MASTER

VIBRATIONS OF CIRCULAR
CYLINDRICAL SHELLS

by

Ho Chung

Components Technology Division

ARGONNE
NATIONAL
LABORATORY

U of C-AUA-USERDA

Base Technology

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PREFACE

The work reported herein was performed as part of the base technology activity under the Flow Induced Vibration Program (189a No. CA054-A; Old No. 02659-A) sponsored by ERDA/RRD. The overall objective of the activity is to develop new and/or improved analytical methods and guidelines for designing LMFBR components to avoid detrimental flow induced vibration.

In this report, a general analytical method is presented for evaluating the natural frequencies and mode shapes of a circular cylindrical shell with homogeneous boundary conditions of any type. The solution is obtained through a direct solution procedure using Sanders' shell equations in which the axial modal displacements are represented as simple Fourier series expressions. Stokes' transformation is used to legitimize the derivatives of the Fourier series. From the exact frequency equation obtained, the numerical results are calculated.

The method is applicable to the dynamic analysis of reactor internal and plant components that can be represented as circular cylindrical shells; the thermal liner of the Fast Test Reactor (FTR) is such a component. The thermal liner is a relatively thin shell concentric with and separated by a narrow fluid-filled gap from the main reactor vessel. The main function of the thermal liner is to shield the reactor vessel from excessive heat from the core and thereby to protect it from possible damage. The thermal liner is subjected to sodium coolant flow, and is thus susceptible to flow induced vibrations. The excitation mechanism may be associated with random pressure fluctuations, fluidelastic instabilities, or resonant vibration resulting from a coincidence between component natural frequencies and vortex-shedding or other flow-related driving frequencies. As an example of the application of the developed method, the determination of

the free vibration modal characteristics of the thermal liner is presented
and discussed.

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NOMENCLATURE

h, R, ℓ	Thickness, radius and length of shell
k	Fourier component in axial direction
m	Number of axial mode number
n	Number of circumferential waves
u, v, w	Axial, circumferential and radial displacement
x, θ	Axial, and circumferential coordinates
t	time
v_o, v_ℓ, w_o, w_ℓ	Unspecified end values
A_{kn}, B_{kn}, C_{kn}	Fourier coefficients for u, v and w
D	Extensional rigidity, $Eh/(1-\nu^2)$
E	Young's modulus
G	Shear modulus
K	Flexural rigidity, $Eh^3/12(1-\nu^2)$
M_x, M_θ, $\bar{M}_{x\theta}$	Bending and twisting moment per unit length
N_x, N_θ, $\bar{N}_{x\theta}$	Membrane force per unit length
$\hat{N}_{x\theta}$	Effective membrane shear force per unit length
$\hat{Q}_x$	Effective transverse shear force per unit length
γ^2	$1/\omega_o^2 = \rho R^2 (1-\nu^2)/E$
δ	$h^2/12\,R^2$
ν	Poisson's ratio
ρ	Mass density
ψ_u, ψ_v, ψ_w	Axial mode shapes corresponding to axial, circumferential, and radial directions. respectively
ω	Circular frequency of shell
ω_o	Lowest extensional frequency of a ring in plain strain $(=\sqrt{E/\{\rho R^2(1-\nu^2)\}}\,)$
Ω	Frequency parameter ($= \omega^2/\omega_o^2$)

NOMENCLATURE (Contd.)

$(\ldots)' = d( )/dx$

$(\ldots)_{,x} = \partial( )/\partial x$

$(\ldots)_{,\theta} = \partial( )/\partial\theta$

$(\ldots)_{,t} = \partial( )/\partial t$

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ABSTRACT

An exact solution method for the free vibration problem of thin circular cylindrical shells is presented. The differential equations of motion are solved directly with the use of simple Fourier series as the modal displacement functions. Stokes' transformation is exploited to obtain correct series expressions for the derivatives of the Fourier series. From this method an explicit expression of the exact frequency equation can be obtained for any kind of boundary conditions. The accuracy of the present method is checked against available data. The proposed method is then used to find the modal characteristics of the thermal liner of the Fast Test Reactor (FTR). The numerical results obtained are compared with finite element method solutions.

I. INTRODUCTION

The vibration of thin circular cylindrical shells has been of great interest to many structural engineers in recent years. Many investigations following the pioneering work of Arnold and Warburton [1,2][*] have been summarized by Leissa [3]. Methods of solution for linear shell vibrations have ranged from approximate energy methods by Arnold and Warburton [1,2] and Sharma [4,5] to the exact solutions as studied by Forsberg [6,7, 8], Warburton [9], Warburton and Higgs [10], and Goldman [11]. Recently, the present author used a modified Rayleigh-Ritz procedure to obtain exact solutions [12]. In this report, the differential equations of shell are solved directly, and an exact frequency equation is obtained.

The problem is treated within the framework of linear, elastic, first-order shell theory. Starting with Sanders' theory [13], the dependent variables are expanded in Fourier series in the <u>circumferential</u> direction. The problem is then considered on a mode-by-mode basis associated with the circumferential Fourier index n. In the <u>axial</u> direction, the variables are built up to form a set of complete orthogonal functions in the form of simple Fourier series which are not required to satisfy the boundary conditions term by term. Stokes' transformation [15] is used to obtain correct series expressions for derivatives of the Fourier series. The boundary conditions which are not satisfied directly by the assumed series are enforced. The unwanted boundary conditions, that are originally unspecified but are implied by the nature of the assumed series, are released to remain unspecified. Through this process, all the boundary conditions of the problem become satisfied by the assumed displacement function in an overall sense. It should be emphasized that the present method can solve problems

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[*] Numbers in brackets designate References at the end of the report.

with any kind of boundary conditions. The direct solution method used in this report differs from those of other contributors [6-11] in the selection of general solutions of shell equations. They chose exponential functions for the modal displacements along the axial direction and substituted them into the equations of motion and enforced the eight specified boundary conditions. This led to an eighth-order algebraic equation and an eighth-order frequency determinant which are coupled together. The simultaneous solution of these two systems of equations involves extremely laborious computation [16]. In the present report, the use of simple Fourier series along the axial direction greatly simplifies the solution procedure and leads to a frequency determinant of low order (≤ 4).

The numerical results from this method are checked with the available exact solutions reported by other contributors. The natural frequencies of the thermal liner model of the Fast Test Reactor (FTR) are calculated and discussed. In particular, the variation of natural frequencies due to the change of boundary conditions is discussed in detail.

II. THEORETICAL BACKGROUND

A. Equations of Motion

First-order liner shell theories, based on Karchhoff's hypothesis, have been given by Donnell, Love, Timoshenko, Novozhilov, Flügge, Vlasov, Sanders, and others. For an excellent summary see Leissa [3]. Sanders' shell equations [13] are said to be "best" for the following reasons [14]: they can be written in general tensor form for arbitrary shell geometry; the principle of virtual work, the minimum energy and the reciprocal relations all apply; the stresses are described by six stress measures that satisfy the equations of equilibrium without approximation; the stress and strain measures obey the uncoupled Love constitutive relations, etc. The celebrated Flügge's equations cannot be given in general tensor form, and they do not satisfy the uncoupled constitutive laws. The present author has chosen Sanders' shell equations

$$R^2 u_{,xx} + \frac{1-\nu}{2}\left(1 + \frac{\delta}{4}\right) u_{,\theta\theta} + R\left\{\frac{1+\nu}{2} - \frac{3(1-\nu)}{8}\delta\right\} v_{,x\theta}$$

$$+ \nu R w_{,x} + \frac{1-\nu}{2} R\delta w_{,x\theta\theta} = \gamma^2 u_{,tt}$$

$$R\left\{\frac{1+\nu}{2} - \frac{3(1-\nu)}{8}\delta\right\} u_{,x\theta} + (1+\delta) v_{,\theta\theta} + \frac{1-\nu}{2} R^2 \left(1 + \frac{9\delta}{4}\right) v_{,xx}$$

$$- \frac{(3-\nu)}{2} R^2 \delta w_{,xx\theta} + w_{,\theta} - \delta w_{,\theta\theta\theta} = \gamma^2 v_{,tt} \tag{1}$$

$$\frac{1-\nu}{2} R\delta u_{,x\theta\theta} + \nu R u_{,x} - \frac{3-\nu}{2} R^2 \delta v_{,xx\theta} + v_{,\theta} + w$$

$$+ \delta\{R^4 w_{,xxxx} + 2R^2 w_{,xx\theta\theta} + w_{,\theta\theta\theta\theta} - v_{,\theta\theta\theta}\} = -\gamma^2 w_{,tt}$$

