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VIRTUAL LINK AND KNOT INVARIANTS FROM NON-ABELIAN YANG-BAXTER 2-COCYCLE PAIRS

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Abstract

Given a set X, we provide the algebraic counterpart of the (mixed) Reidemeister moves for virtual knots and links, with semi-arcs labeled by X: we define (commutative and noncommutative) invariants with values in groups, using "2-cocycles", and we also introduce a universal group $U_{nc}^{fg}(X)$ and functions $\pi_f, \pi_g\colon X\times X \to U_{nc}^{fg}(X)$ governing all 2-cocycles in X. We exhibit examples of computations -of the group and their invariants- achieved using GAP [7].

Introduction and preliminaries

In [6] we constructed an invariant for knots and links using noncommutative 2-cocycles, that is, for (X,σ) a special solution of the Yang-Baxter equation (see definitions of biquandle below) and a map $f\colon X\times X \to G$, where G is a (eventually) non-abelian group, and f satisfies certain equations that we call noncommutative 2-cocycle conditions. In this way a noncommutative version of the state-sum invariant can be defined. In this work we generalize this construction for virtual knots and links. Since a (diagram of a) virtual link has two types of crossings, for a given set X of possible labels for the semi arcs, we need two rules for coloring semi arcs in a crossing, say (X,S) and (X,β), and also we need two types of weights. We consider pairs $f,g\colon X\times X \to G$ that we call "noncommutative 2-cocycle pairs". The strategy consists in three steps:

- Ask invariance under generalized (i.e. classical, virtual or mixed) Reidemeister moves for colorings using some label set X: this step was already done in [2], we recall the notion in Definition 6.
- Ask invariance under generalized Reidemeister moves for products of weights in a given order, for a given pair of weight functions $f,g\colon X\times X \to G$, where G is some (maybe non commutative) group, the map f determines the weights for classical crossings and g for the virtual ones.
- Find a target group G and 2-cocycles f,g with values in G. We solve this problem in a universal way.

Remark 1. If the group is abelian and the map β in Definition 6 is of the form $\beta(x,y) = (a^{-1}y, ax)$, this construction -assuming 2 is invertible- gives the "state-sum" considered also in [2], but there are much more general solutions for "β" , and also there are genuine non-commutative examples, in Section 3 we provide an example of a non-trivial non-

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commutative 2-cocycle whose abelianization is trivial.

Remark 2. After defining the group $U_{nc}^{fg}(X)$ (see Definition 31 in Section 2) that has the property of being the universal target of noncomutative 2-cocycle pairs for a given (X, S, β), one can see that there is a canonical choice of 2-cocycles, so the invariant that *a priori* depends on the set of colorings and a choice of a non-commutative 2-cocycle pair, it is actually determined by intrinsic properties of the set of colorings.

Considering virtual pairs of the form (X,flip,β) we produce non-trivial and *different* ways to generalize the linking number to virtual links or knots (see Example 24 and Remark 26), including the "self linking number" introduced by Kauffman in [11].

The contents of this work are as follows: after recalling the combinatorial definition of a virtual link or knot, we introduce, in *Section 1*, the notion of non-abelian 2-cocycle pair. Using this notion we propose the noncommutative invariant in Definition 20, proving that it is actually an invariant. In *Section 2* we define a group together with a noncommutative 2-cocycle pair that has the universal property as target of noncommutative 2-cocycles. This group is defined in terms of generators and relations, and it is actually computable for virtual pairs of small cardinality. We end by computing invariants of some virtual knots and links using this universal group.

Definition 3. A set theoretical solution of the Yang-Baxter equation is a pair (X, σ) where $\sigma\colon X \times X \to X \times X$ is a bijection satisfying

$$(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma) = (\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})$$

Notation: $\sigma(x, y) = (\sigma^1(x, y), \sigma^2(x, y))$ and $\sigma^{-1}(x, y) = \overline{\sigma}(x, y)$.

A solution (X, σ) is called non degenerated, or *birack* if in addition:

1. (*left invertibility*) for any $x, z \in X$ there exists a unique y such that $\sigma^1(x, y) = z$,
2. (*right invertibility*) for any $y, t \in X$ there exists a unique x such that $\sigma^2(x, y) = t$.

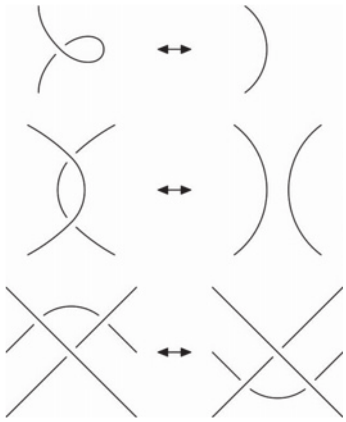
A birack is called *biquandle* if, given $x_0 \in X$, there exists a unique $y_0 \in X$ such that $\sigma(x_0, y_0) = (x_0, y_0)$. In other words, if there exists a bijective map $s\colon X \to X$ such that

$$\{(x, y) : \sigma(x, y) = (x, y)\} = \{(x, s(x)) : x \in X\}$$

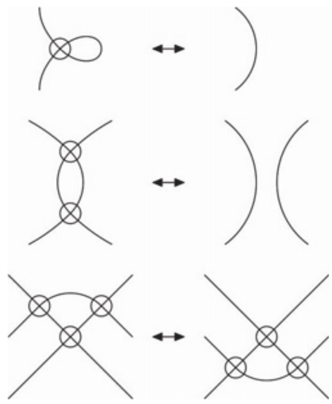
Remark 4. The biquandle condition, from the perspective of knots and links, is introduced in order to have compatibility with first Reidemeister move when coloring with (X, σ). From the algebraic point of view, it can be proven that (X, σ) is a biquandle if and only if its derived rack is a quandle (see Lemma 0.3 in [6] for biquandle equivalent conditions).

Following Kauffman [10], a virtual link or knot can be defined using diagrams with two types of crossings: classical and virtual ones; a virtual crossings will be a 4-valent vertex with a small circle around it. Virtual links/knots may be considered to be equivalence classes of planar virtual knot diagrams under the equivalence relation generated by the three (classical) Reidemeister moves, the virtual moves and a mixed Reidemeister move.

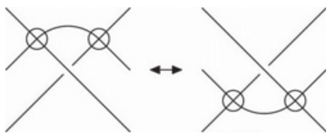
All links and knots considered in this work will be oriented ones. A useful reduction is proved in [2]:



Classical Reidemeister moves:
RI, RII and RIII.



Virtual Reidemeister moves:
vRI, vRII, vRIII



Mixed Reidemeister move: mixed RIII

Lemma 5 (Lemma 2.4, [2])**.** *The classical and virtual II moves, together with one oriented mixed* RIII *or* vRIII *move, imply the other oriented mixed* RIII *and* vRIII *moves. That is, we can reverse the direction of any strand in type mixed* RIII *or* vRIII *move using a sequence of* RII *and* vRII *moves.*

Definition 6. A pair of biquandles (X, S), (X,β), (shortly (X, S,β)) is called a *virtual pair* if $\beta^2 = 1$ and $(1\times\beta)(S\times 1)(1\times\beta) = (\beta\times 1)(1\times S)(\beta\times 1)$. This notion is also called virtual invariant in [1].

