* $\eta > 0$ *such that, except on an event of probability decaying exponentially in* $n^{(d-1)/d}$,

$$\mathrm{MBIS}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \geq \eta n^2 \rho_n^{d+1}. \tag{2.12}$$

Theorem 2.2 shows that in the supercritical case, the projection algorithm is a *constant approximation algorithm*, in the sense that with high probability, its cost stays within a constant factor of being optimal. This property is considered important by computer scientists, even when the constant factor is possibly large. In general, this is the case here; no attempt is made here to give numerical values to the constants η and K, which could be far apart. However, in cases where $d = 2$ and $n\rho_n^d / \log n$ tends to infinity, it is shown (by other means) in Theorem 6.3 of [8] that, for all six problems, one can take any $K > 1$, while for MBW and MBIS one can take any $\eta < 1$, for MSC one can take any $\eta < \frac{5}{6}$, for MCUT and MBIS one can take $\eta = 0.264$, and for MLA one can take $\eta = 0.175$. Thus, when $d = 2$ and $n\rho_n^d$ grows faster than $\log n$, the upper and lower bounds are reasonably close together. The results of this paper show that the behavior of these problems on random geometric graphs is qualitatively the same right down to the critical point.

By analogy with the Beardwood–Halton–Hammersley theorem [1] on the cost for the TSP on random points, analogous results for other such problems [31, 34], and also results on layout problems in the subcritical regime [9, 10], we conjecture that in the supercritical regime, $\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n; \rho_n))/(n^3 \rho_n^{d+1})$ converges in probability to a finite positive limit, and likewise for the other problems. Results of this type appear in [8] for MBW and MVS when $d = 2$ and $n\rho_n^d / \log n \to \infty$, but we have no other results of this type. This contrasts, for example, with the case of the MLA cost of a deterministic regular square lattice where the precise growth rate is known [24]. Techniques of subadditivity, much used in [31, 34], seem not to be applicable here.

Before proving the above results in Section 7, we shall consider layout problems on certain other graphs, first *site percolation* (a discrete analogue of random geometric graphs) and then geometric graphs based on the Poisson process (in which the number of points is randomized). The results on these processes are of some intrinsic interest, and also lead toward our main concern of random geometric graphs.

3. Preliminaries. To start with, we explain informally the various powers of n and ρ_n, arising in the upper bounds of Theorem 2.1. Note first that for the projection ordering, the bandwidth BW and also the vertex separation VS would be expected to behave like the number of points in a slab of width ρ_n, which in turn behaves like $n\rho_n$. The sum-cut SC is the sum of n expressions of this form, and so behaves like $n^2\rho_n$. Both CUT and BIS behave like the number of edges connecting points on the "left" of a given point to points on its "right"; this behaves like the number in a vertical slab (which behaves like $n\rho_n$ as before), multiplied by the typical number of connections from a point in the slab to points in the neighboring slab to its right (which behaves like $n\rho_n^d$), giving overall behavior like $n^2\rho_n^{d+1}$; the linear arrangement cost, using the alternative expression for LA, is given by the sum of n expressions of this form, giving the correct order of magnitude of $n^3\rho_n^{d+1}$ for LA.

One benefit of tackling several layout problems together lies in the following inequalities relating them to one another. For any layout φ on a graph $\mathscr{G}$ with n vertices and maximum degree D,

$$\mathrm{SC}(\varphi) \le \mathrm{LA}(\varphi) \le n\,\mathrm{CUT}(\varphi) \le D \times n\,\mathrm{BW}(\varphi),$$

and hence

$$\mathrm{MSC}(\mathscr{G}) \le \mathrm{MLA}(\mathscr{G}) \le n\mathrm{MCUT}(\mathscr{G}) \le D \times n\mathrm{MBW}(\mathscr{G}). \tag{3.1}$$

Similarly,

$$\mathrm{MSC}(\mathscr{G}) \le n\mathrm{MVS}(\mathscr{G}) \le n\mathrm{MBW}(\mathscr{G}). \tag{3.2}$$

Finally, there is the inequality $\mathrm{MVS}(\mathscr{G}) \le \mathrm{MCUT}(\mathscr{G})$, but we shall not use this.

Except for MBIS, the minimal costs are *monotone* in the sense that if $\mathscr{G}$ is a subgraph of $\mathscr{G}'$, then $\mathrm{MSC}(\mathscr{G}) \le \mathrm{MSC}(\mathscr{G}')$, $\mathrm{MLA}(\mathscr{G}) \le \mathrm{MLA}(\mathscr{G}')$, $\mathrm{MCUT}(\mathscr{G}) \le \mathrm{MCUT}(\mathscr{G}')$, and $\mathrm{MBW}(\mathscr{G}) \le \mathrm{MBW}(\mathscr{G}')$. The cost for MBIS is not monotone, but satisfies

$$\mathrm{MBIS}(\mathscr{G}) \le \mathrm{MCUT}(\mathscr{G}). \tag{3.3}$$

The natural way to find upper bounds on the cost functionals of these problems is to exhibit some particular ordering; the cost of such an ordering is an upper bound for the optimal cost. For *lower* bounds, on the other hand, it is necessary to argue more indirectly. Our main deterministic tool for these is the following result, which gives lower bounds for an arbitrary graph in terms of a measure of its level of connectivity. Recall that a *path* in a graph is a sequence of disjoint vertices with each pair of successive vertices connected by an edge.

LEMMA 3.1. *Suppose $\mathscr{G} = (V, E)$ is a connected graph with n vertices. Suppose k, ν_1, ν_2 are positive integers with $k \le n/2$, such that for any two disjoint subsets A, B of V, with $|A| \ge k$ and $|B| \ge k$, there exists a collection of ν_1 edge-disjoint paths in $\mathscr{G}$, with each path starting in A and ending in B, such that no vertex of $\mathscr{G}$ has more than ν_2 of these paths passing through it. Then*

$$\mathrm{MLA}(\mathscr{G}) \ge (n - 2k)\nu_1, \tag{3.4}$$

$$\mathrm{MSC}(\mathscr{G}) \ge (n - 2k)\nu_1/\nu_2. \tag{3.5}$$

Furthermore, if $\mathscr{G}' = (V', E')$ is a graph with $\mathscr{G}$ as a subgraph, and $n' := |V'|$ satisfies $k + n'/2 + 1 \le n$, then $\mathrm{MBIS}(\mathscr{G}') \ge \nu_1$.

REMARK. An important special case of the above result occurs when $\nu_2 = 1$; in this case, the paths in the condition for the lemma are *vertex*-disjoint.

PROOF. Let φ be an arbitrary ordering on the vertices of $\mathscr{G}$. Let A consist of the first k vertices in the ordering, and let B consist of the last k vertices. Take a collection of ν_1 edge-disjoint paths in $\mathscr{G}$, with each path starting in A and ending in B, such that no vertex of $\mathscr{G}$ has more than ν_2 of these paths passing through it.

Pick a vertex $v \in V\backslash(A \cup B)$. Each of the paths has a first crossing of v, that is, a first edge from a vertex preceding or equaling v in the ordering, to one following v in the ordering. This implies that $\chi(v, \varphi) \ge \nu_1$; summing over all vertices in $V\backslash(A \cup B)$, we obtain (3.4). Moreover, since no vertex is shared by more than ν_2 of the paths, we also have $\Delta(v, \varphi) \ge \nu_1/\nu_2$; summing over all vertices in $V\backslash(A \cup B)$, we obtain (3.5).

