Violation of local uncertainty relations as a signature of entanglement

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Entangled states represent correlations between two separate systems that are too precise to be represented by products of local quantum states. We show that this limit of precision for the local quantum states of a pair of *N*-level systems can be defined by an appropriate class of uncertainty relations. The violation of such local uncertainty relations may be used as an experimental test of entanglement generation.

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I. INTRODUCTION

As more and more experimental realizations of entanglement sources become available, it is necessary to develop efficient methods of testing the entanglement produced by such sources [1–6]. In particular, the output of entanglement sources is usually in a mixed state due to various decoherence effects. For such mixed states, it can be a difficult task to distinguish whether the output is really entangled, or whether it is separable into some mixture of nonorthogonal product states. Although there are simple formal criteria if the complete density matrix is known, the experimental determination of all matrix elements of an output state requires considerable experimental efforts [7]. It is therefore desirable to simplify the verification of entanglement by reducing it to the observation of only a few characteristic statistical properties. One well-known statistical property of entanglement is the violation of Bell's inequalities, and previous experiments often relied on this property as proof of entanglement [4,8]. However, the requirements for Bell's inequality violations are usually more restrictive than the conditions for entanglement [9], and the experiments still require a comparison of at least four different correlation measurements. These complications arise from the fact that Bell's inequalities test the possibility of local hidden variable models. For entanglement verification, it is not necessary to exclude hidden variable models, since entanglement can be defined entirely within the context of conventional quantum theory, without any reverence to alternative models. A more efficient method may therefore be the definition of a boundary between entangled states and nonentangled states in terms of expectation values of special operators called entanglement witnesses [10,11]. Each witness operator defines a statistical limitation for separable states derived directly from the topology of Hilbert space. However, the construction of witnesses that can be tested with only a few local von Neumann measurements is still a highly nontrivial task [6]. Since the experimental verification of optical entanglement typically uses local von Neumann measurements, it may be desirable to express the criteria for entanglement directly in terms of the measurement statistics obtained in such experiments.

In this paper, we therefore propose an alternative approach to entanglement verification based on the observation that entanglement seems to overcome the uncertainty limit by allowing correlations between sets of noncommuting properties of two systems to be more precise than any local definition of these properties could ever be. Since this precision in the correlations between two spatially separated systems is the property that originally leads to the discovery and definition of entanglement [12,13], a quantitative evaluation of local uncertainty violations may provide one of the most precise experimental measures of entanglement. A generalized characterization of entanglement as a suppression of noise below the local quantum limit may also be useful in the study of teleportation errors and related problems of quantum communication [14–16] and in the evaluation of the increased precision achieved by applications of entanglement such as quantum lithography [17] or atomic clock synchronization [18,19].

In the following, we first reformulate the uncertainty principle, adapting it to arbitrary properties of *N*-level systems. This reformulation of uncertainty provides unconditional limitations for the predictabilities of measurement outcomes for any selection of noncommuting physical properties. We can then derive local uncertainty limits valid for all nonentangled states. Since no separable quantum state can overcome these limits, any violation of such local uncertainty relations is an unambiguous proof of entanglement. Some typical examples are provided and the possibility of obtaining a quantitative measure of entanglement from local uncertainties is discussed.

II. SUM UNCERTAINTY RELATIONS FOR *N*-LEVEL SYSTEMS

The use of uncertainty arguments to study entanglement is well known from continuous variable systems [12,20,21]. However, these arguments are based on the conventional product uncertainty of position and momentum. This product uncertainty is based on the observation that an eigenstate of position must have infinite momentum uncertainty and vice versa. In *N*-level systems, all physical properties have upper and lower bounds, making infinite uncertainties impossible. Consequently, the products of *N*-level uncertainties will always be zero if the system is in an eigenstate of one of the properties concerned. This means that the product uncertainties derived from the commutation relations of operators do

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not provide a generally valid uncertainty limit for N-level systems. In order to obtain a quantitative definition of uncertainty limits, it is therefore necessary to reformulate and adapt the uncertainty principle to N-level systems.

