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Violation of Cosmic Censorship in the Gravitational Collapse of a Dust Cloud*

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Abstract. The behaviour of the outgoing light rays in the gravitational collapse of an inhomogeneous spherically symmetric dust cloud is analyzed. It is shown that, for an open subset of initial density distributions, the first singular event, which occurs at the center of symmetry, is the vertex of an infinity of future null geodesic cones which intersect future null infinity. The frequency of the corresponding light rays is infinitely redshifted.

Introduction

Tolman [1] in 1934 found a class of solutions of Einstein's equations which represent inhomogeneous spherically symmetric dust clouds. In 1939 Oppenheimer and Snyder [2] studied, as an idealized model of gravitational collapse, the special case of Tolman's class of solutions which corresponds to a homogeneous spherically symmetric dust cloud. They analyzed the behavior of the outgoing light rays and were thus led to the introduction of the black hole idea. The Oppenheimer and Snyder study, although treating only a very special case, was highly important for providing the intuition which guided the approach to more general problems. In fact, the concept of a trapped surface played a central role in the Penrose–Hawking singularity theorems [3]. Then, a conjecture was introduced, derived again from the Oppenheimer–Snyder example, namely that no singularities which are visible from infinity can develop from regular initial data [3]. This is the weaker form of what is now called "the cosmic censorship conjecture." There is a number of important results, among them the area theorem of black holes [3], which assume the truth of this conjecture. Finally, Penrose [4] introduced a stronger form of the cosmic censorship conjecture which states that any singularities that arise from regular

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** On leave of absence from Max Planck Institute for Astrophysics, D-8046 Garching, Federal Republic of Germany initial data are not even locally visible. This stronger conjecture is verified in the Oppenheimer–Snyder example. When precisely formulated [4], it is equivalent to global hyperbolicity of the spacetime manifold.

Although much has been learned from the example of the homogeneous spherically symmetric dust cloud, the general inhomogeneous Tolman class of solutions has not been given the attention it deserves, despite the fact that it is the only known infinite dimensional family of asymptotically flat solutions of Einstein's equations. The difficulties of such an investigation lie in solving the differential equation for the light rays.

In this paper I treat the general inhomogeneous Tolman class and I give a thorough analysis of the behavior of the outgoing light rays. The results are rather surprising, leading to a different picture of gravitational collapse.

Section 1

We take as our starting point the Tolman solution [1] which gives the evolution of a spherically symmetric dust cloud (that is to say a perfect fluid with equation of state $p=0$) which starts from rest. The description is given in co-moving coordinates τ and R, where R, which takes values in the nonnegative real numbers, labels the spherical shells of dust, and τ is the proper time along the lines $R=\text{const}$, that is, along the world lines of the dust. Let $A(\tau,R)$ be the area of the 2-sphere $\tau=\text{const}$, $R=\text{const}$. One defines the radius $r(\tau,R)$ by setting $A=4\pi r^2$. As it is possible to make an arbitrary relabeling of the spherical dust shells by $R\mapsto f(R)$, we fix the labeling by requiring that on the initial surface $\tau=0$, R coincides with the radius:

$$r(0, R)=R. \tag{1.1}$$

We denote by $\rho(R)$ the initial mass density and by $m(R)$ the initial mass distribution:

$$m(R)=4\pi\int_0^R \rho(S)S^2dS. \tag{1.2}$$

Let $a(R)$ denote the mean density within the sphere of radius R on the initial surface:

$$a(R)=\frac{3m(R)}{4\pi R^3}. \tag{1.3}$$

The radius $r(R,\tau)$ is given by the parametric equations:

$$\tau=\left(\frac{3}{32\pi a}\right)^{1/2}(\eta+\sin\eta), \tag{1.4}$$

$$r=\frac{R}{2}(1+\cos\eta);\quad 0\leqq\eta<\pi.$$

The value $\eta=0$ of the parameter corresponds to the initial surface $\tau=0$ while the value $\eta=\pi$ corresponds to the final singular surface, as we discuss below. On the final singular surface we have $r=0$. We are considering the evolution of the dust cloud in the future of the initial surface (so we take η, $\tau\geqq 0$). Let us denote partial differentiation with respect to τ by a dot and partial differentiation with respect to R

by a prime. We have (from (1.4))

$$\dot{r} = -\left(\frac{2m}{r} - \frac{2m}{R}\right)^{1/2}. \tag{1.5}$$

(From this equation together with (1.1) we verify that we are considering a dust cloud which is initially at rest.) The evolution of the mass density $\varepsilon(\tau, R)$ is given by:

$$\varepsilon = \frac{m'}{4\pi r' r^2} = \frac{\rho}{r'}\left(\frac{R}{r}\right)^2. \tag{1.6}$$

If we define the mass $m(\tau, R_1)$ which at time τ is included within the dust shell $R = R_1$ by $m(\tau, R_1) = 4\pi \int_0^{r(\tau, R_1)} \varepsilon r^2 dr(\tau, R)$, we see that it is conserved:

$$m(\tau, R_1) = 4\pi \int_0^{R_1} \varepsilon r^2 r' dR = 4\pi \int_0^{R_1} \rho(R) R^2 dR = m(R_1). \tag{1.7}$$

Equations (1.4)–(1.6) are formally identical to their analogues in the Newtonian theory but their interpretation here is quite different because the spacetime has the metric

$$dS^2 = -d\tau^2 + e^{2\omega} dR^2 + r^2 d\Sigma^2, \tag{1.8}$$

where $d\Sigma^2$ is the canonical metric of the 2-sphere and

$$e^{\omega} = \frac{r'}{\left(1 - \frac{2m}{R}\right)^{1/2}}. \tag{1.9}$$

We shall assume that $2m/R < 1$, which will be seen below to be equivalent to the assumption that the initial hypersurface contains no trapped surfaces. We also assume that the mass density on the initial surface is smooth, that is, ρ is such that if extended to the negative R-axis as an even function is a C^∞ function on the entire real line. Since we wish to consider only finite dust clouds, we assume in addition that ρ is of compact support. Let R_0 be the radius of the support of ρ. For $R > R_0$ the solution will be the unique spherically symmetric vacuum solution of Einstein's equations, namely, the Schwarzschild solution. The Schwarzschild coordinates are $t(\tau, R)$ and $r(\tau, R)$, where r is the radius defined above and t is defined by requiring 1) that the curves $t = \text{const}$ are orthogonal to the curves $r = \text{const}$, and 2) that $\lim_{R\to\infty} t(\tau, R) = \tau$. In the vacuum region $R > R_0$ we find

$$t = 2m_0 \log \frac{\left(\frac{R}{2m_0} - 1\right)^{1/2} + \tan\frac{\eta}{2}}{\left(\frac{R}{2m_0} - 1\right)^{1/2} - \tan\frac{\eta}{2}}$$

$$+ R\left(\frac{R}{2m_0} - 1\right)^{1/2} \cdot \tfrac{1}{2}(\eta + \sin\eta) + 2m_0\left(\frac{R}{2m_0} - 1\right)^{1/2} \eta,$$

$$\tau = \left(\frac{R^3}{8m_0}\right)^{1/2} (\eta + \sin\eta), \tag{1.10}$$

where m_0 is the total mass of the cloud,

$$m_0 = \int_0^{R_0} 4\pi R^2 \rho(R) dR, \quad m(R) = m_0 \quad \text{for} \quad R \geqq R_0. \tag{1.11}$$

The Schwarzschild time coordinate t exists in the vacuum region only for $\tan(\eta/2) < (R/2m_0 - 1)^{1/2}$ which in view of (1.4) is equivalent to $r > 2m_0$. The transformation of coordinates given by (1.10) and (1.4) reduces the metric in the vacuum region to the standard Schwartzschild form:

$$ds^2 = -\left(1 - \frac{2m_0}{r}\right)dt^2 + \left(1 - \frac{2m_0}{r}\right)^{-1} dr^2 + r^2 d\Sigma^2. \tag{1.12}$$

The surface $\eta = \pi$ or

$$\tau = \tau_s(R) = \left(\frac{3\pi}{32a(R)}\right)^{1/2} \tag{1.13}$$

in the Tolman solution is an essential singularity; not only do the mass density and curvature invariants blow up there, but the metric itself admits no continuous extension.

The only other set where the solution is not regular is the set where $r' = 0$ and $\eta < \pi$. This is the set of points where the spherical shells of dust cross each other. These shell crossing singularities are inessential [5]; although the mass density and curvature invariants blow up there, the metric is in fact continuous. This is seen if we replace the coordinates (τ, R) by the coordinates (τ, r). In these coordinates the Christofel symbols belong to L^∞ locally in the region $0 \leqq \eta < \pi$. The Riemann curvature tensor and the energy momentum tensor are well defined distribution in this region, and therefore we have a weak solution of Einstein's equations in the entire region.

From Eqs. (1.4) we obtain:

$$r' = \tfrac{1}{2}(1 + \cos\eta) - \frac{1}{4}\frac{\sin\eta(\eta + \sin\eta)}{(1 + \cos\eta)}\frac{Ra'}{a}. \tag{1.14}$$

We see from the above equation that on a world line $R = \text{const} > 0$ we have a shell crossing point if and only if $a'(R) > 0$; there is no shell crossing on the central world line $R = 0$. We wish to avoid considering the shell crossing singularities so that we can concentrate on the structure of the essential singularity. We therefore assume the initial mass density ρ is a monotonically non-increasing function of $R: \rho'(R) \leqq 0$ for all R. This implies that the same is true for the initial mean density a and we have no shell crossing.