Suppose $\mathscr{G}' = (V', E')$ is a graph with $\mathscr{G}$ as a subgraph, and $n' := |V'|$ satisfies $k + n'/2 + 1 \le n$. Each ordering on $\mathscr{G}'$ determines a bisection, i.e., a partition (A_0, A_1) of V' with $|(|A_0| - |A_1|)| \le 1$. For $i = 0, 1$, we have $|A_i| \le (n'/2) + 1$, so that

$$|V \cap A_{1-i}| = |V\backslash A_i| \ge n - \frac{n'}{2} - 1 \ge k.$$

Hence there are at least ν_1 disjoint edges connecting $V \cap A_0$ to $V \cap A_1$, and $\mathrm{MBIS}(\mathscr{G}') \ge \nu_1$. □

Finally in this section, we include one probabilistic preliminary result on exponential decay, which will be used in Section 6. Suppose W_i are independent identically distributed Poisson variables and $\varepsilon > 0$. We shall be interested in the rate of decay of $P[\sum_{i=1}^{n}(W_i^2 - E[W_i^2]) > \varepsilon n]$, which is not amenable to standard methods because the square of a Poisson variable does not have a well-behaved moment generating function. We give a near-optimal exponential decay result encompassing a slightly more general setting of triangular arrays of Poisson variables whose means can vary between rows. The proof is deferred to the Appendix.

LEMMA 3.2. *Suppose* $(\lambda_n)_{n\geq 1}$ *is a sequence of positive real numbers satisfying* $\liminf_{n\to\infty} \lambda_n \in (0,\infty]$. *Suppose that for* $n = 1, 2, 3, \ldots$, *the random variables* $W_{1,n}, W_{2,n}, \ldots, W_{n,n}$ *are independent Poisson variables with mean* λ_n. *Let* $\varepsilon > 0$. *Then* $P[\sum_{i=1}^{n}(W_{i,n}^2 - EW_{i,n}^2) > \varepsilon n \lambda_n^2]$ *decays exponentially in* $n^{1/2}(\log n)^{-2}$.

4. Site percolation. Let d be an integer with $d \geq 2$. Let $\mathscr{L}_m$ be the usual d-dimensional hypercubic lattice of side m, that is, the graph with vertex set $V_m = ([0, m) \cap \mathbf{Z})^d$ and with edges between nearest neighbors. We state our result for site percolation on this graph; there is an analogous result for bond percolation.

Given $p \in (0, 1)$, *site percolation with parameter* p on $\mathscr{L}_m$ is obtained by taking a random subset ("outcome") ω of V_m, with each vertex independently included in ω with probability p. Sometimes we shall refer to elements of ω as "open vertices". The induced subgraph of $\mathscr{L}_m$, that is, the maximal subgraph of $\mathscr{L}_m$ with vertex set ω, will be denoted $\mathscr{S}_m$; we write P_p for probability with respect to this process. By a *cluster* we mean the vertex set of a component of $\mathscr{S}_m$. We write $|C|$ (the *size* of C) for the number of vertices in a cluster C.

A similar site percolation process can be generated analogously on the infinite lattice with vertex set $\mathbf{Z}^d$ and edges between nearest neighbors; let $\theta(p)$ denote the probability that the origin lies in an infinite cluster for this process, and set $p_c = \inf\{p\colon \theta(p) > 0\}$, the *critical value* of p. It is well known [15] that $p_c \in (0, 1)$.

We start with some trivial upper bounds on the optimal costs for our six problems, valid for any p. We shall show below that these are of the correct order of magnitude in the supercritical case $p > p_c$.

PROPOSITION 4.1. *Every possible outcome of* $\mathscr{S}_m$ *satisfies the following upper bounds*:

$$\mathrm{MBW}(\mathscr{S}_m) \leq m^{d-1}, \tag{4.1}$$

$$\mathrm{MVS}(\mathscr{S}_m) \leq m^{d-1}, \tag{4.2}$$

$$\mathrm{MSC}(\mathscr{S}_m) \leq m^{2d-1}, \tag{4.3}$$

$$\mathrm{MLA}(\mathscr{S}_m) \leq 2dm^{2d-1}, \tag{4.4}$$

$$\mathrm{MCUT}(\mathscr{S}_m) \leq 2dm^{d-1}, \tag{4.5}$$

$$\mathrm{MBIS}(\mathscr{S}_m) \leq 2dm^{d-1}. \tag{4.6}$$

PROOF. By monotonicity, to prove (4.1) it suffices to consider the case where every vertex is open so that $\mathscr{S}_m = \mathscr{L}_m$. Let φ be the lexicographic ordering on the vertices of $\mathscr{L}_m$. Then $\mathrm{BW}(\varphi) = m^{d-1}$, and (4.1) follows. Then (4.2) and (4.3) follow by (3.2), and (4.5) and (4.4) follow by (3.1). Finally, (4.6) follows from (4.5) and (3.3). □

We now prove that for $p > p_c$ (or for $\theta(p) > p/2$ in the case of MBIS), for each of these problems there is a lower bound within a constant of the upper bound in Proposition 4.1, that holds with high probability.

THEOREM 4.1. (a) *Let $p > p_c$. Then there exists $\eta > 0$ such that, except on an event of probability decaying exponentially in m^{d-1}, we have*

$$\mathrm{MBW}(\mathscr{S}_m) \geq \eta m^{d-1}, \tag{4.7}$$

$$\mathrm{MVS}(\mathscr{S}_m) \geq \eta m^{d-1}, \tag{4.8}$$

$$\mathrm{MSC}(\mathscr{S}_m) \geq \eta m^{2d-1}, \tag{4.9}$$

$$\mathrm{MLA}(\mathscr{S}_m) \geq \eta m^{2d-1}, \tag{4.10}$$

$$\mathrm{MCUT}(\mathscr{S}_m) \geq \eta m^{d-1}. \tag{4.11}$$

(b) *Let $p > p_c$ with $\theta(p) > p/2$. Then there exists $\eta > 0$ such that $P_p[\mathrm{MBIS}(\mathscr{S}_m) < \eta m^{d-1}]$ decays exponentially in m^{d-1}.*

The key to the proof is the following lemma.

LEMMA 4.1. *Let $p \in (p_c, 1]$ and $\varepsilon \in (0, \theta(p)/5)$. For $\delta > 0$, let $E_{\varepsilon,m,\delta}$ denote the event that for site percolation on $\mathscr{L}_m$,* (i) *there is a unique cluster C of size exceeding $(\theta(p) - \varepsilon)m^d$, and* (ii) *for any pair of disjoint subsets A, B of C with $|A| \geq 2\varepsilon m^d$ and $|B| \geq 2\varepsilon m^d$, there are at least δm^{d-1} vertex-disjoint paths in C from A to B.*

Then there exists $\delta = \delta(p, \varepsilon) > 0$, such that $P_p[E^c_{\varepsilon,m,\delta}]$ decays exponentially in m^{d-1}.

PROOF. Take $\tilde{p} \in (0, p)$ such that $\theta(\tilde{p}) > \theta(p) - \varepsilon$. Such a $\tilde{p}$ exists by continuity of the percolation probability above the critical point; see [15, Section 6.3]. Let $E_{\varepsilon,m,0}$ denote the event that there exists a cluster of size exceeding $(\theta(p) - \varepsilon)m^d$. By [26, Theorem 4], there exists $\gamma > 0$ such that, for large enough m,

$$P_{\tilde{p}}[E^c_{\varepsilon,m,0}] < \exp(-\gamma m^{d-1}).$$

Take $\delta > 0$ such that $\delta \log(p/(p - \tilde{p})) < \gamma$. Let F_m denote the event that (i) there is a unique cluster C of size exceeding $(\theta(p) - \varepsilon)m^d$; (ii) this cluster satisfies $|C| < (\theta(p) + \varepsilon)m^d$; and (iii) there exist disjoint subsets A, B of C, each of size at least $2\varepsilon m^d$ such that there exist at most δm^{d-1} vertex-disjoint paths in C from A to B. We need to show that $P_p[F_m]$ is small.

If F_m occurs, then by Menger's theorem ([4, p. 52]), it is possible by removing at most δm^{d-1} vertices to disconnect A from B; to use Menger's theorem directly, add a vertex connected to each vertex of A, and likewise for B, and consider independent (i.e., vertex-disjoint) paths between the two added vertices. This removal of vertices takes us outside the event $E_{\varepsilon,m,0}$ because of the uniqueness of C, and the fact that after removing these vertices no subcomponent of C has size greater than $(\theta(p)+\varepsilon-2\varepsilon)m^d$.