Example 7. If (X, S) is a biquandle and $a \in \mathrm{Aut}(X, S)$, that is, $a\colon X \to X$ is a bijection satisfying $(a\times a)S(a^{-1}\times a^{-1}) = S$, then one can consider $\beta(x, y) = (a^{-1}y, ax)$. It is easy to check that we get a virtual pair in that way.

Not every virtual pair arise as in the above construction, if S is involutive (i.e. $S^2 = \mathrm{Id}$) with $S(x, y) \neq (a^{-1}y, ax)$, then (X, S, S) is a virtual pair. But there are also different examples with non-involutive S, already with $|X| = 3$. The following is an example with cardinal 4:

Example 8. $X = \mathbb{Z}/4\mathbb{Z}$, $S(x, y) = (-y, x + 2y)$

$$\beta(x, y) = \begin{cases} (y, x) & \text{if } x \text{ or } y \text{ is odd} \\ (y + 2, x + 2) & \text{if } x \text{ and } y \text{ are even} \end{cases}$$

1. Non-abelian 2-cocycle pair

1.1. Cocycle equations. We begin this section by introducing the notion of noncommutative 2-cocycle pair. If one analyzes the properties that a general weight (see subsection 1.2) must satisfy in order to generalize the construction given in [6] to the virtual case, then one ends with the following definition:

Definition 9. Let H be a (not necessarily abelian) group and (X,S,β) a virtual pair. A pair of functions $f,g\colon X\times X\to H$ is a *noncommutative 2-cocycle pair* if:

- the pair f,S satisfies:
  - f1) $f(x,y)f(S^2(x,y),z)=f(x,S^1(y,z))f(S^2(x,S^1(y,z)),S^2(y,z))$,
  - f2) $f(S^1(x,y),S^1(S^2(x,y),z))=f(y,z)$,
  - f3) $f(x,s(x))=1$ (recall the map $s\colon X\to X$ from Definition 3),
- the pair g,β satisfies:
  - g1) $g(x,s_\beta(x))=1$ (notice that β involutive implies that (X,β) is a biquandle, hence, there is an associated map $s_\beta\colon X\to X$),
  - g2) $g(x,y)g(\beta(x,y))=1$,
  - g3) $g(x,y)g(\beta^2(x,y),z)=g(x,\beta^1(y,z))g(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))$,
  - g4) $g(y,z)g(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))=g(x,y)g(\beta^1(x,y),\beta^1(\beta^2(x,y),z))$,
  - g5) $g(y,z)g(x,\beta^1(y,z))=g(\beta^2(x,y),z)g(\beta^1(x,y),\beta^1(\beta^2(x,y),z))$,
- and compatibility conditions between f,g,β,S:
  - m1) $g(y,z)=g(S^1(x,y),\beta^1(S^2(x,y),z))$,
  - m2) $g(y,z)g(x,\beta^1(y,z))=g(S^2(x,y),z)g(S^1(x,y),\beta^1(S^2(x,y),z)$,
  - m3) $g(x,\beta^1(y,z))f(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))=f(x,y)g(S^2(x,y),z)$

are satisfied for any $x,y,z\in X$.

Example 10. Let (X,S,β) be as in Example 8, let H be the group with generators $\{a,b,c,d,h\}$ and relations

$$bc=cb,\ c^2=1,\ [h,a]=[h,b]=[h,c]=[h,d]=1,$$

define $f,g\colon X\times X\to H$ by the tables

f	0	1	2	3
0	1	a	1	a
1	b	c	bc	1
2	1	d	1	d
3	b	1	bc	c

g	0	1	2	3
0	1	h	1	h
1	h^{-1}	1	h^{-1}	1
2	1	h	1	h
3	h^{-1}	1	h^{-1}	1

One can check by hand that the pair (f,g) is a 2-cocycle pair, and after Theorem 32 we will see that any other 2-cocycle pair $(\widetilde{f},\widetilde{g})\colon X\times X\to\widetilde{H}$ necessarily factorizes through this pair and a group homomorphism $\rho\colon H\to\widetilde{H}$.

Example 11. If $X=\{1,2\}$ and $S=\beta=$flip, then the cocycle conditions f3, g1 and g2 are

$$f(1,1)=f(2,2)=g(1,1)=g(2,2)=1$$

$$g(1,2)=:h,\ g(2,1)=h^{-1}$$

Call $a = f(1,2)$ and $b = f(2,1)$, conditions f1, f2, g3-g5 and m1 m2 are trivially satisfied and condition m3 is

$$ah = ha, \; bh = hb.$$

So, if one takes H the group freely generated by $\{a, b, h\}$ with relations $ah = ha$ and $bh = hb$, then the pair f, g defined by $g(1,2) = h = g(2,1)^{-1}$, $f(1,2) = a$, $f(2,1) = b$ and $f(1,1) = f(2,2) = g(1,1) = g(2,2) = 1$ is a 2-cocycle pair.

Next we consider some special cases and analyze the general equations for each case.

Some special cases

If (X, S) is a biquandle, $a \in \mathrm{Aut}(X, S)$ and $\beta(x,y) = (a^{-1}y, ax)$, then equations f1,f2,f3 remain the same. An easy computation shows that g3 together with the choice of $y = ax$ gives $g(x,z) = g(ax, az)$ for all x, z. Using this condition, the other equations may be simplified giving the following (equivalent) set:

g0) $g(x,z) = g(ax, az)$
g1) $g(x, ax) = 1$,
g2) $g(x,y)g(a^{-1}y, ax) = 1$,
g3) $g(x,y)g(x,z) = g(x,z)g(x,y)$,
g4) $g(y,z)g(x,y) = g(x,y)g(y,z)$,
g5) $g(y,z)g(x,z) = g(x,z)g(y,z)$,
m1) $g(y,z) = g(S^1(x,y), a^{-1}z)$,
m2) $g(S^2(x,y), z)g(y,z) = g(y,z)g(x, a^{-1}z)$,
m3) $g(x, a^{-1}z)f(ax, ay) = f(x,y)g(S^2(x,y), z)$.

Notice that g3, g4, g5 are automatic if the group is abelian. Actually, when the group is abelian, the equations m1, m2 and m3 can be replaced by

m1)' $g(y,z) = g(S^1(x,y), a^{-1}z)$,
m2)' $g(x,z) = g(S^2(x,y), az)$,
m3)' $f(ax, ay) = f(x,y)$.

Another interesting situation is when the biquandle is given by a quandle, that is $S(x,y) = (y, x \triangleleft y)$. In this case f2 is automatic, and one can make explicit S^1 and S^2 giving the following:

Proposition 12. *If $X = (Q, \triangleleft)$ is a quandle and $a \in \mathrm{Aut}(Q)$, then (f, g) is a non-abelian 2-cocycle pair for $\beta(x,y) = (a^{-1}y, ax)$ and $S(x,y) = (y, x \triangleleft y)$ if and only if they verify the following equations*

f1) $f(x,y)f(x \triangleleft y, z) = f(x,z)f(x \triangleleft z, y \triangleleft z)$,
f3) $f(x,x) = 1$,
g0-m1-m2) $g(x,z) = g(ax, z) = g(x, az) = g(x \triangleleft y, z)$,
g1) $g(x,x) = 1$,
g2) $g(x,y)g(y,x) = 1$,
g3) $g(x,y)g(x,z) = g(x,z)g(x,y)$,
g4) $g(y,z)g(x,y) = g(x,y)g(y,z)$,
g5) $g(y,z)g(x,z) = g(x,z)g(y,z)$,
m3) $g(x,z)f(ax, ay) = f(x,y)g(x \triangleleft y, z)$.