where

$$\gamma^2 = \rho R^2 (1-\nu^2)/E \ , \qquad \delta = h^2/12\, R^2 \ .$$

The geometry of a circular cylindrical shell is shown in Fig. 1. Also shown on the figure are the forces and moments acting on a section of the shell parallel to the coordinate lines.
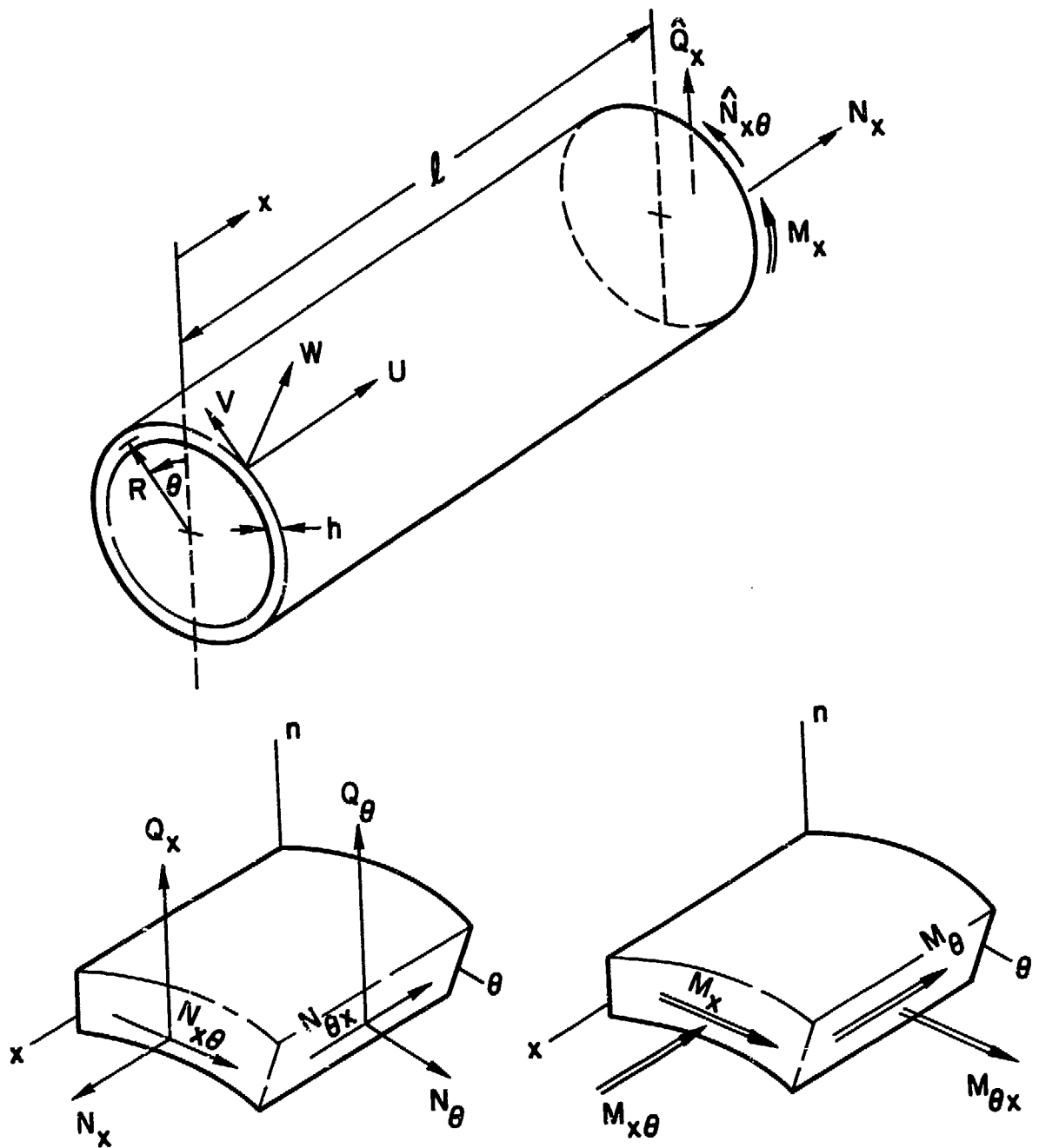
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Fig. 1. Cylindrical Shell with Applied Forces

The boundary conditions applicable to Sanders' theory reduce to prescribing the following quantities at the ends of the cylindrical shell:

$$\left.\begin{array}{ll} N_x \text{ or } u \,, & \hat{N}_{x\theta} \text{ or } v \\[2mm] \hat{Q}_x \text{ or } w \,, & M_x \text{ or } \dfrac{\partial w}{\partial x} \end{array}\right\} \quad x = 0, \ell \tag{2}$$

where

$$\hat{N}_{x\theta} = \bar{N}_{x\theta} + \frac{3}{2R}\bar{M}_{x\theta} \,, \qquad \hat{Q}_x = \frac{\partial M_x}{\partial x} + \frac{2}{R}\frac{\partial \bar{M}_{x\theta}}{\partial \theta} \tag{3}$$

The "effective" forces per unit length $\hat{N}_x$ and $\hat{Q}_x$ are shown in Fig. 1. The force quantities on the left in (2)[*] are associated with the natural boundary conditions while the displacement quantities on the right are associated with the geometric boundary conditions. Further information on these relationships may be found in References [12] and [13]. It will prove to be useful to have the relationships between boundary forces and displacements explicitly written out,

$$N_x = D\left[u_{,x} + \frac{\nu}{R}v_{,\theta} + \frac{\nu}{R}w\right]$$

$$\hat{N}_{x\theta} = \frac{D(1-\nu)}{2}\left[\frac{1}{R}\left(1 - \frac{3}{4}\delta\right)u_{,\theta} + \left(1 + \frac{9}{4}\delta\right)v_{,x} - 3\delta w_{,x\theta}\right]$$

$$\hat{Q}_x = K\left[-\frac{(1-\nu)}{2R^3}u_{,\theta\theta} + \frac{3-\nu}{2R^2}v_{,x\theta} - \frac{2-\nu}{R^2}w_{,x\theta\theta} - w_{,xxx}\right] \tag{4}$$

$$M_x = K\left[\frac{\nu}{R^2}(v_{,\theta} - w_{,\theta\theta}) - w_{,xx}\right]$$

where $D = Eh/(1-\nu^2)$, $K = Eh^3/12(1-\nu^2)$.

B. Modal Functions

The modes of vibration are characterized by the number, n, of circumferential waves and the number, m, of axial waves. For each n,m combination three natural frequencies exist [1,2]. The essential difference between these three vibration forms are the relative amplitudes in the axial, circumferential and radial directions. It can be shown [1,2] that for any n, a

---

[*] Numbers in parentheses designate the equation numbers in the report.

general relation for the displacements in any mode may be written in the following form:

$$u(x,\theta,t) = \psi_u(x)\cos n\theta \sin \omega t$$

$$v(x,\theta,t) = \psi_v(x)\sin n\theta \sin \omega t$$

$$w(x,\theta,t) = \psi_w(x)\cos n\theta \sin \omega t$$

where ψ_u, ψ_v, ψ_w are mode functions corresponding to axial, tangential and radial displacements, respectively. The crucial part of the analysis involves choosing appropriate series forms for these mode functions. The series should be simple in form and at the same time preserve orthogonality properties. It is not necessary that the series satisfy any particular boundary conditions since we are dealing with a general solution at this point.

There are two convenient sets of Fourier series that meet all these requirements for the mode functions along the axial direction. The first set, designated "CSS," can be written in the form

$$\psi_u(x) = A_{on} + \sum_{k=1}^{\infty} A_{kn} \cos \frac{k\pi x}{\ell}$$

$$\psi_v(x) = \sum_{k=1}^{\infty} B_{kn} \sin \frac{k\pi x}{\ell} \tag{6}$$

$$\psi_w(x) = \sum_{k=1}^{\infty} C_{kn} \sin \frac{k\pi x}{\ell}$$

while the second set, designated "SCC," can be written as

$$\psi_u(x) = \sum_{k=1}^{\infty} A_{kn} \sin \frac{k\pi x}{\ell}$$

$$\psi_v(x) = B_{on} + \sum_{k=1}^{\infty} B_{kn} \cos \frac{k\pi x}{\ell} \tag{7}$$

$$\psi_w(x) = C_{on} + \sum_{k=1}^{\infty} C_{kn} \cos \frac{k\pi x}{\ell}$$

The first set represents the exact solution to the shell with simply-supported ends with no axial constraint (SNA-SNA). This "SNA" shell

has boundary conditions at each end of the form

$$N_x = 0 \quad , \quad v = 0 \quad , \quad w = 0 \quad , \quad M_x = 0 \quad . \tag{8}$$

The "CSS" set satisfies these boundary conditions term-by-term.

The second set represents the exact solution to the shell with freely-supported ends with no tangential constraint (FSNT-FSNT). This "FSNT" shell has boundary conditions at each end of the form

$$u = 0 \quad , \quad \hat{N}_{x\theta} = 0 \quad , \quad \hat{Q}_x = 0 \quad , \quad \frac{\partial w}{\partial x} = 0 \quad . \tag{9}$$

The "SCC" set satisfies the above boundary conditions term-by-term. It should be carefully pointed out that by using Stokes' transformation, it is possible to use these sets to find exact solutions for shell problems with any possible combination of homogeneous end conditions. There are sixteen possible sets of homogeneous boundary conditions that can be specified independently at each end [6]; these are listed in Appendix A.

C.   Stokes' Transformation

When differentiating the foregoing series, care must be taken with respect to the end values. For example, when using a sine series to represent a function, the end values of the function are forced to be zero. With Stokes' transformation [15,17],however, the end values of the sine series are released by being defined separately and these values are then included in the successive derivatives of the series.

Consider a function $f(x)$ represented by a Fourier sine series in the open range $0 < x < \ell$ and by values f_o and f_ℓ at the end points,

$$f(o) = f_o \quad , \quad f(\ell) = f_\ell \quad , \quad f(x) = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{\ell} \quad (0 < x < \ell) \tag{10}$$

Since it is not certain that the derivative $f'(x)$ can be represented by term-by-term differentiation of the sine series, the derivative is instead represented by an independent cosine series of the form

$$f'(x) = b_0 + \sum_{k=1}^{\infty} b_k \cos \frac{k\pi x}{\ell} \quad . \tag{11}$$

Stokes' transformation then consists of integrating by parts in the basic definitions of the coefficients to obtain the required relationship between b_k and a_k, as follows

$$
\begin{aligned}
b_k &= \frac{2}{\ell} \int_0^{\ell} f'(x) \cos \frac{k\pi x}{\ell} \, dx \\
&= \frac{2}{\ell} \left[f(x) \cos \frac{k\pi x}{\ell} \right]_0^{\ell} + \frac{2k\pi}{\ell^2} \int_0^{\ell} f(x) \sin \frac{k\pi x}{\ell} \, dx \tag{12} \\
&= \frac{2}{\ell} \left[(-1)^k f_{\ell} - f_0 \right] + \frac{k\pi}{\ell} a_k
\end{aligned}
$$

Similar care must be taken when finding the correct sine series corresponding to $f''(x)$. The complete set of formulas for the sine series are then written as

$$
\left\{
\begin{aligned}
&f(o) = f_0 \qquad f(\ell) = f_{\ell} \\
&f(x) = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{\ell} \qquad (0 < x < \ell)
\end{aligned}
\right.
$$

$$f'(x) = -\frac{f_0 - f_{\ell}}{\ell} - \sum_{k=1}^{\infty} \left[\frac{2}{\ell} \{ f_0 - (-1)^k f_{\ell} \} - \frac{\pi}{\ell} k a_k \right] \cos \frac{k\pi x}{\ell} \qquad (0 \le x \le \ell)$$

$$\tag{13}$$

$$
\left\{
\begin{aligned}
&f''(o) = f_0'' \qquad f''(\ell) = f_{\ell}'' \\
&f''(x) = \left(\frac{\pi}{\ell} \right) \sum_{k=1}^{\infty} k \left[\frac{2}{\ell} \{ f_0 - (-1)^k f_{\ell} \} - \frac{\pi}{\ell} k a_k \right] \sin \frac{k\pi x}{\ell} \qquad (0 < x < \ell)
\end{aligned}
\right.
$$

If the function $f(x)$ is expanded in a Fourier cosine series, similar transformation formulas must be used to establish the correct form of the differentiated series. These formulas for the sine and cosine series are used in the direct solution procedure of the shell equations.

