# Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard

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**Abstract**

Can the vertices of an arbitrary graph $G$ be partitioned into $A \cup B$, so that $G[A]$ is a line-graph and $G[B]$ is a forest? Can $G$ be partitioned into a planar graph and a perfect graph? The NP-completeness of these problems are special cases of our result: if $\mathcal{P}$ and $\mathcal{Q}$ are additive induced-hereditary graph properties, then $(\mathcal{P}, \mathcal{Q})$-colouring is NP-hard, with the sole exception of graph 2-colouring (the case where both $\mathcal{P}$ and $\mathcal{Q}$ are the set $\mathcal{O}$ of finite edgeless graphs). Moreover, $(\mathcal{P}, \mathcal{Q})$-colouring is NP-complete iff $\mathcal{P}$- and $\mathcal{Q}$-recognition are both in NP. This completes the proof of a conjecture of Kratochvíl and Schiermeyer, various authors having already settled many sub-cases.

Kratochvíl and Schiermeyer conjectured in [19] that for any additive hereditary graph properties $\mathcal{P}$ and $\mathcal{Q}$, recognising graphs in $\mathcal{P} \circ \mathcal{Q}$ is NP-hard, with the obvious exception of bipartite graphs (the case where both $\mathcal{P}$ and $\mathcal{Q}$ are the set $\mathcal{O}$ of finite edgeless graphs). They settled the case where $\mathcal{Q} = \mathcal{O}$, and it was natural to extend the conjecture to *induced*-hereditary properties. Berger's result [3] that reducible additive induced-hereditary properties have infinitely many minimal forbidden subgraphs provided support for the extended conjecture.

We prove the extension of the Kratochvíl-Schiermeyer conjecture in this paper. Problems such as the following (for an arbitrary graph $G$) are therefore NP-complete. Can $V(G)$ be partitioned into $A \cup B$, so that $G[A]$ is a line-graph and $G[B]$ is a forest? Can $G$ be partitioned into a planar graph and a perfect graph? For fixed $k, \ell, m$, can $G$ be partitioned into a $k$-degenerate subgraph, a subgraph of maximum degree $\ell$, and an $m$-edge-colourable subgraph?

Garey et al. [15, 22] essentially showed $(\mathcal{O}, \{\text{forests}\})$-colouring to be NP-complete, while Brandstädt et al. [4, Thm. 3] proved the case $(\mathcal{O}, \{P_4, C_4\} - \text{free graphs})$.

Let $\mathcal{P}$ be a property and let $\mathcal{P}^k$ be the product of $\mathcal{P}$ with itself, $k$ times. Brown and Corneil [6, 8] showed that $\mathcal{P}^k$-recognition is NP-hard when $\mathcal{P}$ is the set of perfect graphs and $k \geq 2$, while Hakimi and Schmeichel [17] did the case $\{\text{forests}\}^2$. There was particular interest in $G$-free $k$-colouring (where $\mathcal{P}$ has just one forbidden induced-subgraph $G$). When $G = K_2$ we get graph colouring, one of the best known NP-complete problems, while subchromatic number [2, 13] (partitioning into subgraphs whose components are all cliques) is the case $G = P_3$. Brown [7] proved the case where $G$ is 2-connected, and Achlioptas [1] showed NP-completeness for all $G$. In fact, Achlioptas' proof settles the case $\mathcal{R}^k$ for any irreducible additive induced-hereditary $\mathcal{R}$.

# 1 Preliminaries

We consider only simple finite graphs, referring to [14] and [25] for general definitions in complexity and graph theory. We write $G \leq H$ when $G$ is an induced subgraph of $H$. We identify a graph property with the set of graphs that have that property. A property $\mathcal{P}$ is *additive*, or *(induced-)hereditary*, if it is closed under taking vertex-disjoint unions, or (induced-)subgraphs. The properties we consider contain the null graph $K_0$ and at least one, but not all (finite simple non-null) graphs.