Section 2

The curves $r = \text{const}$ have a normal r_μ with components $\dot{r}$, r' and we have:

$$g^{\mu\nu} r_\mu r_\nu = -\dot{r}^2 + e^{-2\omega} r'^2 = 1 - \frac{2m}{r}. \tag{2.1}$$

It follows that the curves $r = \text{const}$ are timelike, null or spacelike according to whether $r > 2m$, $= 2m$ or $< 2m$.

An outgoing light ray satisfies the differential equation

$$d\tau/dR = e^{\omega(\tau, R)}. \tag{2.2}$$

Along an outgoing light ray we have:

$$\frac{dr}{d\tau} = \dot{r} + r'\frac{dR}{d\tau} = \dot{r} + r'e^{-\omega} = \left(1 - \frac{2m}{R}\right)^{1/2} - \left(\frac{2m}{r} - \frac{2m}{R}\right)^{1/2}. \tag{2.3}$$

It follows that $dr/d\tau > 0$, $= 0$, < 0 according to whether $r > 2m$, $= 2m$, $< 2m$. Thus the surface defined by:

$$r(\tau, R) = 2m(R), \tag{2.4}$$

or equivalently,

$$\tau = \tau_a(R) = \left(\frac{3}{32\pi a(R)}\right)^{1/2}\left[\cos^{-1}\left(\frac{4m(R)}{R} - 1\right) + \sqrt{1 - \left(\frac{4m(R)}{R} - 1\right)^2}\right] \tag{2.5}$$

is the locus of turning points of the outgoing light rays, namely, the set of events where the outgoing light rays stop diverging and start to converge. This is the apparent horizon.

We see from Eq. (2.5) that the condition that $2m(R)/R < 1$ is equivalent to the condition that the apparent horizon lies to the future of the initial hypersurface and thus the initial hypersurface contains no trapped surfaces. We also see that the apparent horizon lies to the past of the singular surface except at the center $R = 0$ where there is a second order contact point:

$$\begin{aligned} \tau_a(0) &= \tau_s(0) = \left(\frac{3\pi}{32\alpha}\right)^{1/2} = \tau_0, \\ \tau_a'(0) &= \tau_s'(0) = 0, \\ \tau_a''(0) &= \tau_s''(0) = \frac{3}{10}\left(\frac{3\pi}{8\alpha^3}\right)^{1/2}\beta, \end{aligned} \tag{2.6}$$

where $\alpha = \rho(0)$ is the central density and $\beta = -\rho''(0)/2$.

Implicitly differentiating (2.4) we obtain:

$$r'dR + \dot{r}d\tau = 2m'dR;$$

therefore the tangent to the apparent horizon curve on the (τ, R) plane is given by:

$$\frac{d\tau}{dR} = \left(1 - \frac{2m}{R}\right)^{-1/2}(r' - 2m') = e^{\omega} - \frac{2m'}{\left(1 - \frac{2m}{R}\right)^{1/2}}, \tag{2.7}$$

and we have:

$$d\tau^2 - e^{2\omega}dR^2 = \frac{4m'}{\left(1 - \frac{2m}{R}\right)}(m' - r'). \tag{2.8}$$

From the above two equations we conclude that the apparent horizon is null and future directed in the vacuum region $R \geqq R_0$, while it is either spacelike or past directed for $0 < R < R_0$.

Any light ray which intersects the apparent horizon goes to the singularity: A light ray either goes to the singularity before reaching the surface of the cloud, or it reaches the surface of the cloud later than the apparent horizon. From this point the light ray will be in Schwarzschild geometry and since it is now later than the event horizon $r = 2m_0$ of the Schwarzschild solution, it will reach the singularity within a finite affine parameter interval.

In the vacuum region the apparent horizon coincides with the event horizon of the Schwarzschild solution $r = 2m_0$. It follows that the outgoing light ray which coincides with the curve $r = 2m_0$ in the vacuum region is the generator of the boundary of the region of the spacetime which can communicate with infinity, that is, the generator of the event horizon. Since the event horizon coincides with the apparent horizon in the vacuum region, any light ray later than the event horizon intersects the apparent horizon inside the cloud.

No light ray can emmanate from the singular surface except from the singular point at the center. Indeed, if a light ray emmantes from the singular surface at some $R_1 > 0$, then by continuity there must exist an $\varepsilon > 0$ such that for $R_1 < R < R_1 + \varepsilon$ the light ray is later than the apparent horizon and earlier than the singular surface (since the apparent horizon is everywhere but at the center earlier than the singular surface). Therefore by Eq. (2.3) $dr/dR < 0$, while $r(R_1) = 0$ and $r(R) > 0$, a contradiction. Thus no point of the singular surface except possibly the singular point at the center is visible to an observer. However, since the possibility of a light ray emmanating from the singular point at the center is not excluded, the strong cosmic censorship conjecture is left open. By the above argument such a light ray must lie to the past of the apparent horizon for $0 < R < \varepsilon$ if ε is sufficiently small.

A light ray which goes to infinity must be earlier than the apparent horizon at $R = R_0$ (the surface of the cloud). Since for $0 < R < R_0$, the apparent horizon is either spacelike or past directed, such a light ray must also be earlier than the apparent horizon for all $0 < R < R_0$. However, because the apparent horizon touches the singular surface at the center, it is possible that the light ray tends to the apparent horizon and therefore also to the singular surface for $R \to 0$. The singular point at the center would then be visible from infinity. We see therefore that, due to the peculiar nature of the singular point at the center, arguments based on comparison between a light ray and the apparent horizon leave open even the weak form of the cosmic censorship conjecture.

We shall now show:

Proposition 1. *If $\rho''(0) = 0$, then strong cosmic censorship holds.*

Proof. We shall show that in this case there exists an $R_1 > 0$ such that for $0 \leqq R \leqq R_1 : \tau_a(R) \leqq \tau_0$. As we remarked above, for a light ray $\tau(R)$ emmanating from the singular point at the center there must exist an $R_2 > 0$ such that $\tau(R) < \tau_a(R)$ for $0 < R < R_2$. So here for $0 < R < \min\{R_1, R_2\}$ we would have $\tau(R) < \tau_0 = \tau(0)$, which is impossible for an outgoing light ray. This would therefore prove the nonexistence of light rays emmanating from the singular point at the center and hence the strong cosmic censorship.

Since $\rho''(0)=0$, there exists a positive constant k such that

$$a(0)-a(R)\leqq kR^4. \tag{2.9}$$

In view of the fact that for $0\leqq x<1$ it holds:

$$\cos^{-1}(2x-1)+\sqrt{1-(2x-1)^2}\leqq\pi-\tfrac{4}{3}x^{3/2}, \tag{2.10}$$

Eq. (2.5) implies:

$$\tau_a\leqq\left(\frac{3}{32\pi a}\right)^{1/2}\left[\pi-\frac{4}{3}\left(\frac{2m}{R}\right)^{3/2}\right]. \tag{2.11}$$

Therefore

$$\tau(R)-\tau_0\leqq\left(\frac{3\pi}{32}\right)^{1/2}\left\{\frac{1}{(a(R))^{1/2}}-\frac{1}{(a(0))^{1/2}}-\frac{64}{9}\left(\frac{2\pi}{3}\right)^{1/2}a(R)R^3\right\}. \tag{2.12}$$

But

$$\frac{1}{(a(R))^{1/2}}-\frac{1}{(a(0))^{1/2}}\leqq\frac{1}{2(a(R))^{3/2}}(a(0)-a(R))\leqq\frac{kR^4}{2(a(R))^{3/2}}. \tag{2.13}$$

So we have:

$$\tau_a(R)-\tau_0\leqq\left(\frac{3\pi}{32}\right)^{1/2}\frac{kR^3}{2(a(R))^{3/2}}\left[R-\frac{128}{9k}\left(\frac{2\pi}{3}\right)^{1/2}(a(R))^{5/2}\right], \tag{2.14}$$

while from (2.9) for $R\leqq(a(0)/2k)^{1/4}$ we have $a(R)\geqq\frac{1}{2}a(0)$. Thus if we take

$$R_1=\min\left\{\left(\frac{a(0)}{2k}\right)^{1/4},\ \frac{128}{9k}\left(\frac{2\pi}{3}\right)^{1/2}\left(\frac{a(0)}{2}\right)^{5/2}\right\}, \tag{2.15}$$

for $0\leqq R\leqq R_1$, it holds $\tau_a(R)\leqq\tau_0$.

While by the above proposition strong cosmic censorship holds in the case $\rho''(0)=0$, we shall show in the following sections that in the generic case $\rho''(0)<0$ strong cosmic censorship is in fact false and that for an open subset of initial densities weak cosmic censorship is false as well.

