Applied Mathematics and Mechanics
(English Edition, Vol.9, No.4, Apr.1988)

Published by SUT, Shanghai, China

VIBRATION ANALYSIS OF ELLIPTICAL COLUMN PARTIALLY SUBMERGED IN WATER*

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(Received Feb.8, 1987)

Abstract

In this paper, a general analytical method based on Ref. [1] is presented to study the bending vibration of an elliptical column partially submerged in water. Besides, it is pointed out that there is a limitation to the method mentioned in Ref. [2]. As a special example, the natural frequencies of circular column submerged in water considering compressibility are calculated, and the extent of compressible effect is given.

I. Introduction

In recent years, a great attention is paid to the vibrational problems of a column in water by some engineers. The equation of column-water coupled vibration is a differential-integral equation. The usually analytical methods are: (1) The column is dealt with rigid body[3]; (2) The function of mode of vibration in water is substituted by that of mode of vibration without water[4]; (3) With the function of vibrational mode without water as a basis, the function of vibrational mode with water is expanded as an unlimited series[1]; (4) The integral term of the differential-integral equation of the coupled system is considered as the non-homogeneous term of the differential equation[2][5,6]. The column with elliptic cross-section were discussed only in Refs.[1 – 3].

The general analytical method presented in this paper starts with the differential-integral equation of column-water coupled vibration given by Ref. [1]. According to the basic idea of Ref. [7], the modal functions of column and water are expanded as the same complete orthogonal series for uncoupling the mode of column and water. Then the mode function is represented as a sum of a uniform convergence series and a linear polynomial, so that the problems of convergence and differentiation of mode series are solved. Finally, we can obtain an exact solution of the mode function and a corresponding frequency equation denoted by limited order determinent of column partically submerged in water with arbitrary boundary condition and with several mediated constraints.

II. Column-Water Coupled Vibration Equation and Its Uncoupling

2.1 Establishment of column-water coupled vibrational equation[1]

Assume that height of an elliptical column (shown in Fig. 1) is H, the elliptical half major axis is a and minor half axis is b, the thickness is d, and the depth of water is h $(h<H)$.

* Projects Supported by the Science Fund of the Chinese Academy of Sciences.

We suppose that the water is not viscous ideal fluid, motion is not curl and the influence of surface wave is often neglected. If the column is vibrating along y direction and water is disturbed. According to the supposition, in the fluid velocity field, there exists a potential function $\phi(\xi,\eta,z,t)$ which satisfies Laplacian equation:

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+\frac{\gamma^2}{2}(\mathrm{ch}2\xi-\cos2\eta)\frac{\partial^2\phi}{\partial z^2}=\frac{1}{c^2}\frac{\gamma^2}{2}(\mathrm{ch}2\xi-\cos2\eta)\frac{\partial^2\phi}{\partial t^2} \tag{2.1}$$

where $\gamma^2=a^2-b^2$, and c is the velocity of sound in water.

$\phi(\xi,\eta,z,t)$ satisfies the following boundary conditions:

$$\frac{\partial\phi}{\partial z}=0,\qquad z=0; \tag{2.2a}$$

$$\frac{\partial\phi}{\partial t}=0,\qquad z=h; \tag{2.2b}$$

$$\frac{\partial\phi}{\partial\xi}=\frac{\partial y}{\partial t}a\cdot\sin\eta,\qquad \xi=\xi_0; \tag{2.2c}$$

$$\phi=0,\qquad \xi=\xi_1\quad(\xi_1\gg\xi_0) \tag{2.2d}$$

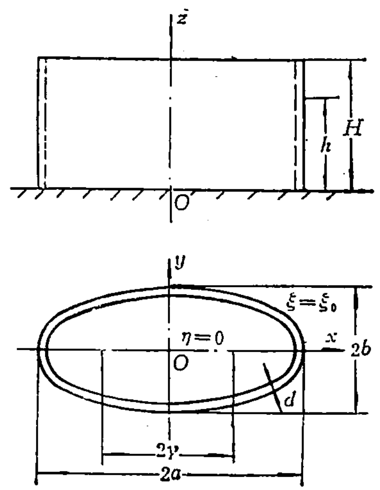


Fig. 1 Elliptical column

where $y=y(z,t)=Y(z)\exp[i\omega t]$ is the displacement along the y direction of the elliptical column, and $\xi_0=\mathrm{th}^{-1}b/a$ is the elliptical surface coordinate.

Using separation of variables, we take

$$\phi(\xi,\eta,z,t)=i\omega H(\xi)G(\eta)Z(z)\exp[i\omega t] \tag{2.3}$$

Substituting it into the above equations, we obtain:

$$\phi(\xi,\eta,z,t)=i\omega\exp[i\omega t]\cdot\frac{2a}{h}\left\{\sum_{s=1}^{j-1}\frac{s_1(\xi_0,q_s)\{B_1^{(1)}\}^2\int_0^h Y(z)\cos\lambda_s z\cdot dz}{s_1'(\xi_0,q_s)\left\{\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\right\}}\cdot\cos\lambda_s z\right.$$

$$\left.+\sum_{s=j}^{\infty}\frac{s_1(\xi_0,-q_s')\{A_1^{(1)}\}^2\int_0^h Y(z)\cos\lambda_s z\cdot dz}{s_1'(\xi_0,-q_s')\left\{\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\right\}}\cos\lambda_s z\right\} \tag{2.4}$$