III. GENERAL FORMULATIONS

The preceding theoretical considerations will now be used to solve cylindrical shell problems. The first set (6) of mode functions, CSS, identically satisfies all the boundary conditions (both geometrical and natural), only for the SNA (simple support without axial constraint) shell. Therefore, substitution of this set into the equations of motion (1) leads to the frequency equation for an SNA shell extensively studied by Dym [18]. For all shells with other boundary conditions, the frequency equation is determined from the specified boundary conditions. Using CSS, the most complicated case (from the point of view of the size of the resulting frequency determinant) involves boundary conditions that are freely-supported without tangential constraint at both ends (FSNT-FSNT). It is interesting to note that for the second set, SCC, all the boundary conditions of the FSNT shell are satisfied on a term-by-term basis and this problem is therefore simple to solve. The most complicated problem for SCC is the SNA shell. In Appendix B, an outline is given of the set of mode functions to use to obtain the simplest frequency equation.

In the following work, details are given only for the solution using the CSS set. However, results for frequencies are shown based on both sets of axial mode functions. For maximum generality of the formulation, the FSNT shell will serve as the base problem for CSS. All other shell problems with different boundary conditions are simplified special cases of this base problem. For this base problem, none of the eight boundary conditions are satisfied by CSS on a term-by-term basis. Thus the constraint conditions to satisfy these boundary conditions lead to an eight-by-eight frequency determinant. Solutions for shells with other boundary conditions lead to a smaller size determinant.

A solution for the vibratory characteristics of an FSNT shell will now be outlined using CSS. The CSS set is given in (6) and their derivatives are obtained through the use of Stokes' transformation (see Appendix C). Substitution of the CSS set and their derivatives into the equations of motion (1) leads to an explicit relation for A_{on} and, in addition, a set of equations in which A_{kn}, B_{kn}, and C_{kn} are coupled together. The solution of these coupled equations allows all the coefficients to be written explicitly in terms of the eight unspecified quantities $\bar{u}_o$, $\bar{u}_\ell$, v_o, v_ℓ, w_o, w_ℓ, $\bar{\bar{w}}_o$, $\bar{\bar{w}}_\ell$ and the frequency parameter Ω as follows

$$A_{on} = \frac{1}{2(a_{21} - \Omega)} [a_1(\bar{u}_o + \bar{u}_\ell) - a_3(v_o + v_\ell) - a_4(w_o + w_\ell)]$$

$$(\Omega \neq a_{21}) \tag{14}$$

$$\begin{Bmatrix} A_{kn} \\ B_{kn} \\ C_{kn} \end{Bmatrix} = \begin{bmatrix} a_\alpha & a_\beta & a_\gamma \\ a_\alpha & b_\beta & b_\gamma \\ a_\alpha & b_\beta & c_\gamma \end{bmatrix} \begin{Bmatrix} a_1[\bar{u}_o + \bar{u}_\ell(-1)^k] - a_3[v_o + v_\ell(-1)^k] \\ - a_5 k[v_o + v_\ell(-1)^k] \\ - a_7 k[v_o + v_\ell(-1)^k] \end{Bmatrix}$$

$$-a_4[w_o + w_\ell(-1)^k]$$

$$-a_7 k[w_o + w_\ell(-1)^k]$$

$$-(a_9 k^3 + a_{10}k)[w_o + w_\ell(-1)^k] + a_9 k[\bar{\bar{w}}_o + \bar{\bar{w}}_\ell(-1)^k] \tag{15}$$

$$(\square_{kn} \neq 0)$$

where

$$\Omega = (\omega/\omega_o)^2 \qquad \omega_o^2 = E/\rho R^2 (1-\nu^2)$$

Other symbols are listed in Appendix D.

The equations (14) and (15) include the four unspecified quantities v_o, v_ℓ, w_o, and w_ℓ which are directly related to the geometrical boundary values

$$v_o = - \frac{2}{\pi \sin n\theta} v(0,\theta) , \qquad v_\ell = \frac{2}{\pi \sin n\theta} v(\ell,\theta)$$

$$w_o = - \frac{2}{\pi \sin n\theta} w(0,\theta) , \qquad w_\ell = \frac{2}{\pi \sin n\theta} w(\ell,\theta) \tag{16}$$

The other four quantities $\bar{u}_o$, $\bar{u}_\ell$, $\bar{\bar{w}}_o$, $\bar{\bar{w}}_\ell$ are associated with the unspecified end forces N_x and moments M_x as follows (see (4)),

$$\left\{\begin{array}{c} N_x(0,\theta) \\ -N_x(\ell,\theta) \end{array}\right\} = -\frac{Eh\ell}{2(1-\nu^2)R^2} \left[a_1\left\{\begin{array}{c} \bar{u}_o \\ \bar{u}_\ell \end{array}\right\} - (a_3 + q_1)\left\{\begin{array}{c} v_o \\ v_\ell \end{array}\right\} - (a_4 + q_4)\left\{\begin{array}{c} w_o \\ w_\ell \end{array}\right\} \right] \cos n\theta \tag{17}$$

$$\left\{\begin{array}{c} M_x(0,\theta) \\ -M_x(0,\ell) \end{array}\right\} = \frac{Eh\ell^2}{2\pi(1-\nu^2)R^2} \left[q_{23}\left\{\begin{array}{c} v_o \\ v_\ell \end{array}\right\} + q_{13}\left\{\begin{array}{c} w_o \\ w_\ell \end{array}\right\} + a_9\left\{\begin{array}{c} \bar{\bar{w}}_o \\ \bar{\bar{w}}_\ell \end{array}\right\} \right] \cos n\theta \tag{18}$$

The equations (14) and (15) may now be rewritten in terms of the eight unspecified boundary quantities v_o, v_ℓ, w_o, w_ℓ, $\tilde{N}{}_x^o$, $\tilde{N}{}_x^\ell$, $\tilde{M}{}_x^o$, $\tilde{M}{}_x^\ell$ for a FSNT shell.

$$A_{on} = \frac{1}{2(a_{21}-\Omega)} \left[q_1(v_o+v_\ell) + q_4(w_o+w_\ell) - \frac{2R^2(1-\nu^2)}{Eh\ell}\left\{ \tilde{N}{}_x^o + \tilde{N}{}_x^\ell \right\} \right]$$
$$(\Omega \neq a_{21}) \tag{19}$$

$$\left\{\begin{array}{c} A_{kn} \\ B_{kn} \\ C_{kn} \end{array}\right\} = \left[\begin{array}{ccc} a_\alpha & a_\beta & a_\gamma \\ a_\beta & b_\beta & b_\gamma \\ a_\gamma & b_\gamma & c_\gamma \end{array}\right] \left\{\begin{array}{c} q_1[v_o + (-1)^k v_\ell] \\ q_2 k[v_o + (-1)^k v_\ell] \\ q_3 k[v_o + (-1)^k v_\ell] \end{array}\right.$$

$$+ q_4[w_o + (-1)^k w_\ell] - \frac{2R^2(1-\nu^2)}{Eh\ell}[\tilde{N}{}_x^o + (-1)^k\tilde{N}{}_x^\ell]$$

$$+ q_5[w_o + (-1)^k w_\ell]$$

$$\left. + q_6 k[w_o + (-1)^k w_\ell] - \frac{2R^2(1-\nu^2)}{Eh\ell} k[\tilde{M}{}_x^o + (-1)^k\tilde{M}{}_x^\ell] \right\} \tag{20}$$

$$(\Box_{kn} \neq 0)$$

where

$$\tilde{N}{}_x^o = \frac{N_x(0,\theta)}{\cos n\theta} \qquad \tilde{N}{}_x^\ell = -\frac{N_x(\ell,\theta)}{\cos n\theta}$$

$$\tilde{M}{}_x^o = -\frac{\pi}{\ell \cos n\theta} M_x(0,\theta) \qquad \tilde{M}{}_x^\ell = \frac{\pi}{\ell \cos n\theta} M_x(\ell,\theta) \tag{21}$$

The boundary conditions appropriate to the FSNT shell are listed in (9). The geometric boundary conditions that must be satisfied are associated with u and $\partial w/\partial x$. Hence for arbitaray n it follows that

$$\psi_u(0) = A_{on} + \sum_{k=1}^{\infty} A_{kn} = 0$$

$$\psi_u(\ell) = A_{on} + \sum_{k=1}^{\infty} A_{kn}(-1)^k = 0$$

$$\psi_w'(0) = \frac{\pi}{\ell} \left[\frac{w_o + w_\ell}{2} + \sum_{k=1}^{\infty} \{w_o + (-1)^k w_\ell + kC_{kn}\} \right] = 0 \tag{22}$$

$$\psi_w'(\ell) = \frac{\pi}{\ell} \left[\frac{w_o + w_\ell}{2} + \sum_{k=1}^{\infty} \{w_o + (-1)^k w_\ell + kC_{kn}\}(-1)^k \right] = 0 \ .$$

The natural boundary conditions that must be forced to zero are $\hat{N}_{x\theta} = 0$ and $\hat{Q}_x = 0$ at both ends. Substitution of the CSS set and their derivatives into these equations, one obtains the following four equations

$$q_1 A_{on} + q_2 \left(\frac{v_o + v_\ell}{2} \right) + q_3 \left(\frac{w_o + w_\ell}{2} \right) + \sum_{k=1}^{\infty} \left[q_1 A_{kn} + q_2 \{v_o + v_\ell(-1)^k + kB_{kn}\} \right.$$
$$\left. + q_3 \{w_o + w_\ell(-1)^k + kC_{kn}\} \right] = 0 \tag{23}$$

$$q_1 A_{on} + q_2 \left(\frac{v_o + v_\ell}{2} \right) + q_3 \left(\frac{w_o + w_\ell}{2} \right) + \sum_{k=1}^{\infty} \left[q_1 A_{kn} + q_2 \{v_o + v_\ell(-1)^k + kB_{kn}\} \right.$$
$$\left. + q_3 \{w_o + w_\ell(-1)^k + kC_{kn}\} \right] (-1)^k = 0 \tag{24}$$

$$q_4 A_{on} + q_5 \left(\frac{v_o + v_\ell}{2} \right) - q_{12} \left(\frac{w_o + w_\ell}{2} \right) + a_9 \left(\frac{\bar{\bar{w}}_o + \bar{\bar{w}}_\ell}{2} \right)$$
$$+ \sum_{k=1}^{\infty} [q_4 A_{kn} + q_5 \{v_o + v_\ell(-1)^k + kB_{kn}\} - q_{12} \{w_o + w_\ell(-1)^k + kC_{kn}\}$$
$$+ a_9 \{\bar{\bar{w}}_o + \bar{\bar{w}}_\ell(-1)^k - w_o k^2 - w_\ell k^2(-1)^k - k^3 C_{kn}\}] = 0 \tag{25}$$

$$q_4 A_{on} + q_5 \left(\frac{v_o + v_\ell}{2} \right) - q_{12} \left(\frac{w_o + w_\ell}{2} \right) + a_9 \left(\frac{\bar{\bar{w}}_o + \bar{\bar{w}}_\ell}{2} \right)$$
$$+ \sum_{k=1}^{\infty} [q_4 A_{kn} + q_5 \{v_o + v_\ell(-1)^k + kB_{kn}\} - q_{12} \{w_o + w_\ell(-1)^k + kC_{kn}\}$$
$$+ a_9 \{\bar{\bar{w}}_o + \bar{\bar{w}}_\ell(-1)^k - w_o k^2 - w_\ell k^2(-1)^k - k^3 C_{kn}\}] (-1)^k = 0 \tag{26}$$