Section 3

Let us be given a nonnegative non-increasing (even) C^∞ function $\rho_1(R)$ of compact support such that $\rho_1''(0)<0$. Let:

$$\begin{aligned}\alpha_1&=\rho_1(0),\\ \beta_1&=-\tfrac{1}{2}\rho_1''(0),\\ R_1^1&=(\alpha_1/\beta_1)^{1/2},\end{aligned} \tag{3.1}$$

and let R_0^1 be the radius of support of $\rho_1(R)$. We set:

$$\rho_0(x)=\frac{1}{\alpha_1}\rho(R):R=xR_1^1. \tag{3.2}$$

Then $\rho_0(x)$ is a nonnegative non-increasing (even) C^∞ function of compact support and

$$\rho_0(0)=1,$$

$$\rho_0''(0)=\frac{(R_1^1)^2}{\alpha_1}\rho_1''(0)=-2, \tag{3.3}$$

and $l=R_0^1/R_1^1$ is the radius of support of $\rho_0(x)$. We then define a 2-parameter family of nonnegative non-increasing (even) C^∞ functions $\rho(R)$ of compact support in the following way: Given α, $\beta>0$ we set:

$$\rho(R)=\alpha\rho_0(x): x=R/R_1,\ R_1=(\alpha/\beta)^{1/2}. \tag{3.4}$$

We have

$$\rho(0)=\alpha,$$

$$\rho''(0)=\frac{\alpha}{R_1^2}\rho_0''(0)=-2\beta, \tag{3.5}$$

and $R_0=lR_1$ is the radius of support of $\rho(R)$. Thus each $\rho_1(R)$ generates a 2-parameter family $\rho(R;\alpha,\beta)$, and this family has a standard member $\rho_0(x)$ which corresponds to $\alpha=\beta=1$. The ratio $R_0/R_1=l$ is common to all members of the family.

The mean density $a(R)$ may be expanded in the form

$$a(R)=\int_0^1 3v^2\rho(Rv)dv. \tag{3.6}$$

In view of (3.4),

$$a(R)=\alpha a_0(x), \tag{3.7}$$

where

$$a_0(x)=\int_0^1 3v^2\rho_0(xv)dv \tag{3.8}$$

is the mean density of the standard. We have

$$a_0'(x)=\int_0^1 3v^3\rho_0'(xv)dv. \tag{3.9}$$

The fact that $\rho_0'=0$, while $\rho_0''(0)=-2$, implies that for sufficiently small x_1 we have $\rho'(x)<0$ for $0<x<x_1$. This together with the fact that $\rho_0'(x)\leqq 0$ for all x implies by (3.9) that $a_0'(x)<0$ for all $x>0$. We also have

$$a_0''(x)=\int_1^1 3v^4\rho_0''(xv)dv. \tag{3.10}$$

From the fact that $a_0(0)=1$ and $a_0'(0)=0$, while by (3.10)

$$a_0''(0)=-\int_0^1 6v^4dv=-\tfrac{6}{5},$$

it follows that

$$a_0(x)=1-\tfrac{3}{5}x^2b(x) \tag{3.11}$$

and

$$a_0'(x) = -\tfrac{6}{5}xc(x)(c(x) = b(x) - \tfrac{1}{2}xb'(x)), \tag{3.12}$$

where $b(0) = c(0) = 1$ and $b(x)$, $c(x) > 0$ for all x (since for all $x > 0$ we have $a_0'(x) < 0$ and $a_0(x) < a_0(0) = 1$). The ratio

$$a(R_0)/\alpha = a_0(l) = \xi \tag{3.13}$$

of the initial mean density of the dust cloud as a whole to the initial central density is common to all members of the 2-parameter family of dust clouds. Let us define

$$\varepsilon = \min_{0 \leqq x \leqq l} b(x). \tag{3.14}$$

According to the foregoing we have $\varepsilon > 0$ and also

$$\varepsilon \leqq b(l) = \frac{1-\xi}{\frac{3}{5}l^2}. \tag{3.15}$$

The expression

$$\frac{Ra'(R)}{a(R)} = \frac{xa_0'(x)}{a_0(x)} = -\frac{6}{5}\frac{x^2c(x)}{(1-\frac{3}{5}x^2b(x))}, \tag{3.16}$$

which enters Eq. (1.14), is common to all members of the family. We also have

$$\frac{2m(R)}{R} = \frac{8\pi}{3}R^2a(R) = \gamma x^2 a_0(x), \tag{3.17}$$

where γ is the dimensionless constant

$$\gamma = \frac{8\pi}{3}\frac{\alpha^2}{\beta}. \tag{3.18}$$

Let us define

$$\eta = \max_{0 \leqq x \leqq l} x^2 a_0(x). \tag{3.19}$$

The requirement $2m(R)/R < 1$ is then equivalent to:

$$\gamma < \gamma_0 = 1/\eta. \tag{3.20}$$

We now define

$$\zeta = \left(\frac{32\pi\alpha}{3}\right)^{1/2}(\tau - \tau_0). \tag{3.21}$$

(τ_0 being given by (2.6).) From (1.4), (3.4), (3.7) and (3.18) we obtain:

$$\zeta = \frac{\eta + \sin\eta}{(a_0(x))^{1/2}} - \pi, \tag{3.22}$$

and

$$\frac{d\tau}{dR} = \frac{1}{2\gamma^{1/2}}\frac{d\zeta}{dx}. \tag{3.23}$$

Expressed in terms of the dimensionless quantities ζ and x the differential equation

(2.2) for the outgoing light rays will be seen to depend only on the standard density $\rho_0(x)$, and on the parameters α, β only through their dimensionless combination γ.

Let

$$\delta = \pi - \eta. \tag{3.24}$$

From (3.22) and (3.12) we obtain:

$$\delta - \sin\delta = g, \tag{3.25}$$

where

$$g = \pi[1 - (1 - \tfrac{3}{5}x^2 b(x))^{1/2}] - (1 - \tfrac{3}{5}x^2 b(x))^{1/2}\zeta. \tag{3.26}$$

Let ϕ be the function

$$\phi(\delta) = \delta - \sin\delta. \tag{3.27}$$

ϕ is a monotone mapping of $[0,\pi]$ onto itself. We define the function χ by

$$\chi(y) = \phi^{-1}(y^3). \tag{3.28}$$

Then χ is a monotone mapping of $[0,\pi^{1/3}]$ onto $[0,\pi]$. The inverse function

$$\chi^{-1}(\delta) = (\delta - \sin\delta)^{1/3} \tag{3.29}$$

has uniformly continuous derivatives of all orders in $[0,\pi]$ and $(\chi^{-1})'(\delta) \geqq \text{const} > 0$. Hence χ is a diffeomorphism of $[0,\pi^{1/3}]$ onto $[0,\pi]$. Since

$$\chi(0) = 0 \quad\text{and}\quad \chi'(0) = 6^{1/3}, \tag{3.30}$$

we have

$$\chi(y) = 6^{1/3} y \psi(y), \tag{3.31}$$

where $\psi \in C^\infty[0,\pi^{1/3}]$ and $\psi > 0$, $\psi(0) = 1$. From (3.25), (3.27), (3.28) and (3.31) we obtain:

$$\delta = \phi^{-1}(g) = \chi(g^{1/3}) = 6^{1/3} g^{1/3} \psi(g^{1/3}). \tag{3.32}$$

We can then express

$$\begin{aligned} 1 - \cos\delta &= p(\chi(g^{1/3}))\cdot\tfrac{1}{2}6^{2/3} g^{2/3} \psi^2(g^{1/3}),\\ \frac{\sin\delta}{1-\cos\delta} &= q(\chi(g^{1/3}))\cdot\frac{2}{6^{1/3} g^{1/3}\psi(g^{1/3})}, \end{aligned} \tag{3.33}$$

where p and q are the functions

$$\begin{aligned} p(\delta) &= \frac{1-\cos\delta}{\tfrac{1}{2}\delta^2},\\ q(\delta) &= \frac{\delta\sin\delta}{2(1-\cos\delta)}. \end{aligned} \tag{3.34}$$

We have p, $q \in C^\infty[0,\pi]$ and $p, q \geqq 0$, $p(0) = q(0) = 1$.