A $(\mathcal{P}, \mathcal{Q})$*-colouring* of $G$ is a partition of $V(G)$ into red and blue vertices, such that the red vertices induce a subgraph $G_{\mathcal{P}} \in \mathcal{P}$, and the blue vertices induce a subgraph $G_{\mathcal{Q}} \in \mathcal{Q}$. The *product* of $\mathcal{P}$ and $\mathcal{Q}$ is $\mathcal{P} \circ \mathcal{Q}$, the set of $(\mathcal{P}, \mathcal{Q})$-colourable graphs. We use $(\mathcal{P}, \mathcal{Q})$-colouring, $(\mathcal{P}, \mathcal{Q})$-partition and $(\mathcal{P} \circ \mathcal{Q})$-recognition interchangeably.

Let $\mathcal{P}$ be an additive induced-hereditary property. Then $\mathcal{P}$ is *reducible* if it is the product of two additive induced-hereditary properties; otherwise it is *irreducible*. It is true, though by no means obvious, that if $\mathcal{P}$ is the product of *any* two properties, then it is also the product of two additive induced-hereditary properties [11].

Now let $\mathcal{P}$ be any induced-hereditary property. The set of minimal forbidden induced-subgraphs for $\mathcal{P}$ is $\mathcal{F}(\mathcal{P}) := \{H \notin \mathcal{P} \mid \forall G < H,\ G \in \mathcal{P}\}$. Note that $\mathcal{F}(\mathcal{O}) = \{K_2\}$, while all other induced-hereditary properties have forbidden subgraphs with at least 3 vertices. $\mathcal{P}$ is also additive iff every graph in $\mathcal{F}(\mathcal{P})$ is connected. Every hereditary property is induced-hereditary, and the product of additive (induced-hereditary) properties is additive (induced-hereditary).

A graph $H$ is *strongly*[1] *uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$*-partitionable* if there is exactly one *ordered* partition $(V_1, \ldots, V_n)$ of $V(H)$ such that for all $i$, $H[V_i] \in \mathcal{P}_i$. More precisely, suppose $V(H) = U_1 \cup \cdots \cup U_n$, where $H[U_i] \in \mathcal{P}_i$ for all $i$. Then there is a permutation $\phi$ of $\{1, \ldots, n\}$ such that, for every $i$:
(a) $V_i = U_{\phi(i)}$;
(b) $P_i = P_{\phi(i)}$.

When the $\mathcal{P}_i$'s are additive induced-hereditary and irreducible, Mihók [21] gave a construction that can easily be adapted (cf. [10, Thm. 5.3], [11], [5]) to give a strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph $H$ with $V_n \neq \emptyset$. We use $H$ to show that $\mathcal{A} \circ \mathcal{B}$-

[1]Without condition (b), $H$ would just be *uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$*-partitionable.*

recognition is at least as hard as $\mathcal{A}$-recognition, when $\mathcal{A}$ and $\mathcal{B}$ are additive induced-hereditary properties (the result is not true for all properties, e.g., $\mathcal{B} := \{G \mid |V(G)| \geq 10\}$).

**1. Theorem.** *Let $\mathcal{A}$ and $\mathcal{B}$ be additive induced-hereditary properties. Then there is a polynomial-time transformation from the $\mathcal{A}$-recognition problem to the $(\mathcal{A}\circ\mathcal{B})$-recognition problem.*

**Proof:** It is clearly enough to prove this when $\mathcal{B}$ is irreducible. For any graph $G$ we will construct (in time linear in $|V(G)|$) a graph $G'$ such that $G \in \mathcal{A}$ if and only if $G' \in \mathcal{A} \circ \mathcal{B}$.

Let $\mathcal{A} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{n-1}$, $\mathcal{B} = \mathcal{P}_n$, where the $\mathcal{P}_i$'s are irreducible additive induced-hereditary properties. Let $H$ be a fixed strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph, with partition $(V_1, \ldots, V_n)$, such that $V_n \neq \emptyset$. Let $v_H$ be some fixed vertex that is not in $V_n$, say $v_H \in V_1$.

For any graph $G$, we construct $G'$ by taking a copy of $G$ and a copy of $H$, and making every vertex of $G$ adjacent to every vertex of $N(v_H) \cap V_n$. By additivity of $\mathcal{A}$, if $G$ is in $\mathcal{A}$, then $G'$ is in $\mathcal{A} \circ \mathcal{B}$.