Proof. All of them are simple specializations to this particular case, except

$$g(x,z) = g(x \triangleleft y, z),$$

(We thank the referee for pointing out this part.) that is the combination of

g5) $g(x,z)g(y,z) = g(y,z)g(x,z)$,
m2) $g(x \triangleleft y, z)g(y,z) = g(y,z)g(x,a^{-1}z)$, and
m1) $g(x,z) = g(x,a^{-1}z)$.

□

Corollary 13. *Let $(Q,\triangleleft)$ be a quandle, $a \in Aut(Q,\triangleleft)$, denote* $\mathrm{Inn}(Q)$ *the subgroup generated by maps of the form $- \triangleleft y$, and consider $G = \langle \mathrm{Inn}(Q), a\rangle$ the subgroup generated by* $\mathrm{Inn}(Q)$ *and a. If G acts transitively on X then $g \equiv 1$.*

Proof. Given $x, z \in X$, the equality

$$g(x \triangleleft y, z) = g(x,z) = g(ax,z)$$

says that

$$g(\tau(x), z) = g(x,z)\ \forall \tau \in G,$$

but if G acts transitively on X, then there exist τ such that $\tau(x) = z$, and so

$$g(x,z) = g(\tau(x),z) = g(z,z) = 1,\ \forall x, z \in X.$$

□

Notice that in Proposition 12, condition m3 is nontrivial even if $g \equiv 1$. We mention another special case:

Corollary 14. *Let $X = \{1,\dots,n\}$ and $a(x) = x+1$ Mod n, then (f,g) is a non-abelian 2-cocycle pair for S=flip and $\beta(x,y) = (a^{-1}y, ax)$ if and only if $g \equiv 1$,*

$$f(x,y) = f(ax,ay),\ f(x,x) = 1$$

and

$$f(x,y)f(x,z) = f(x,z)f(x,y).$$

In particular, for $n = 2$, f is fully (and freely) determined by $f(1,2)$.

Particular case β=flip and H an abelian group

The specialization in this case gives the equations f1, f2, f3 together with

g1) $g(x,x) = 1$,
g2) $g(x,y)g(y,x) = 1$.
m1) $g(y,z) = g(S^1(x,y),z)$,
m2) $g(x,z) = g(S^2(x,y),z)$.

Connected components

Let $(Q,\triangleleft)$ be a quandle and consider the equivalence relation generated by $x \triangleleft y \sim x$, $\forall x, y \in Q$. Recall that Q is called *connected* if there is only one equivalence class.

Generalizing this definition, one can consider, for a biquandle (X, S), the equivalence relation generated by

$$\forall x, y \in X, \quad x \sim S^1(x, y) \text{ and } y \sim S^2(x, y),$$

that is, if $S(x, y) = (y', x')$ then $x \sim x'$ and $y \sim y'$. The equivalence classes are called *connected components*, and the biquandle (X, S) is called *connected* if there is only one class. Clearly if S is given by a quandle then this definition agrees with the previous one.

For a virtual pair (X, S, β) there is also a natural equivalence relation, the one generated by

$$\forall x, y \in X, \quad x \sim S^1(x, y) \sim \beta^1(x, y)$$

and

$$y \sim S^2(x, y) \sim \beta^2(x, y).$$

That is, if $S(x, y) = (y', x')$ and $\beta(x, y) = (y'', x'')$ we are setting $x \sim x' \sim x''$, $y \sim y' \sim y''$.

Definition 15. For a virtual pair (X, S, β), equivalent classes of elements of X are called connected components. The virtual pair (X, S, β) is called connected if there is only one class.

Remark 16. If one is interested in *knots*, then it is clear that one can restrict the attention to connected virtual pairs, because a coloring of a knot only uses elements of the same connected component of X.

Example 17. If the biquandle (X, S) is already connected then (X, S, β) is obviously connected, the same for the biquandle (X, β). With the help of a computer one can check that for cardinal 2,3,5 these are the only cases. For cardinal 4 there are examples of connected virtual pairs (X, S, β) with nonconnected (X, S) and nonconnected (X, β). More precisely, there are 167 (isomorphism classes of) connected virtual pairs of size 4, and 10 of them have disconnected S and β. Similar thing happens in cardinal 6, see table in subsection 2.2.

A straightforward consequence of m1 and m2, in the same spirit of Corollary 13, is:

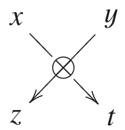
Corollary 18. *If (X, S) is a connected biquandle and β=flip then $g \equiv 1$.*

On the opposite side, if the biquandle is trivial (i.e. $S(x, y) = (y, x)$) then m1) and m2) are trivial, the conditions for g are only $g(x, x) = 1$ and $g(x, y) = g(y, x)^{-1}$, as in Example 11.

Next we will construct an invariant for oriented knots or links from a virtual pair (X, S, β) and a 2-cocycle pair (f, g).

1.2. Weights. Let (X, S, β) be a virtual pair, H a group and $f, g\colon X \times X \to H$ a non-abelian 2-cocycle pair. Let $L = K_1 \cup \cdots \cup K_r$ be a virtual oriented link diagram on the plane, where $K_1, \ldots, K_r$ are connected components, for some positive integer r. A *coloring* of L by X is a rule that assigns an element of X to each semi-arc of L, in such a way that for every regular crossing (figure on the left corresponds to a positive crossing and figure on the right to a negative one):

where $(z,t) = S(x,y)$ and in case of a virtual crossing:

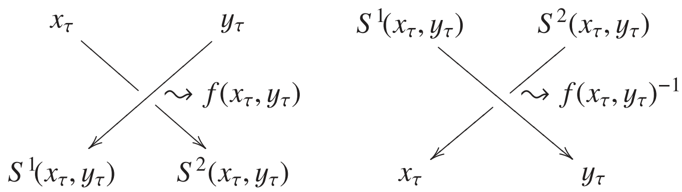


where $(z,t) = \beta(x,y)$.

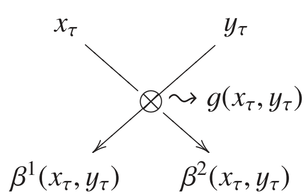
REMARK 19. The conditions for (X, S, β) to be a virtual pair are precisely the compatibility of the set of colorings with the Reidemeister moves (RI, RII, RIII, vRI, vRII, vRIII and mixed RIII), so given (X, S, β) a virtual pair, the number of colorings of a link (or a knot) using (X, S, β) is an invariant of that link (or knot).

Let $C \in Col_X(L)$ be a coloring of L by X and $(b_1, \ldots, b_r)$ a set of base points on the components $(K_1, \ldots, K_r)$. Let $\tau^{(i)} = \{\tau_1^{(i)}, \ldots, \tau_{k(i)}^{(i)}\}$, for $i = 1, \ldots, r$, be the ordered set of regular crossings such that the under-arc belongs to component i or it is virtual crossing involving component i. The order of the set $\tau^{(i)}$ is given by the orientation of the component starting at the base point.