In view of (3.23) and (3.16), (3.17), (3.25) and (3.33), the differential equation for the outgoing light rays [cf. (2.2), (1.9), (1.14) and (3.24)]

$$\frac{d\tau}{dR}=\left(1-\frac{2m}{R}\right)^{-1/2}\left[\frac{1}{2}(1-\cos\delta)-\frac{1}{4}\frac{\sin\delta(\pi-\delta+\sin\delta)}{(1-\cos\delta)}\frac{Ra'}{a}\right], \tag{3.35}$$

when expressed in terms of the dimensionless quantities ζ and x, becomes:

$$\frac{d\zeta}{dx}=6^{2/3}\gamma^{1/2}(1-\gamma x^2 a_0(x))^{-1/2}\times\left[\frac{1}{2}p\psi^2 g^{2/3}+\frac{1}{5}\frac{q}{\psi}g^{-1/3}(\pi-g)\frac{x^2c(x)}{a_0(x)}\right]. \tag{3.36}$$

Our aim is to find a form of the differential equation for the outgoing light rays in which the singular point at the center $x=\zeta=0$ appears as a regular singular point. In order to do this we first introduce in the role of ζ the quantity θ defined by setting

$$\zeta=\frac{3\pi}{10}x^{7/3}\theta. \tag{3.37}$$

Then if we also define the quantity h by setting

$$g=\frac{3\pi}{10}x^2h, \tag{3.38}$$

Eq. (3.36) becomes

$$\frac{d\theta}{dx}=\left(\frac{15}{\pi}\right)^{1/3}\gamma^{1/2}\frac{(1-\gamma x^2a_0(x))^{1/2}}{x}\left[p\psi^2h^{2/3}+\frac{4}{3}\frac{q}{\psi}h^{-1/3}\left(1-\frac{g}{\pi}\right)\frac{c(x)}{a_0(x)}\right]-\frac{7}{3}\frac{\theta}{x}, \tag{3.39}$$

and from (3.26) we have:

$$h=\frac{[1-(1-\frac{3}{5}x^2b(x))^{1/2}]}{\frac{3}{10}x^2}-x^{1/3}(1-\tfrac{3}{5}x^2b(x))^{1/2}\theta. \tag{3.40}$$

Setting now

$$x=u^3. \tag{3.41}$$

and using in the role of γ the dimensionless constant λ defined by

$$\lambda=\left(\frac{15}{\pi}\right)^{1/3}\gamma^{1/2},\quad \lambda<\lambda_0=\left(\frac{15}{\pi}\right)^{1/3}\gamma_0^{1/2}, \tag{3.42}$$

we bring the differential equation for the outgoing light rays to the desired form:

$$\frac{d\theta}{du}-\frac{7(\lambda-\theta)}{u}=\lambda f(u,\theta;\lambda), \tag{3.43}$$

$$f(u,\theta;\lambda)=\frac{1}{u}\left\{\left[1-\left(\frac{\pi}{15}\right)^{2/3}\lambda^2u^6a_0(u^3)\right]^{-1/2}f_1(u,\theta)-7\right\}. \tag{3.44}$$

Here $f_1(u,\theta)$ is a function which is common to all members of the 2-parameter family of dust clouds and is given by:

$$f_1(u,\theta)=3(h(u,\theta))^{2/3}P((g(u,\theta))^{1/3})$$
$$+4(h(u,\theta))^{-1/3}Q((g(u,\theta))^{1/3})\left(1-\frac{g(u,\theta)}{\pi}\right)\frac{c(u^3)}{a_0(u^3)}, \tag{3.45}$$

where P and Q are the functions defined by:

$$P(y)=p(\chi(y))\psi^2(y), \tag{3.46}$$
$$Q(y)=\frac{q(\chi(y))}{\psi(y)},$$

and from (3.38), (3.40) we have

$$g(u,\theta)=\frac{3\pi}{10}u^6h(u,\theta), \tag{3.47}$$

$$h(u,\theta)=\frac{[1-(1-\frac{3}{5}u^6b(u^3))^{1/2}]}{\frac{3}{10}u^6}-u(1-\tfrac{3}{5}u^6b(u^3)^{1/2}\theta. \tag{3.48}$$

For $u\in[0,l^{1/3}]$ the function h is C^∞ and it is also positive for

$$\theta<\sigma(u),$$

where

$$\sigma(u)=\frac{1}{u}\cdot\frac{1}{(1-\frac{3}{5}u^6b(u^3)^{1/2}}\cdot\frac{[1-(1-\frac{3}{5}u^6b(u^3)^{1/2}]}{\frac{3}{10}u^6}$$
$$=\frac{b(u^3)}{u}\cdot\frac{1}{\frac{1}{2}[1+(1-\frac{3}{5}u^6b(u^3))^{1/2}]}\cdot\frac{1}{(1-\frac{3}{5}u^6b(u^3))^{1/2}}. \tag{3.49}$$

We have

$$\sigma(u)\geqq\varepsilon/u, \tag{3.50}$$

where ε is defined by (3.14). The curve $\theta=\sigma(u)$ that is $h=0$ corresponds to the singular surface $\delta=0$ minus the singular point at the center. The functions $h^{2/3}$, $h^{-1/3}$ and also $g^{1/3}$ are then C^∞ functions of u,θ for $0\leqq u\leqq l^{1/3}$, $\theta<\sigma(u)$. Since we are considering outgoing light rays, we restrict ourselves to $\theta\geqq 0$ which corresponds to $\tau\geqq\tau_0$. With this restriction we have

$$0\leqq g(u,\theta)\leqq\pi[1-(1-\tfrac{3}{5}u^6b(u^3))^{1/2}]$$
$$\leqq\pi(1-\xi^{1/2}) \tag{3.51}$$

We conclude that the functions $f_1(u,\theta)$ is C^∞ and nonnegative in the strip $0\leqq u\leqq l^{1/3}$, $0\leqq\theta<\sigma(u)$. Since $h(0,\theta)=1$, $g(0,\theta)=0$, $P(0)=Q(0)=1$ and $c(0)=a_0(0)=1$, we have:

$$f_1(0,\theta)=7 \tag{3.52}$$

It follows that $f(u,\theta;\lambda)$ is C^∞ in the strip $0 \leqq u \leqq l^{1/3}$, $0 \leqq \theta < \sigma(u)$, and we have

$$f(u,\theta;\lambda) \geqq -\frac{7}{u}. \tag{3.53}$$

Section 4

Theorem 1. *For every* $0 < \lambda < \lambda_0$, *there exists an* $a_2 \in]0, l^{1/3}]$ *and a solution* $\theta \in C^\infty[0,a_2]$ *to the differential Eq. (3.43). This solution is nonnegative, satisfies* $\theta(0) = \lambda$, *and it is the only solution of the equation which is continuous at* $u = 0$.

Proof. Consider the linear differential equation obtained from (3.43) by replacing θ in $f(u,\theta;\lambda)$ *by a given continuous function* $\bar{\theta}$ such that $0 \leqq \bar{\theta}(u) < \sigma(u)$:

$$\frac{d\theta}{du} - \frac{7(\lambda-\theta)}{u} = \lambda f(u, \bar{\theta}(u);\lambda). \tag{4.1}$$

There is only one solution of this equation which is continuous at $u = 0$, namely the one which satisfies $\theta(0) = \lambda$. This solution is given by:

$$\theta(u) = \lambda[1 + u \int_0^1 s^7 f(su, \bar{\theta}(su);\lambda) ds]. \tag{4.2}$$

We shall study the nonlinear map T_λ defined by $\theta = T_\lambda(\bar{\theta})$. We first choose an $a_0 \in]0, l^{1/3}]$ such that

$$a_0 < \varepsilon/\lambda. \tag{4.3}$$

We then choose a μ such that

$$\lambda < \mu < \varepsilon/a_0. \tag{4.4}$$

It then follows from (3.50) that for all $u \in [0,a_0]$ we have

$$\sigma(u) > \mu,$$

and therefore $f(u,\theta;\lambda)$ is a C^∞ function in the strip $0 \leqq u \leqq a_0$, $0 \leqq \theta \leqq \mu$.

Let

$$\Gamma(\mu,\lambda) = \sup_{0 \leqq u \leqq a_0} \sup_{0 \leqq \theta \leqq u} f(u,\theta;\lambda). \tag{4.5}$$

and let θ be a given continuous function such that $0 \leqq \theta(u) \leqq \mu$ for $0 \leqq u \leqq a_0$. Then for $u \in [0,a_0]$ we obtain from (4.2):

$$T_\lambda(\theta)(u) \leqq \lambda(1 + \tfrac{1}{8} u\Gamma) \tag{4.6}$$

if $\Gamma > 0$,

$$T_\lambda(\theta)(u) \leqq \lambda$$

if $\Gamma \leqq 0$. So if we choose

$$a_1 \leqq \min\left\{a_0, \frac{8}{\Gamma}\left(\frac{\mu}{\lambda} - 1\right)\right\} \tag{4.7}$$

if $\Gamma > 0$,

$$a_1 \leqq a_0$$

if $\Gamma \leqq 0$, we have $T_\lambda(\theta)(u) \leqq \mu$ for $u \in [0, a_1]$. In addition, we have from (3.53):

$$T_\lambda(\theta)(u) \geqq \lambda\left[1 + u\int_0^1 s^7\left(-\frac{7}{su}\right)ds\right] = 0.$$

Consider now the set V_μ in $C^0[0, a_2]$ consisting of all $\theta \in C^0[0, a_2]$ such that

$$0 \leqq \theta(u) \leqq \mu \quad \text{for all } u \in [0, a_2].$$

V_μ is a closed subset of $C^0[0, a_2]$, and it follows from the above that the map T_λ sends V_μ into itself for all $a_2 \leqq a_1$. Let

$$\Delta(\mu, \lambda) = \sup_{0 \leqq u \leqq a_0} \sup_{0 \leqq \theta \leqq \mu} \left|\frac{\partial f}{\partial \theta}(u, \theta; \lambda)\right|. \tag{4.8}$$

Then if θ_1, $\theta_2 \in \mathrm{V}_\mu$, we obtain from (4.2)

$$\| T_\lambda(\theta_2) - T_\lambda(\theta_1) \| \leqq \tfrac{1}{8} a_2 \lambda \Delta \| \theta_2 - \theta_1 \|, \tag{4.9}$$

where $\| \quad \|$ denotes the supremum norm in $C^0[0, a_2]$. Thus if we choose $a_2 \leqq a_1$ and also

$$a_2 < \frac{8}{\lambda \Delta}, \tag{4.10}$$

the map T_λ is contractive in V_μ. Hence T_λ has a unique fixed point $\theta \in V_\mu$:

$$\theta(u) = T_\lambda(\theta)(u) = \lambda[1 + u\int_0^1 s^7 f(su, \theta(su); \lambda)ds]. \tag{4.11}$$

