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Virasoro Algebra, Vertex Operators, Quantum Sine-Gordon and Solvable Quantum Field Theories

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Abstract

The relationship between the conformal field theories and the soliton equations (KdV, MKdV and Sine-Gordon, etc.) at both quantum and classical levels is discussed. The quantum Sine-Gordon theory is formulated canonically. Its Hamiltonian is the vertex operator with respect to the Feigin-Fuchs-Miura form of the Virasoro algebra with central charge $c \leq 1$. It is found that the quantum conserved quantities of the Sine-Gordon-MKdV hierarchy are expressed as polynomial functions of the Virasoro generators. In other words, an infinite set of mutually commutative polynomial functions of the Virasoro generators is obtained. A very simple recursion formula for the quantum conserved quantities is found for the special case of $\beta_c^2 = 8\pi$ (β_c is the coupling constant in Coleman's theory of quantum Sine-Gordon).

§ 1. Introduction

Conformal field theories in two dimensions ([1], [2]), especially those corresponding to the unitarizable Virasoro algebras ([3]) with central charge $c < 1$, are extremely interesting, since they offer rich examples in which Green's functions are calculable. The relationship between the conformal field theories and solvable lattice models has been extensively investigated ([4]). In order to obtain dynamical understanding of their solvability, we discuss in the present paper the relationship between the conformal field theories and other types of solvable models, i.e., solvable continuum quantum field theories and in particular the quantum Sine-Gordon and the quantum Modified KdV (MKdV) theory ([5], [6]).

Let us recall the condition characterizing solvability in classical particle dynamics. That is the existence of independent and mutually involutive constants of motion (conserved quantities) as many as the degree of freedom (Liouville) ([7]). So we expect a solvable quantum field theory

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is characterized by the existence of an infinite set of quantum commuting operators (conserved quantities) ([5], [6], [8]). Of course, conformal field theories in two dimensions do have an infinite number of conservation laws

$$\partial_{\bar{z}}(T_{zz}z^n)=0, \quad \partial_z(T_{\bar{z}\bar{z}}\bar{z}^n)=0, \qquad n \in \boldsymbol{Z}, \tag{1.1}$$

in which $z=x_1+ix_2$, $\bar{z}=x_1-ix_2$ are coordinates and T_{zz}, $T_{\bar{z}\bar{z}}$ are components of the Energy-Momentum tensor of the system. However, they are a consequence of two invariance principles

$$\partial_\mu T_{\mu\nu}=0, \qquad \text{translational invariance,} \tag{1.2}$$

$$T_{\mu\mu}=0, \qquad \text{scale invariance.} \tag{1.3}$$

Therefore no other dynamical information about the solvability of conformal field theories seems to be contained in eq. (1.1), except that the dynamics is decomposed into the holomorphic and antiholomorphic parts.

At the level of the classical theory, however, there is a hint. Gervais ([9]) pointed out that an infinite set of mutually involutive polynomial functions of the Virasoro generators does exist provided that the Virasoro commutation relations are regarded as the Poisson brackets. These are nothing but the well known infinite set of conserved quantities of the Korteweg de Vries (KdV) equation ([10]), a solvable classical field theory. In this paper we show its quantum version. Namely, we show that *there exists an infinite set of mutually commutative "polynomial" functions of the Virasoro generators.* And *these quantum commuting operators are closely related with the solvable quantum Sine-Gordon (S-G) theory and the MKdV theory,* examples of solvable quantum field theories ([6]). We put a quotation mark on "polynomial" since the ordering of the Virasoro generators must be carefully specified due to their commutation relations in contrast with the classical case in which ordering is immaterial.

This paper is organized as follows. In Section 2 we briefly review the basic results of the classical soliton theories which are necessary for the present purpose. The close relationships with the quantum case are stressed, for example the Miura transformation ([11]) connecting the MKdV and the KdV hierarchy as the Feigin-Fuchs form ([12]) of the Virasoro field and the Sine-Gordon Hamiltonian as the classical limit of vertex operators. In Section 3 the outline of the quantum soliton theories, proposed by the present authors ([5], [6], [8]) is introduced. We show the construction method and the properties of the quantum conserved quantities for the MKdV hierarchy. As a new result we give a proof of the existence of the infinite set of quantum commuting operators, which is valid for most soliton equations. Section 4 and Section 5 are the main part of this paper. In Section 4 we define the integrable Sine-Gordon theory in terms of the vertex operators and the Feigin-Fuchs-Miura form of the Virasoro algebra. The mutually commuting "polynomial" functions of the Virasoro generators are identified with conserved quantities of the quantum S-G theory. In Section 5 we give a method of construction of these mutually commutative "polynomial" functions of the Virasoro generators. Explicit forms of some of its lower members are given and a simple recursion formula is proposed. Rather distinctive features are found when the Virasoro algebras happen to be some members of the discrete unitarizable series and they also correspond to the special values of the coupling constant in Coleman's ([13]) theory of quantum Sine-Gordon equation.

§ 2. Classical Soliton Theories

In order to introduce appropriate notation and to be selfcontained, we summarize the classical soliton theories which are necessary for later sections. Certain new results, for example Proposition 2.5, are contained. For more details see [6].

2-1. Modified Korteweg de Vries (MKdV) equation

The MKdV eq. reads

$$u_t = u_{\sigma\sigma\sigma} - 6u^2 u_\sigma, \tag{2.1.1}$$

in which $u=u(t, \sigma)$ is a real (hermitian) field in $1+1$ dimensions depending on time (t) and space (σ) variables and $u_t = \partial_t u$, $u_\sigma = \partial_\sigma u$, $u_{\sigma\sigma} = \partial_\sigma^2 u, \cdots$, etc. It is assumed that u has continuous σ-derivatives of all orders. Throughout this paper we impose the periodic boundary condition with a period 2π,

$$u(t, \sigma+2\pi) = u(t, \sigma). \tag{2.1.2}$$

An infinite set of polynomial conserved quantities is given as follows

$$I_n = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2} u(\sigma) Y_{2n-1}(u), \qquad n \in N, \tag{2.1.3}$$

in which $Y_n(u)$ is a polynomial in $u, u_\sigma, u_{\sigma\sigma}, \cdots$, defined recursively

$$Y_{n+1} = \partial_\sigma Y_n + u \sum_{k=1}^{n-1} Y_k Y_{n-k}, \qquad Y_1 = -u. \tag{2.1.4}$$

It should be remarked that in eqs. (2.1.3), (2.1.4) and hereafter the *time t is always fixed and suppressed.* It is easy to see that uY_{2n} is a total σ-

derivative giving rise to a trivial conserved quantity. Explicit forms of some lower members of I_n are

(2.1.5)
$$\begin{aligned} I_1&=\frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2} u^2, \\ I_2&=\frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2}(-u^4-u_\sigma^2), \\ I_3&=\frac{1}{2\pi}\int_0^{2\pi} d\sigma\left(u^6+5u^2u_\sigma^2+\frac{1}{2}u_{\sigma\sigma}^2\right), \\ I_4&=\frac{1}{2\pi}\int_0^{2\pi} d\sigma\left(-\frac{5}{2}u^8-35u^4u_\sigma^2+\frac{7}{2}u_\sigma^4-7u^2u_{\sigma\sigma}^2-\frac{1}{2}u_{\sigma\sigma\sigma}^2\right). \end{aligned}$$

Let us denote by P the integral of a local polynomial p in $u, u_\sigma, u_{\sigma\sigma}, \cdots$,