At a positive crossing τ, let x_τ, y_τ be the color on the incoming arcs. The *Boltzmann weight* at a positive crossing τ is $B_{f,g}(\tau, C) = f(x_\tau, y_\tau)$. At a negative crossing τ, denote $S(x_\tau, y_\tau)$ the colors on the incoming arcs. The *Boltzmann weight* at τ is $B_{f,g}(\tau, C) = f(x_\tau, y_\tau)^{-1}$



At a virtual crossing τ, let x_τ, y_τ be the color on the incoming arcs. The *Boltzmann weight* at τ is $B_{f,g}(\tau, C) = g(x_\tau, y_\tau)$.



We will show that a convenient product of these weights is invariant under Reidemeister moves. More precisely, take an oriented component, start at a base point, take the product of Boltzmann weights associated to the crossing whenever it is a virtual crossing, or the crossing is classical but one is going through the under arc.

For a group element $h \in H$, denote $[h]$ the conjugacy class to which h belongs.

DEFINITION 20. The set of conjugacy classes

$$\overrightarrow{\Psi}(L, f, g) = \overrightarrow{\Psi}_{(X,f,g)}(L) = \{[\Psi_i(L, C, f, g)]\}_{\substack{1 \le i \le r \\ C \in Col_X(L)}}$$

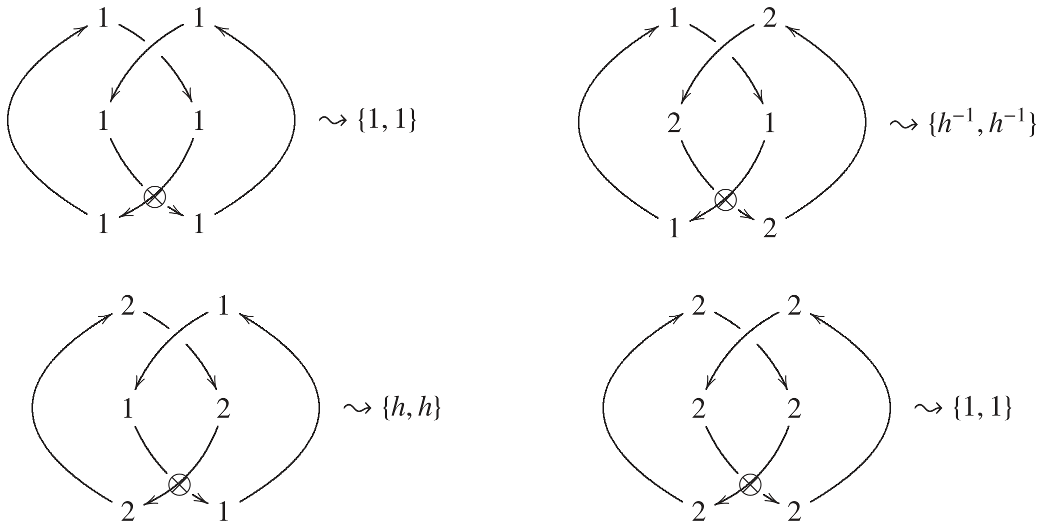
where $\Psi_i(L, C, f, g) = \prod_{j=1}^{k(i)} B_{fg}(\tau_j^{(i)}, C)$ (the order in this product is following the orientation of the component) is called the *conjugacy biquandle cocycle invariant* of the link.

The following is our main theorem:

Theorem 21. *The conjugacy biquandle cocycle Ψ is well defined and then define a knot/link invariant.*

REMARK 22. This invariant clearly generalizes the one constructed in [6] by simply taking β = flip and $g \equiv 1$. On the opposite side, if one chooses $f \equiv 1$ and general g, this invariant will be trivial on classical links or knots, so a nontrivial g may detect virtuality.

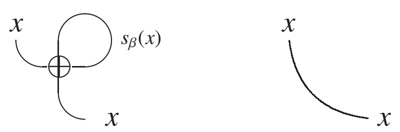
EXAMPLE 23. Take the group $H = \langle h \rangle$, $X = \{1, 2\}$, $S = \beta$ = flip, $f \equiv 1$, $g(1, 1) = g(2, 2) = 1$, $g(1, 2) = g(2, 1)^{-1} = h$. Here we show all possible colorings and the corresponding invariants.



In particular, this link is nontrivial and non classical.

Proof of Theorem 21. We will check the product of weights is invariant under Reidemeister moves. In [6] calculations due to regular crossings can be found, remains to consider virtual and mixed Reidemeister moves. Following Lemma 5 we will check only one orientation of arcs in each Reidemeister move (the rest will be equivalent).

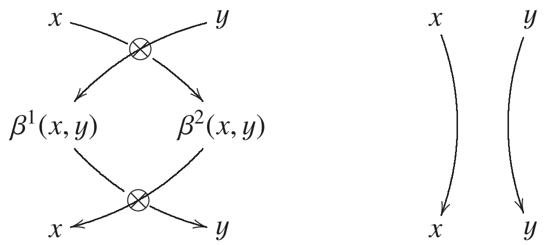
- Virtual Reidemeister type I move:
  $\beta(x, s_\beta(x)) = (x, s_\beta(x))$



the condition (g1) $g(x, s_\beta(x)) = 1$, assures that the factor due to this crossing will not change the product.

- Virtual Reidemeister type II move:

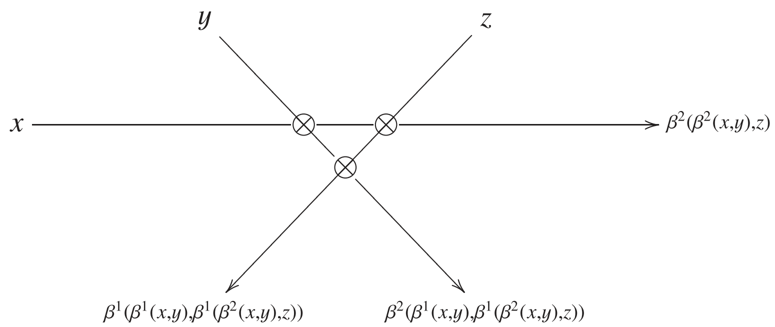
  Take, for example, the following diagram:

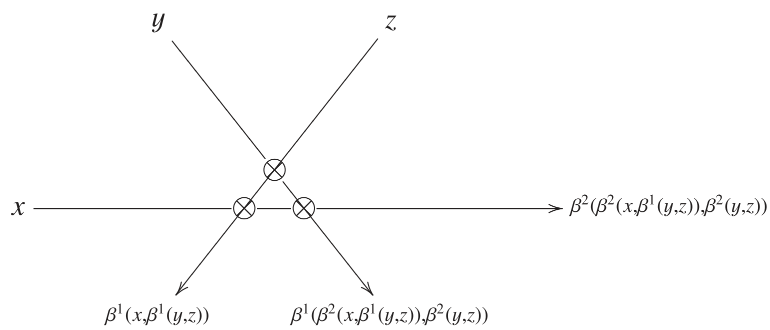


  Condition (g2) assures the product of weights due to these crossings will not change the product.

- Virtual Reidemeister type III move:

  Start by naming the incoming arcs x, y, z, then the outcoming arcs are respectively equal as β is a solution of YBeq.





The product of the weights following the horizontal arc, in the first diagram, is:

$$A_1 = g(x, y)\, g(\beta^2(x, y), z)$$

and in the second diagram is:

$$B_1 = g(x, \beta^1(y, z))\, g(\beta^2(x, \beta^1(y, z)), \beta^2(y, z))$$

$A_1 = B_1$ is item (g3) in Definition 9.