It follows from (4.11) that $\theta(0) = \lambda$, also that $\theta \in C^1[0, a_2]$ and

$$\frac{d\theta}{du} = \lambda[f(u, \theta; \lambda) - 7\int_0^1 s^7 f(su, \theta(su); \lambda)ds]. \tag{4.12}$$

Consequently, θ is a solution of the differential Eq. (3.43). Also, $\theta \in C^n[0, a_2]$ implies $f \in C^n[0, a_2]$, which by (4.12) implies $\theta \in C^{n+1}[0, a_2]$. Therefore $\theta \in C^\infty[0, a_2]$. □

Since the point $u = a_2$, $\theta = \theta(a_2)$ is a regular point of the differential Eq. (3.43), the solution uniquely extends as a C^∞ solution to an interval $[0, b[$where either: 1) $b = \infty$, or 2) b is finite and $\theta(u) \to \sigma(u)$ for $u \to b$. The solution given by the above theorem represents an outgoing light ray n_0 which emmanates from the singular point at the center and which either (case 2) intersects the apparent horizon inside the surface of the cloud $x = l$, in which case it reaches the singular hypersurface, or (case 1) it arrives at the surface of the cloud earlier than the apparent horizon and therefore goes to future null infinity.

Consider now any event (R_1, τ_1) which lies to the future of n_0. The outgoing light ray through this event must also lie to the future of n_0, and since no outgoing light ray emmanates from the singular surface except from the singular point at the center (see Sect. 1), the light ray must tend to the singular point at the center for $R \to 0$. We therefore obtain

Proposition 2. *Through any event to the future of n_0 there passes an outgoing light ray which emmanates from the singular point at the center.*

There is therefore an infinity of outgoing light rays emmanating from the singular point at the center. It is possible to show that all these light rays except n_0 have a second order contact point at the center with the singular surface.

Section 5

Theorem 2. *n_0 is the earliest of all light rays which emmanate from the singular point at the center.*

Proof. We shall show that there exist x_0 and η_0 positive such that for $0 < \eta \leqq \eta_0$ the curve $n_0 - \eta$ defined by $\zeta(x) = \zeta_{n_0}(x) - \eta$ is spacelike for $0 \leqq x < x_0$. The following argument would then prove the theorem: Given any event in the past of n_0, the outgoing light ray $\zeta_1(x)$ through that event must be at $x = x_0$ earlier than $n_0 : \zeta_{n_0}(x_0 - \zeta_1(x_0) \equiv \eta_1 > 0$. We need only consider the case $\eta_1 \leqq \eta_0$. At each $0 \leqq x \leqq x_0$, the spacelike curve $n_0 - \eta_1$ which passes through the same event $(x_0, \zeta_1(x_0))$ must be not earlier than the outgoing light ray $\zeta_1(x)$, therefore in particular $\zeta_1(0) \leqq -\eta_1$ and the light ray emmanates from a regular point on the central world line $x = 0$.

A future directed curve $\zeta(x)$ is spacelike iff

$$\left(\frac{d\tau}{dR} = \right)\frac{1}{2\gamma^{1/2}}\frac{d\zeta}{dx} < e^{\omega(\zeta, x)}.$$

For the curve $n_0 - \eta$ we have:

$$\frac{1}{2\gamma^{1/2}}\frac{d\zeta}{dx} = \frac{1}{2\gamma^{1/2}}\frac{d\zeta_{n_0}}{dx} = e^{\omega(\zeta_{n_0}, x)}.$$

Therefore the curve $n_0 - \eta$ is spacelike at x iff:

$$e^{\omega(\zeta_{n_0}, x)} < e^{\omega(\zeta, x)} = e^{\omega(\zeta_{n_0} - \eta, x)}.$$

It follows that it is enough to show that we can find x_0 and η_0 positive such that for $0 < \eta \leqq \eta_0$ and $0 \leqq x < x_0$,

$$\frac{\partial e^{\omega(\zeta, x)}}{\partial \zeta} < 0$$

for all ζ such that $\zeta_{n_0}(x) - \eta \leqq \zeta \leqq \zeta_{n_0}(x)$. We have (see (1.9)):

$$\frac{\partial e^{\omega}}{\partial \tau} = \frac{\dot{r}'}{\left(1 - \frac{2m}{R}\right)^{1/2}}, \tag{5.1}$$

while from (1.5), (1.4) and (1.14) we obtain:

$$\begin{aligned}
\dot{r}' &= -\left(\frac{2m}{r}-\frac{2m}{R}\right)^{-1/2}\left(\frac{m'}{r}-\frac{mr'}{r^2}-\frac{m'}{R}+\frac{m}{R^2}\right)\\
&= -\left[\frac{2m}{R}\left(\frac{1}{\frac{1}{2}(1-\cos\delta)}-1\right)\right]^{-1/2}\left\{\frac{m'}{R}\left(\frac{1}{\frac{1}{2}(1-\cos\delta)}-1\right)\right.\\
&\quad\left.-\frac{m}{R^2}\left[\left(\frac{1}{\frac{1}{2}(1-\cos\delta)}-1\right)-\frac{1}{4}\frac{\sin\delta(\pi-\phi(\delta))}{\frac{1}{4}(1-\cos\delta)^3}\frac{Ra'}{a}\right]\right\}.
\end{aligned} \tag{5.2}$$

Using the dimensionless time and radial coordinates ζ and x and the standard density and mean density $\rho_0(x)$ and $a_0(x)$ (see Sect. 3) we obtain from (5.1) and (5.2) the expression:

$$\begin{aligned}
\frac{\partial e^{\omega}}{\partial\zeta} &= -\frac{1}{(1-\gamma x^2 a_0(x))^{1/2}}\cdot\frac{1}{\gamma^{1/2}(a_0(x))^{1/2}}\\
&\quad\cdot\frac{1}{(\frac{1}{2}(1-\cos\delta))^{1/2}}\cdot\frac{1}{(1-\frac{1}{2}(1-\cos\delta))^{1/2}}\\
&\quad\cdot\{W_2(\delta,x)-W_1(\delta,x)\},
\end{aligned} \tag{5.3}$$

where we have introduced the functions:

$$\begin{aligned}
W_2(\delta,x) &= \tfrac{3}{4}(1-\tfrac{1}{2}(1-\cos\delta))\rho_0(x),\\
W_1(\delta,x) &= \tfrac{1}{4}a_0(x)\left[(1-\tfrac{1}{2}(1-\cos\delta))+\frac{1}{2}\frac{\sin\delta(\pi-\phi(\delta))}{(1-\cos\delta)^2}\cdot\frac{6x^2c(x)}{5\,a_0(x)}\right].
\end{aligned} \tag{5.4}$$

It follows from expression (5.3) that we have $\partial e^{\omega}/\partial\zeta<0$ iff $W_2>W_1$. Since the function

$$\frac{\sin\delta}{1-\cos\delta}=\left(\frac{1}{\frac{1}{2}(1-\cos\delta)}-1\right)^{1/2}$$

is a monotonically non-increasing function of δ for $\delta\in[0,\pi]$, both $W_2(\delta,x)$ and $W_1(\delta,x)$ are monotonically non-increasing functions of δ at each x. On the other hand, δ is a monotonically decreasing function of ζ at each x. Hence for all ζ such that $\zeta_{n_0}(x)-\eta\leqq\zeta\leqq\zeta_{n_0}(x)$, we have:

$$W_1(\delta,x)\leqq W_1(\delta_{n_0}(x),x)\equiv W_1(x),$$

while

$$W_2(\delta,x)\geqq W_2(\delta_{n_0-\eta}(x),x).$$

Let us define the function

$$q_1(\delta)=\frac{1}{4}\frac{\delta^3\sin\delta}{(1-\cos\delta)^2}. \tag{5.5}$$

q_1 is a nonnegative C^∞ function of δ for $\delta\in[0,\pi]$ and $q_1(0)=1$. By (3.32) we have:

$$\frac{\sin\delta}{(1-\cos\delta)^2}=\frac{4q_1(\delta)}{\delta^3}=\frac{2q_1(\delta)}{3g\cdot(\psi(g^{1/3}))^3},$$

and using (3.38) we obtain:

$$W_1(\delta,x)=\tfrac{1}{4}a_0(x)\left[(1-\tfrac{1}{2}(1-\cos\delta))+\frac{4}{3}\frac{c(x)}{a_0(x)}\cdot\frac{q_1(\delta)\cdot(1-\frac{3}{10}x^2h)}{h\cdot(\psi(g^{1/3}))^3}\right]. \tag{5.6}$$