Therefore, any outcome in F_m can be modified to an outcome in the complement of the (increasing) event $E_{\varepsilon,m,0}$ by removal of at most δm^{d-1} open vertices. It follows by the site percolation version of Theorem 2.45 of [15] that, for large enough m,

$$P_p[F_m] \le \left(\frac{p}{p-\tilde{p}}\right)^{\delta m^{d-1}} P_{\tilde{p}}[E^c_{\varepsilon,m,0}]$$
$$\le \exp[m^{d-1}(\delta \log(\frac{p}{p-\tilde{p}}) - \gamma)],$$

which decays exponentially in m^{d-1} by the choice of δ.

Finally, $P[E^c_{\varepsilon,m,\delta}\backslash F_m]$ also decays exponentially in m^{d-1} by Theorem 4 of [26]. □

PROOF OF THEOREM 4.1. Assume $p > p_c$. Choose $\varepsilon_1 \in (0, \theta(p)/6)$, and $\delta = \delta(p,\varepsilon_1) > 0$ so that $P_p[E^c_{\varepsilon_1,m,\delta}]$ decays exponentially in m^{d-1}. For $\omega \in E_{\varepsilon_1,m,\delta}$, with C denoting the unique cluster of size exceeding $(\theta(p)-\varepsilon_1)m^d$, it follows from monotonicity and (3.5) that

$$\mathrm{MSC}(\mathscr{S}_m) \ge \mathrm{MSC}(C) \ge (\theta(p)-\varepsilon_1-5\varepsilon_1)m^d(\delta m^{d-1}),$$

giving us (4.9). Then (4.10) and (4.11) follow by (3.1), and (4.8) and (4.7) follow by (3.2). This completes the proof of part (a).

For (b), assume additionally that $\theta(p) > p/2$. Take $\varepsilon_2 > 0$ with $4\varepsilon_2 + p/2 < \theta(p)$, and take $\delta > 0$ such that $P[E^c_{\varepsilon_2,m,\delta}]$ decays exponentially in m^{d-1}. If $|\omega|$ denotes the number of open vertices in an outcome ω, then by standard arguments applying Markov's inequality to the moment generating function, $P_p[|\omega| > (p+\varepsilon_2)m^d]$ decays exponentially in m^d. Suppose $|\omega| \le (p+\varepsilon_2)m^d$, with also $\omega \in E_{\varepsilon_2,m,\delta}$, and let C denote the unique cluster of size exceeding $(\theta(p)-\varepsilon_2)m^d$. Then

$$\lceil 2\varepsilon_2 m^d \rceil + \frac{|\omega|}{2} + 1 \le |C|,$$

so by the last part of Lemma 3.1, $\mathrm{MBIS}(\mathscr{S}_m) \ge \delta m^{d-1}$. □

5. Poisson processes with fixed intensity. Let $\lambda > 0$, and recall that $\mathscr{P}_\lambda$ denotes a homogeneous Poisson process on $\mathbf{R}^d$ of intensity λ. Let B_m denote the box $[0,m)^d$. This section is concerned with lower bounds on the costs for layout problems on $\mathscr{G}(\mathscr{P}_\lambda \cap B_m; 1)$, as m tends to infinity running through the integers, with λ fixed satisfying $\lambda > \lambda_c$; later, in Theorem 6.1, we shall give upper bounds of the same order of magnitude as these lower bounds.

THEOREM 5.1. *Let* $\lambda \in (\lambda_c, \infty)$, *and let* $\mathscr{G}_m = \mathscr{G}(\mathscr{P}_\lambda \cap B_m; 1)$. *Then*: (a) *there exists a constant* $\eta > 0$ *such that, except on an event of probability decaying exponentially in* m^{d-1},

$$\mathrm{MBW}(\mathscr{G}_m) \geq \eta m^{d-1}, \tag{5.1}$$

$$\mathrm{MVS}(\mathscr{G}_m) \geq \eta m^{d-1}, \tag{5.2}$$

$$\mathrm{MSC}(\mathscr{G}_m) \geq \eta m^{2d-1}, \tag{5.3}$$

$$\mathrm{MLA}(\mathscr{G}_m) \geq \eta m^{2d-1}, \tag{5.4}$$

$$\mathrm{MCUT}(\mathscr{G}_m) \geq \eta m^{d-1}; \tag{5.5}$$

(b) *if also* $\theta(\lambda) > \frac{1}{2}$, *then there exists a constant* $\eta > 0$ *such that, except on an event of probability decaying exponentially in* m^{d-1},

$$\mathrm{MBIS}(\mathscr{G}_m) \geq \eta m^{d-1}. \tag{5.6}$$

The proof uses a continuum analogue to Lemma 4.1. By a *cluster* in what follows, we mean the vertex set of a component of $\mathscr{G}_m$. For any cluster C, let $|C|$ (the *size* of C) denote the number of vertices it has.

LEMMA 5.1. *Let* $\lambda \in (\lambda_c, \infty)$ *and* $\varepsilon \in (0, \lambda\tilde{\theta}(\lambda)/5)$. *For* $\delta > 0$, *let* $\tilde{E}_{\varepsilon,m,\delta}$ *denote the event that* (i) *there is a unique cluster* C *on* $\mathscr{G}_m$ *of size exceeding* $(\lambda\tilde{\theta}(\lambda) - \varepsilon)m^d$, *and* (ii) *for any pair of disjoint subsets* A, B *of* C *with* $|A| \geq 2\varepsilon m^d$ *and* $|B| \geq 2\varepsilon m^d$, *there are at least* δm^{d-1} *vertex-disjoint paths in* C *from* A *to* B.

Then there exists $\delta = \delta(\lambda, \varepsilon) > 0$, *such that* $P[\tilde{E}^c_{\varepsilon,m,\delta}]$ *decays exponentially in* m^{d-1}.

PROOF. Take $\tilde{\lambda} \in (0, \lambda)$ such that $\tilde{\lambda}\tilde{\theta}(\tilde{\lambda}) > \lambda\tilde{\theta}(\lambda) - \varepsilon$. Such a $\tilde{\lambda}$ exists by continuity of the continuum percolation probability above the critical point; see [23], page 78. Write $P_{\tilde{\lambda}}$, respectively P_λ, for probability with respect to the Poisson process $\mathscr{P}_{\tilde{\lambda}}$, respectively $\mathscr{P}_\lambda$. Let $\tilde{E}_{\varepsilon,m,0}$ denote the event that there exists a cluster of $\mathscr{G}_m$ of size exceeding $(\lambda\tilde{\theta}(\lambda) - \varepsilon)m^d$. By Theorem 1 of [26], there exists $\gamma > 0$ such that, for large enough m,

$$P_{\tilde{\lambda}}[\tilde{E}^c_{\varepsilon,m,0}] < \exp(-\gamma m^{d-1}).$$

Take $\delta > 0$ such that $\delta \log\big(\lambda/(\lambda - \tilde{\lambda})\big) < \gamma$. Let $\tilde{F}_m$ denote the event that (i) there is a unique cluster C of size exceeding $(\lambda\tilde{\theta}(\lambda) - \varepsilon)m^d$; (ii) this cluster satisfies $|C| < (\lambda\tilde{\theta}(\lambda) + \varepsilon)m^d$; and (iii) there exist disjoint subsets A, B of C, each of size at least $2\varepsilon m^d$ such that there exist at most δm^{d-1} vertex-disjoint paths in $\mathscr{G}_m$ from A to B. By the same argument using Menger's theorem as in the proof of Lemma 4.1, any outcome of $\mathscr{P}_\lambda$ in $\tilde{F}_m$ can be modified to an outcome in the complement of the (increasing) event $\tilde{E}_{\varepsilon,m,0}$ by removal of fewer than δm^{d-1} vertices.