where

$$\lambda_s=\frac{2s-1}{2h}\pi$$

$$q_s=\frac{\gamma^2}{4}\left(\frac{\omega^2}{c^2}-\lambda_s^2\right),\qquad \frac{\omega}{c}>\lambda_s;$$

$$q_s'=\frac{\gamma^2}{4}\left(\lambda_s^2-\frac{\omega^2}{c^2}\right),\qquad \frac{\omega}{c}<\lambda_s;$$

$$s_1(\xi,q_s)=\mathrm{Se}_1(\xi,q_s)-\frac{\mathrm{Se}_1'(\xi_1,q_s)}{\mathrm{Ge}_1'(\xi_1,q_s)}\mathrm{Ge}_1(\xi,q_s)$$

$$s_1(\xi,-q_s')=\mathrm{Se}_1(\xi,-q_s')-\frac{\mathrm{Se}_1'(\xi_1,-q_s')}{\mathrm{Ge}_1'(\xi_1,-q_s')}\mathrm{Ge}_1(\xi,-q_s')$$

Se_1 and Ge_1 are modified Mathieu functions, Se_1' and Ge_1' are their first order derivation, $A_{2r+1}^{(1)}$ and $B_{2r+1}^{(1)}$ are the coefficients of Mathieu series[8].

In accordance with Bernoulli's function, it is very easy to determine the y-axis projection of the resultant constituted by the water dynamic pressure of column per unit height along the z direction.

$$p|_{\xi=\xi_0}=-\rho\int_0^{2\pi}\frac{\partial\phi}{\partial t}\bigg|_{\xi=\xi_0}\cdot a\sin\eta d\eta$$

$$=\frac{2\pi a^2\rho\omega^2}{h}\exp[i\omega t]\left\{\sum_{s=1}^{j-1}\frac{s_1(\xi_0,q_s)\{B_1^{(1)}\}^2\int_0^h Y\cos\lambda_s z dz}{s_1'(\xi_0,q_s)\left\{\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\right\}}\cdot\cos\lambda_s z\right.$$

$$\left.+\sum_{s=j}^{\infty}\frac{s_1(\xi_0,-q_s')\{A_1^{(1)}\}^2\int_0^h Y\cos\lambda_s z\cdot dz}{s_1'(\xi_0,-q_s')\left\{\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\right\}}\cos\lambda_s z\right\} \tag{2.5}$$

where ρ is the density of water. When the compressibility of water is ignored, only the second term is retained in expression (2.5), so the derivation in Ref. [2] is wrong in comparison with expression (2.5)[2]. By use of Dalembert's principle, we obtain the equation of free bending vibration of section $O-h$ of column as

$$EJ\frac{d^4Y_1(z)}{dz^4}-\omega^2(\rho_1F+m(z))Y_1(z)=0 \tag{2.6}$$

where $Y_1(z)$ is the mode of the section $O-h$ of column, ω is the corresponding frequency, and ρ_1 is the density of column.

$$m(z)\frac{2\rho_1}{Y_1(z)h}\left[\sum_{s=1}^{\infty}N_s\int_0^h Y_1(z)\cos\lambda_s z dz\right]\cdot\cos\lambda_s z \tag{2.7}$$

$$N_s=\begin{cases}-\dfrac{\pi a^2\rho}{\rho_1F}\dfrac{s_1(\xi_0,q_s)\{B_1^{(1)}\}^2}{s_1'(\xi_0,q_s)\left\{\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\right\}} & (1\leqslant s\leqslant j-1)\\[2ex] -\dfrac{\pi a^2\rho}{\rho_1F}\dfrac{s_1(\xi_0,-q_s')\{A_1^{(1)}\}^2}{s_1'(\xi_0,-q_s')\left\{\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\right\}} & (j\leqslant s\leqslant\infty)\end{cases} \tag{2.8}$$

It can be seen from equation (2.6) that the influence of water to the column is equivalent to the distributed mass $m(z)$ attached to the column.

2.2 Uncoupling the mode of column and water[7]

Equation (2.6) is a differential-integral equation. Based on the idea of Ref.[7], we take

$$Y_1(z)=\sum_{s=1}^{\infty} a_s\cos\lambda_s z \tag{2.9}$$

Substituting it into equation (2.6) and by use of the complete orthogonal property of $\{\cos\lambda_s z\}$in (O,h) interval, we obtain:

$$EJ\frac{d^4Y_1(z)}{dz^4}-\omega^2\left[\rho_1 F\sum_{s=1}^{\infty} a_s(1+N_s)\cos\lambda_s z\right]=0 \tag{2.10}$$

The term $d^4Y_1(z)/dz^4$ can be obtained by the method in the next section, but can't be done by term-by-term derivation. This process from equation (2.6) to equation (2.10) makes column-water coupled differential-integral equation into column-water uncoupled differential equation, so that the equivalent mechanics model of column-water coupled vibration can be established.