Considering the fact that $\delta_{n_0}(x)$, $g_{n_0}(x)$ and $h_{n_0}(x)$ are all continuous functions of x and $h_{n_0}(x)>0$, we conclude from (5.6) that $W_1(x)$ is a continuous function of x and that

$$W_1(0)=\tfrac{1}{4}(1+\tfrac{4}{3})=\tfrac{7}{12}.$$

Therefore, for any $\varepsilon_1>0$ there exists an $x_1>0$ such that for $0\leqq x<x_1$ it holds:

$$W_1(x)<\tfrac{7}{12}+\varepsilon_1. \tag{5.7}$$

Since (see Sect. 3):

$$\phi(\delta)=\pi(1-(a_0(x))^{1/2})-(a_0(x))^{1/2}\zeta,$$

for all ζ such that $\zeta_{n_0}(x)-\eta\leqq\zeta\leqq\zeta_{n_0}(x)$ we have:

$$\phi(\delta)\leqq\phi(\delta_{n_0-\eta})=\phi(\delta_{n_0})+\eta(a_0(x))^{1/2}\leqq\phi(\delta_{n_0})+\eta.$$

Now $\phi(\delta_{n_0}(x))=g_{n_0}(x)$ is a continuous function of x and $g_{n_0}(0)=0$. Therefore, for every $\varepsilon_2>0$, there exists an $x_2>0$ such that for $0\leqq x<x_2$, we have $g_{n_0}(x)<\varepsilon_2$, and thus also

$$\phi(\delta)<\varepsilon_2+\eta. \tag{5.8}$$

Let us define the function

$$q_2(\delta)=\frac{2(1-\cos\delta)}{6^{2/3}(\phi(\delta))^{2/3}}.$$

q_2 is a positive C^∞ function of δ for $\delta\in[0,\pi]$ and $q_2(0)=1$. Let:

$$c_1\equiv\max_{0\leqq\delta\leqq\pi}q_1(\delta). \tag{5.9}$$

Then for $0\leqq x<x_2$ it holds by (5.8):

$$\begin{aligned}1-\tfrac{1}{2}(1-\cos\delta)&\geqq 1-\tfrac{1}{4}6^{2/3}c_1(\phi(\delta))^{2/3}\\&>1-\tfrac{1}{4}6^{2/3}c_1(\varepsilon_2+\eta)^{2/3}.\end{aligned} \tag{5.10}$$

On the other hand, $\rho_0(x)$ is also a continuous function of x and $\rho_0(0)=1$. Therefore for every $\varepsilon_3>0$, there exists an $x_3>0$ such that for $0\leqq x<x_3: \rho_0(x)>1-\varepsilon_3$. We conclude that for $0\leqq x<\min\{x_2,x_3\}$, we have:

$$W_2(\delta,x)(\geqq W_2(\delta_{n_0-\eta}(x),x))>\tfrac{3}{4}(1-\varepsilon_3)[1-\tfrac{1}{4}6^{2/3}c_1(\varepsilon_2+\eta)^{2/3}].$$

Setting $x_0=\min\{x_1,x_2,x_3\}$, we obtain that for $0\leqq x<x_0$ we have $W_2>W_1$ if

$$\tfrac{7}{12}+\varepsilon_1\leqq\tfrac{3}{4}-\tfrac{3}{4}\varepsilon_3-\tfrac{3}{16}\cdot6^{2/3}\cdot c_1\cdot(1-\varepsilon_3)\cdot(\varepsilon_2+\eta)^{2/3}.$$

Thus, we first choose ε_3 such that

$$\tfrac{3}{4}\varepsilon_3<\tfrac{3}{4}-\tfrac{7}{12}=\tfrac{1}{6},$$

that is:

$$\varepsilon_3 < \tfrac{2}{9}. \tag{5.11}$$

Then we choose ε_2 and η_0 such that

$$\tfrac{3}{16}\cdot 6^{2/3}\cdot c_1\cdot(1-\varepsilon_3)\cdot(\varepsilon_2+\eta_0)^{2/3} < \tfrac{1}{6}-\tfrac{3}{4}\varepsilon_3,$$

that is:

$$\varepsilon_2+\eta_0 < \tfrac{1}{6}\left(\frac{4}{c_1}\right)^{3/2}\left(\frac{\frac{2}{9}-\varepsilon_3}{1-\varepsilon_3}\right)^{3/2}. \tag{5.12}$$

Finally, we choose ε_1 such that:

$$\varepsilon_1 \leqq \tfrac{3}{4}\left[\left(\tfrac{2}{9}-\varepsilon_3\right)-\tfrac{1}{4}6^{2/3}c_1(1-\varepsilon_3)(\varepsilon_2+\eta_0)^{2/3}\right]. \tag{5.13}$$

Then if $0<\eta\leqq\eta_0$, for all (x,ζ) such that $0\leqq x<x_0$ and $\zeta_{n_0}(x)-\eta\leqq\zeta\leqq\zeta_{n_0}(x)$, we have $W_2>W_1$ and hence

$$\frac{\partial e^{\omega}(\zeta,x)}{\partial\zeta}<0. \qquad \square$$

By the above theorem, the light ray n_0 is the generator of the boundary of the region of the spacetime which is uniquely predictable by the given data on the initial hypersurface, that is, the generator of the Cauchy horizon.

Section 6

Lemma 1. *In any given 2-parameter family of initial density distributions there exists a* $\lambda_1>0$ *such that for all* $0<\lambda\leqq\lambda_1:\zeta_{n_0}(l;\lambda)<\zeta_a(l,\lambda)$, *that is,* n_0 *arrives at the surface of the cloud earlier than the apparent horizon.*

Proof. We shall first show that given some μ such that

$$0<\mu<\varepsilon/l^{1/3}, \tag{6.1}$$

we can find λ_1, $0<\lambda_1<\mu$, such that for all $0<\lambda\leqq\lambda_1$, we can set

$$a_0=a_1=a_2=l^{1/3}$$

in the proof of Theorem 1. First, if μ satisfies (6.1) we can set $a_0=l^{1/3}$ for any λ satisfying $0<\lambda<\mu$. Now if $\gamma\leqq 1/2\eta$, that is, if

$$\lambda\leqq\left(\frac{15}{\pi}\right)^{1/3}\frac{1}{(2\eta)^{1/2}}, \tag{6.2}$$

we have:

$$\left[1-\left(\frac{\pi}{15}\right)^{2/3}\lambda^2u^6a_0(u^3)\right]^{-1/2}\leqq\left[1-\frac{1}{2\eta}u^6a_0(u^3)\right]^{-1/2}\leqq 2^{1/2}. \tag{6.3}$$

It follows that

$$f(u,\theta;\lambda)\leqq f_2(u,\theta), \tag{6.4}$$

where

$$f_2(u,\theta)=\frac{1}{u}\left\{\left[1-\frac{1}{2\eta}u^6 a_0(u^3)\right]^{-1/2} f_1(u,\theta)-7\right\} \tag{6.5}$$

is a function which is common to all members of the 2-parameter family and is also a C^∞ function of (u,θ) in the strip $0 \leqq u \leqq l^{1/3}$, $0 \leqq \theta < \sigma(u)$.

Let us define

$$\Gamma_2(\mu)=\sup_{0\leqq u\leqq l^{1/3}}\ \sup_{0\leqq\theta\leqq\mu} f_2(u,\theta). \tag{6.6}$$

$\Gamma_2(\mu)$ is independent of λ and is finite for μ satisfying (6.1). Since

$$\Gamma(\mu,\lambda)\leqq\Gamma_2(\mu), \tag{6.7}$$

it follows from (4.7) that we can set $a_1 = l^{1/3}$ if

$$\lambda\leqq\frac{\mu}{1+\frac{l^{1/3}}{8}\Gamma_2(\mu)}. \tag{6.8}$$

Consider now the function $f_3(u,\theta)$ defined by:

$$f_3(u,\theta)=\frac{1}{u}[f_1(u,\theta)-7]. \tag{6.9}$$

f_3 is common to all members of the 2-parameter family and is also a C^∞ function of (u,θ) in the strip $0 \leqq u \leqq l^{1/3}$, $0 \leqq \theta < \sigma(u)$. We have:

$$\frac{\partial f(u,\theta;\lambda)}{\partial\theta}=\left[1-\left(\frac{\pi}{15}\right)^{2/3}\lambda^2u^6a_0(u^3)\right]^{-1/2}\frac{\partial f_3(u,\theta)}{\partial\theta}. \tag{6.10}$$

Let us define

$$\Delta_3(\mu)=\sup_{0\leqq u\leqq l^{1/3}}\ \sup_{0\leqq\theta\leqq\mu}\left|\frac{\partial f_3(u,\theta)}{\partial\theta}\right|. \tag{6.11}$$

$\Delta_3(\mu)$ is independent of λ and is finite for μ satisfying (6.1). In virtue of (6.3) it holds

$$\Delta(\mu,\lambda)\leqq 2^{1/2}\Delta_3(\mu). \tag{6.12}$$

Therefore, by (4.10) we can set $a_2 = l^{1/3}$ if

$$\lambda<\frac{8}{2^{1/2}l^{1/3}\Delta_3(\mu)}. \tag{6.13}$$

In conclusion, if we choose λ_1 satisfying the inequalities (6.2), (6.8) and (6.13), then for all $0<\lambda\leqq\lambda_1$ we can set $a_0=a_1=a_2=l^{1/3}$. It then follows from Theorem 1 that the solution $\theta(u)$ which represents the light ray n_0 belongs to $V_\mu[0,l^{1/3}]$, hence $\theta(u)\leqq\mu$ for every $u\in[0,l^{1/3}]$, and therefore in particular $\theta(l^{1/3})\leqq\mu$, that is:

$$\zeta_{n_0}(l)\leqq\frac{3\pi}{10}l^{7/3}\mu. \tag{6.14}$$

We shall now show that we can choose μ such that for all λ not greater than the corresponding $\lambda_1(\mu)$, we have $\zeta_{n_0}(l;\lambda) < \zeta_a(l;\lambda)$. This would complete the proof of the lemma.