We need a continuum percolation version of Theorem 2.45 of [15]. The proof of this is similar to the one in [15], using the fact that if the points of a Poisson process of rate λ are each independently discarded with probability $(\lambda - \tilde{\lambda})/\lambda$ and retained with probability $\tilde{\lambda}/\lambda$, then we obtain a Poisson process of rate $\tilde{\lambda}$. This result gives us (for large enough m)

$$P_\lambda[\tilde{F}_m] \leq \left(\frac{\lambda}{\lambda - \tilde{\lambda}}\right)^{\delta m^{d-1}} P_{\tilde{\lambda}}[\tilde{E}^c_{\varepsilon,m,0}]$$
$$\leq \exp[m^{d-1}(\delta \log(\frac{\lambda}{\lambda - \tilde{\lambda}}) - \gamma)],$$

which decays exponentially in m^{d-1} by the choice of δ.

Finally, $P[\tilde{E}^c_{\varepsilon,m,\delta} \backslash \tilde{F}_m]$ also decays exponentially in m^{d-1} by Theorem 1 of [26]. $\square$

PROOF OF THEOREM 5.1. Assume $\lambda > \lambda_c$. Choose $\varepsilon_3 \in (0, \lambda\tilde{\theta}(\lambda)/6)$, and $\delta = \delta(\lambda, \varepsilon_3) > 0$, so that $P[\tilde{E}^c_{\varepsilon_3,m,\delta}]$ decays exponentially in m^{d-1}. Suppose $\tilde{E}_{\varepsilon_3,m,\delta}$ occurs, and let C be the vertex set of the unique cluster of size exceeding $(\lambda\tilde{\theta}(\lambda) - \varepsilon_3)m^d$. Then by Lemma 3.1,

$$\text{MSC}(\mathscr{G}_m) \geq \text{MSC}(C) \geq ((\lambda\tilde{\theta}(\lambda) - \varepsilon_3) - 5\varepsilon_3)m^d \delta m^{d-1},$$

giving us (5.3). Then (5.4) and (5.5) follow by (3.1), and (5.2) and (5.1) follow by (3.2).

For (b), assume additionally that $\tilde{\theta}(\lambda) > \frac{1}{2}$. Take $\varepsilon_4 > 0$ with $4\varepsilon_4 + \lambda/2 < \lambda\tilde{\theta}(\lambda)$. Take $\delta > 0$ such that $P[\tilde{E}^c_{\varepsilon_4,m,\delta}]$ decays exponentially in m^{d-1}. By standard arguments, $P[|\mathscr{P}_\lambda \cap B_m| > (\lambda + \varepsilon_4)m^d]$ decays exponentially in m^d. Suppose $\tilde{E}_{\varepsilon_4,m,\delta}$ occurs, and also $|\mathscr{P}_\lambda \cap B_m| \leq (\lambda + \varepsilon_4)m^d$. Let C be the vertex set of the unique cluster of size exceeding $(\lambda\tilde{\theta}(\lambda) - \varepsilon_4)m^d$. Then $\lceil 2\varepsilon_4 m^d \rceil + \frac{1}{2}|\mathscr{P}_\lambda \cap B_m| + 1 \leq |C|$, so by Lemma 3.1, $\text{MBIS}(\mathscr{G}_m) \geq \delta m^{d-1}$.

6. High intensity. In this section, we again consider Poisson processes on the box $B_m = [0, m)^d$. We consider the graph $\mathscr{G}(\mathscr{P}_{\lambda_m} \cap B_m; \rho)$, with ρ fixed but λ_m now allowed to vary with m. We shall be mainly concerned with the case $\lambda_m \to \infty$, but our first result is a set of upper bounds, holding with high probability and valid for λ_m constant as well as $\lambda_m \to \infty$.

THEOREM 6.1. *Suppose* $0 < \liminf_{m\to\infty} \lambda_m \leq \infty$, *and let* $\mathscr{G}_m$ *denote the graph* $\mathscr{G}(\mathscr{P}_{\lambda_m} \cap B_m; 1)$. *Then there exists a constant* K *such that*, *except on an event of probability decaying exponentially in* $\lambda_m m^{d-1}$,

$$\text{MBW}(\mathscr{G}_m) \leq K\lambda_m m^{d-1}, \tag{6.1}$$

$$\text{MVS}(\mathscr{G}_m) \leq K\lambda_m m^{d-1}, \tag{6.2}$$

$$\text{MSC}(\mathscr{G}_m) \leq K\lambda_m^2 m^{2d-1}, \tag{6.3}$$

and, *except on an event of probability decaying exponentially in* $m^{(d-1)/2} (\log m)^{-2}$,

$$\text{MLA}(\mathscr{G}_m) \leq K\lambda_m^3 m^{2d-1}, \tag{6.4}$$

$$\text{MCUT}(\mathscr{G}_m) \le K\lambda_m^2 m^{d-1}, \tag{6.5}$$

$$\text{MBIS}(\mathscr{G}_m) \le K\lambda_m^2 m^{d-1}. \tag{6.6}$$

PROOF. Let φ_{LEX} be the lexicographic ordering on the vertices of $\mathscr{G}_m$ with points simply ordered by their first coordinate (the "projection heuristic" or "projection algorithm" [8]). The result is established by showing that suitable upper bounds hold with high probability for the cost of φ_{LEX}, for each of the six problems in question.

Divide B_m into slabs $S_{0,m}, S_{1,m}, \dots, S_{m-1,m}$ defined by $S_{j,m} = [j, j+1) \times [0,m)^{d-1}$. Then for $i < j$, the points in $S_{i,m}$ precede those in $S_{j,m}$ in the ordering φ_{LEX}. Also, points in $S_{i,m}$ and $S_{j,m}$ are not connected by edges of $\mathscr{G}_m$ for $|i-j| \ge 2$.

Let E_m be the event $\cap_{j=0}^{m-1}\{|\mathscr{P}_{\lambda_m} \cap S_{j,m}| \le 2\lambda_m m^{d-1}\}$, that each slab $S_{j,m}$ contains at most $2\lambda_m m^{d-1}$ points of $\mathscr{P}_{\lambda_m}$. Then $P[E_m^c]$ decays exponentially in $\lambda_m m^{d-1}$. Also, when event E_m occurs, the lexicographic ordering satisfies $\text{BW}(\varphi_{\text{LEX}}) \le 4\lambda_m m^{d-1}$, giving us (6.1); then by (3.2) we have also (6.2) and (6.3).

The proof for (6.4), (6.5), and (6.6) is more involved but is still based on the projection heuristic. For $\mathbf{i} = (i_1, \dots, i_m) \in ([0,m) \cap \mathbf{Z})^d$, set $Q_{\mathbf{i}} = \prod_{k=1}^d [i_k, i_k + 2)$. For each edge $\{X, Y\}$ of $\mathscr{G}_m$, there exists $\mathbf{i} \in V_m$ such that $X \in Q_{\mathbf{i}}$ and $Y \in Q_{\mathbf{i}}$. Let $i \in \{0, 1, 2, \dots, m-1\}$, and define the event

$$F_i = \left\{ \sum_{\mathbf{j} \in (\mathbf{Z} \cap [0,m))^{d-1}} |\mathscr{P}_{\lambda_m} \cap Q_{i,\mathbf{j}}|^2 \le m^{d-1} 4^d (2\lambda_m^2 + \lambda_m) \right\}.$$

For $\mathbf{j} \in (\mathbf{Z} \cap [0,m))^{d-1}$, set $W_{\mathbf{j}} = |\mathscr{P}_{\lambda_m} \cap Q_{i,\mathbf{j}}|$. Observe that $W_{\mathbf{j}}$ is independent of $W_{\mathbf{k}}$ for $\|\mathbf{j} - \mathbf{k}\|_\infty \ge 2$. Taking sets U_r^m to be intersections of various integer translates of $2\mathbf{Z}^{d-1}$ with $[0,m)^{d-1}$, we can (and do) partition $(\mathbf{Z} \cap [0,m))^{d-1}$ into 2^{d-1} pieces $U_1^m, \dots, U_{2^{d-1}}^m$ with $\{W_{\mathbf{j}}, \mathbf{j} \in U_r^m\}$ mutually independent for each r, and with $\lfloor m/2 \rfloor^{d-1} \le |U_r^m| \le \lceil m/2 \rceil^{d-1}$ for each r. Since $EW_{\mathbf{j}}^2 \le 4^d(\lambda_m^2 + \lambda_m)$, we have

$$\begin{aligned} P[F_i^c] &\le P\left[\sum_{\mathbf{j} \in (\mathbf{Z} \cap [0,m))^{d-1}} (W_{\mathbf{j}}^2 - EW_{\mathbf{j}}^2) > m^{d-1} 4^d \lambda_m^2 \right] \\ &\le \sum_{r=1}^{2^{d-1}} P\left[\sum_{\mathbf{j} \in U_r^m} (W_{\mathbf{j}}^2 - EW_{\mathbf{j}}^2) > m^{d-1} \lambda_m^2 \right], \end{aligned}$$

so that by Lemma 3.2, $P[F_i^c]$ decays exponentially in $m^{(d-1)/2}(\log m)^{-2}$, and hence so does $P[\cup_{i=0}^{m-1} F_i^c]$.