III. Free Vibration Characteristics of Column-Water System

3.1 Basic equation

The column-water coupled vibration equation is

$$\left.\begin{aligned} &EJ\frac{d^4Y_1}{dz^4}-\omega^2\left[\rho_1 F\sum_{s=1}^{\infty} a_s(1+N_s)\cos\lambda_s z\right]=0 \qquad (0\leqslant z\leqslant h)\\ &EJ\frac{d^4Y_2}{dz^4}-\omega^2\rho_1 FY_2=0 \qquad (h\leqslant z\leqslant H)\end{aligned}\right\} \tag{3.1}$$

Boundary forces and displacements are

$$M=-EJY'', \quad Q=-EJY''' \tag{3.2}$$

Different mode function are selected according to the water depth. 'G – S' set with water*, 'S – S' set without water**are selected, thst is,

$$\left.\begin{aligned} &Y_1=\sum_{s=1}^{\infty} a_s\cos\lambda_s z \qquad (0\leqslant z\leqslant h)\\ &Y_2=\sum_{s=1}^{\infty} \bar{a}_s\sin\bar{\lambda}_s(H-z) \qquad (h\leqslant z\leqslant H)\end{aligned}\right\} \tag{3.3}$$

where

$$\lambda_s=\frac{2s-1}{2h}\pi, \quad \bar{\lambda}_s=\frac{s\pi}{H-h} \qquad (s=1,2,3,\cdots)$$

3.2 Convergence and differentiation of the modes

Any mode function can be expressed by a set of complete orthogonal series. $\{\cos\lambda_s z\}$,

* 'G – S' set respresents the exact solution for the beam with boundary conditions at one end guided and at the other simply-supported.

** 'S – S' set represents the exact solution for the beam with simply-supported boundary condition at both ends.

$\{\sin\lambda_s z\}$ in expression (3.3) are complete orthogonal series basis. The combination of them converges uniformly only in [0,*h*] or [*h*,*H*] interval as usual, but would not be consistent at the end points, and its derivative can not be represented by term-by-term differentiation of series. However, we can solve those problem of the convergence and differentiation of mode by reform of the series. For example let us consider

$$Y_1(z) \doteq \sum_{s=1}^{\infty} a_s \cos\lambda_s z \qquad (0 \leqslant z \leqslant h)$$

only when $Y_1(h)=0$, the series continuous in interval [0,*h*] and converges uniformly to $Y_1(z)$ and expresses $Y_1'(z)$ term-by-term differentiation. In the case of $Y_1(h) \neq 0$, we take

$$Y_1(z) = \sum_{s=1}^{\infty} a_s \cos\lambda_s z + \frac{z}{h} Y_1(h) - \frac{z}{h} Y_1(h) \tag{3.4a}$$

Expanding the last term right of equality (3.4) as cosine series and adding to the first term, we obtain

$$Y_1(z) = \sum_{s=1}^{\infty} \left\{ a_s + \frac{2}{\lambda_s h}\left[\frac{1}{\lambda_s h} - (-1)^s\right] Y_1(h)\right\} \cos\lambda_s z + \frac{z}{h} Y_1(h) \tag{3.4b}$$

Depending on the definition, the series right of equality (3.4b) converges uniformly in close interval [0,*h*] and its derivative $Y_1'(z)$ can be represented by term-by-term differentiation of the series as follows:

$$Y_1'(z) = \sum_{s=1}^{\infty} \left\{ -a_s\lambda_s - \frac{2}{h}\left[\frac{1}{\lambda_s h} - (-1)^s\right] Y_1(h)\right\} \sin\lambda_s z + \frac{Y_1(h)}{h} \tag{3.5}$$

For the similar reason given above we can obtain the correct form of the successive derivations of series (3.3) (see appendix).

It can be proved that not only the convergence of the series is ensured but it is, indeed, improved by this approach.

3.3 Column-water frequency equation with different water depth

According to the method of derivative of the mode series mentioned in section 3.2 we obtain successive derivatives of series (3.3), and substitute the result into expressions (3.1), (3.2). The result can be written as:

$$\left\{\lambda_s^4 - \frac{\rho_1 F}{EJ}(1+N_s)\omega^2\right\} a_s = d_1 Y_1'(0) + d_2 Q(0) + d_3 Y_1(h) + d_4 Y_1''(h) \tag{3.6a}$$

$$\left\{\bar{\lambda}_s^4 - \frac{\rho_1 F}{EJ}\omega^2\right\} \bar{a}_s = \bar{d}_1 Y_2'(h) + \bar{d}_2 Y_2''(h) + \bar{d}_3 Y_2(H) + \bar{d}_4 M(H) \tag{3.6b}$$

Except for G – S – S boundary condition, expression (3.6) is written as follows:

$$\left.\begin{aligned} a_s &= q_1 Y_1'(0) + q_2 Q(0) + q_3 Y_1(h) + q_4 Y_1''(h) \\ \bar{a}_s &= \bar{q}_1 Y_2(h) + \bar{q}_2 Y_2''(h) + \bar{q}_3 Y_2(H) + \bar{q}_4 M(H) \end{aligned}\right\} \tag{3.7}$$

Generally, for the end-supported case S – G set (simply-supported down the end, guided up the end) must satisfy the boundary conditions:

$$Y_1(0) = M(0) = Y_2'(H) = Q(H) \doteq 0 \tag{3.8a}$$

continuous conditions:

$$Y_1(h)=Y_2(h),\ Y_1'(h)=Y_2'(h),\ Y_1''(h)=Y_2''(h),\ Y_1'''(h)=Y_2'''(h) \qquad (3.8b)$$

Substituting expression (3.3) into expression (3.8) and utilizing equation (3.7), we obtain a frequency equation as:

$$[e_{ij}]_{6\times 6}\{Y_1'(0),\ Q(0),\ Y_1(h),\ Y_1''(h),\ Y_2(H),\ M(H)\}^T=\{0\} \qquad (3.9)$$

and a corresponding frequency determinant $|e_{ij}|_{6\times 6}=0$. (3.10)