(2.1.6)
$$P=\frac{1}{2\pi}\int_0^{2\pi} d\sigma p(u, u_\sigma, u_{\sigma\sigma}, \cdots),$$

and by V the vector space (with complex coefficients) spanned by them

(2.1.7)
$$V=\{P|P \text{ given in eq. (2.1.6)}\}.$$

We introduce the following Poisson bracket for the field $u(\sigma)$,

(2.1.8)
$$\{u(\sigma), u(\sigma')\}=\left(\frac{\beta}{2}\right)^2 2\pi\partial_\sigma\delta(\sigma-\sigma'),$$

in which β is a positive constant. Later it will be interpreted as the coupling constant in the quantum Sine-Gordon theory. In fact, the MKdV equation (2.1.1) is the canonical equation with $(\beta/2)^{-2}I_2$ as the Hamiltonian,

(2.1.9)
$$\partial_t u(\sigma)=\{u(\sigma), (\beta/2)^{-2}I_2\}=u_{\sigma\sigma\sigma}(\sigma)-6u^2u_\sigma(\sigma).$$

The vector space V is closed with respect to the Poisson bracket (2.1.8), i.e.,

(2.1.10)
$$P, Q\in V \Longrightarrow \{P, Q\}\in V.$$

Some of the main results of the classical MKdV eq. are:

Proposition 2.1 (Involution).

(2.1.11)
$$\{I_n, I_m\}=0, \qquad n, m\in N,$$

Proposition 2.2 (Uniqueness).

$$(2.1.12) \qquad P \in V, \qquad \{I_2, P\}=0 \Longrightarrow P=\sum_{n=0}^{\infty} c_n I_n,$$

in which $I_0=\int_0^{2\pi} u d\sigma$ *and* c_n *are constants.*

2-2. Sine (h)-Gordon equation

The Sine-Gordon (S-G) equation (in the light-cone coordinates) reads

$$(2.2.1) \qquad \partial_t \partial_\sigma \phi + \frac{1}{\beta} \sin \beta\phi = 0,$$

whereas the Sinh-Gordon eq. is

$$(2.2.2) \qquad \partial_t \partial_\sigma \phi + \frac{1}{\beta} \sinh \beta\phi = 0.$$

Here β is the same positive constant as appear in eq. (2.1.8). In classical theory it can be absorbed by redefinition of the field. These two equations are closely related with the MKdV eq. In fact, by the identification

$$(2.2.3) \qquad u(\sigma) = \frac{1}{2} \beta \partial_\sigma \phi(\sigma),$$

the conserved quantities $I_n(u)$, eq. (2.13) of the MKdV eq., eq. (2.1.1) are also conserved quantities of the Sinh-Gordon eq. (2.2.2). The Sine-Gordon eq. is, by the same identification (2.2.3), related to another type of MKdV eq.

$$(2.2.4) \qquad u_t = u_{\sigma\sigma\sigma} + 6u^2 u_\sigma,$$

which is obtained from eq. (2.1.1) by replacing u by iu as the Sine-Gordon eq. is derived from the Sinh-Gordon eq. by replacing ϕ by $i\phi$.

Proposition 2.3 (The Sine-Gordon-MKdV hierarchy). *The Sine(h)-Gordon eq. and the MKdV eq. share the same set of polynomial conserved quantities under the identification* (2.2.3).

In the following *we always identify* u *with* $(\beta/2)\partial_\sigma\phi$ by eq. (2.2.3). We impose the following Poisson brackets on $\phi(\sigma)$ in conformity with the Poisson bracket for u, eq. (2.1.8),

$$(2.2.5) \qquad \{\phi(\sigma), \phi(\sigma')\} = -\pi\varepsilon(\sigma-\sigma'), \qquad \{\phi(\sigma), u(\sigma')\} = \frac{\beta}{2} 2\pi\delta(\sigma-\sigma'),$$

in which $\varepsilon(\sigma)$ is the signature function. These relations can be obtained as the Dirac bracket ([14]) when formulated in terms of the Sine-Gordon Lagrangian in the light-cone coordinates. The S-G eq. can be expressed as a canonical equation

(2.2.6) $$\beta\partial_t\partial_\sigma\phi(\sigma)=2u_t(\sigma)=2\{u(\sigma),\ H\},$$

with the Hamiltonian

(2.2.7) $$H=\frac{1}{2\pi\beta^2}\int_0^{2\pi}d\sigma(1-\cos\beta\phi).$$

Then Proposition 2.3 can be expressed simply as

Proposition 2.3′.

(2.2.8) $$\{\mathrm{I}_n(iu),\ H\}=0,\qquad n\in N.$$

It should be remarked that our coupling constant β^2 differs from the conventional one ([13]) β_c^2 by 2π,

(2.2.9) $$\beta_c^2=2\pi\beta^2.$$

Due to the periodic boundary condition on u, eq. (2.1.2), we expand $u(\sigma)$ in a Fourier series

(2.2.10) $$u(\sigma)=\frac{\beta}{2}\sum_{n=-\infty}^{\infty}\alpha_n e^{-in\sigma},$$

in which

(2.2.11) $$(\alpha_n)^\dagger=\alpha_{-n},\qquad n\in Z$$

by reality (hermiticity) of the field u. This fixes the expansion of $\phi(\sigma)$ except for the zero mode

(2.2.12) $$\phi(\sigma)=q+\alpha_0\sigma+i\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-in\sigma}.$$

The Poisson bracket (2.2.5) can be rewritten as

(2.2.13) $$\{\alpha_n,\ \alpha_m\}=-in\delta_{n+m,0},\qquad \{q,\ \alpha_n\}=\delta_{n,0}.$$

If we impose the condition that $e^{i\beta\phi}$ and $e^{-i\beta\phi}$ be periodic in σ with a period 2π, we have

(2.2.14) $$\beta\alpha_0\in Z.$$

2–3. Korteweg de Vries (KdV) equation

The KdV eq.

$$v_t = v_{\sigma\sigma\sigma} + 6vv_\sigma, \tag{2.3.1}$$

is also closely related with the MKdV eq. In fact, the Miura transformation ([11])

$$v = u^2 \pm iu_\sigma, \tag{2.3.2}$$

maps a solution of the MKdV eq. (2.2.4) to that of the KdV eq.

$$v_t - (v_{\sigma\sigma\sigma} + 6vv_\sigma) = (2u \pm i\partial_\sigma)[u_t - (u_{\sigma\sigma\sigma} + 6u^2 u_\sigma)]. \tag{2.3.3}$$

There is another type of Miura transformation

$$v = u^2 \pm u_\sigma, \qquad \text{hermitian}, \tag{2.3.4}$$

connecting the MKdV eq., (2.1.1) to the KdV eq.

$$v_t = v_{\sigma\sigma\sigma} - 6vv_\sigma, \tag{2.3.5}$$

$$v_t - (v_{\sigma\sigma\sigma} - 6vv_\sigma) = (2u \pm \partial_\sigma)[u_t - (u_{\sigma\sigma\sigma} - 6u^2 u_\sigma)]. \tag{2.3.6}$$

The second type of the Miura transformation, which we no longer discuss in the present paper, corresponds to a Virasoro algebra $c > 1$ after quantization.