We claim that

$$\bigcap_{i=0}^{m-1} F_i \subset \{\text{MCUT}(\mathscr{G}_m) \le 2^{2d+1}(2\lambda_m^2 + \lambda_m) m^{d-1}\}. \tag{6.7}$$

To prove this, suppose X, Y, and Z are vertices such that $\{Y, Z\}$ contributes to $\chi(X, \varphi_{\text{LEX}})$, so that $\pi_1(Y) \le \pi_1(X) < \pi_1(Z)$ with π_1 denoting projection onto the first coordinate, and also $\|Y - Z\| \le 1$. Then for some $\mathbf{i} = (i_1, \mathbf{j}) \in (\mathbf{Z} \cap [0, m))^d$, we have $Y \in Q_{\mathbf{i}}$ and $Z \in Q_{\mathbf{i}}$. Furthermore, if i is taken so that X lies in the slab S_i, we must have $i = i_1$ or $i = i_1 - 1$, so that

$$\chi(X, \varphi) \le \sum_{i_1 = i-1}^{i} \sum_{\mathbf{j} \in (\mathbf{Z} \cap [0, m))^{d-1}} |\mathscr{P}_{\lambda_m} \cap Q_{i_1, \mathbf{j}}|^2,$$

and (6.7) follows. This completes the proof of (6.5), and (6.4) follows by (3.1), while (6.6) follows by (3.3). □

We now work toward Theorem 6.2 below, which gives lower bounds of the same form as the upper bounds in Theorem 6.1. For convenience, in Theorem 6.2 we shall take the distance parameter ρ to be $2d$, so that if $\|x\|_\infty \le 2$, then $\|x\| \le \rho$.

LEMMA 6.1. *Suppose λ_m is a sequence with $\lambda_m \to \infty$. Let $\varepsilon \in (0, 1/21)$. Then there exists $\gamma = \gamma(\varepsilon) > 0$ such that, except on an event of probability decaying exponentially in m^{d-1}, the set $\mathscr{P}_{\lambda_m} \cap B_m$ has a subset $\mathscr{R}_m$ with $|\mathscr{R}_m| > (1 - 2\varepsilon)\lambda_m m^d$, such that for any two disjoint sets $A, B \subset \mathscr{R}_m$ with $\min(|A|, |B|) \ge \lambda_m m^d/3$, there exists a collection of at least $\gamma \lambda_m^2 m^{d-1}$ paths in $\mathscr{G}(\mathscr{R}_m; 2d)$ from A to B, such that no point of $\mathscr{R}_m$ has more than λ_m of these paths passing through it.*

PROOF. The proof uses an induced discrete percolation process on the lattice $\mathscr{L}_m$ with vertex set $V_m = ([0, m) \cap \mathbf{Z})^d$, defined as follows. For $\mathbf{i} = (i_1, i_2, \dots, i_d) \in V_m$, let $H_{\mathbf{i}}$ denote the unit volume hypercube $\prod_{r=1}^d [i_r, i_r + 1)$. Let $\mathbf{i}$ be deemed "open" if $\lambda_m(1 - \varepsilon) < |\mathscr{P}_{\lambda_m} \cap H_{\mathbf{i}}| < \lambda_m(1 + \varepsilon)$. The set of open vertices is a realization of site percolation on $\mathscr{L}_m$ with parameter p_m, and $p_m \to 1$ by Chebyshev's inequality. Hence $\theta(p_m) \to 1$ by the continuity of the percolation probability ([15, Theorem 6.35]) or more directly by a Peierls argument.

For $\delta > 0$, let $G_{\varepsilon, m, \delta}$ denote the event that there is a big cluster C of open vertices in V_m, of size at least $(1 - \varepsilon)m^d$, such that for any two disjoint subsets S_1, S_2 of C with $|S_1| \ge \varepsilon m^d$ and $|S_2| \ge \varepsilon m^d$, there are at least δm^{d-1} vertex-disjoint paths in C from S_1 to S_2. By Lemma 4.1, we can (and do) choose $\delta > 0$ such that $P[G^c_{\varepsilon, m, \delta}]$ decays exponentially in m^{d-1}.

Suppose the set of open vertices $\mathbf{i} \in V_m$ induced by $\mathscr{P}_{\lambda_m}$ is an outcome in $G_{\varepsilon, m, \delta}$, and let C be the big cluster as described in the definition of that event. Define the restricted point process

$$\mathscr{R}_m = \mathscr{P}_{\lambda_m} \cap (\cup_{\mathbf{i} \in C} H_{\mathbf{i}}).$$

By definition, $|\mathscr{R}_m| \ge (1 - \varepsilon)^2 \lambda_m m^d > (1 - 2\varepsilon)\lambda_m m^d$.

Let A and B be arbitrary disjoint subsets of $\mathscr{R}_m$ of cardinality at least $\lambda_m m^d/3$. Let elements of A be denoted "red," and let points of B be denoted "green." Let R_m be the set of $\mathbf{i} \in C$ such that $H_{\mathbf{i}}$ contains at least $\varepsilon\lambda_m$ red points, and let G_m be the set of $\mathbf{i} \in C$ such that $H_{\mathbf{i}}$ contains at least $\varepsilon\lambda_m$ green points. We claim that

$$\text{card}(R_m) \geq 3\varepsilon m^d, \qquad \text{card}(G_m) \geq 3\varepsilon m^d. \tag{6.8}$$

Obviously, it suffices to prove the claim for R_m. Suppose it were false. The cardinality of R_m would be less than $3\varepsilon m^d$. Since we are considering only $\mathbf{i} \in C$, which implies $\mathbf{i}$ is open and $H_{\mathbf{i}}$ contains at most $(1+\varepsilon)\lambda_m$ points, the total number of red points in $\cup_{\mathbf{i}\in R_m} H_{\mathbf{i}}$ would be at most $(1+\varepsilon)3\varepsilon\lambda_m m^d$. Also, since $|C| \leq m^d$, the total number of red points in $\cup_{\mathbf{i}\in C\setminus R_m} H_{\mathbf{i}}$ is at most $\varepsilon\lambda_m m^d$. Thus the total number of red points would be at most $((1+\varepsilon)3\varepsilon + \varepsilon)\lambda_m m^d$, and hence less than $7\varepsilon\lambda_m m^d$, which is a contradiction by the conditions on ε and $|A|$. So the claim (6.8) is true.