We can obtain the exact coupled natural frequency of column-water from expression (3.10) and its corresponding mode from expressions (3.9), (3.7), (3.3). Here, we can establish a most general frequency equation (3.9) to solve the problem with S – G boundary condition by G – S – S set. The frequency equations with the other boundary conditions and constrained conditions can be obtained by deleting inapplicable rows and column from determinant (3.9). If there is a constraint at $z=h$ which makes $Y_1(h)=0$, expression (3.9) will be reduced to

$$[e_{ij}]_{5\times 5}\{Y_1'(0),\ Q(0),\ Y_1''(h),\ Y_2(H),\ M(H)\}^T=\{0\} \qquad (3.11)$$

If the height of column is the same depth of water, expression (3.9) will be reduced to:

$$[e_{ij}]_{4\times 4}\{Y_1'(0),\ Q(0),\ Y_1(h),\ M(h)\}^T=\{0\} \qquad (3.12)$$

3.4 The frequency equations for several cases of boundary conditions

(1) C – F (clamped-free beam)

$$[e_{ij}]_{4\times 4}\{Q(0),\ Y_1(h),\ Y_1''(h),\ Y_2(H)\}^T=\{0\} \qquad (3.13)$$

(2) F – F (free-free beam)

$$[e_{ij}]_{4\times 4}\{Y_1'(0),\ Y_1(h),\ Y_1''(h),\ Y_2(H)\}^T=\{0\} \qquad (3.14)$$

(3) G – S (guided-simply-supported beam)

$$[e_{ij}]_{2\times 2}\{Y_1(h),\ Y_1''(h)\}^T=\{0\} \qquad (3.15)$$

If the height of column is equal to the depth of water, the frequency equation can not be reduced from expression (3.9), but it becomes a simple analytical expression given by expression (3.6), that is

$$\left\{\lambda_s^4-\frac{\rho_1 F}{EJ}(1+N_s)\omega^2\right\}=0 \qquad (3.16)$$

then

$$\omega_s=\frac{\lambda_s^2}{\sqrt{\rho_1 F(1+N_s)/EJ}} \qquad (s=1,2,3,\cdots) \qquad (3.17)$$

Without water expression (3.17) will be reduced to

$$\omega_s=\lambda_s^2/(\rho_1 F/EJ)^{\frac{1}{2}} \qquad (s=1,2,3,\cdots) \qquad (3.18)$$

Because of $N_s>0$, the wet frequency of column in water is lower than the dry frequency of column without water. Here, N_S reflects the effect of added water mass. By the way, the 'general solution' may lose the partial solution in Refs.[2,5,6] in which the integral term is considered as the non-homogeneous term of differential equation. The general solution is

$$Y_1(z)=D_1\cos kz+D_2\sin kz+D_3\mathrm{ch}kz+D_4\mathrm{sh}kz+\sum_{s=1}^{\infty}\frac{E_s}{1-\frac{h}{2}E_s}\left(\sum_{i=1}^{4}D_iI_s^{(i)}\right)\cos\lambda_s z \tag{a}$$

here

$$k^4=\frac{\rho_1F\omega^2}{EJ} \tag{b}$$

The G – S boundary condition must satisfy:

$$Y_1'(0)=Y_1''(0)=Y_1(h)=Y_1''(h)=0 \tag{c}$$

Substituting expression (a) into expression (c), the series term right of equation (a) will vanish naturally since $\cos\lambda_s z$ satisfies (c). Therefore, the effect of water can not be reflected. The result is agreeable with that obtained from expression (3.14) for the day frequency of column without water. This result is wrong.

IV. Numerical Results and Discussions

In order to show the reliability of this method and consider the effect of compressibility of water quantitatively the vibrational characteristics of cantilever circular column in water is calculated.

Example 1 Variation of natural frequencies due to radius of column. The material properties are: $H=h=20$m, $d=0$, $E=2.94\times10^{10}$pa, $\rho_1=2450$kg/m^3, $\rho=1000$kg/m^3, $c=1438.027$m/s, ω_i^A , ω_i^W, ω_i^{CW} stand for the i-th natural frequencies in air, in water neglected compressibility and considered compressibility. Table 1 shows that the result in this paper is in agreement with that in Ref. [6], so this method is reliable. Because of the effects of added mass of water and the energy dissipation considering of the compressibility of water, these make $\omega_i^A>\omega_i^W>\omega_i^{CW}$. With the decrease of a/h, $|\omega_i^W-\omega_i^A|/\omega_i^W$ becomes larger and $|\omega_i^{CW}-\omega_i^W|/\omega_i^{CW}$ becomes smaller. These show that the smaller the radius, the greater the effect of water and the lower

Table 1 Variation of natural frequencies due to radius of column

a/h	1st order frequency rad/s				$\frac{\lvert\omega_1^W-\omega_1^A\rvert}{\omega_1^W}$ %	$\frac{\lvert\omega_1^{CW}-\omega_1^W\rvert}{\omega_1^{CW}}$ %
	ω_1^A	ω_1^W	ω_1^W [6]	ω_1^{CW}		
0.003	0.9135	0.7734	0.7763	0.7734	18.11	0.00
0.025	7.612	6.517	6.647	6.517	16.81	0.00
0.040	12.180	10.050	10.782	10.050	15.95	0.00

a/h	2nd order frequency rad/s				$\frac{\lvert\omega_2^W-\omega_1^A\rvert}{\omega_1^W}$ %	$\frac{\lvert\omega_1^{CW}-\omega_1^W\rvert}{\omega_1^{CW}}$ %
	ω_2^A	ω_2^W	ω_2^W [6]	ω_1^{CW}		
0.003	5.725	4.847	4.864	4.847	18.11	0.00
0.025	47.706	40.845	41.635	40.843	16.79	0.00
0.040	78.329	65.834	67.492	65.816	15.94	0.03