The Poisson bracket of the u field defines a Poisson bracket for the v field through the Miura transformation, eq. (2.3.2),

$$\{v(\sigma), v(\sigma')\} = 2\pi\left(\frac{\beta}{2}\right)^2 [2(v(\sigma) + v(\sigma'))\partial_\sigma \delta(\sigma - \sigma') + \partial_\sigma^3 \delta(\sigma - \sigma')]. \tag{2.3.7}$$

This gives rise to a Virasoro algebra at the Poisson bracket level ([9]) up to normalization if an appropriate constant is added.

The KdV eq. has also an infinite set of polynomial conserved quantities $K_n(v)$,

$$K_n(v) = \frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2} v(\sigma) Z_{2n+1}(v), \qquad n = 0, 1, 2, 3, \cdots, \tag{2.3.8}$$

in which $Z_n(v)$ is defined recursively

$$Z_{n+1} = \partial_\sigma Z_n + v \sum_{k=1}^{n-1} Z_k Z_{n-k}, \qquad Z_1 = 1. \tag{2.3.9}$$

The explicit forms of the lower members of the conserved quantities are (we impose the periodic boundary condition on v, too.)

$$\begin{aligned} K_0 &= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2} v, \\ K_1 &= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2} v^2, \\ K_2 &= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2}(-v_\sigma^2+2v^3), \\ K_3 &= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2}(v_{\sigma\sigma}^2-10vv_\sigma^2+5v^4), \\ K_4 &= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \frac{1}{2}(-v_{\sigma\sigma\sigma}^2+14vv_{\sigma\sigma}^2-70v^2v_\sigma^2+14v^5), \cdots. \end{aligned} \tag{2.3.10}$$

Theorems for the KdV hierarchy corresponding to those of the MKdV case, Propositions 2.1 and 2.2 also hold true for appropriate Poisson brackets. And, in particular, *for the Poisson bracket* (2.3.7), *the set* $K_n(v)$ *gives the infinite set of mutually involutive polynomial functions of the Virasoro generators* ([9]), which are nothing but the Fourier components v_n of the v field.

If we substitute v by the Miura transformation eq. (2.3.2), the polynomial conserved quantities $K_n(v)$ of the KdV eq. (2.3.1) give rise to the polynomial conserved quantities of the MKdV eq. (2.2.4). In fact, we have

Proposition 2.4.

$$K_n(v=u^2 \pm iu_\sigma)=I_{n+1}(iu), \qquad n=0, 1, 2, \cdots,. \tag{2.3.11}$$

The r.h.s. of the above equation is a real element of the vector space V, eq. (2.1.7). Therefore, the polynomial conserved quantities $K_n(v)$, $n=0, 1, 2, \cdots$, can also be uniquely characterized by the condition that they give rise to real elements in V.

Proposition 2.5. *If the integral of a local polynomial in* $v, v_\sigma, v_{\sigma\sigma}, \cdots$, *after substitution by the Miura transformation,* (2.3.2), *is a real element of* V, *then it is a polynomial conserved quantity of the MKdV eq.* (2.2.4).

Although the proof is quite simple, we give its outline here, since it serves as a simplified model of the corresponding theorem Proposition 4.1 in the quantum case. In fact we have a Poisson bracket relation

(2.3.12) $$\left\{(u^2+iu_\sigma)(\sigma), \int_0^{2\pi} d\sigma' e^{i\beta\phi(\sigma')}\right\}=0, \quad \forall\sigma \in [0, 2\pi].$$

If $F(v)$ is an integral of a local polynomial in v, v_σ, $v_{\sigma\sigma}$, $\cdots$, and if it is a real element of V,

(2.3.13) $$F(v)^\dagger = F(v), \qquad v=u^2+iu_\sigma,$$

we have

(2.3.14) $$\left\{F(v), \int_0^{2\pi} d\sigma' e^{+i\beta\phi(\sigma')}\right\}=0.$$

and

(2.3.15) $$\left\{F(v), \int_0^{2\pi} d\sigma' e^{-i\beta\phi(\sigma')}\right\}=0.$$

The second equality follows from the first by complex (hermitian) conjugation. Then we have

(2.3.16) $$\{F(v), H\}=0,$$

and it is a polynomial conserved quantity of the MKdV-Sine-Gordon hierarchy.

§ 3. Quantum Soliton Theory, the MKdV Hierarchy

In this section we introduce various concepts and notation of the integrable quantum field theories ([6]) which constitute the background for the developments in Section 4 and Section 5. The main result is that the soliton theories can be quantized as continuum theories in such a way that the infinite set of polynomial conserved quantities survives, Proposition 3.6. In other words, quantum soliton theories have an infinite set of quantum commuting operators. For details see [6], [8].

Canonical quantization of the $u(\phi)$ field is achieved by replacing the Poisson brackets (2.2.13) by the following commutation relations

(3.1) $$[\alpha_n, \alpha_m]=\hbar n\delta_{n+m,0}, \qquad [q, \alpha_n]=i\hbar\delta_{n,0}.$$

Since we are interested in the relationship between the classical and quantum theories all the $\hbar$ (Planck's constant divided by 2π) dependence are displayed explicitly. In terms of the field $u(\sigma)$, the commutation relation reads

(3.2) $$[u(\sigma), u(\sigma')]=i\hbar' 2\pi\partial_\sigma\delta(\sigma-\sigma'),$$

in which $\hbar'$ is defined by

$$\hbar'=\left(\frac{\beta}{2}\right)^2\hbar. \tag{3.3}$$

It should be remarked that $\hbar$ in [5], [6] should be identified with $\hbar'$ in this paper.

We interpret α_m, $m>0$ ($m<0$) as annihilation (creation) operators and α_0 as a zero mode (momentum operator) and q its conjugate coordinate operator. The vacuum $|0>$ and a state $|0;\mu>$ with momentum μ are defined as follows

$$\alpha_n|0>=0, \qquad n\geq 0. \tag{3.4}$$

$$\alpha_n|0;\mu>=0, \quad n>0, \quad \alpha_0|0;\mu>=\mu|0;\mu>, \quad \mu\in \boldsymbol{R}. \tag{3.5}$$

The Fock space is spanned by the basis

$$\alpha_{-n_1}\alpha_{-n_2}\cdots\alpha_{-n_k}|0;\mu>, \quad n_j>0, \quad \mu\in \boldsymbol{R}, \tag{3.6}$$

and $u(\sigma)$ is an operator acting on this space.

Next let us define *normal ordering*, which we denote by : :. Suppose a monomial in α_n is given

$$\alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_k}, \tag{3.7}$$

its normal ordering is defined by putting all the annihilation operators to the right of the creation operators

$$:\alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_k}:=\alpha_{n'_1}\cdots\alpha_{n'_j}\alpha_{n'_{j+1}}\cdots\alpha_{n'_k}, \tag{3.8}$$

in which $(n'_1, \cdots, n'_k)$ is a reordering of $(n_1, \cdots, n_k)$ such that

$$n'_1, n'_2, \cdots, n'_j\leq 0 \quad \text{and} \quad n'_{j+1}, \cdots, n'_k\geq 0. \tag{3.9}$$

Since creation (annihilation) operators commute among themselves their ordering is immaterial. In this way we can define normal ordering for products of field operators at the same (different) point or their integrals as well.

$$:u^n(\sigma):, \quad :u(\sigma_1)u_{\sigma\sigma}(\sigma_2):, \quad \int_0^{2\pi}d\sigma\ :u^n u^m_{\sigma\sigma}(\sigma):, \text{ etc.} \tag{3.10}$$

Let us denote by $\hat{V}$ the quantum counterpart of the vector space V, eq. (2.1.7), defined in Section 2–1, (3.11) $$\hat{V}=\{:P:\,|\,P \text{ given in eq. (2.1.6)}\}.$$

Proposition 3.1. *The vector space $\hat{V}$ is closed with respect to the commutation relation* (3.2), *i.e.*,

(3.12) $$\forall :P:,\ :Q: \in \hat{V} \Longrightarrow [:P:,\ :Q:] \in \hat{V}.$$

Therefore, we look for the infinite set of quantum commuting operators in this space. It should be remarked that quantum closure is not an automatic consequence of classical closure and quantization. For example, it breaks down for the Nonlinear Schrödinger equation in the conventional quantization ([8]). Supposing that the coefficients of P, $Q \in V$ has no $\hbar'$ dependence, we have the following correspondence between the quantum and classical quantities.