The product of the weights following the arc labeled by y, in the first diagram, is:

$$A_2 = g(x,y)\, g(\beta^1(x,y), \beta^1(\beta^2(x,y),z))$$

and in the second, is:

$$B_2 = g(y,z)\, g(\beta^2(x,\beta^1(y,z)), \beta^2(y,z))$$

$A_2 = B_2$ is item (g4) in Definition 9.

The product of the weights following the arc labeled by z, in the first diagram, is:

$$A_3 = g(\beta^2(x,y),z)\, g(\beta^1(x,y), \beta^1(\beta^2(x,y),z))$$

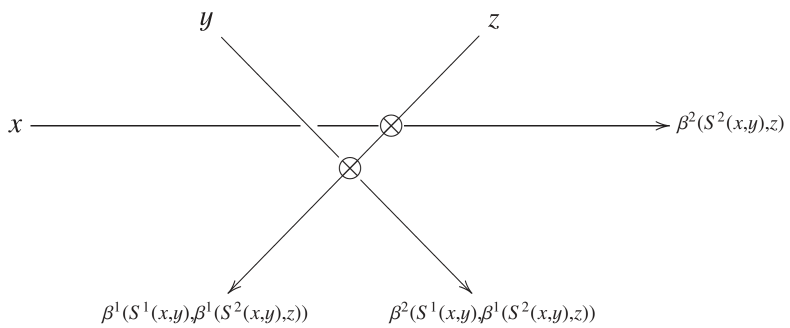
and in the second, is:

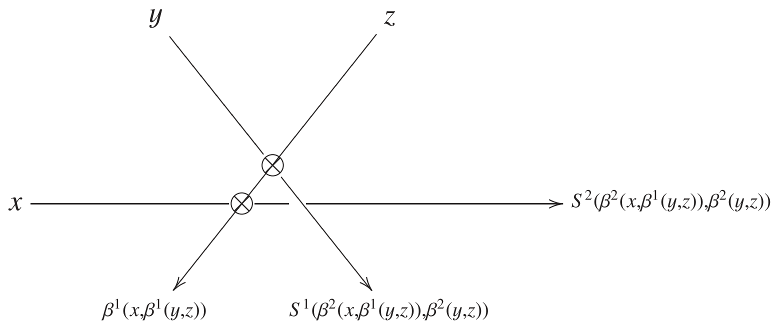
$$B_3 = g(y,z)\, g(x,\beta^1(y,z))$$

$A_3 = B_3$ is item (g5) in Definition 9.

- Mixed virtual Reidemeister type III move:

  Start by naming, in both diagrams, x, y, z the incoming arcs. The outcoming arcs are respectively equal as (X, S, β) is a virtual pair.





The product of the weights following the arc labeled by x in the first diagram is:

$$A_1 = f(x,y)\, g(S^2(x,y),z)$$

and in the second diagram is:

$$B_1 = g(x,\beta^1(y,z)) f(\beta^2(x,\beta^1(y,z)), \beta^2(y,z))$$

$A_1 = B_1$ is item (m3) in Definition 9.

The product of the weights following the arc labeled by y in the first diagram is:

$$A_1 = g(S^1(x,y), \beta^1(S^2(x,y),z))$$

and in the second diagram is:

$$B_1 = g(y, z)$$

$A_1 = B_1$ is item (m1) in Definition 9.

The product of the weights following the arc labeled by z in the first diagram is:

$$A_1 = g(S^2(x, y), z)\, g(S^1(x, y), \beta^1(S^2(x, y), z))$$

and in the second diagram is:

$$B_1 = g(y, z)\, g(x, \beta^1(y, z))$$

$A_1 = B_1$ is item (m2) in Definition 9.

This shows that the product of the weights does not change under generalized Reidemeister moves. A change of base points causes cyclic permutations of Boltzmann weights, and hence the invariant is defined up to conjugacy. □

Example 24. In [8] the authors mention that there are several ways to generalize the notion of linking number to the virtual case. For 2-component links, they give two independent versions of the linking number: the invariant $lk_{\frac{1}{2}}$ may be computed as a sum of signs of real crossings where the first component passes over the second one. Similarly, $lk_{\frac{2}{1}}$ is defined by exchanging the components in the definition of $lk_{\frac{1}{2}}$.

In our context, previous definitions can be achieved in the following way: take a two component (virtual) link. Take (X, S, β) the virtual pair with $S = \beta =$ flip and f, g a 2-cocycle pair with $g = 1$. Take two different elements $1, 2 \in X$. Color "the first" component with color 1 and "the second" component with color 2. The invariant for the *second* component will be $f^{lk_{\frac{1}{2}}}(2, 1)$. The invariant for the first component will be $f^{lk_{\frac{2}{1}}}(1, 2)$. Recall (see Example 11) that for X=$\{1, 2\}$, S=β=flip , cocycle pairs can be obtained considering $G = Free\{a, b\} \times Free\{h\}$ and $f, g\colon X \times X \to G$ defined by

$$f(1, 1) = f(2, 2) = g(1, 1) = g(2, 2) = 1$$

$$g(1, 2) = h,\ g(2, 1) = h^{-1}$$

$$f(1, 2) = a,\ f(2, 1) = b$$

Example 25. Take $X = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$, $S =$ flip and β given by

$$\beta(0, 0) = (1, 1),\ \beta(1, 1) = (0, 0),$$

$$\beta(0, 1) = (0, 1),\ \beta(1, 0) = (1, 0)$$

One can check that this rule can be written as $\beta(x, y) = (y - 1, x + 1)$ so it is an involutive biquandle, and also one can easily check that the coloring rule for (X, S, β) is the rule of "changing the color when going trough a virtual crossing and not changing the color when the crossing is classical", just as in [11]. If one considers the 2-cocycle equations then (see Corollary 14) we are lead to $g \equiv 1$ and a group $H = \langle a \rangle$ with $f\colon X \times X \to H$ satisfying

$$f(1,0) = f(0,1) = a,\ f(0,0) = 1,\ f(1,1) = 1.$$

If one uses this cocycle pair for a classical 2-component link, then exponent of a is the linking number. So, if one uses this cocycle pair for virtual knots or links, one gets a different generalization of the linking number to the virtual case (see [11] for the notion of "self-linking number").

Remark 26. Given $(X, S = \text{flip})$, the condition for an involutive β to be compatible with S, in the sense that (X, S, β) is a virtual pair, is non-trivial. Nevertheless, there are plenty of examples; for instance, if $|X| = 7$, there are 3456 involutive solutions, 1959 of them are compatible with $S = \text{flip}$.