a/h	3rd order frequency rad/s				$\frac{\lvert\omega_3^W - \omega_1^A\rvert}{\omega_1^W}$	$\frac{\lvert\omega_1^{CW} - \omega_1^W\rvert}{\omega_1^{CW}}$
	ω_3^A	ω_3^W	ω_3^W [6]	ω_1^{CW}	%	%
0.003	16.029	13.573	13.555	13.572	18.09	0.00
0.025	133.578.	114.689	116.382	114.652	16.50	0.03 .
0.040	213.725	185.263	189.634	184.969	15.30	0.16

the effect of compressibility. With the increase of i-th number of mode, $|\omega_i^W-\omega_i^A|/\omega_i^W$ reduces and $|\omega_i^{CW}-\omega_i^W|/\omega_i^{CW}$ increases. These show that the effect of water on higher number of mode is smaller and the effect of compressibility on higher number of mode is greater. From this example, the effect of compressibility of water on slender can be neglected.

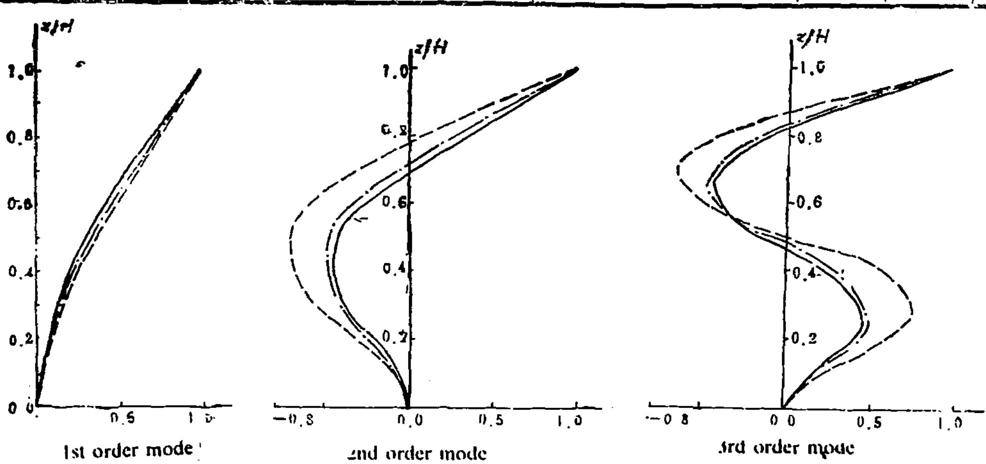
Example 2 Variation of natural frequencies due to thickness of column. The material properties are: $H=h=80$m, $a=10$m, $E=2.94\times10^{10}$pa, $\rho_1=2450\text{kg/m}^3$, $\rho=1000\text{kg/m}^3$, $c=1438.027$m/s, the thicknnes of column is d. Table 2 shows the natural frequencies. Fig.2 and Fig.3 show the vibrational mode of displacement of column and hydrodynamics, when $d/a=0.01$.

From table 2 the thinner the thickness of column, the greater the effect of water and its compressibility. When the thickness of column is quite thin, the effect ot compressibility of water on natural frequency can not be neglected.

From Fig.2, the difference of basic mode of column between in air and in water is small, so the

Table 2 Variation of natural frequencies due to thickness of column

d/a	1st frequency rad/s			$\frac{\lvert\omega_1^{CW}-\omega_1^W\rvert}{\omega_1^{CW}}$	2nd frequency rad/s			$\frac{\lvert\omega_2^{CW}-\omega_2^W\rvert}{\omega_2^{CW}}$	3rd frequency rad/s			$\frac{\lvert\omega_3^{CW}-\omega_3^W\rvert}{\omega_3^{CW}}$
	ω_1^A	ω_1^W	ω_1^{CW}	%	ω_2^A	ω_2^W	ω_2^{CW}	%	ω_3^A	ω_3^W	ω_3^{CW}	%
1.0	9.515	8.470	8.468	0.024	59.63	52.96	52.38	1.228	166.97	150.35	148.18	1.45
0.5	10.639	9.157	9.154	0.035	66.67	57.27	56.37	1.594	186.68	163.11	160.18	1.83
0.01	13.390	3.553	3.546	0.197	83.91	22.72	22.40	1.340	234.91	67.889	63.057	7.66

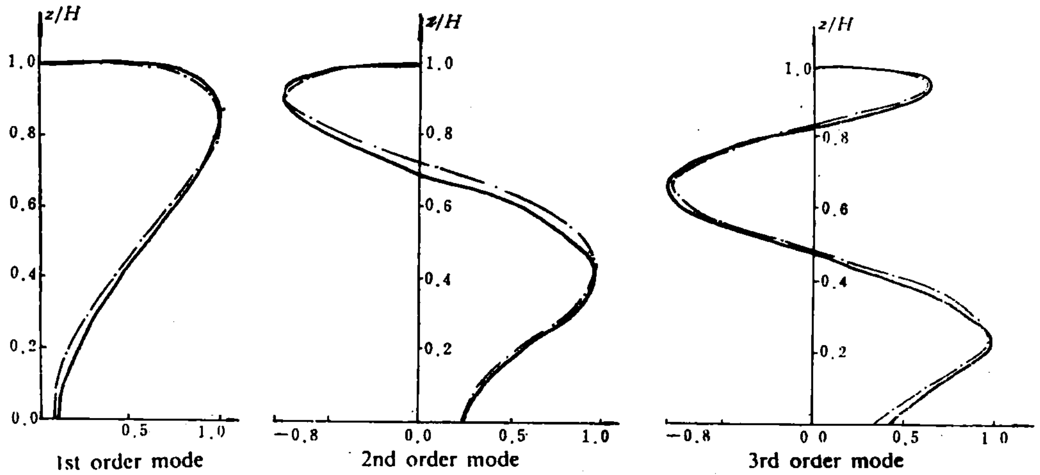


1st order mode 2nd order mode 3rd order mode

(---in air, —·— in incompressible water, —— in compressible water)

Fig. 2 Mode of cylinder with thin thickness result[4] has alittle error in replacing the mode with water by that without water. The effect of compressibility of water on higher number of mode is greater.