Proposition 3.2 (Correspondence).

(3.13) $$[:P:,\ :Q:]_{(1)}=i:\{P,\ Q\}:,$$

in which the l.h.s. is the coefficient of the lowest term of a polynomial in $\hbar'$;

(3.14) $$[:P:,\ :Q:]=\sum_{k=1}\hbar'^{k}[:P:,\ :Q:]_{(k)}.$$

In proving Propositions 3.1, 3.2 and other relations the techniques of conformal field theories and/or string theories are quite useful. Let us explain its essence briefly. We introduce a fictitious "time" variable τ and define a complex variable z by

(3.15) $$z=e^{\tau+i\sigma},$$

and consider the field theory defined on the complex plane

(3.16) $$u(z)=\left(\frac{\beta}{2}\right)\sum_{n}\alpha_{n}z^{-n},$$

instead of the original theory defined on the circle S^1. Then $\forall P \in V$, eq. (2.1.6) can be expressed as

(3.17) $$P=\oint_{0}\frac{dz}{z}p(u,\ Du,\ D^{2}u,\ \cdots,),$$

in which

(3.18) $$\oint=\frac{1}{2\pi i}\oint, \qquad D=iz\frac{\partial}{\partial z}.$$

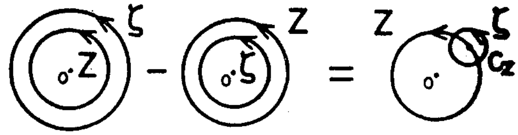

Fig. 1

(The quantum expression has : : on both sides). The integration contour should encircle the origin once and otherwise arbitrary due to the analyticity. By a simple deformation of integration contours, we get the following general expression for the commutator $[:P:, :Q:]$, (Fig. 1),

$$[:P:, :Q:]=\oint_o \frac{dz}{z}\oint_{c_z}\frac{d\zeta}{\zeta}R:p(\zeta):\, :q(z):, \tag{3.19}$$

in which c_z is a small contour encircling the point z and R denotes the radial ordering ([15]). For two local operators $a(\zeta)$, $b(z)$ with coordinates z and ζ their radial ordering is defined by

$$Ra(\zeta)b(z)=\begin{array}{ll} a(\zeta)b(z) & \text{if } |\zeta|>|z|, \\ b(z)a(\zeta) & \text{if } |\zeta|<|z|. \end{array} \tag{3.20}$$

Generalization to the products of three or more local operators is obvious. In order to calculate the commutator, eq. (3.19), we expand the radial ordered product $R:p(\zeta):\, :q(z):$ by Wick's theorem ([16]). Namely we expand a radial ordered product of local operators into a sum of a normal ordered product times propagators. The simplest example of Wick's theorem is

$$Ru(\zeta)u(z)=:u(\zeta)u(z):+\hbar'\Delta(\zeta, z), \tag{3.21}$$

in which the propagator $\Delta(\zeta, z)$ is defined by

$$\hbar'\Delta(\zeta, z)=\langle 0|Ru(\zeta)u(z)|0\rangle=\hbar'\frac{\zeta z}{(\zeta-z)^2}. \tag{3.22}$$

Then the ζ integration is the standard residue calculus. For more details see [6].

The infinite set of quantum commuting operators $\hat{I}_n$, $n=1, 2, 3, \cdots$, are constructed as follows. We start from the ansatz

$$\begin{aligned} &\hat{I}_1\equiv :I_1:, \qquad \hat{I}_2\equiv :I_2: \\ &\hat{I}_n\equiv :I_n:+\sum_{k=1}^{n-1}\hbar'^k :I_n^{(k)}:, \qquad n\geq 3, \end{aligned} \tag{3.23}$$

in which "quantum correction" terms $:I_n^{(k)}:\in\hat{V}$ are to be determined.

They should satisfy the irreducibility condition;

(3.24) $\qquad I_n^{(k)}$ does not contain any of I_m as a part.

By using Proposition 3.2 and the results of the classical theory, Proposition 2.2 etc., we get the following results.

Proposition 3.3 (Uniqueness) ([6]). *The one parameter ($\hbar'$) family of conditions $[\hat{I}_2, \hat{I}_n]=0$ gives an overdetermined set of equations for $I_n^{(k)}$ and its solution, if exists, is unique.*

Proposition 3.4 (Commutativity) ([6]).

(3.25) $$[\hat{I}_2, \hat{I}_n]=[\hat{I}_2, \hat{I}_m]=0 \Longrightarrow [\hat{I}_n, \hat{I}_m]=0.$$

By explicit calculation with an aid of a formula manipulation computor programs ([17]), we have established the existence of $\hat{I}_n$, $n=3, 4, 5, 6$, and obtained their explicit forms.

In order to prove the existence of the infinite set of quantum commuting operators, we assume the following lemma, which is *purely classical.*

Lemma 3.5. *Let $P_3, P_4, \cdots, P_N, Q \in V$ be of even order and odd weight (for the definition of the order and the weight see* [6]). *If they satisfy*

(3.26) $$\{I_2, P_n\}+\{I_n, Q\}=0, \qquad 3\leq n\leq N,$$

then

(3.27) $$^{\exists}R \in V \quad \textit{and} \quad Q=-\{I_2, R\}, \quad P_n=\{I_n, R\}$$

and vice versa. Here R is of even order and even weight and it is unique modulo $\sum_m c_m I_m$.

We have verified the Lemma for weight $(Q)\leq 15$.

Proposition 3.6 (Existence). *If the Lemma holds in the classical theory, then there exists a one-parameter ($\hbar'$) family of an infinite set of quantum commuting operators $\hat{I}_n$, $n=1, 2, 3, \cdots$, which reduce to the classical ones in the limit $\hbar'\to 0$.*

The proof is obtained by a slight modification of the proof of Proposition 3.4 (Commutativity) with the help of the lemma. The outline is as follows. Assuming that the operators $\hat{I}_n$, $3\leq n\leq N$, satisfying the condition $[\hat{I}_2, \hat{I}_n]$

$=0$ are found, we try to find an operator $\hat{I}_{N+1}$ commuting with $\hat{I}_2$, $[\hat{I}_2, \hat{I}_{N+1}]=0$. Let us suppose that an operator $\tilde{I}_{N+1}$ satisfying a weaker condition

(3.28) $$[\hat{I}_2, \tilde{I}_{N+1}]=\hbar'^L \text{ and higher order in } \hbar',\ L\geq 2,$$

is found. This means that the polynomials $I_{N+1}^{(k)}$, $1\leq k\leq L-2$, in the ansatz