The sets R_m and G_m need not be disjoint. But by (6.8) we can (and do) take R'_m and G'_m to be disjoint with $R'_m \subset R_m$ and $G'_m \subset G_m$, with

$$\text{card}(R'_m) \geq \varepsilon m^d, \qquad \text{card}(G'_m) \geq \varepsilon m^d. \tag{6.9}$$

Let $\tilde{C}$ be an "expanded" version of the subgraph of $\mathscr{L}_m$ induced by the vertex set C, in which each vertex $\mathbf{i}$ of C is replaced by $\mathscr{P}_{\lambda_m}(H_{\mathbf{i}})$ "offspring," and adjacency amongst offspring is inherited from that between parents. This graph is isomorphic to a subgraph of $\mathscr{G}(\mathscr{R}_m; 2d)$, since the choice of distance parameter $2d$ means that any two Poisson points in adjacent unit hypercubes are connected by an edge of $\mathscr{G}(\mathscr{R}_m; 2d)$. Take such an isomorphism and let "red" and "green" colorings in $\tilde{C}$ be determined by this isomorphism and the colorings on points of $\mathscr{R}_m$. Note that each vertex of C has at least $\lceil\varepsilon\lambda_m\rceil$ "offspring" since $1-\varepsilon > \varepsilon$.

Suppose π is a path in C which starts at a point of R'_m and ends at a point of G'_m. Then it is possible to find at least $\lceil\varepsilon\lambda_m\rceil^2$ edge-disjoint paths of the expanded graph $\tilde{C}$ following the same route as π, which furthermore each start at a red vertex and end at a green one, and such that each vertex of $\tilde{C}$ has at most $\varepsilon\lambda_m$ of these paths passing through it. This can be proved by induction on the length of π.

By definition of the event $G_{\varepsilon,m,\delta}$, we can take $\lceil\delta m^{d-1}\rceil$ vertex-disjoint paths in C from points of R'_m to points of G'_m. Each of these corresponds to at least $\lceil\varepsilon\lambda_m\rceil^2$ edge-disjoint paths in $\tilde{C}$ following the same route, starting at red vertices and ending at green ones. Using the isomorphism between $\tilde{C}$ and a subgraph of $\mathscr{G}(\mathscr{R}_m; 2d)$, this gives us a total of at least $\delta\varepsilon^2\lambda_m^2 m^{d-1}$ edge-disjoint paths in $\mathscr{G}(\mathscr{R}_m; 2d)$, each starting in A and ending in B. Moreover, no vertex in $\tilde{C}$ has more than λ_m of these paths passing through it, and taking $\gamma = \delta\varepsilon^2$ gives us the result. $\square$

THEOREM 6.2. *Suppose $(\lambda_m)_{m\geq 1}$ is a sequence with $\lambda_m \to \infty$, and let $\mathscr{G}_m$ denote the graph $\mathscr{G}(\mathscr{P}_{\lambda_m} \cap B_m; 2d)$. Then there exists $\eta > 0$ such that, except on an event of probability decaying exponentially in m^{d-1},*

$$\text{MBW}(\mathscr{G}_m) \geq \eta\lambda_m m^{d-1}, \tag{6.10}$$

(6.11) $$\mathrm{MVS}(\mathscr{G}_m) \geq \eta \lambda_m m^{d-1},$$

(6.12) $$\mathrm{MSC}(\mathscr{G}_m) \geq \eta \lambda_m^2 m^{2d-1},$$

(6.13) $$\mathrm{MLA}(\mathscr{G}_m) \geq \eta \lambda_m^3 m^{2d-1},$$

(6.14) $$\mathrm{MCUT}(\mathscr{G}_m) \geq \eta \lambda_m^2 m^{d-1},$$

(6.15) $$\mathrm{MBIS}(\mathscr{G}_m) \geq \eta \lambda_m^2 m^{d-1}.$$

PROOF. Choose $\varepsilon_5 \in (0, 1/21)$, and $\gamma = \gamma(\varepsilon_5)$ as in Lemma 6.1. Assume from now on that the outcome of $\mathscr{P}_{\lambda_m}$ is such that $\mathscr{P}_{\lambda_m} \cap B_m$ has a subset $\mathscr{R}_m$ with $|\mathscr{R}_m| > (1-2\varepsilon_5)\lambda_m m^d$, such that for any two disjoint sets $A, B \subset \mathscr{R}_m$ with $\min(|A|, |B|) \geq \lambda_m m^d/3$, there exists a collection of at least $\gamma\lambda_m^2 m^{d-1}$ paths in $\mathscr{G}(\mathscr{R}_m; 2d)$ from A to B, such that no point of $\mathscr{R}_m$ has more than λ_m of these paths passing through it; by Lemma 6.1, the probability that this fails to occur decays exponentially in m^{d-1}.

By Lemma 3.1, in these circumstances,

$$\begin{aligned}\mathrm{MLA}(\mathscr{G}(\mathscr{R}_m; 2)) &\geq \left(1 - 2\varepsilon_5 - \left(\tfrac{2}{3} + \varepsilon_5\right)\right) \lambda_m m^d (\gamma \lambda_m^2 m^{d-1}) \\ &\geq \varepsilon_5 \gamma \lambda_m^3 m^{2d-1}.\end{aligned}$$

This gives us (6.13), and (6.14) follows by (3.1). By Lemma 3.1 again,

$$\mathrm{MSC}(\mathscr{G}(\mathscr{R}_m; 2)) \geq \varepsilon_5 \gamma \lambda_m^3 m^{2d-1}/\lambda_m.$$

This gives us (6.12), and (6.11) and (6.10) follow by (3.2).

For MBIS, use the fact that as long as $|\mathscr{P}_{\lambda_m} \cap B_m| \leq (1+\varepsilon_5) m^d \lambda_m$, by the choice of ε_5 we have

$$\lceil \lambda_m m^d/3 \rceil + \tfrac{1}{2}|\mathscr{P}_{\lambda_m} \cap B_m| + 1 \leq |\mathscr{R}_m|,$$

and therefore by the last part of Lemma 3.1, $\mathrm{MBIS}(\mathscr{G}_m) \geq \gamma\lambda_m^2 m^{d-1}$. □

7. Fixed numbers of points. At last we can prove the results announced in Section 2, concerned with graphs of the form $\mathscr{G}(\mathscr{X}_n; \rho_n)$ with $\rho_n \to 0$ and $n\rho_n^d$ tending to a possibly infinite limit λ. We obtain these from the corresponding results on Poisson processes by coupling the process $\mathscr{X}_n$ to a Poisson process with a slightly higher or lower density of points.

For the case $\lambda < \infty$, the coupling goes as follows. Take $\lambda_1 < \lambda < \lambda_2$. Set $m_n = \lceil \rho_n^{-1} \rceil$ and $m_n' = \lfloor \rho_n^{-1} \rfloor$. Let M_n and M_n' be Poisson variables with mean $\lambda_1 m_n^d$ and $\lambda_2 (m_n')^d$, respectively, independent of $(X_1, X_2, X_3, \ldots)$. Then $P[M_n > n]$ and $P[M_n' < n]$ decay exponentially in n.

Set $m_n \mathscr{X}_n = \{m_n X_i \colon 1 \leq i \leq n\}$, and set

$$\mathscr{P}_n = \{m_n X_i \colon 1 \leq i \leq M_n\}, \qquad \mathscr{P}_n' = \{m_n' X_i \colon 1 \leq i \leq M_n'\},$$

which are Poisson processes, on B_{m_n} with intensity λ_1 and on $B_{m_n'}$ with intensity λ_2, respectively.

If $M_n \le n$, then $\mathscr{G}(\mathscr{P}_n;1)$ is a subgraph of $\mathscr{G}(m_n\mathscr{X}_n;m_n\rho_n)$, which is isomorphic to $\mathscr{G}(\mathscr{X}_n;\rho_n)$. Hence by monotonicity,

(7.1) $P[\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n;\rho_n)) < \mathrm{MLA}(\mathscr{G}(\mathscr{P}_n;1))]$ decays exponentially in n,

and likewise for MBW, MCUT, MSC, and MVS. Similarly,

(7.2) $P[\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n;\rho_n)) > \mathrm{MLA}(\mathscr{G}(\mathscr{P}'_n;1))]$ decays exponentially in n,

and likewise for MBW, MCUT, MSC and MVS.