From Fig.3, the effect of compressibility of water on higher number of hydrodynamic mode is greater.



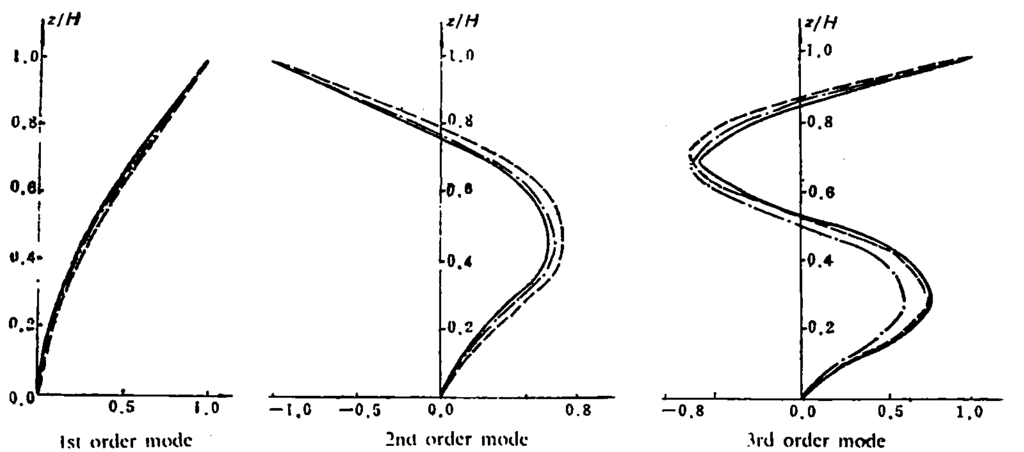
(—·—incompressible water, ——compressible water)

Fig. 3 Hydrodymics in free vibration

Example 3 Variation of natural frequencies due to water depth. The material properties are: $H=80\text{m}$, $a=10\text{m}$, $d=5\text{m}$, $E=2.94\times10^{10}\text{pa}$, $\rho_1=2450\text{kg/m}^3$, $\rho=1000\text{kg/m}^3$, $c=1438.027\text{m/s}$. Table 3 shows the law of vibration of natural frequencies due to water depth. Fig.4, Fig.5 show the mode of displacement of column and mode of hydrodynamics when $h/H=0.5$.

From Table 3, the deeper the water, the greater the effect of added mass of water and the compressibility.

Fig.4 and Fig.5 show the same law in Fig2, Fig.3.

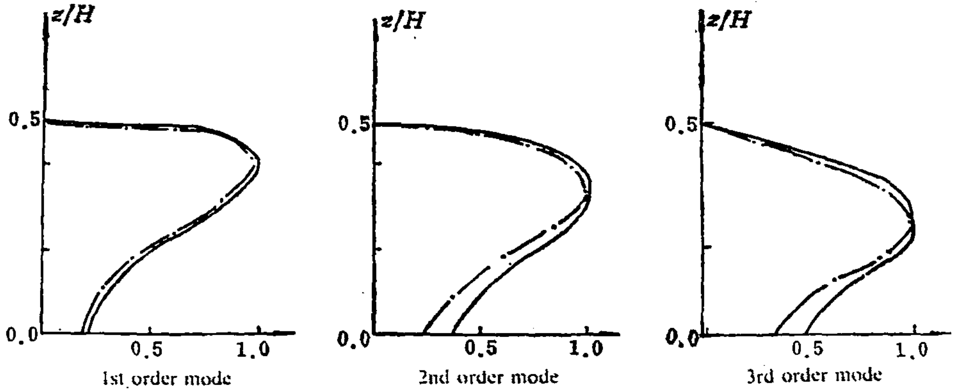


(---in air, ·—— in compressible water, — in compressible water)

Fig. 4 Mode of circular column submerged in water

Table 3 Variation of natural frequencies due to water depth.

h/H	1st frequency rad/s			$\frac{\lvert\omega_1^{CW}-\omega_1^W\rvert}{\omega_1^{CW}}$	2nd frequency rad/s			$\frac{\lvert\omega_2^{CW}-\omega_2^W\rvert}{\omega_2^{CW}}$	3rd frequency rad/s			$\frac{\lvert\omega_3^{CW}-\omega_3^W\rvert}{\omega_3^{CW}}$
	ω_1^A	ω_1^W	ω_1^{CW}	%	ω_2^A	ω_2^W	ω_2^{CW}	%	ω_3^A	ω_3^W	ω_3^{CW}	%
0	10.639				66.671				186.68			
0.5		10.572	10.572	0.00		62.635	62.140	0.80		172.35	170.93	0.83
1.0		9.157	9.154	0.03		57.287	56.368	1.59		163.11	156.01	4.5



(—·— incompressible water, — compressible water)

Fig. 5 Hydrodvmics in free vibration

V. Conclusion

(1) The existence of water reduces the natural frequency and changes the mode shape of the column in air. Usually, the deeper the water and the more slender the column and the thinner the thickness and the lower the number of mode, the greater the effect of added mass of water. The fundamental mode with water is similar to that in air, but the higher modes are not.