(3.29) $$\tilde{I}_{N+1}=:I_{N+1}:+\sum_{k=1}^{L-1}\hbar'^k:I_{N+1}^{(k)}:+\text{higher orders},$$

are uniquely determined and

(3.30) $$[\hat{I}_2, \tilde{I}_{N+1}]=\hbar'^L:(\{I_2, I_{N+1}^{(L-1)}\}+Q):+\text{higher orders}.$$

Here Q is given by

(3.31) $$:Q:=[\hat{I}_2, :I_{N+1}^{(L-3)}:]_{(3)}+[\hat{I}_2, :I_{N+1}^{(L-2)}:]_{(2)},$$

and $I_{N+1}^{(L-1)}$ is yet to be determined. Next we consider the Jacobi identity

(3.32) $$[\hat{I}_2, [\tilde{I}_{N+1}, \hat{I}_n]]+[\tilde{I}_{N+1}, [\hat{I}_n, \hat{I}_2]]+[\hat{I}_n, [\hat{I}_2, \tilde{I}_{N+1}]]=0, \quad 3\leq n\leq N,$$

in which the second term vanishes by assumption. Then we expand $[\tilde{I}_{N+1}, \hat{I}_n]$ in powers of $\hbar'$ and denote

(4.33) $$[\tilde{I}_{N+1}, \hat{I}_n]=\sum_{k=2}^{L-1}\hbar'^k[\tilde{I}_{N+1}, \hat{I}_n]_{(k)}+\hbar'^L:(\{I_{N+1}^{(L-1)}, I_n\}+P_n): \\ +\text{higher orders},$$

in which $:P_n:$ is a known element of $\hat{V}$ expressed in terms of $\hat{I}_n$ and I_{N+1}, $I_{N+1}^{(k)}$, $1\leq k\leq L-2$;

$$:P_n:=[:I_{N+1}:, :I_n:]_{(L)}+\cdots,.$$

By substituting eqs. (3.30) and (3.33) into the Jacobi identity, eq. (3.32), we repeat the same argument as given in the proof of the commutativity ([6]), and obtain

$$[\tilde{I}_{N+1}, \hat{I}_n]_{(k)}=0, \qquad 1\leq k\leq L-1,$$

as the coefficients of $\hbar'^k$, $1\leq k\leq L$, in the Jacobi identity. From the coefficient of $\hbar'^{L+1}$ we get an identity

(3.34) $$:\{I_2, \{I_{N+1}^{(L-1)}, I_n\}+P_n\}:+:\{I_n, \{I_2, I_{N+1}^{(L-1)}\}+Q\}:=0,$$

which reduces to

$$\{I_2, P_n\}+\{I_n, Q\}=0, \qquad 3\leq n\leq N, \tag{3.35}$$

after using the Jacobi identity for the Poisson bracket. Lemma 3.5 tells that

$$Q=-\{I_2, R\}, \qquad \exists R\in V. \tag{3.27}$$

Therefore, by choosing

$$I_{N+1}^{(L-1)}=R, \tag{3.36}$$

we can get

$$[\hat{I}_2, \tilde{I}_{N+1}]=\hbar'^{L+1} \quad \text{and higher orders.} \tag{3.37}$$

By repeating this process for finite steps we can get $\hat{I}_{N+1}$ commuting with $\hat{I}_2$. Q.E.D.

It should be remarked that the main theorems Proposition 3.3 (Uniqueness), 3.4 (Commutativity) and 3.6 (Existence) on the infinite set of quantum commuting operators are the direct consequences of the corresponding theorems in the classical theory, Propositions 2.1, 2.2 and Lemma 3.5 in terms of Proposition 3.1 (Quantum Closure) and 3.2 (Correspondence). In other words, *the proofs for Propositions* 3.3, 3.4 *and* 3.6 *are quite universal and they hold true for an arbitrary classically integrable system* (i.e., having an infinite set of polynomial conserved quantities) *provided that the quantized system satisfies the quantum closure and the correspondence theorem.*

The explicit forms of $\hat{I}_n$, $n\leq 6$ can be found in [6]. For the special value of $\hbar'=1$, (the "genuine" quantum theory?) they are especially neat

$$\hat{I}_1=\oint\frac{dz}{z}:u^2:, \qquad \hat{I}_2=\oint\frac{dz}{z}\frac{1}{2}:\{-u^4-(Du)^2\}:,$$

$$\hat{I}_3(\hbar'=1)=\oint\frac{dz}{z}:\{(D^2u)^2+15u^2(Du)^2+u^6\}:,$$

$$\hat{I}_4(\hbar'=1)=\oint\frac{dz}{z}:\{(D^3u)^2+28u^2(D^2u)^2-7(Du)^4+70u^4(Du)^2+u^8\}\times\left(-\frac{5}{2}\right):,$$

$$\hat{I}_5(\hbar'=1)=\oint\frac{dz}{z}:\{(D^4u)^2+45u^2(D^3u)^2-100u(D^2u)^3-90(Du)^2(D^2u)^2+210u^4(D^2u)^2-315u^2(Du)^4+210u^6(Du)^2+u^{10}\}\times 7:, \tag{3.38}$$

$$\hat{I}_6(\hbar'=1)=\oint\frac{dz}{z}:\{(D^5u)^2+66u^2(D^4u)^2-660u(D^2u)(D^3u)^2$$
$$-165(Du)^2(D^3u)^2+495^4(D^3u)^2+220(D^2u)^4$$
$$-2200u^3(D^2u)^3-5940u^2(Du)^2(D^2u)^2+924u^6(D^2u)^2$$
$$+297(Du)^6-3465u^4(Du)^4+495u^8(Du)^2+u^{12}\}$$
$$\times(-21):, \cdots .$$

In Section 5 we will derive a simple expression (recursion formula) for $\hat{I}_n(\hbar'=1)$, $\forall n \in N$, eq. (5.17).

§ 4. Vertex Operators and Quantum Sine-Gordon Theory

In this section we formulate the quantum Sine-Gordon theory as an additional member of the integrable quantum MKdV hierarchy. Our approach is canonical. We consider the field operator ϕ defined on the complex plane

$$\phi(z)=q-i\alpha_0 \ln z+i\sum_{n\neq 0}\frac{\alpha_n}{n}z^{-n}. \tag{4.1}$$

The canonical commutation relation for the field ϕ is completely fixed by those for α_n and q given in Section 3,

$$[\alpha_n, \alpha_m]=\hbar n\delta_{n+m,0}, \qquad [q, \alpha_n]=i\hbar\delta_{n,0}. \tag{3.1}$$

The central problem is to define a quantum Hamiltonian $\hat{H}$ corresponding to the classical one H

$$H=\frac{1}{2\pi\beta^2}\int_0^{2\pi}d\sigma(1-\cos\beta\phi(\sigma))=\frac{1}{\beta^2}\oint\frac{dz}{z}\left(1-\frac{1}{2}(e^{i\beta\phi(z)}+e^{-i\beta\phi(z)})\right), \tag{4.2}$$

in such a way as to admit an infinite set of polynomial commuting operators. The answer is given by Propositions 4.1 and 4.2 which are quantum version of Propositions 2.5 and 2.3′ respectively.

