Vibrations of Cylindrical Shell Structures Filled With Layered Viscoelastic Material

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Abstract. A thin-walled shell and a thick-walled mass (cylinder) in contact with it, made of a different material, are structural elements of many machines, apparatus, and structures. The paper considers forced steady-state vibrations of cylindrical shell structures filled with a layered viscoelastic material. The study aims to determine the damping properties of vibrations of a structurally inhomogeneous cylindrical mechanical system under the influence of harmonic loads. The dynamic stress-strain state of a three-layer cylindrical shell filled with a viscoelastic material under the action of internal time-harmonic pressure is investigated. The oscillatory processes of the filler and the bonded shell satisfy the Lamé equations. At the contact between the shell and the filler, the rigid contact conditions are satisfied. Dependences between stresses and strains for a linear viscoelastic material are presented in the form of the Boltzmann-Voltaire integral. The method of separation of variables, the method of the theory of potential functions (special functions), and the Gauss method are used to solve this problem. Based on the analysis of the numerical results, it was found that the dependence of the resonant amplitude of the shell displacements on the viscous properties of the filler is 12-15%. Analysis of the results obtained shows that the study of vibrations of shells containing fillers according to the rod theory will lead to rather large erroneous results (up to 20%).

1 Introduction

Acceleration of scientific and technological progress, intensive development of the national economy will lead to the need to create new systems and technologies that should work in wide ranges of speeds, temperatures, pressures, and loads. At the same time, they must be convenient for operation, comfortable for humans, environmentally friendly, silent, etc. Such systems, first of all, include a variety of vehicles: airplanes, helicopters, automobiles,

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etc. For such systems, one of the topical aspects is the problem of suppressing vibrations arising under the influence of external loads [1-4].

A thin-walled shell and a thick-walled mass (cylinder) made of different material in contact with its inner or outer surfaces are structural elements in many machines, apparatus, and structures [5,6]. The works [7, 8, 9] are devoted to calculating the stress-strain state of the dynamic processes of thin-walled shells and the elastic or viscoelastic filler contained in them. In addition, these issues affect the literature on solid fuel engines to one degree or another. For example, the design of solid-propellant rocket engine structures has led to a long-term study of a structural system consisting of a hollow viscoelastic cylinder enclosed in a thin body (viscoelastic shell) [10, 11]. A large number of problems of axisymmetric configuration under plane deformation have been formulated and solved. In this case, the quasi-static linear theory of viscoelasticity was usually used.

The aim of the work is to study the damping properties of vibrations of a structurally inhomogeneous mechanical system under the influence of local loads.

To study the damping properties of vibrations of a mechanical system, taking into account the influence of the geometric and physical-mechanical parameters of the shell, the method of separated variables, the method of the theory of potential functions (special functions), and the Gauss method are used.

2 Methods

2.1 Formulation of the problem

Consider a long viscoelastic multilayer cylinder with a circular cavity of constant radius $a$ and with a radius of the outer surface $b$ (total radius), along which the cylinder is attached to an elastic (or viscoelastic) shell (Figure 1). Some layers (elements) of the cylinder can be massless. In this case, they are characterized by stiffness coefficients. The material of the multilayer cylinder is a typical highly filled polymer, the physical and mechanical characteristics of which are determined by the nature of the binder, the adhesion of the binder to the multilayer filler. The system "cylinder - body (shell)" is subject to time-varying uniform normal (vibration) pressure.

The dynamic stress state of a three-layer cylindrical shell filled with a viscoelastic material under the action of internal dynamic pressure is considered.

![Fig.1. Three-layer construction (h₁- first layer, h₂- second layer, 3- filler)](image)

The equations of small vibrations of a layered cylinder have the form [12]

$$\ddot{u} + (\tilde{\lambda} + \tilde{\mu}) \frac{\partial^2 u}{\partial t^2} = \rho_k \frac{a^2 \ddot{u}}{\partial x^2}, (k =1, 2, 3... N-1)$$

(1)
If a k=N, then (1) describes the equation of motion of the shell. Here \( \overline{u}_k^{i}(u_{rk}, u_{\theta k}, u_{\chi k}) \) and \( \overline{u}_N^{i}(w, v, u) \) - displacement vectors, is the density of the k-th material.