In its most general form, the uncertainty principle states that it is never possible to simultaneously predict the measurement outcomes for all observables of the system. In terms of quantum theory, the relevant observables of the system are represented by a set of Hermitian operators $\{\hat{A}_i\}$. The uncertainty of $\hat{A}_i$ for any given quantum state is then defined as the statistical variance of the randomly fluctuating measurement outcomes,

$$\delta A_i^2 = \langle \hat{A}_i^2 \rangle - \langle \hat{A}_i \rangle^2. \tag{1}$$

This positive property of the quantum state can only be zero if the quantum state is an eigenstate of $\hat{A}_i$, representing perfect predictability of the measurement outcome. We can therefore conclude that a quantum state with zero uncertainty in all the properties $\hat{A}_i$ must be a simultaneous eigenstate of all the operators in $\{\hat{A}_i\}$. If there is no such simultaneous eigenstate, there must be a nontrivial lower limit $U>0$ for the sum of the uncertainties,

$$\sum_i \delta A_i^2 \geqslant U. \tag{2}$$

The limit U is defined as the absolute minimum of the uncertainty sum for any quantum state. It therefore represents a universally valid limitation of the measurement statistics of quantum systems.

Since U represents a global minimum, it may be difficult to determine its value in cases where the operators $\hat{A}_i$ have a complicated form. However, there are a number of significant cases where this limit is fairly easy to determine. For N-level systems, one such fundamental limit can be obtained using the spin algebra of the corresponding spin $l=(N-1)/2$ system with

$$(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2)|\psi\rangle = l(l+1)|\psi\rangle \tag{3}$$

for any state $|\psi\rangle$. The expectation values of $\hat{L}_i$ define a vector with a maximal length equal to the extremal eigenvalues of $\pm l$ along any axis. We therefore obtain the uncertainty limit

$$\delta L_x^2 + \delta L_y^2 + \delta L_z^2 = \underbrace{\langle \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \rangle}_{=l(l+1)} - \underbrace{\langle \hat{L}_x \rangle^2 + \langle \hat{L}_y \rangle^2 + \langle \hat{L}_z \rangle^2}_{\leqslant l^2} \geqslant l. \tag{4}$$

This uncertainty relation defines an absolute limit to the precision of spin variables in any N-level system. For the commonly studied case of two-level systems, the spin variables are often expressed in terms of the normalized Pauli matrices $\hat{\sigma}_i$, which have eigenvalues of ± 1 instead of $\pm 1/2$. The sum uncertainty relation for the Pauli matrices is then given by

$$\delta\sigma_1^2 + \delta\sigma_2^2 + \delta\sigma_3^2 \geqslant 2. \tag{5}$$

This uncertainty relation provides a quantitative description of the fact that only a single-spin component of a two-level system can have a well-defined value. It is also possible to formulate an uncertainty relation for only two spin components by noting that $\delta\sigma_i^2 \leqslant 1$. This simplified uncertainty relation reads

$$\delta\sigma_1^2 + \delta\sigma_2^2 \geqslant 1. \tag{6}$$

This is indeed the correct uncertainty minimum. For general spin l systems, such a simple derivation of the $\hat{L}_x$-$\hat{L}_y$ uncertainty is not possible, since the maximal uncertainty of $\hat{L}_z$ is equal to l^2 and therefore exceeds the uncertainty limit for all three spin components. Nevertheless, there exists an uncertainty limit of $\hat{L}_x$ and $\hat{L}_y$ for any value of l, since $\hat{L}_x$ and $\hat{L}_y$ do not have any common eigenstates. For $l=1$, we have determined this limit by optimizing the spin squeezing properties of states with average spins in the x-y plane. The result reads

$$\delta L_x^2 + \delta L_y^2 \geqslant \frac{7}{16}. \tag{7}$$

In the $\hat{L}_z$ basis, the minimum uncertainty state of this relation is given by

$$|\phi\rangle = \frac{\sqrt{5}}{4} e^{-i\phi}|-1\rangle + \frac{\sqrt{6}}{4}|0\rangle + \frac{\sqrt{5}}{4} e^{+i\phi}|+1\rangle. \tag{8}$$

It may be interesting to note that this minimum uncertainty state has an $\hat{L}_z$ uncertainty of $\delta L_z^2 = 5/8$, so that the total of all three spin uncertainties exceeds the limit set in relation (4) by 1/16. Relation (7) is therefore more than just a truncated version of Eq. (4).