The last two equations contain the inexpedient $\bar{\bar{w}}_o$, $\bar{\bar{w}}_\ell$. Therefore, the equations (25) and (26) are rewritten in terms of $\tilde{M}{}^o_x$ and $\tilde{M}{}^\ell_x$ as for the case of equations (14) and (15).

$$q_4 A_{on} + q_3 \left(\frac{v_o + v_\ell}{2}\right) - q_{18}\left(\frac{w_o + w_\ell}{2}\right) - \frac{2R^2(1-\nu^2)}{Eh\ell}\left(\frac{\tilde{M}{}^o_x + \tilde{M}{}^\ell_x}{2}\right) + \sum_{k=1}^{\infty} q_4 A_{kn}$$

$$+ q_{23}k B_{kn} + q_3\{v_o + v_\ell(-1)^k + kB_{kn}\} + q_{13}k C_{kn} - a_9\{w_o + w_\ell(-1)^k$$

$$+ kC_{kn}\}k^2 - q_{18}\{w_o + w_\ell(-1)^k + kC_{kn}\} - \frac{2R^2(1-\nu^2)}{Eh}\{\tilde{M}{}^o_x + \tilde{M}{}^\ell_x(-1)^k\} = 0.$$

$$\tag{27}$$

$$q_4 A_{on} + q_3 \left(\frac{v_o + v_\ell}{2}\right) - q_{18}\left(\frac{w_o + w_\ell}{2}\right) - \frac{2R^2(1-\nu^2)}{Eh\ell}\left(\frac{\tilde{M}{}^o_x + \tilde{M}{}^\ell_x}{2}\right) + \sum_{k=1}^{\infty} q_4 A_{kn}$$

$$+ q_{23}k B_{kn} + q_3\{v_o + v_\ell(-1)^k + kB_{kn}\} + q_{13}k C_{kn} - a_9\{w_o + w_\ell(-1)^k$$

$$+ kC_{kn}\}k^2 - q_{18}\{w_o + w_\ell(-1)^k + kC_{kn}\} - \frac{2R^2(1-\nu^2)}{Eh\ell}\{\tilde{M}{}^o_x + \tilde{M}{}^\ell_x(-1)^k\}(-1)^k = 0.$$

$$\tag{28}$$

The frequency determinant may now be constructed. Substitution of the relations for the coefficients A_{on}, A_{kn}, B_{kn}, and C_{kn} (19,20) into the eight constraint conditions due to the geometric and the natural boundary conditions (22–24, 27, 28) leads to the homogeneous matrix equation

$$[e_{i,j}]\{\tilde{N}{}^o_x, \tilde{N}{}^\ell_x, \tilde{M}{}^o_x, \tilde{M}{}^\ell_x, v_o, v_\ell, w_o, w_\ell\}^T = \{0\} \tag{29}$$

where $i,j = 1,2, \ldots 8$.

It is interesting to compare this equation to the result previously obtained by the present author [12] through a Rayleigh-Ritz procedure in conjunction with Lagrange multipliers. By redefining the unspecified end forces $\tilde{N}_x$, $\tilde{M}_x$ as proportional to the Lagrange multipliers $\tilde{N}{}^o_x = \frac{\lambda_1}{\pi R}$, $\tilde{N}{}^\ell_x = \frac{\lambda_2}{\pi R}$, $\tilde{M}{}^o_x = \frac{\lambda_3}{R\ell}$, and $\tilde{M}{}^\ell_x = \frac{\lambda_4}{R\ell}$, equation (29) becomes identical to the matrix equation in reference [12]. For a nontrivial solution of (29), the determinant

of the matrix must vanish,

$$\left|e_{i,j}\right| = 0 \tag{30}$$

resulting in a characteristic equation whose eigenvalues determine the natural frequencies of the shell. The corresponding eigenvectors determine the mode shapes. This frequency determinant is symmetric, and is based on the exact satisfaction of the boundary conditions. The natural frequency appears as a parameter $\Omega = (\omega/\omega_o)^2$ in this equation. Each element of this frequency determinant is an infinite series, and is listed in Appendix E. From the development of these equations it may be shown that there are two restrictions involved. The first restriction is $\Omega \neq \frac{1-\nu}{2}\left(1+\frac{\delta}{4}\right)n^2$. For axisymmetric (n = 0) shell vibrations, this becomes $\Omega \neq 0$; thereby invalidating (19) for rigid body motion. However, conditions associated with this rigid body motion are usually determined by inspection. For non-axisymmetric motion (n $\neq$ 0), the case of $\Omega = \frac{1-\nu}{2}\left(1+\frac{\delta}{4}\right)n^2$ is automatically rejected on the grounds that it leads to rigid body motion with non-zero frequency. The second restriction, $\square_{kn} \neq 0$, may be shown to imply that this CSS set cannot be used to solve the FSNT shell. This is not a serious drawback since it is easy to show that the second (SCC) set offers a very simple solution to this case.

IV. FREQUENCY DETERMINANT

In this section, the frequency determinant is derived for several cases of shell boundary conditions. The frequency determinant (30) is based on the boundary conditions of "freely-supported with no tangential constraints" at both ends (FSNT-FSNT). None of the FSNT-FSNT boundary conditions are satisfied by the CSS set of displacement functions on a term-by-term basis. The requirement to satisfy these boundary conditions by the CSS set in an overall sense led to the eight-by-eight frequency determinant (30). From this determinant, one can also obtain the frequency equation for any combination of boundary conditions among sixteen possible cases (See Appendix A) without going through the detailed procedures required to formulate the corresponding algebraic eigenvalue problems. This can be done by deleting inapplicable rows and columns from the frequency determinant (30). Therefore, solutions for shells with other boundary conditions lead to a smaller size determinant than for the FSNT-FSNT shell. For certain boundary conditions, the second set, SCC, leads to a simpler frequency determinant than the CSS set. The better choice of the mode displacement functions is suggested in Appendix B. Indeed, if the order of the frequency determinant obtained by the use of one set is "P" the one obtained by the other set is "8-P" for the same set of boundary conditions. Therefore, one can always obtain the exact frequency determinant with the order ≤ 4 for any case of boundary conditions.

A. <u>Simply-Supported Shell with No Axial Constraints at Both Ends (SNA-SNA)</u>

The simplest case of boundary conditions is, of course, the SNA shell since the CSS set of axial mode functions satisfies all the boundary conditions,

$$N_x = 0 , \quad v = 0 , \quad w = 0 , \quad M_x = 0 \quad (x = 0, \ell) \tag{31}$$

on a term-by-term basis. It should be noted that the boundary values

u, $\hat{N}_{x\theta}$, $\hat{Q}_x$, and $\frac{\partial w}{\partial x}$ at both ends are unspecified by this set. The frequency equation for the SNA shell is directly obtained from the equations of motion as a three-by-three determinant. The elements of the frequency determinant are no longer infinite series but reduce to algebraic terms. Past work on this problem includes that of Arnold and Warburton [1,2] and Dym [18]. The frequency equation is as follows

$$
\begin{vmatrix}
S_{11} & S_{12} & S_{13} \\
 & S_{22} & S_{23} \\
\text{Symm.} & & S_{33}
\end{vmatrix} = 0 \quad .
\tag{32}
$$

B. <u>Simply-Supported Shell with Axial Constraints at Both Ends (SS-SS)</u>

As a typical case involving the enforcement of geometric boundary conditions and the release of unwanted natural boundary conditions, consider the simply-supported shell with axial constraints at both ends. The boundary conditions are

$$
u = 0 , \quad v = 0 , \quad w = 0 , \quad M_x = 0 \quad (x = 0, \ell) \quad .
\tag{33}
$$

The CSS set automatically satisfies the last three equations and the unwanted natural boundary conditions $N_x = 0$ at both ends (see Eq. (31)). The frequency equation can be found from the geometrical constraint conditions of $u = 0$ at both ends, and at the same instance, leaving the end forces N_x unspecified. This is effectively done with (30) by retaining the rows and columns associated with $\tilde{N}_x^o$, $\tilde{N}_x^\ell$. The end result is the exact two-by-two frequency determinant

$$
\begin{vmatrix}
e_{1,1} & e_{1,2} \\
\text{symm.} & e_{2,2}
\end{vmatrix} = 0 \quad .
\tag{34}
$$

The elements of the determinant are listed in Appendix E. It is interesting to note that the second set, SCC, would involve the enforcement and release of geometric and natural boundary conditions, and lead to a six-by-six determinant.

C. Shell with Clamped Ends (C-C)

Consider the shell clamped at both ends. The boundary conditions are

$$u = 0 \; , \quad v = 0 \; , \quad w = 0 \; , \quad \frac{\partial w}{\partial x} = 0 \qquad (x = 0, \ell) \quad . \tag{35}$$

Among these conditions, the four geometric boundary conditions $u = 0$ and $\frac{\partial w}{\partial x} = 0$ at both ends are not automatically satisfied by the CSS set. Instead, this CSS set unnecessarily satisfied $N_x = 0$, $M_x = 0$ at the ends. The frequency determinant is obtained from (30) by retaining the rows and columns associated with $\overset{\sim o}{N_x}$, $\overset{\sim \ell}{N_x}$, $\overset{\sim o}{M_x}$, $\overset{\sim \ell}{M_x}$,

$$\begin{vmatrix} e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} \\ & e_{2,2} & e_{2,3} & e_{2,4} \\ & & e_{3,3} & e_{3,4} \\ \text{Symm.} & & & e_{4,4} \end{vmatrix} = 0 \quad . \tag{36}$$

If the second set, SCC, is used, one would obtain a different four-by-four frequency determinant.