Conversely, if $G' \in \mathcal{A} \circ \mathcal{B} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, then it has an ordered partition $(W_1, \ldots, W_n)$ with $W_i \in \mathcal{P}_i$ for each $i$. Since the $\mathcal{P}_i$'s are induced-hereditary, $G'[W_i] \in \mathcal{P}_i$ implies $G'[W_i \cap V(H)] \in \mathcal{P}_i$. Then[2] $(W_1 \cap V(H), \ldots, W_n \cap V(H)) = (V_1, \ldots, V_n)$; in particular, $v_H \in W_1$.

Suppose some $z \in V(G)$ is in $W_n$. Now $(V_1 \setminus \{v_H\}, V_2, \ldots, V_{n-1}, V_n \cup \{z\})$ is a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $(H - v_H) + z \cong H$. Then $(V_1 \setminus \{v_H\}, V_2, \ldots, V_{n-1}, V_n \cup \{v_H\})$ is a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $H$ that is different from $(V_1, \ldots, V_n)$ (since $V_n \neq \emptyset$), a contradiction.

Thus no vertex of $G$ is in $W_n$, and so $G \leq G'[W_1 \cup \cdots \cup W_{n-1}] \in \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{n-1} = \mathcal{A}$, and $G \in \mathcal{A}$ as required. □

# 2 NP-hardness

We will prove the main result by transforming a version of $p$-IN-$r$-SAT to $(\mathcal{P}, \mathcal{Q})$-colouring, where $p$ and $r$ are fixed integers depending on $\mathcal{P}$ and $\mathcal{Q}$. We recall that $p$-IN-$r$-SAT is the problem of determining whether an arbitrary formula with clauses of size $r$ has a valid truth assignment that sets exactly $p$ literals to TRUE in each clause? Schaefer [24] showed this to be NP-complete even for formulae with all literals unnegated, for any fixed $p$ and $r$, so long as $1 \leq p < r$ and $r \geq 3$. We restate this version as:

$p$-IN-$r$-COLOURING
**Instance**: an $r$-uniform hypergraph.
**Problem**: is there a set of vertices $U$ such that, for each hyper-edge $e$, $|U \cap e| = p$?

[2] Up to some permutation of the subscripts as in (a), (b).