(2) The natural frequency considering the compressibility is lower than that neglecting compressibility. Usually, the deeper the water and the squatter the column and the thinner the thickness and the higher the number of mode, the greater the effect of compressibility of water. In most cases the compressibility can be neglected, but only in the case of $\omega_s h/c \geqslant (2s-1)/5$ (s in the number of mode), the effect of the compressibility becomes significant.

(3) This analytical method presented in this paper can be used to solve the problem of free bending vibration of an elliptical column partically submerged in water. The elliptical column is with different water depth, arbitrary boundary condition and with several intermediate constraints and in the case of $h<H$. The effect of compressibility of water can be shown quantitatively. Besides, it is pointed out that there is a limitation to the method in Ref.[2].

Appendix

$$Y_1=\sum_{s=1}^{\infty} a_s \cos\lambda_s z=\sum_{s=1}^{\infty} \{a_s+s_1 Y_1(h)\}\cos\lambda_s z+\frac{z}{h}Y_1(h)$$

$$Y_1'=\sum^{\infty}\left\{-a_s\lambda_s+(-1)^s\frac{2}{h}Y_1(h)+(-1)^s s_1\, Y_1'(0)\right\}\sin\lambda_s z+\frac{h-z}{h}Y_1'(0)$$

$$Y_1''=\sum_{s=1}^{\infty}\left\{-\lambda_s^2 a_s+(-1)^s\frac{2}{h}\lambda_s Y_1(h)-\frac{2}{h}Y_1'(0)+s_1 Y_1''(h)\right\}\cos\lambda_s z+\frac{z}{h}Y_1''(h)$$

$$Y_1'''=\sum_{s=1}^{\infty}\left\{\lambda_s^3 a_s-(-1)^s\frac{2}{h}\lambda_s^2 Y_1(h)+\frac{2}{h}\lambda_s Y_1'(0)+\frac{2}{h}(-1)^s Y_1''(h)\right.$$
$$\left.+(-1)^s s_1 Y_1'''(0)\right\}\sin\lambda_s z+\frac{h-z}{z}Y_1'''(0)$$

$$Y_1''''=\sum_{s=1}^{\infty}\left\{\lambda_s^4 a_s-(-1)^s\frac{2}{h}\lambda_s^3 Y_1(h)+\frac{2}{h}\lambda_s^2 Y_1'(0)+(-1)^s\frac{2}{h}\lambda_s Y_1''(h)-\frac{2}{h}Y_1'''(0)\right\}\cos\lambda_s z$$

$$Y_2=\sum_{s=1}^{\infty}\bar{a}_s\sin\bar{\lambda}_s\bar{z}$$
$$=\sum_{s=1}^{\infty}\left\{\bar{a}_s+s_2\left[Y_2(h)-(-1)^s Y_2(H)\right]\right\}\sin\bar{\lambda}_s\bar{z}+\frac{\bar{z}}{h'}Y_2(H)+\frac{h'-\bar{z}}{h'}Y_2(h)$$

$$Y_2'=\sum_{s=1}^{\infty}\left\{\bar{a}_s\bar{\lambda}_s-\frac{2}{h'}\left[Y_2(h)-(-1)^s Y_2(H)\right]\right\}\cos\bar{\lambda}_s\bar{z}-\frac{1}{h'}[Y_2(h)-Y_2(H)]$$

$$Y_2''=\sum_{s=1}^{\infty}\left\{-\bar{\lambda}_s^2\bar{a}_s+\frac{2}{h'}\bar{\lambda}_s[Y_2(h)-(-1)^s Y_2(H)]+s_2[Y_2''(h)-(-1)^s Y_2''(H)]\right\}\sin\bar{\lambda}_s\bar{z}$$
$$+\frac{\bar{z}}{h'}Y_2''(h)+\frac{h'-\bar{z}}{h'}Y_2''(H)$$

$$Y_2'''=\sum_{s=1}^{\infty}\left\{-\bar{\lambda}_s^3\bar{a}_s+\frac{2}{h'}\bar{\lambda}_s^2[Y_2(h)-(-1)^s Y_2(H)]-\frac{2}{h'}[Y_2''(h)-(-1)^s Y_2(H)]\right\}\cos\bar{\lambda}_s\bar{z}$$
$$-\frac{1}{h'}[Y_2''(h)-Y_2''(H)]$$

$$Y_2''''=\sum_{s=1}^{\infty}\left\{\bar{\lambda}_s^4\bar{a}_s-\frac{2}{h'}\bar{\lambda}_s^3[Y_2(h)-(-1)^s Y_2(H)+\frac{2}{h'}\bar{\lambda}_s[Y_2''(h)-(-1)^s Y_2''(H)]\right\}\sin\bar{\lambda}_s\bar{z}$$

where

$$s_1=\frac{2}{\lambda_s h}\left[\frac{1}{\lambda_s h}-(-1)^s\right],\quad s_2=-\frac{2}{\bar{\lambda}_s h'},\quad h'=H-h,\quad \bar{z}=H-z$$