D. Shell with Free Ends (F-F)

As a typical case involving the enforcement of natural boundary conditions and the release of unwanted geometrical boundary conditions consider a free-free shell with boundary conditions

$$N_x = 0 \; , \quad \hat{N}_{x\theta} = 0 \; , \quad \hat{Q}_x = 0 \; , \quad M_x = 0 \qquad (x = 0, \ell) \quad . \tag{37}$$

From t.. properties of CSS it is apparent that the tangential and radial displacem. ts, v and w respectively, are identically zero at the ends. Therefore, t. releasing procedure is required to remove these unwanted geometric bouna ~y conditions. By defining v and w separately at the end points (17) a. including these values in the subsequent differentiation via Stokes' transform tion, the exact frequency equation can be found from

the enforcement of natural boundary conditions $\hat{N}_{x\theta} = 0$, $\hat{Q}_x = 0$ at the ends. The result is obtained from (30) by retaining the rows and columns associated with v_o, v_ℓ, w_o, w_ℓ

$$
\begin{vmatrix}
e_{5,5} & e_{5,6} & e_{5,7} & e_{5,8} \\
 & e_{6,6} & e_{6,7} & e_{6,8} \\
 & & e_{7,7} & e_{7,8} \\
\text{Symm.} & & & e_{8,8}
\end{vmatrix} = 0 \quad .
\tag{38}
$$

The use of the second set also leads to another four-by-four determinant.

E. $\underline{\text{Clamped-Free Shell (C-F)}}$

This case involves the enforcement and release of geometrical and natural boundary conditions. For this non-symmetric set of boundary conditions

$$
u = 0 \ , \quad v = 0 \ , \quad w = 0 \ , \quad \frac{\partial w}{\partial x} = 0 \quad (x = 0)
$$

$$
N_x = 0 \ , \quad \hat{N}_{x\theta} = 0 \ , \quad \hat{Q}_x = 0 \ , \quad M_x = 0 \quad (x = \ell) \quad ,
\tag{39}
$$

the SSS set leads to a four-by-four frequency determinant. The $u = 0$ and $\partial w/\partial x = 0$ conditions at $x = 0$ must be enforced with the release of the end force and moment N_x, M_x. The $\hat{N}_{x\theta} = 0$ and $\hat{Q}_x = 0$ conditions at $x = \ell$ must be enforced with the release of the end displacements v, w. Retaining the rows and columns in (30) associated with N_x^o, M_x^o, v_ℓ, w_ℓ leads to

$$
\begin{vmatrix}
e_{1,1} & e_{1,3} & e_{1,6} & e_{1,8} \\
 & e_{3,3} & e_{3,6} & e_{3,8} \\
 & & e_{6,6} & e_{6,8} \\
\text{Symm.} & & & e_{8,8}
\end{vmatrix} = 0 \quad .
\tag{40}
$$

If the second set, SCC, were used, a different four-by-four frequency determinant would be obtained.

V. DISCUSSIONS OF RESULTS

A. Convergence and Accuracy

The frequency equations derived in the former chapter involve infinite series of algebraic terms. Before any numerical results are calculated, manipulations should be performed to make the rate of convergence of the infinite series terms as close to each other as possible. The convergence of certain sums can be accelerated by subtracting off a series with a known sum [19].

As a typical example, consider the following sequence of manipulations for a simple series

$$
\begin{aligned}
\sum_{m=1}^{\infty} \frac{m^2}{m^4-\Omega} &= \sum_{m=1}^{\infty} \frac{\Omega}{(m^4-\Omega)m^2} + \sum_{m=1}^{\infty} \frac{1}{m^2} \\
&= \sum_{m=1}^{\infty} \frac{\Omega^2}{(m^4-\Omega)m^6} + \Omega \sum_{m=1}^{\infty} \frac{1}{m^6} + \sum_{m=1}^{\infty} \frac{1}{m^2} \\
&= \sum_{m=1}^{\infty} \frac{\Omega^2}{(m^4-\Omega)m^6} + \frac{\pi^6}{945}\Omega + \frac{\pi^2}{6} \ .
\end{aligned}
\tag{41}
$$

Through the above process, the order of the series has been improved from m^{-2} to m^{-10}, and accordingly, the rate of convergence will be greatly accelerated.

The infinite series in the shell frequency determinant were treated in the same way as in the above example. In this way, fast convergence was obtained and the same upper limit could be used for all series in the frequency determinant; typically an upper limit of 50 terms was used. Frequency results were obtained from the frequency determinant by substituting values of ω/ω_o and monitoring the determinant until it vanished.

Convergence and accuracy of the present method is described in Tables 1 and 2. In order to facilitate comparison with the exact solutions reported by other contributors who used Flügge's and Donnell's theory,

Table 1. Convergence of Natural Frequency Parameters, ω/ω_o

Number of Terms	Case 1		Case 2		Case 3		Case 4	
	Sanders	Donnell	Sanders	Donnell	Sanders	Donnell	Sanders	Donnell
5	.01523	.01555	.05880	.0684	.3161	.3230	.3589	.3594
7	.01520	.01552	.05851	.0681	.3134	.3203	.3586	.3591
10	.01517	.01550	.05823	.06792	.3122	.3191	.3585	.3590
20	.01515	.01547	.05795	.06767	.3119	.3188	.3585	.3590
30	.01513	.01545	.05788	.06760	.3119	.3188	.3585	.3590
40	.01512	.01544	.05785	.06758	.3119	.3188	.3585	.3590
50	.01510	.01542	.05784	.06758	.3118	.3188	.3585	.3590
100	.01509	.01541	.05784	.06757	.3118	.3188	.3585	.3590
150	.01509	.01541	.05784	.06757	.3118	.3188	.3585	.3590
200	.01509	.01541	.05784	.06757	.3118	.3188	.3585	.3590
300	.01508	.01541	.05784	.06757	.3118	.3188	.3585	.3590
500	.01508	.01541	.05784	.06757	.3118	.3188	.3585	.3590
Ref. [19]	.01508*	.01541	.05787*	.06757	.3117*	.3188	.3583*	**

Case 1: Clamped-Clamped Shell ($\ell/R = 10$, $R/h = 500$, $\nu = 0.3$, $n = 4$)

Case 2: Clamped-Clamped Shell ($\ell/R = 10$, $R/h = 20$, $\nu = 0.3$, $n = 2$)

Case 3: Clamped-Clamped Shell ($\ell/R = 2$, $R/h = 20$, $\nu = 0.3$, $n = 3$)

Case 4: Free-Free Shell   ($\ell/R = 5$, $R/h = 20$, $\nu = 0.3$, $n = 1$)

*Solutions based on Flügge's theory [7,8].

**No exact solutions are available from literature.

Table 2. Comparison of Natural Frequency Parameters, ω/ω_o

n	m	Boundary Condition	Shell Parameters			Present Method		Others*	
			ℓ/R	R/h	ν	Sanders	Donnell	Flügge	Donnell
4	1	C-C	10	500	0.3	.01508	.01541	.01508	.01541
2	1	C-C	10	20	0.3	.05784	.06757	.05787	.06757
3	1	C-C	2	20	0.3	.3118	.3188	.3117	.3188
1	1	F-F	5	20	0.3	.3585	.3590	.3583	
0	1	F-F	8.1	20	0.3	.3669		.3671	
2	5	F-F	8.67	500	0.3	.4472		.4472	
2	1	C-F	1.14	20	0.3	.3076		.3082	
2	2	C-F	2.88	20	0.3	.3081		.3082	
2	3	C-F	5.07	20	0.3	.3079		.3082	
4	1	SS-SS	10	500	0.3	.01508		.01507	
1	1	SNAT-SNAT	30	500	0.3	.00702		.00700	
1	3	SNAT-SNAT	5	500	0.3	.3699		.3698	
0	1	SNA-SNA	2π	10	0.3	.296081	.295804	.296081**	.295804
2	1	SNA-SNA	2π	10	0.3	.095812	.119730	.095812**	.119730
4	1	SNA-SNA	2π	10	0.3	.425921	.454809	.425921**	.454808

*These results are due to Forsberg [6,7,8], Warburton [9,10] and Dym [18].

**These solutions are based on Sanders' theory [18].

the present method was also applied to Donnell's equations. In general, the solution approaches the exact frequency from above as the number of terms included in the series increases. However, as shown in Table 1, the use of more than 50 terms did not improve the solutions significantly. Table 2 compares the results of the present method with the available exact solutions. In all cases there was excellent comparison for a wide variety of boundary conditions and geometric parameters. For shells clamped at both ends, the solutions of the present method using Donnell's theory are the same as those reported by Forsberg [8]. The frequency parameters for the SNA-SNA shell by the use of Sanders' as well as Donnell's theory are also identical to the results given by Dym [18]. For other boundary conditions, F-F, C-F, SS-SS and SNAT-SNAT, the available solutions are all based on Flügge's shell theory. The differences between solutions obtained from different thin shell theories are usually negligible as revealed in Table 2.

Mode shapes for a particular natural frequency can always be investigated by substituting the calculated frequency parameter into the frequency matrix and finding the relative values of unspecified end forces and values, and finally evaluating the coefficients of the assumed modal displacement functions. Comparison of mode shapes between the present method and others is not attempted in this report.

B. Vibrations of Thermal Liner

The thermal liner of the Fast Test Reactor (FTR) is a thin, circular cylindrical shell situated concentrically inside the main reactor vessel. The annular gap between these two cylindrical vessels is occupied by sodium coolant. Since the thermal liner is subjected to sodium coolant flow, it is susceptible to flow-induced vibrations. A vibration problem

may occur if the natural frequencies of the thermal liner are close to the frequency of some vibration source in the reactor vessel. The support conditions of the thermal liner may be best represented as rigidly fixed at the bottom and free at the top: clamped-free (C-F) boundary conditions.

A series of experiments had been performed as a part of an effort to investigate the modal characteristics of the thermal liner [20]. The experimental model is approximately 1/14 scale of the prototype thermal liner for the FTR, giving the following dimensions and material properties:

Thickness $\qquad$ $h = 0.058$ in.

Radius (to the center

of thickness) $\qquad$ $R = 8.505$ in

Length $\qquad$ $\ell = 20.125$ in.

Material density $\qquad$ $\rho = 701 \times 10^{-6}$ lb-sec^2/in.^{4}

Young's modulus $\qquad$ $E = 26.5 \times 10^6$ psi

Poisson's ratio $\qquad$ $\nu = 0.3$

The experimental model was intended to simulate a clamped-free shell; however, insufficient clamping ($\partial w/\partial x \neq 0$) and imperfect axial constraint ($u \neq 0$) at the bottom has been judged a probable cause for disagreement between the experimental results and the analytical predictions of natural frequencies. Therefore, in this study the sensitivity of the natural frequencies to changes in the bottom-boundary conditions is investigated using the exact present solution method. The accuracy of the solutions is compared to the results obtained by two finite element computer codes: SAP IV [21] and NASTRAN [22]. The boundary conditions considered are:

(1) C-F : Clamped-Free

(2) SS-F : Simply Supported-Free

(3)   CNA-F: Clamped with No Axial Constrain-Free

(4)   SNA-F: Simply Supported with No Axial Constrain-Free.