PROOF OF THEOREM 2.1 WHEN $\lambda < \infty$. Suppose $n\rho_n^d \to \lambda \in (0,\infty)$. Choose $\lambda_2 \in (\lambda,\infty)$ and define $\mathscr{P}'_n$ as above. We use the fact that $m'_n \sim \rho_n^{-1}$. By Theorem 6.1, along with the MBW, MVS, and MSC analogues to (7.2), there exists $K_1 > 0$ such that, except on an event of probability decaying exponentially in $(m'_n)^{d-1}$ (i.e., exponentially in ρ_n^{1-d}),

$$\mathrm{MBW}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \le \mathrm{MBW}(\mathscr{G}(\mathscr{P}'_n;1)) \le K_1\rho_n^{1-d},$$

$$\mathrm{MVS}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \le \mathrm{MVS}(\mathscr{G}(\mathscr{P}'_n;1)) \le K_1\rho_n^{1-d},$$

$$\mathrm{MSC}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \le \mathrm{MSC}(\mathscr{G}(\mathscr{P}'_n;1)) \le K_1\rho_n^{1-2d}.$$

Arguing the same way using the second half of Theorem 6.1, we have, except on an event of probability decaying exponentially in $\rho_n^{(1-d)/2}|\log\rho_n|^{-2}$, that

$$\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \le K_1\rho_n^{1-2d}, \qquad \mathrm{MCUT}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \le K_1\rho_n^{1-d},$$

and so by (3.3),

$$\mathrm{MBIS}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \le K_1\rho_n^{1-d}.$$

These six inequalities can be converted into (2.1)–(2.6), using the assumption that $n\rho_n^d \to \lambda \in (0,\infty)$; for example, $K_1\rho_n^{1-d} \sim (K_1/\lambda)n\rho_n$ so (2.1) follows from the first of the above six inequalities. $\square$

PROOF OF THEOREM 2.2 WHEN $\lambda \in (\lambda_c,\infty)$. Suppose $n\rho_n^d \to \lambda \in (0,\infty)$. Choose $\lambda_1 \in (\lambda_c,\lambda)$ and define $\mathscr{P}_n$ as before in this section. Note that $m_n \sim \rho_n^{-1}$. By Theorem 5.1, along with (7.1), there exists $\eta_1 > 0$ such that, except on an event of probability decaying exponentially in ρ_n^{1-d},

$$\mathrm{MBW}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \ge \mathrm{MBW}(\mathscr{G}(\mathscr{P}_n;1)) \ge \eta_1\rho_n^{1-d},$$

$$\mathrm{MVS}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \ge \mathrm{MVS}(\mathscr{G}(\mathscr{P}_n;1)) \ge \eta_1\rho_n^{1-d},$$

$$\mathrm{MSC}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \ge \mathrm{MSC}(\mathscr{G}(\mathscr{P}_n;1)) \ge \eta_1\rho_n^{1-2d},$$

$$\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \ge \mathrm{MLA}(\mathscr{G}(\mathscr{P}_n;1)) \ge \eta_1\rho_n^{1-2d},$$

$$\mathrm{MCUT}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \ge \mathrm{MCUT}(\mathscr{G}(\mathscr{P}_n;1)) \ge \eta_1\rho_n^{1-d}.$$

Then (2.8)–(2.10) follow using the assumption that $n\rho_n^d \to \lambda \in (0,\infty)$.

Now assume also that $\tilde{\theta}(\lambda) > \frac{1}{2}$, and consider the bisection problem. Using the continuity of the continuum percolation probability above the critical point, take λ_1 in the above coupling, and $\varepsilon_6 \in (0, \lambda_1\tilde{\theta}(\lambda_1)/5)$, such that $\lambda_1\tilde{\theta}(\lambda_1)-3\varepsilon_6 > \lambda/2$. Let M_n and $\mathscr{P}_n$ be as above. By Lemma 5.1, there exists $\delta > 0$ such that, except on an event of probability decaying exponentially in ρ_n^{1-d}, the graph $\mathscr{G}(\mathscr{P}_n; 1)$ includes a cluster C of size at least $(\lambda_1\tilde{\theta}(\lambda_1) - \varepsilon_6)m^d$, such that for any two subsets of C of size at least $2\varepsilon_6 m_n^d$, there are at least δm_n^{d-1} edge-disjoint paths connecting them.

Since $n \sim \lambda m_n^d$, for large n we have $\lceil 2\varepsilon_6 m_n^d \rceil + \frac{n}{2} + 1 \le (\lambda_1\tilde{\theta}(\lambda_1) - \varepsilon_6)m_n^d$, so by the last part of Lemma 3.1, $\mathrm{MBIS}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \ge \delta m_n^{d-1}$, giving us (2.12). $\square$

PROOF OF THEOREM 2.1 WHEN $\lambda = \infty$. In the case $n\rho_n^d \to \infty$ (and $\rho_n \to 0$), we use a slightly different coupling which goes as follows. With $m_n' = \lfloor \rho_n^{-1} \rfloor$ as before, let N_n' be Poisson with mean $2n$, independent of $(X_1, X_2, \ldots)$. Define the point process

$$\mathscr{Q}_n' = \{m_n' X_i \colon 1 \le i \le N_n'\},$$

which is a Poisson process of rate $2n(m_n')^{-d}$ on $B_{m_n'}$. Since $P[N_n' < n]$ decays exponentially in n, a similar argument to the proof of (7.2) gives us

$$P[\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n; \rho_n)) > \mathrm{MLA}(\mathscr{G}(\mathscr{Q}_n'; 1))] \quad \text{decays exponentially in } n, \tag{7.3}$$

and likewise for MBW, MCUT, MSC and MVS.

We use the fact that $m_n' \sim \rho_n^{-1}$. By Theorem 6.1, along with (7.3) and analogues for the other monotone problems, there exists a constant K such that, except on an event of probability decaying exponentially in $(n\rho_n^d)\rho_n^{1-d}$,

$$\mathrm{MBW}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le \mathrm{MBW}(\mathscr{G}(\mathscr{Q}_n'; 1)) \le K(n\rho_n^d)\rho_n^{1-d},$$

$$\mathrm{MVS}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le \mathrm{MVS}(\mathscr{G}(\mathscr{Q}_n'; 1)) \le K(n\rho_n^d)\rho_n^{1-d},$$

$$\mathrm{MSC}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le \mathrm{MSC}(\mathscr{G}(\mathscr{Q}_n'; 1)) \le K(n\rho_n^d)^2\rho_n^{1-2d},$$

and except on an event of probability decaying exponentially in $\rho_n^{(1-d)/2} |\log \rho_n|^{-2}$,

$$\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le \mathrm{MLA}(\mathscr{G}(\mathscr{Q}_n'; 1)) \le K(n\rho_n^d)^3\rho_n^{1-2d},$$

$$\mathrm{MCUT}(\mathscr{G}(\mathscr{X}_n; \rho_n)) \le \mathrm{MCUT}(\mathscr{G}(\mathscr{Q}_n'; 1)) \le K(n\rho_n^d)^2\rho_n^{1-d}.$$

This gives us (2.1)–(2.5), and (2.6) follows from (2.5) using (3.3). $\square$

PROOF OF THEOREM 2.2 WHEN $\lambda = \infty$. Assume $n\rho_n^d \to \infty$ and $\rho_n \to 0$. Changing an earlier definition slightly, let $m_n = \lceil 2d\rho_n^{-1} \rceil$. Let $\varepsilon_7 = 1/22$, let N_n be Poisson with mean $n(1-\varepsilon_7)$, independent of $(X_1, X_2, X_3, \ldots)$, and let $\mathscr{P}_n = \{m_n X_i \colon 1 \le i \le N_n\}$. Then, except on an event with probability decaying exponentially in n, we have $n(1-\varepsilon_7) \le N_n \le n$ and $\mathscr{G}(\mathscr{P}_n; 2d)$ is a subgraph of $\mathscr{G}(m_n\mathscr{X}_n; m_n\rho_n)$, which is isomorphic to $\mathscr{G}(\mathscr{X}_n; \rho_n)$. Also, $\mathscr{P}_n$ is a Poisson process on B_{m_n} of rate $(1-\varepsilon_7)nm_n^{-d}$, which is asymptotic to $(2d)^{-d}(1-\varepsilon_7)n\rho_n^d$.