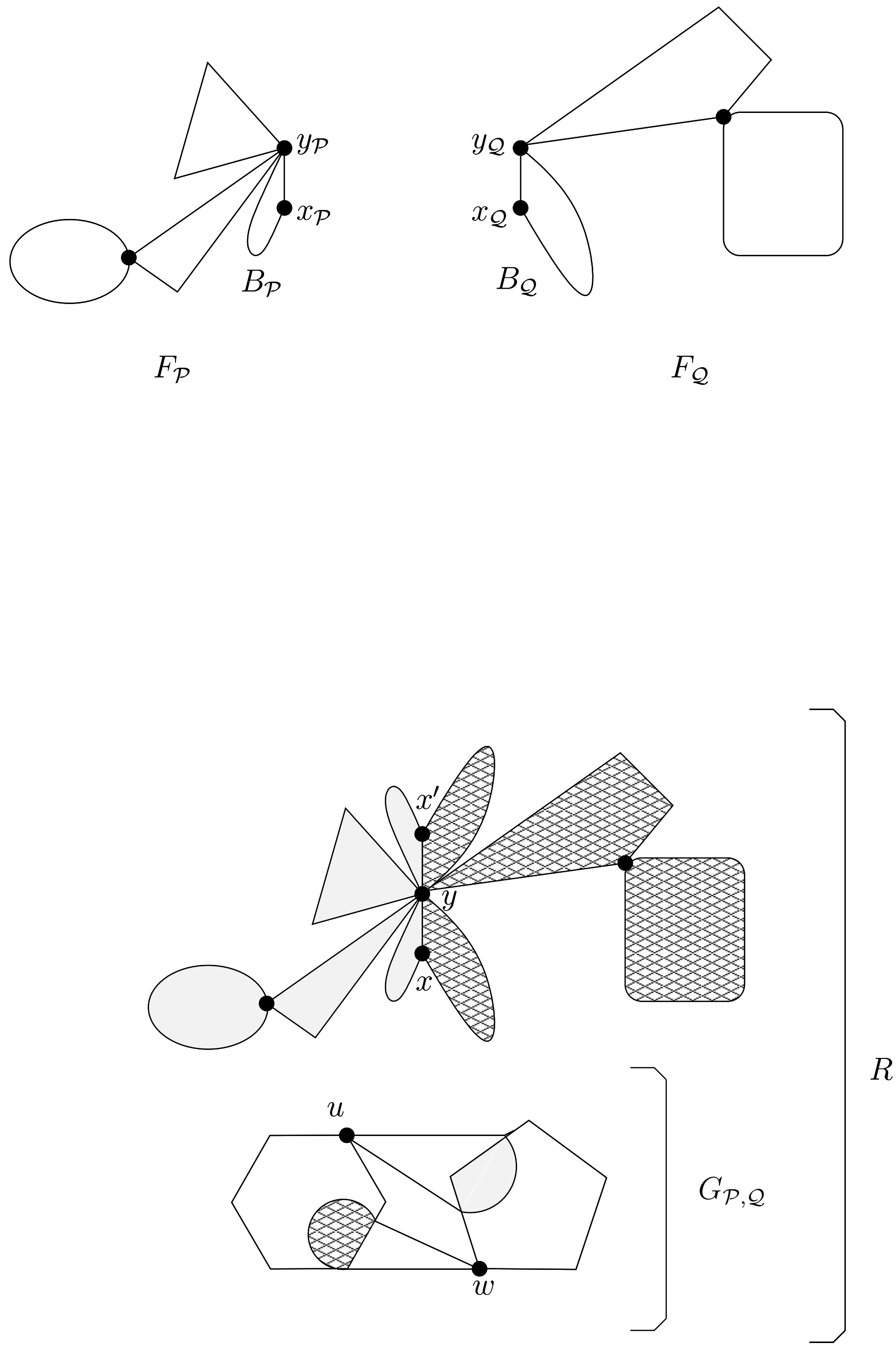


Figure 1: The forbidden graphs $F_{\mathcal{P}}$ and $F_{\mathcal{Q}}$, and the replicator gadget $R$.
The shaded neighbours of $u$ in $G_{\mathcal{P},\mathcal{Q}}$ are connected to the other shaded vertices in $R$.
The hatched neighbours of $w$ in $G_{\mathcal{P},\mathcal{Q}}$ are connected to the other hatched vertices in $R$.

**2. Theorem.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be additive induced-hereditary properties, $\mathcal{P} \circ \mathcal{Q} \neq \mathcal{O}^2$. Then $(\mathcal{P} \circ \mathcal{Q})$-recognition is NP-hard. Moreover, it is NP-complete iff $\mathcal{P}$- and $\mathcal{Q}$-recognition are both in NP.*

**Proof:** We will prove the first part. For the second part, one direction is easy, while the other follows from Theorem 1. Also by Theorem 1 (and by the well-known NP-hardness of recognising $\mathcal{O}^3$ [18]), we need only consider the case where $\mathcal{P}$ and $\mathcal{Q}$ are irreducible. By Theorem 1 there is a strongly uniquely $(\mathcal{P}, \mathcal{Q})$-colourable graph $G_{\mathcal{P},\mathcal{Q}}$ that we use to "force" vertices to be in $\mathcal{P}$ or $\mathcal{Q}$.

More formally, let the unique partition be $V(G_{\mathcal{P},\mathcal{Q}}) = U_{\mathcal{P}} \cup U_{\mathcal{Q}}$. Choose $u \in U_{\mathcal{P}}$. If $G_{\mathcal{P},\mathcal{Q}} \leq H$, and $v \notin V(G_{\mathcal{P},\mathcal{Q}})$ satisfies $N(v) \cap U_{\mathcal{Q}} = N(u) \cap U_{\mathcal{Q}}$, then in any $(\mathcal{P}, \mathcal{Q})$-colouring of $H$, $v$ must be in the $\mathcal{P}$-part[3]; otherwise, in $G_{\mathcal{P},\mathcal{Q}}$ we could transfer $u$ over to the $\mathcal{Q}$ part, giving us a different $(\mathcal{P}, \mathcal{Q})$-colouring. Similarly we choose $w \in U_{\mathcal{Q}}$, whose neighbours we use to force vertices to be in $\mathcal{Q}$. $G_{\mathcal{P},\mathcal{Q}}$ is our first gadget.