In Table 3 the solutions of the present method are compared to the SAP IV and NASTRAN results for the thermal liner with clamped-free boundary conditions. The finite element grid used with the two computer codes is the same and consists of 10 divisions vertically and 9 divisions over a quarter of the shell circumferentially. Apparently the present exact method always gives lower values compared to the two finite element computer codes. The solutions of SAP IV are better than those of NASTRAN, as shown by their closeness to the results of the present method, because SAP IV uses a more accurate quadrilateral element.

The natural frequencies for the thermal liner model in vacuum are calculated by the present method for the above four cases of boundary conditions, and summarized in Table 4 and Fig. 2. The frequencies become smaller as the boundary conditions change from Case (1) to Case (4) for a given circumferential and an axial mode number, n and m. The effect of clamping ($\partial w/\partial x = 0$) can be evaluated by comparing the frequencies for the two pairs of boundary conditions: C-F and SS-F, CNA-F and SNA-F. The natural frequencies for SS-F are smaller than those for C-F by about less than 1%; thus they are not distinguishable in Fig. 2. The differences between frequencies for CNA-F and SNA-F are also small (< 1.5%), except for the modes of n = 0 to 3 with m = 1. The influence of axial constraint (u = 0) is evident in Fig. 2 and, in contrast to the effect of clamping, is significant throughout most of the region of interest. Note that the frequency curve for m = 1 is greatly shifted down for n < 5; moreover, the lowest natural frequencies for shell-type vibration (n ≥ 2) occur at n = 2 for CNA-F and SNA-F, in contrast to n = 4 for C-F and SS-F. However, for higher axial half-waves (m = 2,3), the differences are less than 20%

Table 3. Comparison of Natural Frequencies (Hz) for the Thermal
Liner with Clamped=Free Boundary Conditions

n	m	Present Method	SAP IV	NASTRAN
1	1	855.10		856.3
2	1	403.72	405.1	410.1
3	1	223.34	225.6	232.2
4	1	171.77	174.3	180.5
5	1	199.16	201.7	206.2
6	1	268.86	272.0	275.5
7	1	361.92	366.2	370.1
8	1	472.54	478.4	483.5
9	1	599.03		614.0
3	2	928.28		943.2
4	2	644.48		671.4
5	2	494.69	505.8	529.3
6	2	442.00	454.3	478.0
7	2	464.59	476.7	496.9
8	2	539.45		567.2
9	2	648.34		673.2

Table 4. Natural Frequencies (Hz) of Thermal Liner

for Various Boundary Conditions

| n	m	C-F	SS-F	CNA-F	SNA-F
0	1	1497.90	1497.90	0	0
1	1	855.10	853.85	12.50	0
2	1	403.72	402.59	27.79	21.57
3	1	223.34	222.61	61.81	58.93
4	1	171.77	171.34	112.92	111.32
5	1	199.16	198.63	179.68	178.68
6	1	268.86	268.72	261.69	261.03
7	1	361.92	361.82	358.81	358.34
8	1	472.54	472.45	470.97	470.63
9	1	599.03	598.94	598.15	597.89
10	1	740.81	740.77	740.32	740.12
0	2	2351.51	2346.69	1497.90	1497.90
1	2	2318.98	2318.98	2077.48	2077.24
2	2	1437.11	1435.35	1408.98	1407.79
3	2	928.28	925.39	844.59	842.44
4	2	644.48	641.43	552.22	548.87
5	2	494.69	491.99	418.54	414.14
6	2	442.00	439.89	390.44	385.77
7	2	464.59	463.06	434.21	430.04
8	2	539.45	538.31	522.17	518.70
9	2	648.34	647.41	638.16	635.28
10	2	781.15	780.37	774.87	772.44
1	3	3076.05	3070.64	2723.53	2723.02
2	3	2487.60	2484.64	2479.20	2476.84
3	3	1834.82	1830.35	1811.52	1808.06
4	3	1367.64	1361.83	1323.62	1318.80
5	3	1057.12	1050.67	1001.38	995.01
6	3	864.82	858.43	808.93	801.14
7	3	767.65	761.85	720.27	711.69
8	3	750.67	745.75	715.49	707.05
9	3	798.18	794.11	774.21	766.53
10	3	893.60	890.18	877.72	871.04
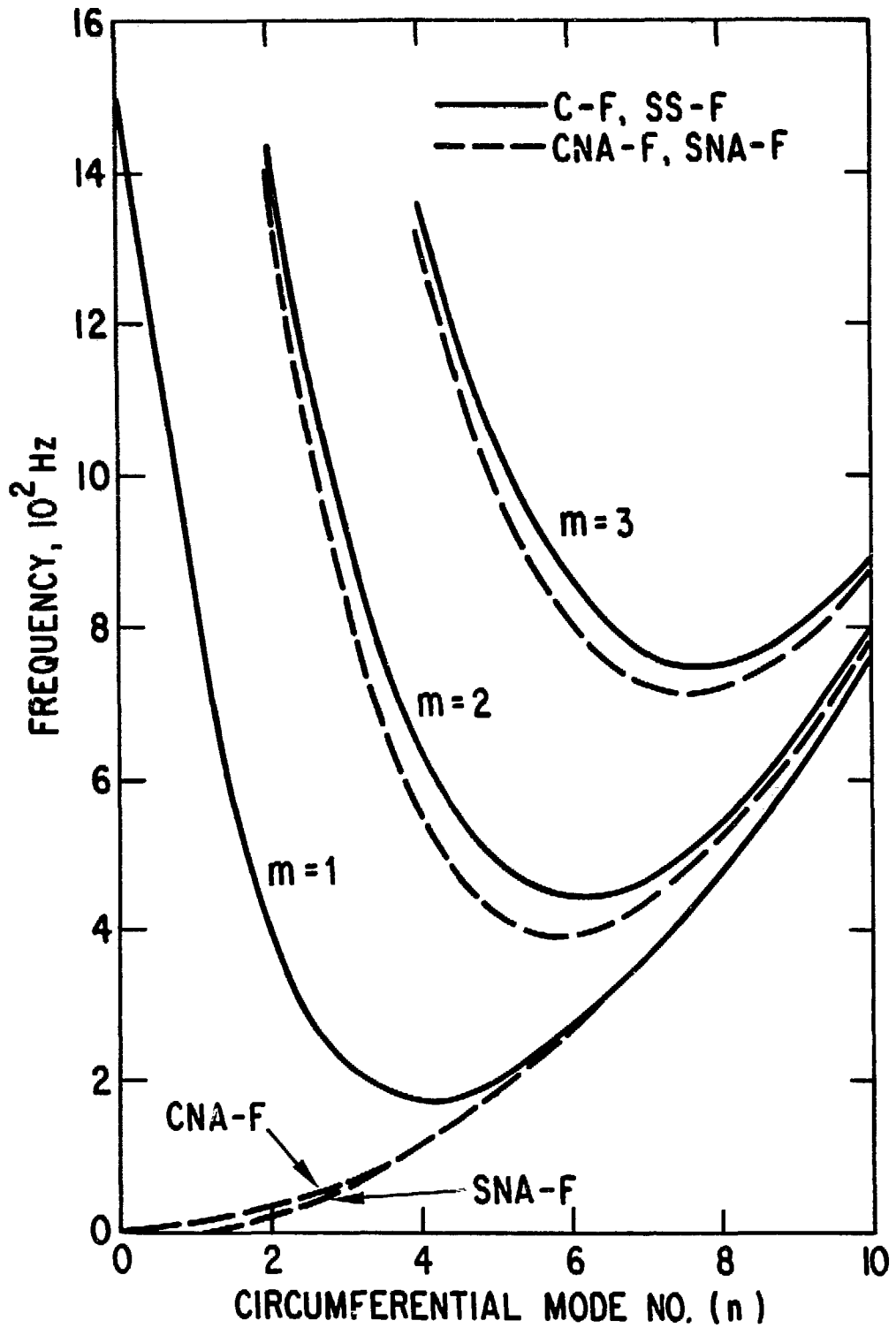
Fig. 2. Natural Frequencies of Thermal Liner Model

throughout the entire range of circumferential wave numbers n. A comparative study of the natural frequencies between the analytical predictions and experimental results is not considered to be within the scope of the present report and is reported elsewhere [20].

VI. CONCLUSIONS

In this report, a general solution procedure for the free vibration of a circular cylindrical shell is presented. The present method offers distinct computational advantages over other exact solution methods [6-11]. Table 5 compares the present method and other exact solution procedures. An uncoupled explicit expression of frequency determinant is obtained by directly solving Sanders' shell equations. The frequency equation is obtained in the process of satisfying all boundary conditions exactly by the assumed simple Fourier series. This frequency equation involves infinite series which are evaluated by the use of appropriate numerical techniques and a computer. The order of frequency determinant may vary from 1 to 8 depending on the boundary conditions to be satisfied. However, an appropriate choice of displacement functions (i.e., the CSS or SCC set) can always furnish a frequency determinant of low order (≤ 4) for any given boundary conditions. The frequency determinant obtained by others [6-11] is always eight-by-eight and coupled with eighth-order algebraic equations resulting from the shell equations. The elements of frequency determinant consist of transcendental functions in place of the infinite series of the present method. However, it should be noted that the transcendental function is evaluated as a series in the present day computer.

In conclusion, the method presented in this report is a relatively simple, exact-solution method for the free vibration problem of circular cylindrical shells.