The boundary conditions at the contact between the layers are

\[
\sigma_r^i = \sigma_r^{i+1}; \tau_{\rho x}^i = \tau_{\rho x}^{i+1}; \tau_{\rho \varphi}^i = \tau_{\rho \varphi}^{i+1} \\
u_r^i = u_r^{i+1}; u_{\theta}^i = u_{\theta}^{i+1}; u_{\chi}^i = u_{\chi}^{i+1}
\]

(2)

where index \( i \) matches the layer number \( i = 1,2,3,\ldots,N; \ r = r_j \). The layers are numbered starting with the outer one. If between \( i - \) oh and \( i + l - \) the layer contains a massless cylinder, then the boundary conditions take the form (at \( r = r_j \)):

\[
\sigma_{rri} = k_r(u_{ri} - u_{r(i+1)}), \tau_{r\theta i} = k_\theta(u_{\theta i} - u_{\theta(i+1)})
\]

\[
\tau_{\rho x i} = k_x(u_{\chi i} - u_{\chi(i+1)}),
\]

(3)

where \( k_r, k_\theta, k_x \) are interaction coefficients. As an example, consider a cylindrical filler \( r = R_0 \) and \( r = R_1 \), bonded sheath.

In the structure (figure 1), the left end (\( z = 0 \)) is rigidly clamped, and the right end (\( z = l \)) is free. When studying forced vibrations, it is assumed that a load is applied to the inner surface

\[
r = R_0; \sigma_{rz} = \sigma_{r\varphi} = 0, \sigma_{rr} = Q_o(t) \]

\[
Q_o(t) = P_0e^{-\rho t}, \quad r = R_0; \sigma_r = \tau_{r\varphi} = \tau_{rz} = 0
\]

(4)

where \( \rho \) is the frequency of external influences; \( P_0 \) is the amplitude of external influences; \( R_0 \) and \( R_b \) are inner and outer radii of the cylinder under consideration.

As an example, consider the dynamic stressed state of a half-cylinder under the action of internal pressure, i.e., consider the change in the resonance region depending on the parameters at different viscosities. The relationship between stresses and strains for a linear viscoelastic material can be represented as

\[
\sigma_{ijk} = \lambda_k(1 - R_{jk})\theta_k \delta_{ij} + 2\mu_k(1 - R_{jk})\varepsilon_{ijk}
\]

(5)

Here, \( \lambda_k, \mu_k \) are operator modulus of elasticity \( k \)-th element, \( \varepsilon_{ijk} \) is strain tensor elements \( k \)-th element, \( \theta \) is volumetric deformation

\[
R_\lambda(k) f(t) = \int_0^t R_{\lambda k}(t - \tau) f(\tau) d\tau; R_{\mu k} f(t) = \int_0^t R_{\mu k}(t - \tau) f(\tau) d\tau
\]

\( f(t) \) is time derivative; \( R_{\lambda k}(t - \tau), R_{\mu k}(t - \tau) \) are relaxation kernels; \( \lambda_{0k}, \mu_{0k} \) are instantaneous coefficients of the modulus of elasticity (\( k = 1, \ldots, N \)) [13].

The equations of motion of this structure satisfy the Lamé equations in a cylindrical coordinate system and are solved in the displacement potentials [14]. Let us introduce three scalar potential functions \( \phi_k, \psi_k, \chi_k \), which are solutions of equations [14].

\[
\nabla^2 \phi_k = \int_{-\infty}^{t} R_{\lambda k}(t - \tau) \nabla^2 \phi_k(\tau) d\tau = \frac{1}{c_{0k}^2} \frac{\partial^2 \phi_k}{\partial t^2},
\]

\[
\nabla^2 \psi_k = \int_{-\infty}^{t} R_{\mu k}(t - \tau) \nabla^2 \psi_k(\tau) d\tau = \frac{1}{c_{0k}^2} \frac{\partial^2 \psi_k}{\partial t^2},
\]

\[\]
\[ \nabla^2 \chi_k - \int_{-\infty}^{t} R(t-\tau) \nabla^2 \chi_k(\tau) d\tau = \frac{1}{c_{10k}^2} \frac{\partial^2 \chi_k}{\partial t^2}, \]  

(6)

where \( c_{10k}^2 = \frac{\lambda_{0k} + 2\mu_{0k}}{p_k} \), \( c_{\perp k}^2 = \mu_0/\rho_k \) is the velocity of propagation of longitudinal and transverse waves, respectively.