$$d_1=-\frac{2}{h}\lambda_s^2,\quad d_2=-\frac{2}{EJh},\quad d_3=(-1)^s\frac{2}{h}\lambda_s^3,\quad d_4=-(-1)^s\frac{2}{h}\lambda_s$$

$$\bar{d}_1=\frac{2}{h'}\bar{\lambda}_s^3,\quad \bar{d}_2=-\frac{2}{h'}\bar{\lambda}_s,\quad \bar{d}_3=-(-1)^s\frac{2}{h'}\bar{\lambda}_s^3,\quad \bar{d}_4=-(-1)^s\frac{2}{EJh'}\bar{\lambda}_s$$

$$c_1=\lambda_s^4-\frac{\rho_1 F}{EJ}(1+N_s)\omega^2,\quad c_2=\bar{\lambda}_s^4-\frac{\rho_1 F}{EJ}\omega^2,\quad q_i=\frac{d_i}{c_1},\quad \bar{q}_i=\frac{d_i}{c_2}\qquad (i=1\sim 4)$$

$$e_{11}=\sum_{s=1}^{\infty}q_1,\quad e_{12}=\sum_{s=1}^{\infty}q_2,\quad e_{13}=\sum_{s=1}^{\infty}(q_s+s_1),\quad e_{14}=\sum_{s=1}^{\infty}q_4$$

$$e_{21}=\sum_{s=1}^{\infty}\left[\lambda_s^2 q_1+\frac{2}{h}\right],\quad e_{22}=\sum_{s=1}^{\infty}\lambda_s^2 q_2,\quad e_{23}=\sum_{s=1}^{\infty}\left[\lambda_s^2 q_3-(-1)^s\frac{2}{h}\lambda_s\right],\quad e_{24}=\sum_{s=1}^{\infty}[\lambda_s^2 q_4-s_1]$$

$$e_{31}=\sum_{s=1}^{\infty}[(-1)^s\lambda_s q_1-s_1],\quad e_{32}=\sum_{s=1}^{\infty}[(-1)^s\lambda_s q_2]$$

$$e_{33}=\sum_{s=1}^{\infty}\left[(-1)^s\lambda_s q_3-\frac{2}{h}+\bar{\lambda}_s\bar{q}_1-\frac{2}{h'}\right]-\frac{1}{h'},\quad e_{34}=\sum_{s=1}^{\infty}[(-1)^s\lambda_s q_4+\bar{\lambda}_s\bar{q}_2]$$

$$e_{35}=\sum_{s=1}^{\infty}\left[\bar{\lambda}_s\bar{q}_3+(-1)^s\frac{2}{h'}\right]+\frac{1}{h'},\quad e_{36}=\sum_{s=1}^{\infty}\bar{\lambda}_s\bar{q}_4$$

$$e_{41}=\sum_{s=1}^{\infty}\left[(-1)^s\lambda_s^3 q_1+(-1)^s\frac{2}{h}\lambda_s\right],\quad e_{42}=\sum_{s=1}^{\infty}\left[(-1)^s\lambda_s^3 q_2-\frac{s_1}{EJ}\right]$$

$$e_{43}=\sum_{s=1}^{\infty}\left[(-1)^s\lambda_s^3 q_3-\frac{2}{h}\lambda_s^2+\bar{\lambda}_s^3\bar{q}_1-\frac{2}{h'}\bar{\lambda}_s^2\right]$$

$$e_{44}=\sum_{s=1}^{\infty}\left[(-1)^s\lambda_s^3 q_4+\frac{2}{h}+\bar{\lambda}_s^3\bar{q}_2+\frac{2}{h'}\right]+\frac{1}{h'},\quad e_{45}=\sum_{s=1}^{\infty}\left[\bar{\lambda}_s^3\bar{q}_3+(-1)^s\frac{2}{h'}\bar{\lambda}_s^2\right]$$

$$e_{46}=\sum_{s=1}^{\infty}\left[\bar{\lambda}_s^3\bar{q}_4+\frac{2}{h'}(-1)^s\frac{1}{EJ}\right]+\frac{1}{h'EJ},\quad e_{53}=\sum_{s=1}^{\infty}\left[(-1)^s\bar{q}_1\bar{\lambda}_s-\frac{2}{h'}(-1)^s\right]-\frac{1}{h'}$$

$$e_{54}=\sum_{s=1}^{\infty}(-1)^s\bar{q}_2\bar{\lambda}_s,\quad e_{55}=\sum_{s=1}^{\infty}\left[(-1)^s\bar{q}_3\bar{\lambda}_s+\frac{2}{h'}\right]+\frac{1}{h'},\quad e_{56}=\sum_{s=1}^{\infty}(-1)^s\bar{q}_4\bar{\lambda}_s$$

$$e_{63}=\sum_{s=1}^{\infty}\left[(-1)^s\bar{\lambda}_s^3\bar{q}_1-(-1)^s\frac{2}{h'}\bar{\lambda}_s^2\right],\quad e_{64}=\sum_{s=1}^{\infty}\left[(-1)^s\bar{\lambda}_s^3\bar{q}_2+(-1)^s\frac{2}{h'}\right]+\frac{1}{h'}$$

$$e_{65}=\sum_{s=1}^{\infty}\left[(-1)^s\bar{\lambda}_s^3\bar{q}_3+\frac{2}{h'}\bar{\lambda}_s^2\right],\quad e_{66}=\sum_{s=1}^{\infty}\left[(-1)^s\bar{\lambda}_s^3\bar{q}_4+\frac{2}{h'EJ}\right]-\frac{1}{h'EJ}$$

The others are zero.

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