III. LOCAL UNCERTAINTY LIMITS

It is now possible to apply the sum uncertainty relations to define the correlation limit for separable states. In general, a pair of quantum systems A and B can be characterized by the operator properties $\hat{A}_i$ and $\hat{B}_i$ with the sum uncertainty relations given by

$$\sum_i \delta A_i^2 \geqslant U_A,$$

$$\sum_i \delta B_i^2 \geqslant U_B. \tag{9}$$

It may be worth noting that the two Hilbert spaces of systems A and B do not need to have the same dimension. In principle, local uncertainty limits can be derived for any $N \times M$ system. Nor is it necessary that there exists any specific relation between the properties $\hat{A}_i$ and $\hat{B}_i$ other than that there is exactly one property $\hat{A}_i$ in A for every property $\hat{B}_i$ in B. The operator properties $\hat{A}_i + \hat{B}_i$ then define a set of joint

properties of the two systems that can determined by local measurements of $\hat{A}_i$ and $\hat{B}_i$, respectively. For product states, the measurement values are uncorrelated and the uncertainties of $\hat{A}_i+\hat{B}_i$ are equal to the sum of the local uncertainties,

$$\delta(A_i+B_i)^2=\delta A_i^2+\delta B_i^2$$

$$\text{for } \hat{\rho}=\rho(A)\otimes\rho(B). \tag{10}$$

Therefore, the measurement statistics of product states are limited by the uncertainty relation

$$\sum_i \delta(A_i+B_i)^2\geqslant U_A+U_B. \tag{11}$$

Moreover, this uncertainty limit also applies to all mixtures of product states, since the uncertainties of a mixture are always equal to or greater than the averaged uncertainties of the components. For the general case of $\hat{\rho}=\Sigma_m p_m\hat{\rho}_m$ and an arbitrary property $\hat{S}$, this relation between the uncertainties of a mixture and the uncertainty of its components can be obtained from

$$\begin{aligned}\delta S^2&=\sum_m p_m \mathrm{tr}\{\rho_m(\hat{S}-\langle\hat{S}\rangle)^2\}\\&=\sum_m p_m\Big(\underbrace{(\mathrm{tr}\{\rho_m\hat{S}^2\}-\mathrm{tr}\{\rho_m\hat{S}\}^2)}_{=\delta S_m^2}+\underbrace{(\mathrm{tr}\{\rho_m\hat{S}\}-\langle\hat{S}\rangle)^2}_{\geqslant 0}\Big)\\&\geqslant\sum_m p_m\delta S_m^2.\end{aligned} \tag{12}$$

It follows from this result that the uncertainty relation (11) for product states also applies to a mixture of product states,

$$\sum_i \delta(A_i+B_i)^2\geqslant U_A+U_B$$

$$\text{for any } \hat{\rho}=\sum_m p_m\hat{\rho}_m(A)\otimes\hat{\rho}_m(B). \tag{13}$$

Any violation of this uncertainty limit therefore proves that the quantum state cannot be separated into a mixture of product states. However, entangled states can overcome this limitation, since entanglement describes correlations that are more precise than the ones represented by mixtures of product states. The violation of any local uncertainty relation of the form (13) is therefore a sufficient condition for the existence of entanglement.