Table 5. Comparisons of the Present Method with

the Other Exact Solution Method

Items	Present Method	Others[*]
1. Frequency Determinant	Uncoupled explicit expression	Coupled with 8th order algebraic equation
2. Order of Frequency Determinant	Varies (1 ~ 4) dependent on boundary conditions	Fixed (8)
3. Element of Frequency Determinant	Infinite series (rate of convergence can be accelerated)	Transcendental Functions (also series in computer computation)

[*]References [6-11]

APPENDIX A

Boundary Conditions of Circular Cylindrical Shell

Case	Code	Description	Boundary Conditions			
1	C	Clamped	w	$\partial w/\partial x$	u	v
2	CNA	Clamped with no Axial Constraints	w	$\partial w/\partial x$	N_x	v
3	CNT	Clamped with no Tangential Constraints	w	$\partial w/\partial x$	u	$\hat{N}_{x\theta}$
4	CNAT	Clamped with no Axial, Tangential Constraints	w	$\partial w/\partial x$	N_x	$\hat{N}_{x\theta}$
5	SS	Simply-Supported	w	M_x	u	v
6	SNA	Simply-Supported with no Axial Constraints (Shear Diaphragm)	w	M_x	N_x	v
7	SNT	Simply-Supported with no Tangential Constraints	w	M_x	u	$\hat{N}_{x\theta}$
8	SNAT	Simply-Supported with no Axial, Tangential Constraints	w	M_x	N_x	$\hat{N}_{x\theta}$
9	FS	Freely-Supported	$\hat{Q}_x$	$\partial w/\partial x$	u	v
10	FSNA	Freely-Supported with no Axial Constraints	$\hat{Q}_x$	$\partial w/\partial x$	N_x	v
11	FSNT	Freely-Supported with no Tangential Constraints	$\hat{Q}_x$	$\partial w/\partial x$	u	$\hat{N}_{x\theta}$
12	FSNAT	Freely-Supported with no Axial, Tangential Constraints	$\hat{Q}_x$	$\partial w/\partial x$	N_x	$\hat{N}_{x\theta}$
13	FAT	Free with Axial, Tangential Constraints	$\hat{Q}_x$	M_x	u	v
14	FT	Free with Tangential Constraints	$\hat{Q}_x$	M_x	N_x	v
15	FA	Free with Axial Constraints	$\hat{Q}_x$	M_x	u	$\hat{N}_{x\theta}$
16	F	Free	$\hat{Q}_x$	M_x	N_x	$\hat{N}_{x\theta}$

APPENDIX B

Shell Boundary Conditions to be Satisfied

on using the First (CSS) or the Second (SCC) Set

Case	Code	First Set (CSS)	Second Set (SCC)	Better Choice
1	C	u $\partial w/\partial x$	v w	1, 2
2	CNA	$\partial w/\partial x$	v w N_x	1
3	CNT	u $\partial w/\partial x$ $\hat{N}_{x\theta}$	w	2
4	CNAT	$\partial w/\partial x$ $\hat{N}_{x\theta}$	w N_x	1, 2
5	SS	u	v w M_x	1
6	SNA	$-$	v w N_x M_x	1
7	SNT	u $\hat{N}_{x\theta}$	w M_x	1, 2
8	SNAT	$\hat{N}_{x\theta}$	w N_x M_x	1
9	FS	u $\partial w/\partial x$ $\hat{Q}_x$	v	2
10	FSNA	$\partial w/\partial x$ $\hat{Q}_x$	v N_x	2
11	FSNT	u $\partial w/\partial x$ $\hat{N}_{x\theta}$ $\hat{Q}_x$	$-$	2
12	FSNAT	$\partial w/\partial x$ $\hat{N}_{x\theta}$ $\hat{Q}_x$	N_x	2
13	FAT	u $\hat{Q}_x$	v M_x	2
14	FT	$\hat{Q}_x$	v N_x M_x	1
15	FA	u $\hat{N}_{x\theta}$ $\hat{Q}_x$	M_x	2
16	F	$\hat{N}_{x\theta}$ $\hat{Q}_x$	N_x M_x	2

APPENDIX C

First Set (CSS) of Displacement Functions and Their Derivatives

$$u(x,\theta) = \left(A_{on} + \sum_{k=1}^{\infty} A_{kn}\cos\frac{k\pi x}{\ell}\right)\cos n\theta \qquad (0 \leq x \leq \ell)$$

$$u_{,x} = -\left(\frac{\pi}{\ell}\right)\sum_{k=1}^{\infty} A_{kn}\, k \sin\frac{k\pi x}{\ell}\cos n\theta \qquad (0 < x < \ell)$$

$$u_{,x}(0,\theta) = -\left(\frac{\pi^2}{2\ell}\right)\bar{u}_o\cos n\theta$$

$$u_{,x}(\ell,\theta) = \left(\frac{\pi^2}{2\ell}\right)\bar{u}_\ell\cos n\theta$$

$$u_{,xx} = \left(\frac{\pi}{\ell}\right)^2\left[\frac{\bar{u}_o+\bar{u}_\ell}{2} + \sum_{k=1}^{\infty} \{\bar{u}_o + \bar{u}_\ell(-1)^k - k^2 A_{kn}\}\cos\frac{k\pi x}{\ell}\right]\cos n\theta \qquad (0 \leq x \leq \ell)$$

$$v(x,\theta) = \sum_{k=1}^{\infty} B_{kn}\sin\frac{k\pi x}{\ell}\sin n\theta \qquad (0 < x < \ell)$$

$$v(0,\theta) = -\frac{\pi}{2}v_o\sin n\theta$$

$$v(\ell,\theta) = \frac{\pi}{2}v_\ell\sin n\theta$$

$$v_{,x} = \left(\frac{\pi}{\ell}\right)\left[\frac{v_o+v_\ell}{2} + \sum_{k=1}^{\infty}\{v_o + v_\ell(-1)^k + kB_{kn}\}\cos\frac{k\pi x}{\ell}\right]\sin n\theta \qquad (0 \leq x \leq \ell)$$

$$v_{,xx} = -\left(\frac{\pi}{\ell}\right)^2\left[\sum_{k=1}^{\infty}\{v_o k + v_\ell k(-1)^k + k^2 B_{kn}\}\sin\frac{k\pi x}{\ell}\right]\sin n\theta \qquad (0 < x < \ell)$$

$$v_{,xx}(0,\theta) = -\frac{\pi^3}{2\ell^2}\bar{\bar{v}}_o\sin n\theta$$

$$v_{,xx}(\ell,\theta) = \frac{\pi^3}{2\ell^2}\bar{\bar{v}}_\ell\sin n\theta$$

$$w(x,\theta) = \sum_{k=1}^{\infty} C_{kn}\sin\frac{k\pi x}{\ell}\cos n\theta \qquad (0 < x < \ell)$$

$$w(0,\theta) = -\frac{\pi}{2}\,w_{o}\cos n\theta$$

$$w(\ell,\theta) = \frac{\pi}{2}\,w_{\ell}\cos n\theta$$

$$w_{,x} = \left(\frac{\pi}{\ell}\right)\left[\frac{w_{o}+w_{\ell}}{2} + \sum_{k=1}^{\infty}\{w_{o} + w_{\ell}(-1)^{k} + kC_{kn}\}\cos\frac{k\pi x}{\ell}\right]\cos n\theta \qquad (0 \le x \le \ell)$$

$$w_{,xx} = -\left(\frac{\pi}{\ell}\right)^{2}\left[\sum_{k=1}^{\infty}\{w_{o}k + w_{\ell}k(-1)^{k} + k^{2}C_{kn}\}\sin\frac{k\pi x}{\ell}\right]\cos n\theta \qquad (0 < x < \ell)$$

$$w_{,xx}(0,\theta) = -\frac{\pi^{3}}{2\ell^{2}}\,\bar{\bar{w}}_{o}\cos n\theta$$

$$w_{,xx}(\ell,\theta) = \frac{\pi^{3}}{2\ell^{2}}\,\bar{\bar{w}}_{\ell}\cos n\theta$$

$$w_{,xxx} = \left(\frac{\pi}{\ell}\right)^{3}\left[\frac{\bar{\bar{w}}_{o}+\bar{\bar{w}}_{\ell}}{2} + \sum_{k=1}^{\infty}\{\bar{\bar{w}}_{o} + \bar{\bar{w}}_{\ell}(-1)^{k} - w_{o}k^{2} - w_{\ell}k^{2}(-1)^{k}\right.$$
$$\left. - k^{3}C_{kn}\}\cos\frac{k\pi x}{\ell}\right]\cos n\theta \qquad (0 \le x \le \ell)$$

$$w_{,xxxx} = \left(\frac{\pi}{\ell}\right)^{4}\left[\sum_{k=1}^{\infty}\{-\bar{\bar{w}}_{o}k - \bar{\bar{w}}_{\ell}k(-1)^{k} + w_{o}k^{3} + w_{\ell}k^{3}(-1)^{k}\right.$$
$$\left. + k^{4}C_{kn}\}\sin\frac{k\pi x}{\ell}\right]\cos n\theta \qquad (0 < x < \ell)$$

$$w_{,xxxx}(0,\theta) = \bar{\bar{\bar{w}}}_{o}\cos n\theta$$

$$w_{,xxxx}(\ell,\theta) = \bar{\bar{\bar{w}}}_{\ell}\cos n\theta$$

APPENDIX D

Symbols used in Equations (14)-(28) and Appendix E

$$a_1 = (\pi R/\ell)^2 \ , \quad a_{21} = \frac{1-\nu}{2}\left(1 + \frac{\delta}{4}\right) n^2 \ , \quad a_2 = a_{21} - \Omega$$

$$a_3 = - \frac{\pi R}{\ell}\left[\frac{1+\nu}{2} - \frac{3(1-\nu)}{8}\,\delta\right] n \ , \quad a_4 = - \frac{\pi R}{\ell}\left[\nu - \frac{(1-\nu)}{2}\,\delta n^2\right]$$

$$a_5 = \left(\frac{\pi R}{\ell}\right)^2 \frac{1-\nu}{2}\left(1 + \frac{9}{4}\,\delta\right) \ , \quad a_{61} = (1 + \delta)n^2 \ , \quad a_6 = a_{61} - \Omega$$

$$a_7 = \left(\frac{\pi R}{\ell}\right)^2 \frac{3-\nu}{2}\,\delta n \ , \quad a_8 = (1 + \delta n^2)n \ , \quad a_9 = \left(\frac{\pi R}{\ell}\right)^4 \delta$$

$$a_{10} = 2\left(\frac{\pi R}{\ell}\right)^2 \delta n^2 \ , \quad a_{11} = 1 + \delta n^4 - \Omega \ , \quad a_{12} = 1 + \delta n^4$$

$$q_1 = \frac{\pi R}{\ell} \frac{1-\nu}{2}\left(1 - \frac{3}{4}\,\delta\right) n \ , \quad q_2 = -a_5 \ , \quad q_3 = -\left(\frac{\pi R}{\ell}\right)^2 \frac{3(1-\nu)}{2}\,\delta n$$

$$q_4 = -\left(\frac{\pi R}{\ell}\right) \frac{1-\nu}{2}\,\delta n^2 \ , \quad q_5 = -a_7 \ , \quad q_6 = -(a_9 m^2 + q_{12})$$

$$q_{12} = \left(\frac{\pi R}{\ell}\right)^2 (2-\nu)\delta n^2 \ , \quad q_{13} = q_{23}n \ ,$$

$$q_{18} = 2\left(\frac{\pi R}{\ell}\right)^2 (1-\nu)\delta n^2 \ , \quad q_{23} = -\left(\frac{\pi R}{\ell}\right)^2 \nu\delta n$$

$$s_{11} = a_1 k^2 + a_{21} - \Omega \ , \quad s_{12} = a_3 k \ , \quad s_{13} = a_4 k$$

$$s_{22} = a_5 k^2 + a_{61} - \Omega \ , \quad s_{23} = a_7 k^2 + a_8 \ , \quad s_{33} = a_9 k^4 + a_{10} k^2 + a_{12} - \Omega$$