Let us consider the following bilinear form of the field $\phi(z)$

$$\begin{aligned} L(z;\lambda)&=\frac{1}{\hbar}\left(\frac{1}{2}:(D\phi)^2:+i\lambda D^2\phi-\frac{\lambda^2}{2}\right),\\ &=\frac{1}{\hbar'}\left(\frac{1}{2}:u^2:+i\frac{\lambda\beta}{2}Du-\frac{\lambda^2}{2}\left(\frac{\beta}{2}\right)^2\right), \end{aligned} \tag{4.3}$$

in which λ is a real constant to be determined later. Let us call $L(z;\lambda)$ "*Virasoro field*", since its moments

(4.4) $$L_n(\lambda)\equiv\oint\frac{dz}{z}L(z;\lambda)z^n, \qquad n\in \boldsymbol{Z},$$

satisfy the Virasoro algebra

(4.5) $$[L_n, L_m]=(n-m)L_{n+m}+\frac{c}{12}n(n^2-1)\delta_{n+m,o},$$

with the central charge

(4.6) $$c=c(\lambda)=1-\frac{12\lambda^2}{\hbar}, \qquad \lambda\in\boldsymbol{R}.$$

Due to hermiticity of the field $\phi(\sigma)$, we have

(4.7) $$(L(z;\lambda))^\dagger=L(y;-\lambda), \qquad y=(\bar{z})^{-1},$$

(4.8) $$(L_n(\lambda))^\dagger=L_{-n}(-\lambda),$$

in which $\bar{z}$ is the complex conjugate of z.

Eq. (4.3) is a quantum version of the Miura transformation, eq. (2.3.2), introduced in Section 2. In conformal field theories, the above form of the Virasoro fields is called the Feigin-Fuchs form ([12]).

Next we consider a *vertex operator* $z^{-\beta\lambda}:e^{i\beta\phi(z)}:$,

(4.9) $$z^{-\beta\lambda}:e^{i\beta\phi(z)}:\equiv\exp\left[-\beta\sum_{n<o}\frac{\alpha_n}{n}z^{-n}\right]e^{i\beta q}z^{\beta(\alpha_0-\lambda)}\exp\left[-\beta\sum_{n>o}\frac{\alpha_n}{n}z^{-n}\right].$$

This gives a single valued holomorphic function of z on $\boldsymbol{C}-\{0\}$, provided we restrict the zero-mode part of the Fock space as follows

(4.10) $$\beta(\mu-\lambda)\in\boldsymbol{Z},$$

in which μ is the eigenvalue of α_o, eq. (3.5). This vertex operator is a primary field ([1]) with respect to the Virasoro field $L(z,\lambda)$,

(4.11) $$[L_n(\lambda), z^{-\beta\lambda}:e^{i\beta\phi(z)}:]=z^n\left(z\frac{\partial}{\partial z}+(n+1)\Delta\right)z^{-\beta\lambda}:e^{i\beta\phi(z)}:, \quad n\in\boldsymbol{Z},$$

in which the conformal dimension Δ is given by

(4.12) $$\Delta\equiv\Delta(\beta,\lambda)=\frac{1}{2}\beta^2\hbar+\lambda\beta.$$

We impose the condition of unit conformal dimension, $\Delta=1$, which determines λ,†)

$$\lambda=\frac{1}{\beta}\left(1-\frac{1}{2}\beta^2\hbar\right). \tag{4.13}$$

In this case the topological condition (4.10) reads

$$\beta\mu-1+\frac{1}{2}\beta^2\hbar\in \boldsymbol{Z} \tag{4.14}$$

which indicates a quantum deviation $\frac{1}{2}\beta^2\hbar=2\hbar'$ from the classical value, e.q (2.2.14). By integrating eq. (4.11) around the origin and using $\Delta=1$, we get a well-known relation in string theories and conformal field theories

$$\left[L_n(\lambda),\oint\frac{dz}{z}z^{1-\beta\lambda}:e^{i\beta\phi(z)}:\right]=0,\qquad n\in \boldsymbol{Z}, \tag{4.15}$$

This corresponds to the classical result, eq. (2.3.12).

Let us summarize by introducing appropriate notation. We have a Virasoro field $L^+(z)$;

$$\begin{aligned}L^+(z)\equiv L(z;\ \lambda)&=\frac{1}{\hbar}\left(\frac{1}{2}:(D\phi)^2:+i\frac{1}{\beta}\left(1-\frac{1}{2}\beta^2\hbar\right)D^2\phi\right.\\&\qquad\left.-\frac{1}{2\beta^2}\left(1-\frac{1}{2}\beta^2\hbar\right)^2\right)\\&=\frac{1}{\hbar'}\left(\frac{1}{2}:u^2:+i\frac{1}{2}(1-2\hbar')Du-\frac{1}{8}(1-2\hbar')^2\right),\end{aligned} \tag{4.16}$$

together with its conjugate,

$$\begin{aligned}L^-(z)&=L(z;\ -\lambda)\\&=\frac{1}{\hbar'}\left(\frac{1}{2}:u^2:-i\frac{1}{2}(1-2\hbar')Du-\frac{1}{8}(1-2\hbar')^2\right),\end{aligned} \tag{4.17}$$

corresponding to the Miura transformations, (2.3.2). Both have the same central charge

$$c=13-\frac{3}{\hbar'}-12\hbar',\qquad \hbar'\equiv\frac{\beta^2}{4}\hbar. \tag{4.18}$$

We have operators $V_1(z;\ \beta)$, $V_2(z;\ \beta)$ defined by

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†) Similar formulas have been discussed in the context of quantum Liouville theory ([18]).

(4.19a) $$V_1(z;\beta)\equiv z^{1-\beta\lambda}:e^{i\beta\phi(z)}:=z^{2\hbar'}:e^{i\beta\phi(z)}:$$

(4.19b) $$V_2(z;\beta)\equiv z^{1-\beta\lambda}:e^{-i\beta\phi(z)}:=z^{2\hbar'}:e^{-i\beta\phi(z)}:.$$

They are related by hermitian conjugation,

(4.20) $$(V_1(z;\beta))^\dagger=V_2(y;\beta),\qquad y=(\bar{z})^{-1}.$$

The Virasoro field $L^+(z)$ commutes with the integral of V_1,

(4.21) $$\left[L^+(z),\oint\frac{d\zeta}{\zeta}V_1(\zeta;\beta)\right]=0,\qquad \forall z\in C-\{0\},$$

and its hermitian conjugate is

(4.22) $$\left[L^-(z),\oint\frac{d\zeta}{\zeta}V_2(\zeta;\beta)\right]=0,\qquad \forall z\in C-\{0\}.$$

This means $z^{-1}V_2(z)$ is a primary field with unit dimension with respect to the Virasoro algebra L^-. Obviously, the classical limits of the operators V_1 and V_2 are

(4.23) $$V_1(z;\beta)\longrightarrow e^{i\beta\phi(z)},\quad V_2(z;\beta)\longrightarrow e^{-i\beta\phi(z)},\quad \hbar\longrightarrow 0.$$

Thus we adopt the following hermitian operator as the Hamiltonian of the quantum Sine-Gordon theory

(4.24) $$\hat{H}=\frac{1}{\beta^2}\oint\frac{dz}{z}\left(1-\frac{1}{2}(V_1+V_2)\right).$$

For the conserved quantities of the quantum Sine-Gordon theory defined by the Hamiltonian $\hat{H}$, eq. (4.24), we have the following main results of this section

Proposition 4.1. *If an operator $F(L^+)$ a functional of the Virasoro field $L^+(z)$ is hermitian, then it commutes with the Hamiltonian $\hat{H}$,* eq. (4.24), *of the quantum Sine-Gordon theory*

(4.25) $$F(L^+)^\dagger=F(L^+)\Longrightarrow[F(L^+),\hat{H}]=0.$$

Namely, it gives a conserved quantity of the quantum S-G theory.