1.3. Cohomologous pairs. From the following lemma we propose the notion of cohomologous 2-cocycle pair:

Lemma 27. *Let $f, g\colon X \times X \to G$ be a 2-cocycle pair and $\lambda\colon X \to G$ be a map. If one defines*

$$f_\lambda(x,y) := \lambda(x) f(x,y) \lambda(S^2(x,y))^{-1}$$

$$g_\lambda(x,y) := \lambda(x) g(x,y) \lambda(\beta^2(x,y))^{-1}$$

then

- *f_λ always satisfies f1,*
- *f_λ satisfies f2 $\iff$ $\lambda(y) = \lambda(S^1(x,y))$ for all x, y,*
- *f_λ satisfies f3 $\iff$ $\lambda(x) = \lambda(s_S(x))$ for all x,*
- *g_λ satisfies g1 $\iff$ $\lambda(y) = \lambda(s_\beta(y))$ for all y,*
- *g_λ always satisfies g3,*
- *$\lambda(y) = \lambda(\beta^1(x,y))$ for all x, y $\iff$ $\lambda(\beta^2(x,y)) = \lambda(x)$ for all x, y.*
- *If $g(x,y) \equiv 1$, then 1_λ verifies g2 $\iff$ $\lambda(x)\lambda(\beta^2(x,y))^{-1}\lambda(\beta^1(x,y))\lambda(y)^{-1} = 1\ \forall x, y$.*
- *If $\lambda(y) = \lambda(\beta^1(x,y))\ \forall x, y$, then g_λ satisfies g2 $\iff$ $[\lambda(x), g(x,y)][g(x,y), \lambda(y)] = 1$ for all x, y, where the brackets denote the commutator. If also $\lambda(x)$ commutes with $g(x,y)$ for all y then $g_\lambda = g$.*

Definition 28. Let H be a group, (X, S, β) be a virtual pair. Two 2-cocycle pairs (f, g) and $(\tilde{f}, \widetilde{g})$ are called cohomologous if $g = \widetilde{g}$ and there exists $\lambda\colon X \to H$ such that

$$\tilde{f}(x,y) = \lambda(x) f(x,y) [\lambda(S^2(x,y))]^{-1}$$

with λ satisfying

- $\lambda(x) = \lambda(s_S(x))$,
- $\lambda(y) = \lambda(S^1(x,y))$,
- $\lambda(y) = \lambda(\beta^1(x,y))$,
- for all x, and y, $\lambda(x)$ commutes with $g(x,y)$.

From Lemma 27 above one can easily prove the following:

Proposition 29. *If (f, g) is a 2-cocycle pair and a map $\lambda : X \to H$ satisfies the four conditions given in Definition 28, then the pair (f_λ, g) is a two cocycle pair which is coho-*

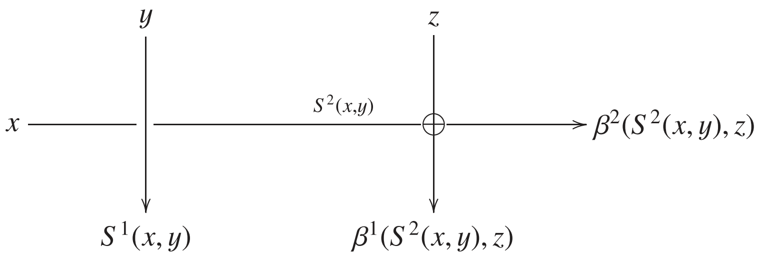
mologous to (f, g).

And one can also prove the expected result:

Proposition 30. *If* $(f, g), (\widetilde{f}, g)$ *are two cohomologous noncommutative 2-cocycle pairs then*

$$[\Psi_i(L, \mathcal{C}, f, g)] = [\Psi_i(L, \mathcal{C}, \widetilde{f}, g)].$$

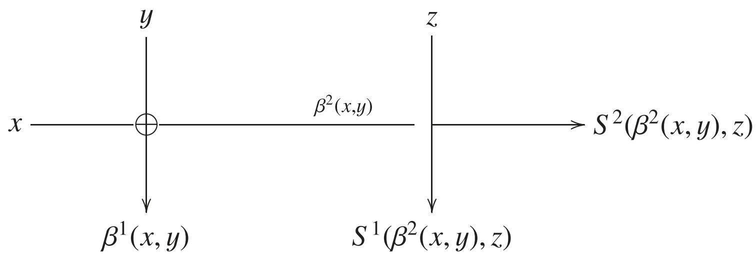
Proof. Let us suppose $\widetilde{f}(x, y) = \gamma(x) f(x, y)[\gamma(S^2(x, y))]^{-1}$. Take a link L, pick a connected component K and a base point. If every crossing in K is virtual it is obvious. If every crossing in K is classical see [6]. If K has both, virtual and classical crossings:



the product of weights for the horizontal line is:

$$\widetilde{f}(x, y) g(S^2(x, y), z) = \lambda(x) f(x, y)[\lambda(S^2(x, y))]^{-1} g(S^2(x, y), z) =$$

$$\lambda(x) f(x, y) g(S^2(x, y), z)[\lambda((S^2(x, y))]^{-1} = \lambda(x) f(x, y) g(S^2(x, y), z)[\lambda(\beta^2(S^2(x, y), z))]^{-1}$$



the product of weights for the horizontal line is:

$$g(x, y)\widetilde{f}(\beta^2(x, y), z) = g(x, y)\lambda(\beta^2(x, y)) f(\beta^2(x, y), z)[\lambda(S^2(\beta^2(x, y), z))]^{-1}$$

hence

$$g(x, y)\lambda(x) f(\beta^2(x, y), z)[\lambda(S^2(\beta^2(x, y), z))]^{-1}$$

$$= \lambda(x) g(x, y) f(\beta^2(x, y), z)[\lambda(S^2(\beta^2(x, y), z))]^{-1}$$

□

2. Universal noncommutative 2-cocycle pair

2.1. Universal construction. Given a virtual pair (X, S, β) we shall define a group together with a universal 2-cocycle pair in the following way:

Definition 31. Let $U_{nc}^{fg} = U_{nc}^{fg}(X, S, \beta)$ be the group freely generated by symbols $(x, y)_f$ and $(x, y)_g$ with relations

f1) $(x,y)_f \,(S^2(x,y),z)_f = (x,S^1(y,z))_f \,(S^2(x,S^1(y,z)),S^2(y,z))_f$
f2) $(S^1(x,y),S^1(S^2(x,y),z))_f = (y,z)_f$
f3) $(x, s(x))_f = 1$
g1) $(x, s_\beta(x))_g = 1$
g2) $(x,y)_g \,(\beta(x,y))_g = 1$
g3) $(x,y)_g \,(\beta^2(x,y),z)_g = (x,\beta^1(y,z))_g \,(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))_g$
g4) $(y,z)_g \,(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))_g = (x,y)_g \,(\beta^1(x,y),\beta^1(\beta^2(x,y),z))_g$
g5) $(y,z)_g \,(x,\beta^1(y,z))_g = (\beta^2(x,y),z)_g \,(\beta^1(x,y),\beta^1(\beta^2(x,y),z))_g$
m1) $(y,z)_g = (S^1(x,y),\beta^1(S^2(x,y),z))_g$
m2) $(y,z)_g \,(x,\beta^1(y,z))_g = (S^2(x,y),z)_g \,(S^1(x,y),\beta^1(S^2(x,y),z)_g$
m3) $(x,\beta^1(y,z))_g \,(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))_f = (x,y)_f \,(S^2(x,y),z)_g$.