$$a_\alpha = (s_{22}s_{33} - s_{23}^2)/\square_{kn} \ , \quad a_\beta = (s_{13}s_{23} - s_{12}s_{33})/\square_{kn}$$

$$a_\gamma = (s_{12}s_{23} - s_{13}s_{22})/\square_{kn} \ , \quad b_\beta = (s_{11}s_{33} - s_{13}^2)/\square_{kn}$$

$$b_\gamma = (s_{12}s_{13} - s_{11}s_{23})/\square_{kn} \ , \quad c_\gamma = (s_{11}s_{22} - s_{12}^2)/\square_{kn}$$

$$\square_{kn} = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{vmatrix}$$

APPENDIX E

Elements of Frequency Determinant Using the First

Set (CSS) of Displacement Functions

$$e_{1,1} = \frac{1}{2a_2} + \sum_{k=1}^{\infty} a_\alpha \quad , \quad e_{1,2} = \frac{1}{2a_2} + \sum_{k=1}^{\infty} a_\alpha (-1)^k \quad ,$$

$$e_{1,3} = \sum_{k=1}^{\infty} a_\gamma k \quad , \quad e_{1,4} = \sum_{k=1}^{\infty} a_\gamma k (-1)^k \quad ,$$

$$e_{1,5} = \frac{q_1}{2a_2} + \sum_{k=1}^{\infty} \{q_1 a_\alpha + q_2 a_\beta k + q_3 a_\gamma k\}$$

$$e_{1,6} = \frac{q_1}{2a_2} + \sum_{k=1}^{\infty} \{q_1 a_\alpha + q_2 a_\beta k + q_3 a_\gamma k\}(-1)^k$$

$$e_{1,7} = \frac{q_4}{2a_2} + \sum_{k=1}^{\infty} \{q_4 a_\alpha + q_5 a_\beta k + q_6 a_\gamma k\}$$

$$e_{1,8} = \frac{q_4}{2a_2} + \sum_{k=1}^{\infty} \{q_4 a_\alpha + q_5 a_\beta k + q_6 a_\gamma k\}(-1)^k$$

$$e_{2,2} = e_{1,1} \quad , \quad e_{2,3} = e_{1,4} \quad , \quad e_{2,4} = e_{1,3} \quad , \quad e_{2,5} = e_{1,6} \quad ,$$

$$e_{2,6} = e_{1,5} \quad , \quad e_{2,7} = e_{1,8} \quad , \quad e_{2,8} = e_{1,7}$$

$$e_{3,3} = \sum_{k=1}^{\infty} c_\gamma k^2 \quad , \quad e_{3,4} = \sum_{k=1}^{\infty} c_\gamma k^2 (-1)^k \quad ,$$

$$e_{3,5} = \sum_{k=1}^{\infty} \{q_1 c_\alpha k + q_2 c_\beta k^2 + q_3 c_\gamma k\}$$

$$e_{3,6} = \sum_{k-1}^{\infty} \{q_1 c_\alpha k + q_2 c_\beta k^2 + q_3 c_\gamma k\}(-1)^k$$

$$e_{3,7} = \frac{1}{2} + \sum_{k=1}^{\infty} \{q_4 c_\alpha k + q_5 c_\beta k^2 + q_6 c_\gamma k^2 + 1\}$$

$$e_{3,8} = \frac{1}{2} + \sum_{k=1}^{\infty} \{q_4 c_\alpha k + q_5 c_\beta k^2 + q_6 c_\gamma k^2 + 1\}(-1)^k$$

$$e_{4,4} = e_{3,3} \quad , \quad e_{4,5} = e_{3,6} \quad , \quad e_{4,6} = e_{3,5} \quad , \quad e_{4,7} = e_{3,8} \quad ,$$

$$e_{4,8} = e_{3,7}$$

$$e_{5,5} = \frac{q_1^2}{2a_2} + \frac{q_2}{2} + \sum_{k=1}^{\infty} \{q_1^2 a_\alpha + q_2^2 b_\beta k^2 + q_3^2 c_\gamma k^2 + 2q_1 q_2 a_\beta k$$

$$+ 2q_2 q_3 b_\gamma k^2 + 2q_1 q_3 a_\gamma k + q_2\}$$

$$e_{5,6} = \frac{q_1^2}{2a_2} + \frac{q_2}{2} + \sum_{k=1}^{\infty} \{q_1^2 a_\alpha + q_2^2 b_\beta k^2 + q_3^2 c_\gamma k^2 + 2q_1 q_2 a_\beta k$$

$$+ 2q_2 q_3 b_\gamma k^2 + 2q_1 q_3 a_\gamma k + q_2\}(-1)^k$$

$$e_{5,7} = \frac{q_1 q_4}{2a_2} + \frac{q_3}{2} + \sum_{k=1}^{\infty} \{q_4(q_1 a_\alpha + q_2 b_\alpha k + q_3 c_\alpha k) + q_5(q_1 a_\beta k + q_2 b_\beta k^2 + q_3 c_\beta k^2)$$

$$+ q_6(q_1 a_\gamma k + q_2 b_\gamma k^2 + q_3 c_\gamma k^2) + q_3\}$$

$$e_{5,8} = \frac{q_1 q_4}{2a_2} + \frac{q_3}{2} + \sum_{k=1}^{\infty} \{q_4(q_1 a_\alpha + q_2 b_\alpha k + q_3 c_\alpha k) + q_5(q_1 a_\beta k + q_2 b_\beta k^2$$

$$+ q_3 c_\beta k^2) + q_6(q_1 a_\gamma k + q_2 b_\gamma k^2 + q_3 c_\gamma k^2) + q_3\}(-1)^k$$

$$e_{6,6} = e_{5,5} \quad , \quad e_{6,7} = e_{5,8} \quad , \quad e_{6,8} = e_{5,7}$$

$$e_{7,7} = \frac{q_4^2}{2a_2} - \frac{q_{18}}{2} + \sum_{k=1}^{\infty} \{q_4(q_4 a_\alpha + q_5 b_\alpha k + q_6 c_\alpha k) + q_5(q_4 a_\beta k + q_5 b_\beta k^2 + q_6 c_\beta k^2)$$

$$+ q_6(q_4 a_\gamma k + q_5 b_\gamma k^2 + q_6 c_\gamma k^2) - a_9 k^2 - q_{18}\}$$

$$e_{7,8} = \frac{q_4^2}{2a_2} - \frac{q_{18}}{2} + \sum_{k=1}^{\infty} \{q_4(q_4 a_\alpha + q_5 b_\alpha k + q_6 c_\alpha k) + q_5(q_4 a_\beta k + q_5 b_\beta k^2 + q_6 c_\beta k^2)$$

$$+ q_6(q_4 a_\gamma k + q_5 b_\gamma k^2 + q_6 c_\gamma k^2) - a_9 k^2 - q_{18}\}(-1)^k$$

$$e_{8,8} = e_{7,7}$$

REFERENCE.;

1. Arnold, R. N., and Warburton, G. B., "Flexural Vibrations of the Walls of Thin Cylindrical Shells Having Freely Supported Ends," Proc. Roy. Soc. London, A197, 1948, pp. 238-256.

2. Arnold, R. N., and Warburton, G. B., "The Flexural Vibrations of Thin Cylinders," Proc. Inst. of Mech. Engrs., A167, 1953, pp. 62-80.

3. Leissa, A. W., "Vibration of Shells," 1973, NASA SP-288.

4. Sharma, C. B., and Johns, D. J., "Free Vibration of Cantilever Circular Cylindrical Shells - A Comparative Study," J. of Sound and Vibration, Vol. 25, No. 3, 1972, pp. 433-449.

5. Sharma, C. B., "Calculation of Natural Frequencies of Fixed-Free Circular Cylindrical Shells," J. of Sound and Vibration, Vol. 35, No. 1, 1974, pp. 55-76.

6. Forsberg, K., "Influence of Boundary Conditions on the Modal Characteristics of Thin Cylindrical Shells," AIAA Journal, Vol. 2, No. 12, Dec. 1964, pp. 2150-2157.

7. Forsberg, K., "Axisymmetric and Beam-Type Vibrations of Thin Cylindrical Shells," AIAA Journal, Vol. 7, No. 2, Feb. 1969, pp. 221-227.

8. Forsberg, K., "A Review of Analytical Methods used to Determine the Modal Characteristics of Cylindrical Shells," NASA CR-613, June 1965.

9. Warburton, G. B., "Vibration of Thin Cylindrical Shells," Journal of Mechanical Engineering Science, Vol. 7, No. 4, 1965, pp. 399-407.

10. Warburton, G. B., and Higgs, J., "Natural Frequencies of Thin Cantilever Cylindrical Shells," J. of Sound and Vibration, Vol. 11, No. 3, 1970, pp. 335-338.

11. Goldman, R. L., "Mode Shapes and Frequencies of Clamped-Clamped Cylindrical Shells," AIAA Journal, Vol. 12, No. 12, Dec. 1974, pp. 1755-1756.

12. Chung, Ho, "A General Method of Solution for Vibrations of Cylindrical Shells," Ph.D. Thesis, Tufts University (March 1974).

13. Sanders, J. L., Jr., "An Improved First Approximation Theory for Thin Shells," Rept. 24, June 1959, NASA.

14. Budiansky, B., and Sanders, J. L., Jr., "On the Best First-order Linear Shell Theory," 1963 Progress in Applied Mechanics - The Prager Anniversary Volume, 1963, 129-140.

15. Bromwich, T. J. I'a, "An Introduction to the Theory of Infinite Series," 2nd Edition Rev., MacMillan, London, 1955.

16. Kraus, H., Thin Elastic Shells, John Wiley & Sons, Inc., 1967.

17. Budiansky, B., and Di Prima, R. C., "Bending Vibrations of Uniform Twisted Beams," J. of Mathematics and Physics, Vol. 39, No. 4, December 1960, pp. 237-245.

18. Dym, C. L., "Some New Results for the Vibrations of Circular Cylinders," J. of Sound and Vibration, Vol. 29, No. 2, 1973, pp. 189-205.

19. Jolley, L. B. W., Summation of Series, 2nd Edition Rev., Dover, New York, 1961.

20. Mulcahy, T. M., Turula, P., Chung, H., and Jendrzejczyk, J. A., "Analytical and Experimental Study of Two Concentric Cylinders Coupled by a Fluid Gap," Technical Memorandum ANL-CT-75-36, Argonne National Laboratory, April 1975.

21. Bathe, K-J., Wilson, E. L., and Peterson, F. E., "SAP IV - A Structural Analysis Program for Static and Dynamic Response of Linear Systems," National Technical Information Service, EERC 73-11, June 1973.

22. McCormick, C. W., The NASTRAN User's Manual (level 15), National Aeronautics and Space Administration (1972).