### 2.2 Solution methods

The solution of integro-differential equations (6) is sought in the form

\[ \varphi_k(r, \theta, z, t) = \sum_{n=0}^{\infty} \alpha_{kn} (\alpha_k r) \left\{ \begin{array}{l} \cos n\theta \\ \sin n\theta \end{array} \right\} \cos (\gamma_k x + \delta) e^{ipt}, \]

\[ \psi_k(r, \theta, z, t) = \sum_{n=0}^{\infty} \psi_{kn}(\beta_k r) \left\{ \begin{array}{l} \sin n\theta \\ \cos n\theta \end{array} \right\} \sin (\gamma_k x + \delta) e^{ipt}, \]

\[ \chi_k(r, \theta, z, t) = \sum_{n=0}^{\infty} \chi_{kn}(\beta_{\perp k} r) \left\{ \begin{array}{l} \cos n\theta \\ \sin n\theta \end{array} \right\} \sin (\gamma_k x + \delta) e^{ipt}, \]  

(7)

where \( \varphi_n, \psi_n, \chi_n \) are amplitudes of longitudinal and transverse potentials, \( \alpha_k \) and \( \beta_k \) are wave numbers; \( \gamma_k \) is complex wavenumber, \( \delta_k \) is phase shift, \( p \) is frequency of external loads.

Substituting (7) into (6), we obtain the Helmholtz equations [15]. Their solutions are expressed through the Bessel functions \( J_n \) and \( N_n \):

\[ \varphi_{nk} = [A_{1nk} J_n(\alpha_{tk} r) + A_{2nk} N_n(\alpha_{tk} r)]; \]

\[ \psi_{nk} = [B_{1nk} J_n(\beta_{tk} r) + B_{2nk} N_n(\beta_{tk} r)]; \]

\[ \chi_{nk} = [C_{1nk} J_n(\beta_{\perp k} r) + C_{2nk} N_n(\beta_{\perp k} r)], \]

(8)

where \( A_{1nk}, A_{2nk}, B_{1nk}, B_{2nk}, C_{1nk}, C_{2nk} \) are arbitrary constants, \( \alpha_{tk}, \beta_{tk} \) are wave numbers of longitudinal and shear waves, which are complex quantities,

\[ \alpha_{tk}^2 = \mu_{tk}^2 - \gamma_k^2, \beta_{tk}^2 = \mu_{tk}^2 - \gamma_{\perp k}^2, \mu_{tk} = p/c_{tk} \Gamma_{k1}, \mu_{tk} = p/c_{tk} \Gamma_{k1}, \]

\[ \Gamma_{k1} = 1 - \Gamma_{k1}^2(p) - i \Gamma_{k2}^2(p); \Gamma_{k2} = 1 - \Gamma_{k2}^2(p) - i \Gamma_{k3}^2(p), \]

\[ \Gamma_{k1}^2(p) = \int_{0}^{p} R_{k1}(\tau) \cos p \tau d\tau, \Gamma_{k2}^2(p) = \int_{0}^{p} R_{k2}(\tau) \cos p \tau d\tau, \]

\[ \Gamma_{k3}^2(p) = \int_{0}^{p} R_{k1}(\tau) \sin p \tau d\tau, \Gamma_{k4}^2(p) = \int_{0}^{p} R_{k2}(\tau) \sin p \tau d\tau, \]

the cosine and sine of the Fourier image of the relaxation kernels, respectively; \( p \) is a real value. The calculations used the Koltunov-Rzhanitsyn three-parameter relaxation kernel: \( R_k(t) = A_k e^{-\beta_k t} + \tau_1^{-\alpha_k} \). Continuous and shear wave potentials \( \varphi_k, \psi_k, \chi_k \) is related to the components of the displacement vector, in a cylindrical coordinate system, by dependencies [16]:

\[ u_{\varphi x} = \frac{\partial \varphi_k}{\partial x} + \frac{1}{\mu_{tk}} \left( \mu_{tk} \chi_k + \frac{\partial^2 \chi_k}{\partial x^2} \right), \]

\[ u_{\varphi \theta} = \frac{\partial \varphi_k}{\partial \theta} + \frac{1}{\mu_{tk}} \left( \frac{\partial \chi_k}{\partial r} + \frac{\partial^2 \chi_k}{\partial \theta \partial x} \right). \]
\[ u_{rk} = \frac{\partial \psi_k}{\partial r} + \frac{\partial \phi_k}{\partial \theta} + \frac{1}{\mu_{tk}} \frac{\partial^2 \chi_k}{\partial x \partial r} \]

In this case, the expressions for stresses and mixed are complex quantities. Actual stresses are determined by the following expressions:

\[ Re \left[ (\sigma_{rr}^{(R)} + i\sigma_{rr}^{(I)}) e^{ipt} \right] = \sigma_{rr}^{(R)} \cos pt - \sigma_{rr}^{(I)} \sin pt \]

Suppose that a liquid or gas from the inner side acts on the structure with harmonic pressure. Then this pressure can be represented as

\[ p = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_k \sin \alpha_k x \cos n \theta e^{ipt} \quad (7) \]

If we consider a structure located on the xOy plane, then in the shell and filler, the radial displacement has the form:

\[ w = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} W_{kn} \sin \alpha_k x \cos n \theta e^{ipt}, \]

\[ u_r = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \alpha_k \left[ A_{1kn} J_n(\gamma_{rk}r) + A_{2kn} N_n(\gamma_{rk}r) \right] + \frac{\gamma_{ik}^2}{\mu_{tk}} \left[ C_{1kn} J_n(\gamma_{rk}r) + C_{2kn} N_n(\gamma_{rk}r) \right] \right\} \sin \alpha_k x \cos n \theta e^{ipt}, \quad (8) \]

where \( \gamma_{ik}^2 = \mu_i^2 - \alpha_i^2, \gamma_{tk}^2 = \mu_t^2 - \alpha_t^2, \mu_l = p/\bar{a}_l \mu_t = p/\bar{a}_t \).

If \( \gamma_{ik}^2 < 0, \gamma_{tk}^2 < 0 \), then instead of \( J_n(z), N_n(z) \) use accordingly \( l_n(z), K_n(z) \).

Thus, for each layer of a cylindrical body (for an extended cylindrical layer) solutions can be written in terms of displacements in the following form:

\[ u_r = \sum_{n=0}^{\infty} \left\{ F_n \frac{dH_n^{(1)}(ar)}{dr} + D_{2n} \gamma_p H_{n+1}^{(1)}(\beta r) + M_{1n} \frac{d^2H_n^{(1)}(\beta r)}{dr^2} \right\} \left( \cos \frac{n \theta}{\sin n \theta} \right) \sin(\gamma x + \delta) e^{ipt} \]

\[ u_{\theta} = \sum_{n=0}^{\infty} \left\{ -F_n n H_n^{(1)}(ar)/r + D_{2n} \gamma_p H_{n+1}^{(1)}(\beta r) - M_{1n} \frac{dH_n^{(1)}(\beta r)}{dr} \right\} \left( \sin \frac{n \theta}{\cos n \theta} \right) \sin(\gamma x + \delta) e^{ipt}, \quad (9) \]

\[ u_z = \sum_{n=0}^{\infty} \left\{ -F_n i \gamma_p H_n^{(1)}(ar) - D_{2n} \left[ \frac{dH_n^{(1)}(\beta r)}{dr} + \frac{n+1}{r} H_n^{(1)}(\beta r) \right] \right\} \left( \cos \frac{n \theta}{\sin n \theta} \right) \cos(\gamma x + \delta) e^{ipt} \]

Displacement expressions (9) contain three arbitrary constants \( F_n, D_{2n}, M_{1n} \) and satisfy the equations of motion (1) for all values of the frequency \( p \). For a short spatial cylinder, the number of arbitrary constants will be six. If the obtained solutions in terms of potentials are expressed with the help of special Bessel and Neumann functions, and also the boundary conditions are used, then we obtain a system of inhomogeneous algebraic equations from 12 unknowns.