An *end-block* of a graph $G$ is a block of $G$ that contains at most one cut-vertex of $G$; in particular, if $G$ has no cut-vertices, then $G$ is itself an end-block. Let $B_{\mathcal{P}}$ be an end-block of $F_{\mathcal{P}} \in \mathcal{F}(\mathcal{P})$, chosen to have the least number of vertices among all the end-blocks of all the graphs in $\mathcal{F}(\mathcal{P})$ (see Figure 1). Because $\mathcal{P}$ is additive and non-trivial, $F_{\mathcal{P}}$ is connected and has at least two vertices, so $B_{\mathcal{P}}$ has $k \geq 2$ vertices. The point to note is that, if $H$ is a graph in $\mathcal{P}$, then adding an end-block with fewer than $k$ vertices produces another graph in $\mathcal{P}$.

Let $y_{\mathcal{P}}$ be the unique cut-vertex contained in $B_{\mathcal{P}}$ (if $B_{\mathcal{P}} = F_{\mathcal{P}}$, pick $y_{\mathcal{P}}$ arbitrarily), and let $x_{\mathcal{P}}$ be a vertex of $B_{\mathcal{P}}$ adjacent to $y_{\mathcal{P}}$. Let $F'_{\mathcal{P}}$ be the graph obtained by adding an extra copy of $B_{\mathcal{P}}$ (incident to the same cut-vertex $y_{\mathcal{P}}$), and let $x'_{\mathcal{P}}$ be a vertex in this new copy that is adjacent to $y_{\mathcal{P}}$.

Similarly, we choose $B_{\mathcal{Q}}$ to be an end-block of $F_{\mathcal{Q}} \in \mathcal{F}(\mathcal{Q})$, minimal among the end-blocks of graphs in $\mathcal{F}(\mathcal{Q})$; we add a copy of $B_{\mathcal{Q}}$, and pick $x_{\mathcal{Q}}$, $y_{\mathcal{Q}}$ and $x'_{\mathcal{Q}}$ as above. We identify $x_{\mathcal{P}}$ with $x_{\mathcal{Q}}$, $y_{\mathcal{P}}$ with $y_{\mathcal{Q}}$, $x'_{\mathcal{P}}$ with $x'_{\mathcal{Q}}$, and label the identified vertices $x, y, x'$.

Finally, we force all the vertices of $F'_{\mathcal{P}}$ (except for $x, y, x'$) to be in $\mathcal{P}$, and all the vertices of $F'_{\mathcal{Q}}$ (except for $x, y, x'$) to be in $\mathcal{Q}$. That is, we add a copy of $G_{\mathcal{P},\mathcal{Q}}$, and make every vertex of $F'_{\mathcal{P}} - \{x, y, x'\}$ adjacent to every vertex of $N(u) \cap U_{\mathcal{Q}}$, and every vertex of $F'_{\mathcal{Q}} - \{x, y, x'\}$ adjacent to every vertex of $N(w) \cap U_{\mathcal{P}}$ (cf. Figure 1).

It can be checked that the resulting gadget $R$ (for 'replicator') has the following properties:

*Claim 1.* In a $(\mathcal{P}, \mathcal{Q})$-colouring of $R$, if $x$ is in $\mathcal{P}$, then $y$ is in $\mathcal{Q}$ and $x'$ is in $\mathcal{P}$; similarly, if $x$ is in $\mathcal{Q}$, then $y$ is in $\mathcal{P}$ and $x'$ is in $\mathcal{Q}$. So $x$ and $x'$ always have the same colour, that is different from that of $y$. Moreover, there is at least one colouring (in fact, exactly one) in which $x$ and $x'$ are in $\mathcal{P}$, and at least one in which both are in $\mathcal{Q}$.

[3]To be precise, we mean that $v$ is coloured the same as $u$: if $\mathcal{P} = \mathcal{Q}$ then a $(\mathcal{P}, \mathcal{Q})$-colouring is also a $(\mathcal{Q}, \mathcal{P})$-colouring, but we adopt the convention that the $\mathcal{P}$-part is the part containing $u$.

*Claim 2.* Let $H$ be an arbitrary graph, and let $H_R$ be a graph obtained by identifying some vertex $z \in H$ with the vertex $x \in R$ (so this becomes a cut-vertex in $H_R$). Then a red-blue colouring of $H_R$ is a $(\mathcal{P}, \mathcal{Q})$-colouring iff it is a $(\mathcal{P}, \mathcal{Q})$-colouring of $H$ and a $(\mathcal{P}, \mathcal{Q})$-colouring of $R$.