IV. VIOLATION OF LOCAL UNCERTAINTY RELATIONS

To illustrate how entanglement can overcome the local uncertainty limit defined by Eq. (13), it may be useful to consider the properties of maximally entangled states. Using the Schmidt bases $|n\rangle_\mathrm{A}$ and $|n\rangle_\mathrm{B}$ for *A* and *B*, these states can be written as

$$|E_{\max}\rangle_{\mathrm{A;B}}=\frac{1}{\sqrt{N}}\sum_n |n;n\rangle_{\mathrm{A;B}}. \tag{14}$$

Such maximally entangled states appear to violate the uncertainty principle because any property of system *A* can be determined by a corresponding measurement on system *B*. That is, a measurement of an eigenvalue of $\hat{A}_i$ in *A* projects the quantum state in *B* into the eigenstate of $-\hat{B}_i$ with the same eigenvalue as the one obtained for $\hat{A}_i$ in *A*. This means that, for any set of operators $\hat{A}_i$ in *A*, there is a set of corresponding operators $-\hat{B}_i$ in *B* such that the measurement result of $\hat{A}_i$ is always equal to the measurement result of $-\hat{B}_i$. In more formal terms, $|E_{\max}\rangle_{\mathrm{A;B}}$ is a simultaneous eigenstate of all $\hat{A}_i+\hat{B}_i$ with eigenvalues of zero [22]. Maximally entangled states can thus have a total uncertainty of zero in all properties $\hat{A}_i+\hat{B}_i$, maximally violating the uncertainty relation (13), with

$$(\hat{A}_i+\hat{B}_i)|E_{\max}\rangle_{\mathrm{A;B}}=0 \quad \text{and} \quad \sum_i \delta(A_i+B_i)^2=0$$

$$\text{for } \langle n|\hat{B}_i|n'\rangle=-\langle n'|\hat{A}_i|n\rangle. \tag{15}$$

Experimentally, it is then possible to evaluate how close a given mixed state output is to an intended maximally entangled state by measuring the remaining uncertainty due to imperfections in the entanglement generation process. To obtain a quantitative estimate of the quality of entanglement generation, the measured uncertainty can be compared with the uncertainty limit of $U_A+U_B=2U$ for separable states. Specifically, the relative violation of local uncertainty may be defined as

$$C_{\mathrm{LUR}}=1-\frac{\sum_i \delta(A_i+B_i)^2}{2U}. \tag{16}$$

Since some amount of entanglement is necessary to overcome the uncertainty limit, C_{LUR} provides a quantitative estimate of the amount of entanglement verified by the violation of local uncertainty. In particular, it may be interesting to determine the minimal amount of entanglement necessary to obtain a given value of local uncertainty violation C_{LUR} for various local uncertainty relations. Once such relations are known, it will be possible to obtain reliable estimates of entanglement from local uncertainty violations without additional assumptions about the quantum state.

V. APPLICATION TO ENTANGLEMENT BETWEEN TWO SPIN-1 SYSTEMS

In general, any uncertainty relation of the type given by Eq. (2) can be used to define an uncertainty limit for nonentangled states according to relation (13). However, in most

cases it will be convenient to define the limit in a highly symmetric way. This can be achieved for any N-level system by using the spin uncertainty (4). The local uncertainty relation for separable states of two spin $l=(N+1)/2$ systems is given by

$$\delta[L_x(A)+L_x(B)]^2+\delta[L_y(A)+L_y(B)]^2 + \delta[L_z(A)+L_z(B)]^2 \geqslant 2l. \quad (17)$$

Any state that violates this uncertainty relation must be entangled. The optimal result of zero total uncertainty is obtained for the singlet state, defined by

$$[\hat{L}_i(A)+\hat{L}_i(B)]|\text{singlet}\rangle_{A;B}=0. \quad (18)$$

Experimental methods of generating such singlet states for three-level systems ($l=1$) have been realized using optical parametric down-conversion to create photons entangled in their spatial degrees of freedom [1,2], or to create entanglement between the polarization properties of a pair of two photon states [3,4]. The relative violation of local uncertainties defined by Eq. (16) may serve as an easily accessible quantitative measure of the achievements represented by these experiments.