Proposition 4.2. *If two hermitian operators $F(L^+)$, $G(L^+)$, both functionals of the Virasoro field $L^+(z)$, with their $\hbar'$ dependence, belongs to $\hat{V}$, eq.* (3.11), *then they commute. In other words, the polynomial conserved quantities of the quantum Sine-Gordon theory are mutually commutative.*

(4.26) $F(L^+)$, $G(L^+) \in \hat{V}$, *and hermitian* $\Longrightarrow [F(L^+), G(L^+)]=0.$

Proof. If $[F, G]\neq 0$, we find $[F, G]$ is also a polynomial conserved quantity of the quantum S-G theory by using the Jacobi identity. Then it gives rise to a polynomial conserved quantity of the classical S-G theory in the $\hbar'\to 0$ limit. However, it is easy to see that $[F, G]$ cannot contain terms of the form $\oint dz/z :u^m(z):$. So it cannot be a polynomial conserved quantity in the classical limit. Therefore, F and G must commute.

The simplest example of the hermitian operators is the zeroth moment of $L^\pm(z)$,

$$L_0^\dagger=\oint\frac{dz}{z}L^+(z)=\frac{1}{2\hbar'}\oint\frac{dz}{z}\left(:u^2:-\frac{1}{4}(1-2\hbar')^2\right), \tag{4.27}$$

which is the quantum version of K_0, eq. (2.3.10), and it corresponds to $\hat{I}_1$, eq. (3.26), up to an additive constant. In the next section we try to obtain the infinite set of quantum commuting operators for the Sine-Gordon-MKdV hierarchy by the analogy with the classical theory. Namely we define appropriate "products" for the Virasoro fields $L^\pm(z)$ and their derivatives $D^nL^\pm(z)$ and express the infinite set of quantum commuting operators in terms of "polynomials" in the Virasoro fields and their derivatives just as in the classical case, eq. (2.3.10).

The quantum Sine-Gordon theory has a richer structure than the classical one. For example at one particular value of the coupling constant the number of conserved quantities suddenly increases enormously.

Proposition 4.3. *At* $\beta^2\hbar=2$ *or* $\hbar'=1/2$, *we have* $L^+(z)=L^-(z)=$ $:u^2(z):$. *Therefore an arbitrary functional of* $L^+(z)$ *is a conserved quantity of the quantum Sine-Gordon theory. Most of them do not commute with each other.*

At this point the Virasoro algebra has a unit central charge $c=1$ and it consists of one free boson. In Coleman's theory of quantum Sine-Gordon ([13]) this corresponds to $\beta_c^2=4\pi$, eq. (2.2.9), ($\hbar=1$ in his paper) and the system is equivalent to a free massive fermion system. In both ways the sudden appearance of a huge number of conserved quantities is understandable, although more detailed comparison is yet to be made.

Before closing this section let us make one interesting remark. A vertex operator $z^{\beta\lambda}:e^{-i\beta\phi(z)}:$, which is closely related with V_2, is a primary field with dimension $\Delta'=\beta^2\hbar-1=4\hbar'-1$ with respect to the Virasoro algebra L^+. This operator has the degeneracy at the third level ([1]), [2]).

§ 5. Infinite dimensional quantum commuting operators as "polynomials" in Virasoro fields

In the previous section we have obtained a new characterization, i.e., hermiticity of a functional of the Virasoro fields, of the infinite set of quantum commuting operators of the Sine-Gordon-MKdV hierarchy. In this section we show explicitly that the polynomial conserved quantities of the quantum Sine-Gordon-MKdV hierarchy can be obtained as "polynomials" in the Virasoro field and its derivatives in complete analogy with the classical case (see § 2–3). This fact suggests very close relation between the conformal field theories and the solvable quantum field theories. It also suggests a huge group (infinite dimensional) of dynamical symmetries generated by the Virasoro field.

Due to its commutation relation, eq. (4.5), it is not trivial to define a "normal product" for the Virasoro fields. Let us start by introducing a suitable and well defined product of two local operators $A(z)$ and $B(z)$. By a local operator we understand a normal ordered polynomial in the field u and its derivatives; for example,

$$A(z) = :a(u, Du, D^2u, \cdots)(z):, \text{ etc.} \tag{5.1}$$

Therefore it has a Laurent expansion

$$A(z) = \sum_{n=-\infty}^{\infty} A_n z^{-n}. \tag{5.2}$$

Of course the Virasoro fields $L^{\pm}(z)$ are local operators. From two local operators $A(z)$ and $B(z)$ we define a third local operator $\overline{AB}(z)$ by

$$\overline{AB}(z) \equiv \oint_{c_z} \frac{d\zeta}{\zeta} \frac{z}{\zeta - z} RA(\zeta)B(z), \tag{5.3}$$

in which c_z is a small contour encircling z and R denotes radial ordering. By using Wick's theorem we find that the r.h.s. is a local operator. This product has the following properties

(5.4) i) $\overline{AI} = \overline{IA} = A$, I; identity operator.

ii) If we denote $\overline{\mathrm{AB}}(z) = \sum_n (\overline{AB})_n z^{-n}$, then we have.

$$(\overline{AB})_n = \sum_{l=1}^{\infty} A_{-l} B_{n+l} + \sum_{l=0}^{\infty} B_{n-l} A_l. \tag{5.5}$$

Namely it defines a kind of normal ordering with respect to the indices.

(5.6) iii) $\overline{AB} \neq \overline{BA}$, (non-commutative)

(5.7) iv) $\oint \frac{dz}{z}(\overline{AB}(z) - \overline{BA}(z)) = -\left[\oint \frac{dz}{z} A(z), \oint \frac{d\zeta}{\zeta} B(\zeta)\right].$

(5.8) v) $D_z(\overline{AB}(z)) = (\overline{DA)B}(z) + \overline{A(DB})(z)$,

i.e. $D = D_z = iz\partial/\partial z$ is a derivation.

vi) Products of three operators can be defined and it is non-associative

(5.9)
$$\overline{\overline{AB}C} \neq \overline{A\overline{BC}},$$

but they satisfy the identity

(5.10)
$$\overline{\overline{AB}C} - \overline{\overline{BA}C} = \overline{A\overline{BC}} - \overline{B\overline{AC}}.$$

vii) In the classical limit ($\hbar \to 0$) it reduces to the ordinary product of functions

(5.11)
$$\lim_{\hbar \to 0} \overline{AB}(z) = A(z)B(z).$$

Next we define symmetric products of several local operators in the following way

(5.12.a)
$$\langle AB \rangle = \frac{1}{2}(\overline{AB} + \overline{BA})$$

(5.12.b)
$$\langle ABC \rangle = (3!)^{-1}(\overline{\overline{AB}C} + \overline{\overline{AC}B} + \overline{\overline{BA}C} + \overline{\overline{BC}A} + \overline{\overline{CA}B} + \overline{\overline{CB}A}),$$

(5.12.c)
$$\langle A_1 A_2 \cdots A_N \rangle = (N!)^{-1} \sum_P \overline{\overline{\overline{A_{P_1} A_{P_2}} A_{P_3}} \cdots A_{P_N}},$$

in which summation is taken over all permutations. It is obvious that the order of $A_1, \cdots, A_N$ is immaterial. $D = iz\partial/\partial z$ is also a derivation for the symmetric products. Therefore by using the symmetric products one can define "polynomials" of the Virasoro fields.