**3 Results and Discussion**

The resulting aggregate stresses are shown in Figures 2-6. The relationship between the amplitude of the shear stress and the frequency of the forced stress is shown in Figure 2, for different \( \delta_{x/\pi} : 1)0.3; 2)0.2; 3)0.1, \ R_0/R = 0.5 \) и \( \gamma_{p} = 0.33.\)
Here, $\delta_1$ is logarithmic decrement, which is determined by expression: $\delta_1 = 2\pi \ln|\omega_{i1}/\omega_{R1}|$. Figure 3 shows that with a decrease in the damping decrement coefficient, the corresponding amplitude-frequency characteristics increase.
Fig. 4. Relationship between the shear stress amplitude and the forced stress frequency

Fig. 5. The relationship between the amplitude of the loop voltage and the frequency of forced voltages

Figure 4 shows the change in the first resonance peak depending on the frequency for different values of the ratio of the radii (R is the outer radius, $R_0$ inner radius of the filler) of the cylindrical body. The considered mechanical system is dissipatively homogeneous [17, 18]. It is seen that with an increase in the ratio of the radii, the corresponding amplitude-frequency characteristics of the loop voltage decrease.

In the second resonance peak $\gamma_p R_0 = 0.70$. The third resonance peak shows the amplitude-frequency characteristics of the loop voltage at different values of the damping decrement [19].

Figure 4 shows the change in the amplitude of the tangential stresses of the aggregate depending on the frequency for different values of the rheological parameters of the material. (1. $A=0.078$; 2. $A=0.036$). The considered mechanical system is dissipatively inhomogeneous (for a shell $R_0=0$) [20, 21]. It can be seen that the amplitudes of displacements, depending on the frequency, take on their maximum value at different
values of the natural vibrations of mechanical systems. Figure 5 shows the changes in the amplitude of the contour stresses in the shell depending on the dimensionless frequencies, i.e., the resonance curves are presented. Curves 1,2,3 indicate the stresses between the cylindrical aggregate and the contact surface at the point $x = l/2$ at different values of the damping decrement: 1) 0.3; 2) 0.2; 3) 0.1 and at $\gamma_p = 0.33$.

![Graph showing resonant curves]

**Fig.6.** The relationship between the amplitude of the radial stress and the frequency of forced stresses

The first resonance peak was obtained at $h/R = 0.95$, the second - at $h/R = 0.75$, and the third - at $h/R = 0.5$. It is seen that at $\delta_1/\pi = 0.1(A = 0.078, \alpha = 0.5)$, even if the maximum values of the loop stresses are reduced by 20-25%, they retain their intensity. Exactly the same - the same tendency persists for radial stresses.

Figure 6 shows the change in the amplitude of the radial stresses of the filler depending on the frequency for different values of the rheological parameters of the shell and filler material (1. $A = 0.078$; 2. $A = 0.036$, $A = 0.018$). The considered mechanical system is dissipatively homogeneous. It was found that the amplitudes of displacements, their maximum value are taken at the first resonant frequency of oscillations of mechanical systems. For a dissipatively inhomogeneous mechanical system, the displacement amplitudes take their maximum value at the second or third resonance frequencies.

The obtained numerical results were compared with the numerical results obtained from the analytical solution [22, 23] for the same values of parameters. The difference in results was up to 12%.

The research results can also be used to develop a new design for drying cotton seeds [24, 25], as well as in improving the energy efficiency and reliability of the power supply [26, 27].

Here it was also possible to consider the control of body vibrations by introducing servo constraints [28-30], which will be considered later.

### 4 Conclusions

Based on the analysis of numerical results, it was established:

1. The dependence of the resonant amplitude of displacements of the shell on the viscous properties of the filler is 12-15%.
2. Analysis of the results obtained shows that studies of vibrations of shells containing a filler, according to the rod theory, will lead to rather large erroneous results (up to 20%).

Reference

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