In order to minimize the experimental effort involved in characterizing the entanglement of three-level systems, it is also possible to use the local uncertainty limit based on relation (7),

$$\delta[L_x(A)+L_x(B)]^2+\delta[L_y(A)+L_y(B)]^2 \geqslant \frac{7}{8}. \quad (19)$$

This inequality requires only two measurement settings corresponding to 18 measurement probabilities for its verification. For comparison, the experimental verification of a Bell's inequality violation reported in Ref. [4] required four settings and 36 measurement probabilities. Moreover, the optimization of the Bell's inequality violations required measurements at additional settings, while the measurement settings for the local uncertainty relation (19) are defined by the symmetry of the experimental setup and do not have to be varied. Unfortunately, the measurement data given in Ref. [4] are not sufficient to allow an analysis of the local uncertainties of this entanglement source. However, the measurement result was interpreted using a simplified noise model given in the $\hat{L}_x$ basis by

$$\hat{\rho}=p_s(|\text{singlet}\rangle\langle\text{singlet}|)+\frac{(1-p_s)}{3}(|+1;-1\rangle \times\langle+1;-1|+|0;0\rangle\langle 0;0|+|-1;+1\rangle\langle -1;+1|), \quad (20)$$

that is, the correlation along the x axis of the spin is assumed to be perfect, while the other two correlations fluctuate with

$$\delta[L_x(A)+L_x(B)]^2=0,$$

$$\delta[L_y(A)+L_y(B)]^2=\frac{4}{3}(1-p_s),$$

$$\delta[L_z(A)+L_z(B)]^2=\frac{4}{3}(1-p_s). \quad (21)$$

For this model, the relative violation of the local uncertainty relation (19) is equal to

$$C_{\text{LUR}}=\frac{32p_s-11}{21}. \quad (22)$$

Using the value of $p_s=0.69$ reported in Ref. [4], the relative violation of relation (19) achieved in this experiment should be equal to $C_{\text{LUR}}=0.53$. It might be interesting to compare this value with direct measurements of local uncertainty violations in future experiments.

VI. UNCERTAINTY VIOLATION AND CONCURRENCE IN 2×2 SYSTEMS

For two-level systems, the uncertainty relations (5) and (6) define two different criteria for entanglement verification. The local uncertainty relation based on Eq. (5) reads

$$\delta[\sigma_1(A)+\sigma_1(B)]^2+\delta[\sigma_2(A)+\sigma_2(B)]^2 + \delta[\sigma_3(A)+\sigma_3(B)]^2 \geqslant 4. \quad (23)$$

This uncertainty relation is useful in order to identify the level of singlet-state entanglement in a noisy mixture. It includes all three Pauli matrices and is therefore not sensitive to any anisotropy in the noise distribution. The local uncertainty relation based on Eq. (6) reads

$$\delta[\sigma_1(A)+\sigma_1(B)]^2+\delta[\sigma_2(A)+\sigma_2(B)]^2 \geqslant 2. \quad (24)$$

This local uncertainty relation can be tested with only two measurement settings. It may therefore be useful in cases where it is necessary to test for entanglement with only a limited number of measurements. Since one of the three Pauli matrices is not considered, this condition for separability is sensitive to noise anisotropies. In particular, it corresponds to Eq. (23) if the uncertainty in $\hat{\sigma}_3(A)+\hat{\sigma}_3(B)$ is close to 2, and is more difficult to violate otherwise.

While a precise characterization of experimentally generated quantum states is very difficult, a measurement of the uncertainties can provide a comparatively simple test of an essential entanglement property. A complete illustration of the many kinds of errors in entanglement generation, which can increase the uncertainty levels and thus degrade the entanglement, is beyond the scope of this paper. In fact, the uncertainty limits presented above are useful precisely because they do not require a full characterization of the statistics given by the complete density matrix. Nevertheless, it may be useful to look at one specific example to illustrate the relationship between the uncertainty boundaries and the actual entanglement of the density matrix. The most simple

case is given by a mixture of a maximally noisy state and the intended maximally entangled state often referred to as a Werner state [9],

$$\hat{\rho} = (1-p_s)\frac{1}{4}\hat{1} + p_s|\text{singlet}\rangle\langle\text{singlet}|. \tag{25}$$