*Proof of Claim 2.* The "only if" follows from the induced-heredity of $\mathcal{P}$ and $\mathcal{Q}$. For the converse we need to show, without loss of generality, that if every red component of $H$ and of $R$ is in $\mathcal{P}$, then every red component $C$ of $H_R$ is in $\mathcal{P}$. If $x \notin C$, then $C$ must be a red component of $H$ or of $R$.

If $x \in C$, then $C$ is formed from a red component $C_H$ of $H$ containing $z$, and a red component $C_R$ of $R$ containing $x$. Since $x$ is red, by Claim 1, $y$ is blue, so $C_R \subseteq B_{\mathcal{P}} - y_{\mathcal{P}}$. Now $B_{\mathcal{P}}$, on $k$ vertices, was a smallest possible end-block among the forbidden subgraphs for $\mathcal{P}$. Since $C_H$ is in $\mathcal{P}$, adding an end-block $C_R$ (or successively adding a sequence of end-blocks) on at most $k-1$ vertices produces another graph in $\mathcal{P}$.

We thus have a gadget that "replicates" the colour of $x$ on $x'$, while preserving valid colourings.

Let $H_{\mathcal{P}}$ be a forbidden subgraph for $\mathcal{P}$ with the least possible number of vertices, say $p+1$; similarly choose $H_{\mathcal{Q}} \in \mathcal{F}(\mathcal{Q})$ on $q+1$ vertices, where $q+1$ is as small as possible, so any graph on at most $p$ (resp. $q$) vertices is in $\mathcal{P}$ (resp. $\mathcal{Q}$). Since $\mathcal{P}$ and $\mathcal{Q}$ are not both $\mathcal{O}$, $p+q \geq 3$, and so $p$-IN-$(p+q)$-COLOURING is NP-complete. We will construct a third gadget to transform this to $(\mathcal{P}, \mathcal{Q})$-colouring.

We start with an independent set $S$ on $p+q$ vertices, $\{x_1, \ldots, x_{p+q}\}$. For every $(p+1)$-subset of $S$, say $T_j = \{x_1, \ldots, x_{p+1}\}$, add a disjoint copy of $H_{\mathcal{P}}$ whose vertices are labeled $x_1^j, \ldots, x_{p+1}^j$. For each $i = 1, \ldots, p+1$, use a new copy $R_{i,j}$ of $R$ to ensure that $x_i$ and $x_i^j$ are always coloured the same; to do this, identify the vertices $x$ and $x'$ of $R_{i,j}$ with $x_i$ and $x_i^j$. For every $(q+1)$-subset of $S$ we add a copy of $H_{\mathcal{Q}}$ in the same manner. Thus every vertex $x_i \in S$ will have $\ell = \binom{p+q-1}{p} + \binom{p+q-1}{q}$ 'shadow vertices' $x_i^1, \ldots, x_i^\ell$ from copies of $H_{\mathcal{P}}$ and $H_{\mathcal{Q}}$. Call this gadget $N$ (for 'pin cushion' — the copies of $H_{\mathcal{P}}$ and $H_{\mathcal{Q}}$ being stuck into the independent set $S$ by 'pins' or 'replicators').

In a $(\mathcal{P}, \mathcal{Q})$-colouring of $N$, no $p+1$ vertices of $S$ can be in $\mathcal{P}$, and no $q+1$ vertices can be in $\mathcal{Q}$, so exactly $p$ vertices of $S$ are in $\mathcal{P}$, and exactly $q$ are in $\mathcal{Q}$. Conversely, suppose that exactly $p$ vertices of $S$ are coloured red, and the other $q$ are blue; colour each vertex $x_i^j$ the same as $x_i$, $1 \leq i \leq p+q$, $1 \leq j \leq \ell$. Then each copy of $H_{\mathcal{P}}$ has at most $p$ red and at most $q$ blue vertices, giving it a valid $(\mathcal{P}, \mathcal{Q})$-colouring. The colouring on the rest of each gadget $R_{i,j}$ is then forced, and we have a $(\mathcal{P}, \mathcal{Q})$-colouring of all of $N$.