Denote f_{xy} and g_{xy} the class in U_{nc}^{fg} of $(x, y)_f$ and $(x, y)_g$ respectively. We also define $\pi_f, \pi_g \colon X \times X \to U_{nc}^{fg}$ by

$$\pi_f, \pi_g \colon X \times X \to U_{nc}^{fg}$$

$$\pi_f(x, y) := f_{xy},$$

$$\pi_g(x, y) := g_{xy}$$

The following is immediate from the definitions:

Theorem 32. *Let (X, S, β) be virtual pair:*

- *The pair of maps $\pi_f, \pi_g \colon X \times X \to U_{nc}^{fg}$ is a noncommutative 2-cocycle pair.*
- *Let H be a group and $f, g \colon X \times X \to H$ a noncommutative 2-cocycle pair, then there exists a unique group homomorphism $\rho \colon U_{nc}^{fg} \to H$ such that $f = \rho \circ \pi_f$ and $g = \rho \circ \pi_g$*

$$\begin{tikzcd} X \times X \arrow[r, "f"] \arrow[d, "\pi_f"'] & H \\ U_{nc}^{fg} \arrow[ur, dashed, "\rho"'] & \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "g"] \arrow[d, "\pi_g"'] & H \\ U_{nc}^{fg} \arrow[ur, dashed, "\rho"'] & \end{tikzcd}$$

Remark 33. U_{nc}^{fg} is functorial. That is, if $\phi \colon (X, S, \beta) \to (Y, S', \beta')$ is a morphism of virtual pairs, namely ϕ satisfy

$$(\phi \times \phi)S(x_1, x_2) = S'(\phi x_1, \phi x_2), \qquad (\phi \times \phi)\beta(x_1, x_2) = \beta'(\phi x_1, \phi x_2)$$

then, ϕ induces a (unique) group homomorphism $U_{nc}^{fg}(X) \to U_{nc}^{fg}(Y)$ satisfying

$$f_{x_1x_2} \mapsto f_{\phi x_1 \phi x_2} \qquad g_{x_1x_2} \mapsto g_{\phi x_1 \phi x_2}$$

Proof. One needs to prove that the assignment $f_{x_1x_2} \mapsto f_{\phi x_1 \phi x_2}$ and $g_{x_1x_2} \mapsto g_{\phi x_1 \phi x_2}$ are compatible with the relations defining $U_{nc}^{fg}(X)$ and $U_{nc}^{fg}(Y)$ respectively, and this is clear since $(\phi \times \phi) \circ S = S \circ (\phi \times \phi)$ and $(\phi \times \phi) \circ \beta = \beta \circ (\phi \times \phi)$. □

REMARK 34. In order to produce an invariant of a knot or link, given a solution (X, S, β), we need to produce a coloring of the knot/link by X, and then find a noncommutative 2-cocycle, but since U_{nc}^{fg} is functorial, given X we always have the universal 2-cocycle pair $f, g \colon X \times X \to U_{nc}^{fg}(X)$, and hence, the information given by the invariant was already included in the combinatorics of the colorings.

Also, if $\phi \colon X \to X$ is a bijection commuting with S and β, then, given a coloring and its invariant calculated with the universal cocycle, we may apply ϕ to each color and get another coloring, and this will produce the same invariant pushed by ϕ in U_{nc}^{fg}.

EXAMPLE 35. Computations of Example 11 show that for $X = \{1, 2\}$ and $S = \beta = \text{flip}$, $U_{nc}^{fg}(X) \cong \text{Free}(a, b) \times \text{Free}(h)$ where $(1,1)_f = (2,2)_f = (1,1)_g = (2,2)_g = 1$, $(1,2)_f = a$, $(2,1)_f = b$, $(1,2)_g = h$, $(2,1)_g = h^{-1}$.

EXAMPLE 36. If $X = \{1, 2\}$, $S(x, y) = (y + 1, x + 1) \pmod 2$ and $\beta = \text{flip}$, then

$$U_{nc}^{fg}(X) \cong \text{Free}(c)$$

where $c = (1,1)_f = (2,2)_f$, $1 = (1,2)_f = (2,1)_f$, and $(x,y)_g = 1$ for all $x, y \in X$. This virtual pair does not give the same information as the previous example (since for instance $g \equiv 1$), but it gives a different way to generalize the linking number to virtual links.

2.2. Some examples of virtual pairs of small cardinality. Using GAP, the list of biquandles and involutive solutions, one can easily compute the list of (isomorphism classes of) virtual pairs of small cardinality. We show the total amount of them in the following table. The amount grows very fast, for cardinal 6 the computer takes too long to compute all virtual pairs, so we put on the table only partial cases for $n = 6$. The notation (S, i_a) is for virtual pairs with biquandle S and involutive β of the form $\beta(x, y) = (a^{-1}y, ax)$, with $a \in \text{Aut}(X, S)$. Notice that for each S there are as many isomorphism classes of pairs (S, i_a) as conjugacy classes of $\text{Aut}(X, S)$.

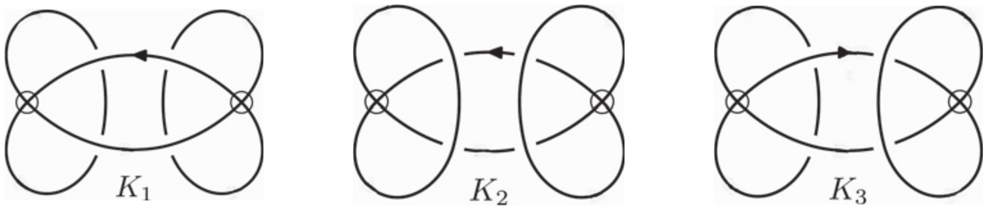
n	2	3	4	5	6
all virtual pairs	4	90	3517	46658	
virtual pairs (S, i_a)	4	38	325	41278	111151
connected virtual pairs	3	26	167	138	836
conn. virtual pairs with non conn. S and non conn β	0	0	10	0	84

The complete list in each case can be found in `http://mate.dm.uba.ar/~mfarinat/papers/GAP/virtual`.

3. Some virtual knots/links and their n.c. invariants

3.1. Kishino's knots. We begin with an example of colorings:

Example 37. Let S be the dihedral quandle, that is $X = \{1, 2, 3\}$ and $S(x, y) = (y, x \triangleleft y)$ where $x \triangleleft y = 2y - x$ (mod 3). $\mathrm{Aut}(X, S)$ can be identified with the dihedral group $D_3 = S_3$. There are three conjugacy classes in D_3, a set of representatives is $\{\mathrm{Id}, (2, 3), (1, 2, 3)\}$. For $a \in \mathrm{Aut}(X, S)$ denote i_a the involutive biquandle given by $i_a(x, y) = (a^{-1}(y), a(x))$. In the following table we write the number of colorings of the Kishino's knots using the corresponding virtual pair, so we see that they are all different.



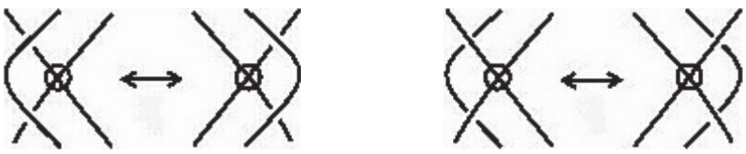
#colorings	(S, i_{id})	$(S, i_{(2,3)})$	$(S, i_{(1,2,3)})$
K_1	9	3	9
K_2	3	9	3
K_3	3	3	3

Moreover, for $X = \{1, 2, 3, 4\}$ with S given by

$$\begin{array}{llll} S(1,1) = (1,1) & S(1,2) = (2,4) & S(1,3) = (4,2) & S(1,4) = (3,3) \\ S(2,1) = (3,4) & S(2,2) = (4,1) & S(2,3) = (2,3) & S(2,4) = (1,2) \\ S(3,1) = (4,3) & S(3,2) = (3,2) & S(3,3) = (1,4) & S(3,4) = (2,1) \\ S(4,1) = (2,2) & S(4,2) = (1,3) & S(4,3) = (3,1) & S(4,4) = (4,4) \end{array}$$

and β = flip, then the number of colorings of K_3 is 16, so K_3 is also nontrivial.