By Theorem 6.2, there is a constant $\eta_2 > 0$ such that, except on an event with probability decaying exponentially in ρ_n^{1-d},

$$\begin{aligned}
\mathrm{MBW}(\mathscr{G}(\mathscr{X}_n;\rho_n)) &\geq \mathrm{MBW}(\mathscr{G}(\mathscr{P}_n;2d)) \geq \eta_2(n\rho_n^d)\rho_n^{1-d},\\
\mathrm{MVS}(\mathscr{G}(\mathscr{X}_n;\rho_n)) &\geq \mathrm{MVS}(\mathscr{G}(\mathscr{P}_n;2d)) \geq \eta_2(n\rho_n^d)\rho_n^{1-d},\\
\mathrm{MSC}(\mathscr{G}(\mathscr{X}_n;\rho_n)) &\geq \mathrm{MSC}(\mathscr{G}(\mathscr{P}_n;2d)) \geq \eta_2(n\rho_n^d)^2\rho_n^{1-2d},\\
\mathrm{MLA}(\mathscr{G}(\mathscr{X}_n;\rho_n)) &\geq \mathrm{MLA}(\mathscr{G}(\mathscr{P}_n;2d)) \geq \eta_2(n\rho_n^d)^3\rho_n^{1-2d},\\
\mathrm{MCUT}(\mathscr{G}(\mathscr{X}_n;\rho_n)) &\geq \mathrm{MCUT}(\mathscr{G}(\mathscr{P}_n;2d)) \geq \eta_2(n\rho_n^d)^2\rho_n^{1-d}.
\end{aligned}$$

By Lemma 6.1, there exists $\gamma > 0$ such that, except on an event with probability decaying exponentially in ρ_n^{1-d}, there exists $\mathscr{R}_n \subset \mathscr{P}_n$ with $|\mathscr{R}_n| \geq (1-3\varepsilon_7)n$, such that if A and B are disjoint subsets of $\mathscr{R}_n$, each of cardinality at least $\lceil n(1-\varepsilon/2)/3\rceil$, then there exist at least $\gamma(n\rho_n^d)^2\rho_n^{1-d}$ edge-disjoint paths in $\mathscr{G}(\mathscr{R}_n;2d)$ from A to B. By the choice of ε_7 we have

$$\lceil n(1-\varepsilon_7)/3\rceil + (n/2) + 1 \leq \frac{5n}{6} \leq |\mathscr{R}_n|,$$

so that by the last part of Lemma 3.1, $\mathrm{MBIS}(\mathscr{G}(\mathscr{X}_n;\rho_n)) \geq \gamma n^2\rho_n^{d+1}$. □

APPENDIX

Proof of Lemma 3.2. Suppose $(\lambda_n)_{n\geq 1}$ satisfies $\liminf_{n\to\infty}\lambda_n \in (0,\infty]$, and $W_{1,n}, W_{2,n}, \dots, W_{n,n}$ are independent Poisson variables with mean λ_n. We are to prove that $P[\sum_{i=1}^n (W_{i,n}^2 - EW_{i,n}^2) > \varepsilon n\lambda_n^2]$ decays exponentially in $n^{1/2}(\log n)^{-2}$.

We shall use Azuma's inequality (see [31], [34], or [33]), which says that if $(M_0, M_1, \dots, M_n)$ is a discrete-time martingale with M_0 a constant, and if $c_1, \dots, c_n$ are constants with $|M_j - M_{j-1}| \leq c_j$ almost surely for each j, then for all $t \geq 0$,

$$P[|M_n - M_0| \geq t] \leq 2\exp\left(-\frac{1}{2}t^2 / \sum_{j=1}^n c_j^2\right).$$

Choose $c > 0$ so that $\liminf(c\lambda_n) > 1$. Then for large enough n,

$$\text{(A.1)}\quad P[W_{1,n} \geq c\lambda_n \log n] \leq \frac{E[\exp(W_{1,n})]}{\exp(c\lambda_n\log n)} = \exp\{\lambda_n(e-1-c\log n)\} \leq n^{-1}.$$

Define a sequence of integers $(\xi_n)_{n\geq 2}$ by

$$\text{(A.2)}\qquad P[W_{1,n} \geq \xi_n] > n^{-1} \geq P[W_{1,n} \geq \xi_n + 1], \qquad n \geq 2.$$

Then by (A.1), $\xi_n \leq c\lambda_n \log n$ for large enough n. Hence,

$$P[W_{1,n} = \xi_n] = P[W_{1,n} = \xi_n + 1](\xi_n + 1)/\lambda_n \leq (2c\log n)/n.$$

By Azuma's inequality applied to the martingale with successive increments given by the independent random variables $W_{i,n}^2 \mathbf{1}_{\{W_{i,n} < \xi_n\}} - EW_{i,n}^2 \mathbf{1}_{\{W_{i,n} < \xi_n\}}$, which are uniformly bounded by ξ_n^2, we obtain for large enough n that

$$\begin{aligned} P\left[\sum_{i=1}^{n} (W_{i,n}^2 \mathbf{1}_{\{W_{i,n} < \xi_n\}} - EW_{i,n}^2) \geq \varepsilon n \lambda_n^2 \right] &\leq 2\exp\left\{-\frac{(\varepsilon n \lambda_n^2)^2}{2n\xi_n^4}\right\} \\ &\leq 2\exp\left\{-\frac{\varepsilon^2 n}{2(c\log n)^4}\right\}. \end{aligned} \tag{A.3}$$

Next, observe that by Markov's inequality applied to the moment generating function of a binomial random variable,

$$\begin{aligned} &P\left[\sum_{i=1}^{n} \mathbf{1}_{\{W_{i,n} \geq \xi_n\}} > n^{1/2}(\log n)^{-2}\right] \\ &\quad \leq \exp(-n^{1/2}(\log n)^{-2})(1 + (e-1)P[W_{1,n} \geq \xi_n])^n \\ &\quad \leq \exp\{-n^{1/2}(\log n)^{-2} + (e-1)(2c\log n + 1)\}, \end{aligned} \tag{A.4}$$

which decays exponentially in $n^{1/2}(\log n)^{-2}$.

For each n, let $(Z_{i,n}, i \geq 1)$ be independent variables with $\mathscr{L}(Z_{i,n}) = \mathscr{L}(W_{i,n} | W_{i,n} \geq \xi_n)$. Then by (A.2),

$$E[\exp(Z_{i,n})] \leq \frac{E[\exp(W_{i,n})]}{P[W_{i,n} \geq \xi_n]} \leq \exp(\lambda_n(e-1) + \log n),$$

so that

$$\begin{aligned} &P\left[\sum_{i=1}^{n} W_{i,n}^2 \mathbf{1}_{\{W_{i,n} \geq \xi_n\}} > \varepsilon n \lambda_n^2 \,\middle|\, \sum_{i=1}^{n} \mathbf{1}_{\{W_{i,n} \geq \xi_n\}} \leq n^{1/2}(\log n)^{-2}\right] \\ &\quad \leq P\left[\sum_{i=1}^{n^{1/2}(\log n)^{-2}} Z_{i,n}^2 > \varepsilon n \lambda_n^2\right] \\ &\quad \leq P\left[\sum_{i=1}^{n^{1/2}(\log n)^{-2}} Z_{i,n} > \varepsilon^{1/2} n^{1/2} \lambda_n\right] \\ &\quad \leq \exp\{n^{1/2}(\log n)^{-2}(\lambda_n(e-1) + \log n) - \varepsilon^{1/2}\lambda_n n^{1/2}\}, \end{aligned} \tag{A.5}$$

which decays exponentially in $\lambda_n n^{1/2}$. Combining (A.3), (A.4) and (A.5), we obtain the desired rate of exponential decay. □

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