Here, the parameter p_s represents the fraction by which the intended entangled state exceeds the background noise. For pairs of two-level systems, the amount of entanglement of any quantum state can be expressed in terms of the concurrence C [23,24]. For Werner states, the concurrence is $C = \max\{(3p_s-1)/2,0\}$. It is interesting to compare this precise measure of the total entanglement of the two systems with the relative violations of local uncertainty defined by relation (23). Since the Werner state is completely isotropic, the uncertainties of each component $\hat{\sigma}_i(A)+\hat{\sigma}_i(B)$ are given by

$$\delta[\sigma_i(A)+\sigma_i(B)]^2 = 2(1-p_s). \tag{26}$$

Therefore, relation (23), which gives equal weight to each component, appears to be optimally suited as a measure of entanglement for this class of states. This expectation is indeed confirmed by the relative violation of local uncertainty, which is in this case precisely equal to the concurrence,

$$C_{\text{LUR}} = 1 - \frac{1-p_s}{3} = C. \tag{27}$$

This result shows that for some class of states, the concurrence is exactly equal to the amount of noise suppression achieved in the total spin variables. It is an interesting question how large this class of states is. At present, we would like to note that it is straightforward to extend the result to arbitrary mixtures of Bell states. In general, it seems to be quite significant that the relative violation of uncertainty can be used as an estimate of the concurrence, even though the experimental effort involved in any precise determination of the concurrence greatly exceeds the effort required to measure the relative violation of local uncertainty.

In this context, it may also be interesting to consider uncertainty relation (24), which requires only two measurement settings. Clearly, this uncertainty limit is more difficult to overcome because it does not include the correlations in the third component $\hat{\sigma}_3(A)+\hat{\sigma}_3(B)$. As a result, the relative violation of this uncertainty for Werner states is lower than the concurrence C by

$$C'_{\text{LUR}} = 1 - \frac{1-p_s}{2} = C - \frac{1-C}{2}. \tag{28}$$

However, since the relative violation of Eq. (24) is always lower than the relative violation of Eq. (23), C'_{LUR} may provide a useful lower bound for an experimental estimate of the concurrence using only two measurement settings.

VII. FURTHER POSSIBILITIES AND OPEN QUESTIONS

As explained in Sec. II, uncertainty relations can be formulated for any operator set $\{\hat{A}_i\}$. It is therefore possible to optimize the choice of operators in the local uncertainty relation with respect to a given physical situation. In particular, it may be possible to classify entangled states according to the types of local uncertainty relations they violate. In any case, it should be kept in mind that the examples given here are far from complete.

As mentioned in Sec. III, local uncertainty relations can also be formulated for $N\times M$ entanglement, where the dimensionality of the two Hilbert spaces is different. One application of this possibility may be the investigation of multipartite entanglement, where it allows the formulation of bipartite uncertainty limits for various partitions of the multipartite system [25].

As noted in the Introduction, local uncertainties may also be useful as a characterization of the increased precision obtained from entanglement in applications such as teleportation, lithography, and clock synchronization [16–19,22]. On the other hand, quantum information protocols usually define entanglement with respect to distillability by local operations and classical communication. This raises the question how the two concepts are related to each other. Does the distillation of entanglement actually decrease the uncertainty in the nonlocal correlation, or does it merely redistribute the quantum fluctuations [26]?

These are just a few of the questions raised by the possibility of quantifying the violation of local uncertainty relations by entangled states. A systematic classification of local uncertainty relations may thus provide many new insights into the physical properties of entangled states.

VIII. CONCLUSIONS

In conclusion, we have generalized the uncertainty principle to uncertainty sums of arbitrary sets of physical properties and derived local uncertainty relations valid for all separable states of a pair of N-level quantum systems. Any violation of these local uncertainty relations indicates that the two systems are entangled. The relative violation of a local uncertainty provides a quantitative measure of this entanglement property and may be used to evaluate experimental entanglement generation processes. It should also be possible to obtain valid estimates of the total entanglement from uncertainty measurements. Specifically, the relative violation of local uncertainty is actually equal to the concurrence for some 2×2 cases. In more general cases, it may be possible to identify the minimal amount of entanglement necessary to obtain the observed level of local uncertainty violation, thus establishing a more precise relation between the local uncertainty violation and the total entanglement of the system. Local uncertainty relations may thus provide an interesting starting point for further investigations into the physical properties of entanglement.

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