In order to facilitate comparison with the MKdV case, let us introduce fields $\mathscr{L}(z)$ by shift and scaling of the Virasoro fields

(5.13a)
$$\mathscr{L}(z) = \hbar'\left(L^+(z) + \frac{1-c}{24}\right)$$

(5.13b) $$=\frac{1}{2}:u^2(z):+i\left(\frac{1}{2}-\hbar'\right)Du(z).$$

By explicit calculation in terms of formula manipulation computer programs ([17]), we have found that the following expressions are hermitian. They are integrals of local "polynomials" (symmetric products in the field $\mathscr{L}(L^{\pm})$).

(5.14a) $$1^\circ \quad \oint\frac{dz}{z}\,\mathscr{L}(z)=\oint\frac{dz}{z}\,\frac{1}{2}:u^2(z):,$$

(5.14b) $$2^\circ \quad \oint\frac{dz}{z}\langle\mathscr{L}^2\rangle=\oint\frac{dz}{z}\left\{\frac{1}{4}:u^4:+\left(\frac{3}{2}\hbar'-\frac{1}{4}-\hbar'^2\right):(Du)^2:\right\},$$

(5.14c) $$3^\circ \quad \oint\frac{dz}{z}\left(\langle\mathscr{L}^3\rangle+\frac{\hbar'}{12}(c+2)\langle(D\mathscr{L})^2\rangle\right),$$

(5.14d) $$4^\circ \quad \oint\frac{dz}{z}\left(\langle\mathscr{L}^4\rangle+\frac{\hbar'}{3}(c+2)\langle\mathscr{L}(D\mathscr{L})^2\rangle\right.$$
$$\left.+\frac{\hbar'^2}{180}(c+2)\left(c-\frac{1}{2}\right)\langle(D^2\mathscr{L})^2\rangle-\frac{\hbar'^2}{24}(c+2)\langle(D\mathscr{L})^2\rangle\right),$$

(5.14e) $$5^\circ \quad \oint\frac{dz}{z}\left(\langle\mathscr{L}^5\rangle+\frac{5}{6}\hbar'(c+2)\langle\mathscr{L}^2(D\mathscr{L})^2\rangle\right.$$
$$+\frac{\hbar'^2}{72}(c+2)(2c-11)\langle\mathscr{L}(D^2\mathscr{L})^2\rangle$$
$$-i\frac{\hbar'^2}{24}(c+2)\langle(D\mathscr{L})^3\rangle$$
$$+\frac{\hbar'^3}{45360}(c+2)(15c^2-184c+902)\langle(D^3\mathscr{L})^2\rangle$$
$$-\frac{25}{72}\hbar'^2(c+2)\langle\mathscr{L}(D\mathscr{L})^2\rangle$$
$$-\frac{5\hbar'^3}{2592}(c+2)(8c-31)\langle(D^2\mathscr{L})^2\rangle$$
$$\left.+\frac{\hbar'^3}{12960}(c+2)(34c+763)\langle(D\mathscr{L})^2\rangle\right).$$

Here c is the central charge of the Virasoro algebra

(4.18) $$c=13-\frac{3}{\hbar'}-12\hbar'.$$

Obviously they yield integrals of local "polynomials" in the Virasoro field $L^+(z)$ when eq. (5.13a) is substituted. Then *the coefficients of these hermitian "polynomial" in the Virasoro fields $L^+(z)$ are polynomials of the central charge c only.* Several remarks are in order.

i) Classical limit. In the classical limit

$$\hbar' \longrightarrow 0, \qquad c \longrightarrow -3/\hbar' \tag{5.15}$$

all the the expressions (5.14a)~(5.14e) coincide exactly with those of (2.3.6a)~(2.3.6e) with the identification $v=2\mathscr{L}$, which is obvious by comparison of their definitions. Of course there are other "quantum correction" terms which vanish in the classical limit.

ii) The second member $\oint dz/z \langle \mathscr{L}^2 \rangle$, eq. (5.14b), defines the quantum MKdV theory in the Sine-Gordon-MKdV hierarchy. In the present case we fixed the Hamiltonian of the quantum Sine-Gordon theory, then the Hamiltonians of the other members of the hierarchy are uniquely determined. In the previous paper, as reviewed in Section 3, we did not consider the Sine(h)-Gordon theory and fixed the Hamiltonian of the quantum MKdV theory $\hat{I}_2$ to have the same form as the classical theory. For a given value of $\hbar'$ the difference can be absorbed by redefinition of the field, see [6].

iii) It is interesting to note

$$\oint \frac{dz}{z} \langle \mathscr{L}^2 \rangle = \hat{I}_2, \qquad \text{for } \hbar' = 1. \tag{5.16}$$

Therefore, at this point, the quantum commuting operators of the MKdV-hierarchy eq. (3.26), and those of the Sine-Gordon-MKdV hierarchy eq. (5.14) coincide due to the uniqueness.

iv) In fact at $\hbar'=1$ or $c=-2$ drastic simplification takes place and only $\langle \mathscr{L}^n \rangle$ terms survive. We have a simple relation

$$\oint \frac{dz}{z} \langle \mathscr{L}^n \rangle = \hat{I}_n, \quad \text{for } \hbar' = 1, \quad n \in N, \tag{5.17}$$

up to a constant factor which we have verified for $n=1\sim 6$.

v) We do not yet have a proof for an arbitrary $n \in N$ that a hermitian operator can always be constructed as a "polynomial" in $\mathscr{L}$ of degree n.

In summary we have the following results.

Proposition 5.1. *A Virasoro algebra with central charge $c<1$ can be realized in terms of the single boson field of the Sine-Gordon-MKdV hierarchy in the Feigin-Fuchs-Miura form. Then the infinite set of quantum* *commuting operators of the Sine-Gordon-MKdV hierarchy is obtained as hermitian "polynomials" in the Virasoro fields.*

Proposition 5.2. *In the special case of $\hbar'=1$ ("genuine" quantum theory?) corresponding to $c=-2$, the quantum commuting operators take a simple form*

$$(5.17) \qquad \oint \frac{dz}{z}\langle \mathscr{L}^n \rangle, \quad \mathscr{L}=\frac{1}{2}:u^2:-\frac{1}{2}Du, \quad n \in N.$$

This is *the quantum recursion formula* for the infinite set of commuting operators for the Sine-Gordon-MKdV hierarchy.

The central charge $c=-2$ is the "lowest"†) member ($m=1$) of the discrete series of unitarizable Virasoro algebras ([3]) $c=1-6/m(m+1)$ and it corresponds to $\beta_c^2=8\pi$ in Coleman's theory ([13]) of quantum S-G eq. This is the maximal value of the coupling constant β_c for which a ground state exists. Another special value in Colemans' theory $\beta_c^2=4\pi$ corresponds to $c=1$, which was discussed in Section 4, can be considered as the "highest" member ($m=\infty$) of the discrete series. In both cases clear distinction can be seen at the level of the infinite set of commuting operators. We expect some distinctive features for the other values of the central charge of the discrete series but so far we have failed to observe clear evidences.

We strongly suspect that the infinite set of quantum commuting operators as hermitian "polynomials" of the Virasoro fields is *the intrinsic property of the Virasora algebra* rather than the special property of the Feigin-Fuchs-Miura form as has been shown here. In other words we think that their commutativity is the consequence of the Virasoro commutation relations only. If confirmed, it would provide a deeper connection between the conformal field theories and the integrable quantum field theories. And a very interesting problem is to identify the huge group of symmetries which contain the above infinite set of "polynomials" of the Virasoro generators as a commuting subalgebra.

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