The apparent horizon is given by $\zeta = \zeta_a(x)$, where (see (3.22))

$$\zeta_a(x) = \pi\left[\frac{1}{(a_0(x))^{1/2}} - 1\right] - \frac{\phi(\delta_a(x))}{(a_0(x))^{1/2}}, \tag{6.15}$$

and $\delta_a(x)$ is defined by (see (3.17))

$$\tfrac{1}{2}(1 - \cos\delta_a) = \gamma x^2 a_0(x). \tag{6.16}$$

Let $k \in]0, \pi/2[$ be the solution of the equation

$$\phi(k) \equiv k - \sin k = \frac{\pi}{2}(1 - \xi^{1/2}), \tag{6.17}$$

where ξ is defined by (3.13). Since by (6.16) it holds

$$\tfrac{1}{2}(1 - \cos\delta_a) \leqq \gamma\eta,$$

if we impose the condition

$$\gamma \leqq \frac{1}{2\eta}(1 - \cos k),$$

that is,

$$\lambda \leqq \left(\frac{15}{\pi}\right)^{1/3} \frac{1}{(2\eta)^{1/2}}(1 - \cos k)^{1/2}, \tag{6.18}$$

we have:

$$\tfrac{1}{2}(1 - \cos\delta_a) \leqq \tfrac{1}{2}(1 - \cos k).$$

Therefore $\delta_a \leqq k$ and

$$\phi(\delta_a) \leqq \phi(k) = \frac{\pi}{2}(1 - \xi^{1/2}).$$

We then obtain from (6.15):

$$\zeta_a(l) = \pi\left(\frac{1}{\xi^{1/2}} - 1\right) - \frac{\phi(\delta_a(l))}{\xi^{1/2}} \geqq \frac{\pi}{2}\left(\frac{1}{\xi^{1/2}} - 1\right). \tag{6.19}$$

It follows from (6.14) and (6.19) that if we choose μ such that

$$\mu < \tfrac{5}{3}\left(\frac{1}{\xi^{1/2}} - 1\right) l^{-7/3}, \tag{6.20}$$

we have $\zeta_{n_0}(l) < \zeta_a(l)$. □

Note. $\lambda_1(\mu)$ is subject to the inequality (6.18) in addition to the inequalities of the first part of the proof.

Lemma 2. *In any given 2-parameter family of initial density distributions there exists a* $\lambda_2 < \lambda_0$, $\lambda_2 \to 0$ *for* $\xi \to 1$, *such that for all* $\lambda_2 \leqq \lambda < \lambda_0$ *all outgoing light rays which emmanate from the singular point at the center intersect the apparent horizon in the interior of the cloud.*

Proof. Let x_0 be the point at which the function $x^2 a_0(x)$ attains its maximum value η in the interval $0 \leqq x \leqq l$. We have $0 < x_0 \leqq l$. By (6.15),

$$\zeta_a(x_0) = \pi\left[\frac{1}{(a_0(x_0))^{1/2}} - 1\right] - \frac{\phi(\delta_a(x_0))}{(a_0(x_0))^{1/2}}. \tag{6.21}$$

Let $k_1 \in]0, \pi[$ be the solution of the equation

$$\phi(k_1) = \pi[1 - (a_0(x_0))^{1/2}]. \tag{6.22}$$

Since $\xi \leqq a_0(x_0) < 1$ we have $k_1 \to 0$ for $\xi \to 1$. Let γ_2 and λ_2 be defined by:

$$\gamma_2 = \frac{1}{2\eta}(1 - \cos k_1), \quad \lambda_2 = \left(\frac{15}{\pi}\right)^{1/3} \gamma_2^{1/2}, \tag{6.23}$$

$\gamma_2, \lambda_2 \to 0$ for $\xi \to 1$. For all $\lambda_2 \leqq \lambda < \lambda_0$ we then have:

$$\tfrac{1}{2}(1 - \cos \delta_a(x_0)) = \gamma\eta \geqq \tfrac{1}{2}(1 - \cos k_1). \tag{6.24}$$

Therefore $\delta_a(x_0) \geqq k_1$ and

$$\phi(\delta_a(x_0)) \geqq \phi(k_1) = \pi[1 - (a_0(x_0))^{1/2}]. \tag{6.25}$$

Hence, by (6.21):

$$\zeta_a(x_0; \lambda) \leqq 0. \tag{6.26}$$

Now all outgoing light rays $\zeta(x)$ such that $\zeta(x) \to 0$ for $x \to 0$ have $\zeta(x_0) > 0$. It follows that all such light rays intersect the apparent horizon at some $x_1 \in]0, x_0[$. □

Section 7

Lemma 3. *In any given 2-parameter family of initial density distributions,* $\zeta_{n_0}(x,; \lambda)$ *is a monotonically increasing function of* λ *at each* x.

Proof. We shall show that $\theta_\lambda(u)$ is a monotonically increasing function of λ at each u. Let $\lambda_1 > \lambda_2$, and let $[0, b[$ be the interval of existence of $\theta_{\lambda_1}(u)$. We shall show that for all $u \in [0, b[$ we have $\theta_{\lambda_1}(u) > \theta_{\lambda_2}(u)$. Let M be the union of all subintervals $[0, b'[$ such that $\theta_{\lambda_1}(u) > \theta_{\lambda_2}(u)$ for all $u \in [0, b'[$. Since $\theta_{\lambda_1}(0) = \lambda_1 > \theta_{\lambda_2}(0) = \lambda_2$, and both θ_{λ_1} and θ_{λ_2} are continuous at $u = 0$, we have $0 \in M$ and therefore M is not empty. We shall demonstrate below that $[0, b_1[\subset M$ and $b_1 < b$ implies $b_1 \in M$ from which follows $M = [0, b[$, and hence the lemma.

Let us define

$$j(u; \lambda) = \left[1 - \left(\frac{\pi}{15}\right)^{2/3} \lambda^2 u^6 a_0(u^3)\right]^{-1/2}. \tag{7.1}$$

$j(u,\lambda)$ is a monotonically increasing function of λ for all $u>0$. $\theta_\lambda(u)$ satisfies the differential equation (see (3.43), (3.44))

$$\frac{d\theta_\lambda}{du} = -\frac{7\theta_\lambda}{u} + \frac{\lambda}{u} j(u;\lambda) f_1(u,\theta_\lambda). \tag{7.2}$$

Let $[0,b_1[\subset M$, $b_1<b$, and let us take u_1, u_0 such that $0<u_1<u_0\leqq b_1$. Then since (7.2) gives

$$\theta_\lambda(u_1) = \theta_\lambda(u_0) + \int_{u_1}^{u_0} \frac{7\theta_\lambda(u)}{u} du - \int_{u_1}^{u_0} \frac{\lambda}{u} j(u;\lambda) f_1(u,\theta_\lambda(u))\,du, \tag{7.3}$$

we obtain:

$$\begin{aligned}\theta_{\lambda_1}(u_1) - \theta_{\lambda_2}(u_1) &= \theta_{\lambda_1}(u_0) - \theta_{\lambda_2}(u_0) + \int_{u_1}^{u_0} \frac{7}{u}(\theta_{\lambda_1}(u) - \theta_{\lambda_2}(u))\,du \\ &- \int_{u_1}^{u_0} \left[\frac{\lambda_1 j(u,\lambda_1)}{u} f_1(u,\theta_{\lambda_1}(u)) - \frac{\lambda_2 j(u;\lambda_2)}{u} f_1(u,\theta_{\lambda_2}(u)) \right] du.\end{aligned} \tag{7.4}$$

We express the integrant of the second integral on the right in (7.4) in the form

$$\frac{1}{u}[\lambda_1 j(u;\lambda_1) - \lambda_2 j(u;\lambda_2)]\cdot f_1(u,\theta_{\lambda_1}(u))$$

$$+\lambda_2 j(u;\lambda_2)\cdot\frac{1}{u}[f_1(u,\theta_{\lambda_1}(u)) - f_1(u,\theta_{\lambda_2}(u))].$$

Now $\lambda_1 j(u;\lambda_1) - \lambda_2 j(u;\lambda_2) > 0$ for all $u>0$. Let us define

$$A(u) = \sup_{0\leqq\theta\leqq\theta_{\lambda_1}(u)} \left|\frac{\partial f_1(u,\theta)}{\partial\theta}\right|. \tag{7.5}$$

$A(u)$ is a continuous function of u for $u\in[0,b[$ and is positive for positive u. For all $u\in[0,b_1]$ we have

$$|f_1(u,\theta_{\lambda_1}(u)) - f_1(u,\theta_{\lambda_2}(u))| \leqq A(u)|\theta_{\lambda_1}(u) - \theta_{\lambda_2}(u)| = A(u)(\theta_{\lambda_1}(u) - \theta_{\lambda_2}(u)).$$