Now, given a $(p+q)$-uniform hypergraph $\mathcal{H}$, we stick a copy of $N$ onto every hyperedge. The resulting graph is $(\mathcal{P}, \mathcal{Q})$-colourable iff $\mathcal{H}$ has a $p$-IN-$(p+q)$-COLOURING. □

# 3 New directions

How far can the main result be extended? Uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs exist even in many cases where the $\mathcal{P}_i$'s are not additive [12]; however, this includes finite $\mathcal{P}_i$'s, so the existence of uniquely colourable graphs does not guarantee NP-hardness.

It may be useful to restate the result as follows: if the graphs in $\mathcal{F}(\mathcal{P})$ and $\mathcal{F}(\mathcal{Q})$ are all connected, then $(\mathcal{P}, \mathcal{Q})$-colouring is NP-hard. This is also true if the graphs in $\mathcal{F}(\mathcal{P})$ and $\mathcal{F}(\mathcal{Q})$ are all disconnected, since $G \in \mathcal{P} \circ \mathcal{Q} \Leftrightarrow \overline{G} \in \overline{\mathcal{P}} \circ \overline{\mathcal{Q}}$, where $\overline{\mathcal{P}}$ is defined by $\mathcal{F}(\overline{\mathcal{P}}) := \{\overline{H} \mid H \in \mathcal{F}(\mathcal{P})\}$.

A natural problem to tackle next would be classifying the complexity of $\mathcal{R}^k$-recognition, where $\mathcal{R}$ has both connected and disconnected minimal forbidden induced-subgraphs. One of the simplest such cases is $\mathcal{R} = (\mathcal{O} \cup \mathcal{K})$, where $\mathcal{K}$ is the set of all cliques: $\mathcal{F}(\mathcal{O} \cup \mathcal{K}) = \{P_3, \overline{P_3}\}$. Gimbel et al. [16] noted that $G \in \mathcal{O}^k \Leftrightarrow nG \in (\mathcal{O} \cup \mathcal{K})^k$ (where $n = |V(G)|$); so $(\mathcal{O} \cup \mathcal{K})^k$-recognition is NP-complete for $k \geq 3$ (and, in fact, polynomial for $k = 1, 2$).

Another natural problem is $(\mathcal{P}, \mathcal{Q})$-colouring, where all graphs in $\mathcal{F}(\mathcal{P})$ are connected, and all those in $\mathcal{F}(\mathcal{Q})$ are disconnected. In all problems, it may make sense to restrict attention to hereditary properties with finitely many forbidden subgraphs.

Another class of problems often considered in the literature is $(\mathcal{D} : \mathcal{P})$-recognition: given a graph $G$ in the domain $\mathcal{D}$, is $G$ in $\mathcal{P}$? This is just $(\mathcal{D} \cap \mathcal{P})$-recognition; if $\mathcal{D}$ and $\mathcal{P}$ are both additive induced-hereditary, then so is $\mathcal{D} \cap \mathcal{P}$, with $\mathcal{F}(\mathcal{D} \cap \mathcal{P}) = \min_{\leq}(\mathcal{F}(\mathcal{D}) \cup \mathcal{F}(\mathcal{P}))$. We leave it as an open question, for reducible $\mathcal{P}$, to determine when $\mathcal{D} \cap \mathcal{P}$ is also reducible; Mihók's characterisations [20, 21] of reducibility may be useful in finding an answer.

# 4 Notes and acknowledgements

The most important part of the proof is the 'replicator' gadget. Phelps and Rödl [23, Thm. 6.2] and Brown [7, Thm. 2.3] used different gadgets to perform similar roles. The forcing technique of Theorem 1 was first used in [19, Thm. 2] and [5, Lemma 3].

Contacts with Lozin were very helpful, as they spurred the author to look at ($K_m$-free, $K_n$-free)-colouring, not knowing it had been settled in [9]. Kratochvíl and Schiermeyer [19] proved a special case of Theorem 2 that covered the case $m = 2$; ($K_2$-free, $K_n$-free)-colouring; I started my proof for general $m$ and $n$ by adapting theirs, and ended up strengthening and simplifying it considerably.

I would like to thank Bruce Richter for many helpful conversations, detailed comments that improved the presentation of the paper, and for spotting a flaw in my original 'pin cushion' gadget. The result here forms part of the Ph.D. thesis that I am writing under his supervision. I would also like to thank the Canadian government for fully funding my studies through a Commonwealth Scholarship.

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