Setting then

$$z(u) = \theta_{\lambda_1}(u) - \theta_{\lambda_2}(u), \tag{7.6}$$

we obtain from (7.4):

$$z(u_1) < z(u_0) + \int_{u_1}^{u_0} \frac{7z(u)}{u} du + \int_{u_1}^{u_0} \frac{\lambda_2 j(u,\lambda_2)}{u} A(u) z(u)\,du. \tag{7.7}$$

This integral inequality implies

$$z(u_0) > z(u_1)\exp\Big(-\int_{u_1}^{u_0} \bar{A}(u)\,du\Big), \tag{7.8}$$

where we have defined

$$\bar{A}(u) = \frac{1}{u}[7 + \lambda_2 j(u,\lambda_2) A(u)]. \tag{7.9}$$

Setting in (7.8) $u_0 = b_1$ we obtain $\theta_{\lambda_1}(b_1) - \theta_{\lambda 2}(b_1) > 0$, and since both θ_{λ_1} and θ_{λ_2} are continuous at $u = b_1$ we have $b_1 \in M$. □

From (6.15) and (6.16) it is obvious that, in any given 2-parameter family of density distributions, $\zeta_a(x;\lambda)$ is a monotonically decreasing function of λ at each x. Lemmas 1, 2 and 3 result in:

Theorem 3. *In any given 2-parameter family of initial density distributions there exists a* λ_c, $0 < \lambda_c < \lambda_0$, $\lambda_c \to 0$ *for* $\xi \to 1$, *such that 1) for all* $0 < \lambda < \lambda_c$, n_0 *goes to future null infinity, 2) for* $\lambda = \lambda_c$, n_0 *coincides with the event horizon, and 3) for all* $\lambda_c < \lambda < \lambda_0$ *we have a regular event horizon.*

Proof. By Lemma 3, $\zeta_{n_0}(l;\lambda)$ is a continuous increasing function of λ, while $\zeta_a(l;\lambda)$ is a continuous decreasing function of λ. On the other hand by Lemma 1, there exists $\lambda_1 > 0$ such that for all $0 < \lambda \leqq \lambda_1 : \zeta_{n_0}(l;\lambda) < \zeta_a(l;\lambda)$, while by Lemma 2 there exists $\lambda_2 < \lambda_0$, $\lambda_2 \to 0$ for $\xi \to 1$, such that for all $\lambda_2 \leqq \lambda < \lambda_0 : \zeta_{n_0}(l;\lambda) > \zeta_a(l;\lambda)$. Therefore there exists λ_c, $\lambda_1 < \lambda_c < \lambda_2$, such that:

$$\zeta_{n_0}(l;\lambda_c) = \zeta_a(l;\lambda_c), \tag{7.10}$$

and for all $0 < \lambda < \lambda_c$ we have:

$$\zeta_{n_0}(l;\lambda) < \zeta_a(l;\lambda), \tag{7.11}$$

while for all $\lambda_c < \lambda < \lambda_0$ we have:

$$\zeta_{n_0}(l;\lambda) > \zeta_a(l;\lambda). \tag{7.12}$$

Equation (7.10) implies that n_0 coincides with the event horizon, (7.11) implies that n_0 goes to future null infinity and (7.12) implies that the event horizon is in the past of n_0 and is therefore regular by Theorem 2.

Note. The fact that $\lambda_c \to 0$ for $\xi \to 1$ means that the region of naked singularities tends to zero for a sequence of standard initial density distributions tending to homogeneity.

Section 8

Consider a light ray which is emitted from the central world line $R = 0$ and which is received by an observer following the world line $R = R_1$ at proper time τ_1:

$$\tau = \tau(R;\tau_1), \quad \tau(0;\tau_1) = \tau_0, \quad \tau(R_1;\tau_1) = \tau_1 .$$

The light ray was emitted from the central world line at proper time τ_0. Since the light ray satisfies the differential equation

$$d\tau/dR = e^{\omega(\tau,R)},$$

if we define

$$g(R) = \frac{\partial\tau(R;\tau_1)}{\partial\tau_1}, \tag{8.1}$$

then g satisfies the differential equation

$$\frac{dg}{dR}=\left(\frac{\partial e^{\omega}}{\partial\tau}\right)_{\tau=\tau(R;\tau_1)}\cdot g$$

together with the condition $g(R_1)=1$. Therefore

$$g(R)=\exp\left[-\int_R^{R_1}\frac{\partial e^{\omega}}{\partial\tau}(\tau(S;\tau_1),S)dS\right]. \tag{8.2}$$

Let us denote the frequency of emission of the light from the central world line by ω_e and the frequency of its reception by the observer following the world line $R=R_1$ by ω_r. We have:

$$\frac{\omega_r}{\omega_e}=\frac{d\tau_0}{d\tau_1}=g(0)=\exp\left[-\int_0^{R_1}\frac{\partial e^{\omega}}{\partial\tau}(\tau(R;\tau_1),R)dR\right]. \tag{8.3}$$

Thus ω_r/ω_e is a function of τ_1. Let $\tau_1(n_0)$ be the proper time at which the observer following the world line $R=R_1$ receives the light ray n_0. For $\tau_1<\tau_1(n_0)$ the light ray arriving at $R=R_1$ at $\tau=\tau_1$ lies to the past of n_0. From the proof of Theorem 2 we know that $\partial e^{\omega}/\partial\tau<0$ in a band $0\leqq R<R^{\star}$, $\tau_{n_0}(R)-\varepsilon\leqq\tau\leqq\tau_{n_0}(R)$. On the other hand $|\partial e^{\omega}/\partial\tau|$ is bounded in the complement of the band in the past of n_0, since this is nowhere near the singular surface. It follows that the integral in the exponential in (8.3) is uniformly bounded from above for $\tau_1<\tau_1(n_0)$ and therefore $\omega_r/\omega_e\geqq$ const >0 for $\tau_1<\tau_1(n_0)$. On the other hand, all the light rays arriving at $R=R_1$ at $\tau=\tau_1>\tau_1(n_0)$ come from the singular point on the central world line (Proposition 2). It follows that $\omega_r/\omega_e=\partial\tau_0/\partial\tau_1=0$ for $\tau_1>\tau_1(n_0)$.

Consider now an observer following a world line $r=r_0$ in the vacuum region outside the cloud and receiving light rays emitted from the center of the cloud. In the vacuum region the equation of the light rays is

$$t-t_0=r-r_0+2m_0\log\left(\frac{r-2m_0}{r_0-2m_0}\right), \tag{8.4}$$

where t_0 is the Schwarzschild coordinate time at which the light ray is received at r_0. Let τ_1 denote the value of τ at which the light ray intersects the line $R=R_0$, that is, the surface of the cloud. As long as $r_1\equiv r(\tau_1,R_0)>2m_0$, t_0 is an analytic function of τ_1 and using (1.4), (1.10) and (8.4) we find:

$$\frac{dt_0}{d\tau_1}=\left(1-\frac{2m_0}{r_1}\right)^{-1}\left[\left(1-\frac{2m_0}{R_0}\right)^{1/2}+\left(\frac{2m_0}{r_1}-\frac{2m_0}{R_0}\right)^{1/2}\right]. \tag{8.5}$$

Also, the relation between the Schwarzschild coordinate time t_0 and the proper time T of the observer following the world line $r=r_0$ is

$$dT=\left(1-\frac{2m_0}{r_0}\right)^{1/2}dt_0. \tag{8.6}$$

Consider now a dust cloud satisfying the condition of case 1) in Theorem 3. The fact that in this case n_0 goes to future null infinity implies that there exists a constant $k<1$ such that $2m_0/r_1\leqq k$ for all $T<T(n_0)$, the proper time at which n_0 is received at

$r = r_0$. Thus, by (8.5) and (8.6):

$$\frac{d\tau}{dT} > \frac{(1-k)}{(1+k^{1/2})} > 0 \quad \text{for all} \quad T < T(n_0).$$

Denoting by τ_0 the proper time of emission of the light rays from the central world line, the ratio of the frequency of reception of the light by the observer following the world line $r = r_0$ to its frequency of emission at the center is equal to:

$$\frac{d\tau_0}{dT} = \frac{d\tau_0}{d\tau_1} \cdot \frac{d\tau_1}{dT},$$

and $d\tau_0/d\tau_1$ is given by (8.3), setting $R_1 = R_0$. We conclude:

Theorem 4. *Consider a dust cloud satisfying the condition of the case 1 of Theorem 3 and consider an observer following a world line $r = r_0$ in the vacuum region who receives light rays emitted from the central world line $R = 0$ of the dust cloud. If ω_e is the frequency of emission of a light ray at the center and ω_r the frequency of its reception by the observer, then the ratio ω_r/ω_e is a discontinuous function of the proper time T of the observer; this function is smooth and greater than a positive constant for $T < T(n_0)$ and is zero for $T > T(n_0)$, where $T(n_0)$ is the proper time at which the light ray n_0 is received by the observer.*

Note Added. After this work was completed I was informed that singularities similar to those analyzed in this paper have been found in computer studies by Douglas M. Eardley and Larry Smarr, "Time functions in numerical relativity: Marginally bound dust collapse," Phys. Rev. **D19**, 2239 (1979).

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