3.2. Links. It is worth to notice that [1] computes virtual pairs of small cardinality. In that work, some classes of virtual pairs are considered, the so called essential pairs, and the welded pairs. Recall that there are "forbidden" Reidemeister moves:

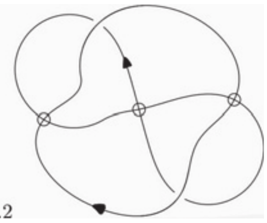


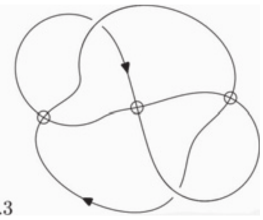
These moves are not allowed in virtual knots, and if one uses both forbidden moves, then one can "unknot" every knot/link. Essential virtual pairs are pairs that *do not* satisfy those forbidden moves, in welded pairs a forbidden move is allowed (see [1] for details). In this work we consider *all* virtual pairs, that's why we have more virtual pairs that in [1]. in particular, for n=2, the trivial example (flip, flip) is not considered in [1], and one can easily see that the number of colorings doesn't give any interesting information, just if the link is connected or not, but the 2-cocycle invariant is highly nontrivial, as we show next.

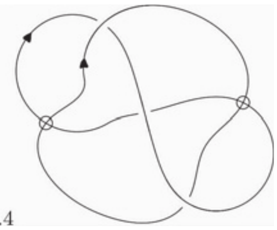
From the list of 51 virtual links provided by A. Bartholomew (these are 2-component links with 4, 5 or 6 crossings), coloring with ($X = \{1,2\}, S = \beta =$ flip) (see Example 35) and computing the invariant (coloring will not distinguish these links), leaves 18 classes. To refine this, color with ($X = \{1,2\}, S =$ antiflip, $\beta =$ flip) (i.e. Example 36) and compute the invariant. Using both invariants leaves 38 classes. Furthermore, color with $X = \{1,2,3,4\}$ and all possible virtual pairs (without computing the invariant) and get 47 classes.

We exhibit three examples of links from this list, and their invariants:

EXAMPLE 38.


$v2.2$


$v2.3$


$v3.4$

(X, S, β)	U_{nc}^{fg}	$v2.2$	$v2.3$	$v3.4$
$(\{1,2\}$,flip,flip$)$	$\langle a,b\rangle \times \langle h\rangle$	$4\{1,1\}$	$2\{a^{-1},b^{-1}\}$, $2\{1,1\}$	$2\{a^{-1},b^{-1}\}$, $2\{1,1\}$
$(\{1,2\}$,a-flip,flip$)$	$\langle c\rangle$	$4\{1,1\}$	$2\{c^{-2},1\}$, $2\{1,1\}$	$2\{c^{-1},c^{-1}\}$, $2\{1,1\}$

The content of the above table is the following:

Using $X = \{1,2\}$ with $S = \beta =$ flip (see Example 35) and computing the invariant gives 4 colorings to each $v2.2, v2.3$ and $v3.4$. For every coloring the invariant of $v2.3$ and $v3.4$ gives $\{a^{-1},b^{-1}\}$ twice and $\{1,1\}$ twice, but the same computation for $v2.2$ gives always $\{1,1\}$.

To be able to distinguish $v2.3$ from $v3.4$, consider same set X but $S =$ antiflip and $\beta =$ flip (see Example 36), again there are 4 possible colorings for each link. The invariant gives: $\{c^{-1},c^{-1}\}$ twice and $\{1,1\}$ twice for $v2.3$ and $\{c^{-2},1\}$ twice and $\{c^{-1},c^{-1}\}$ twice for $v3.4$, and always $\{1,1\}$ for $v2.2$.

REMARK 39. The exponent of a (or b) is the first generalization of linking number to the virtual case (in the sense that if the link is classical then it gives the linking number). Using the second virtual pair, the exponent of c is a different generalization of the linking number.

A non-commutative example

Consider S given by the quandle $\{1,2,3,4\}$ with operation

$$- \triangleleft 1 = - \triangleleft 2 = (3,4),$$

$$- \triangleleft 3 = - \triangleleft 4 = (1,2)$$

that is, $S(x,y) = (y, x \triangleleft y)$, and β the involutive solution

$\beta(x,y) = (l_x(y), r_y(x))$ where

$$l_1 = l_2 = (1,2),$$

$$l_3 = l_4 = (1,2)(3,4),$$

and $r_i = l_i$ $(i = 1,2,3,4)$. This is the pair number 248 in the list vp4 in [5].

Using the relations of $U_{nc}^{fg} = U_{nc}^{fg}(X,S,\beta)$ one can easily see that

$$\begin{array}{ccccccccccccccc}
1 & = & (1,1)_f & = & (2,2)_f & = & (3,3)_f & = & (4,4)_f & = & (1,1)_g & = & (1,2)_g \\
 & = & (2,1)_g & = & (2,2)_g & = & (3,3)_g & = & (3,4)_g & = & (4,3)_g & = & (4,4)_g \\
a & := & (1,2)_f & = & (2,1)_f, & & b & := & (1,3)_f & = & (2,3)_f & & \\
c & := & (1,4)_f & = & (2,4)_f, & & d & := & (3,1)_f & = & (4,2)_f & & \\
e & := & (3,2)_f & = & (4,1)_f, & & f & := & (3,4)_f & = & (4,3)_f & & \\
 & & h & := & (1,3)_g & = & (1,4)_g & = & (2,3)_g & = & (2,4)_g & & \\
 & & h^{-1} & = & (3,1)_g & = & (3,2)_g & = & (4,1)_g & = & (4,2)_g & &
\end{array}$$

So, we have 7 generators, and if one (or a computer) writes the list of all relations in terms of a,b,c,d,e,f,h, one gets

$$b = ac,\ b = ca,\ c = ab,\ c = ba,$$

$$ab = ba,\ ac = ca,\ ah = ha,\ bc = cb,$$

$$bh = hc,\ ch = hb,$$

$$d = ef,\ d = fe,\ e = df,\ e = fd,\ dd = ee,$$

$$df = fd,\ dh = he,\ ef = fe,$$

$$eh = hd,\ fh = hf.$$

If one solves b, and d in terms of a,c,e,f,h, equations above translate into

$$b = ac,\ d = ef,$$

$$ac = ca,\ c = aac,\ c = aca,\ aac = aca,$$

$$ah = ha,\ acc = cac,\ ach = hc,$$

$$ch = hac,\ ch^{-1} = h^{-1}ac,$$

$$ef = fe,\ e = eff,\ e = fef,\ efef = ee,$$

$$eff = fef,\ efh = he,$$

$$eh = hef,\ fh = hf.$$

One can easily see that $a^2 = 1$, $f^2 = 1$, $a = [h,c]$ $(=hch^{-1}c^{-1})$, $f = [e^{-1},h]$.

Remark 40. Let G be a group, $a,c,h \in G$ and assume

$$a^2 = 1,\ a = [h,c],\ [h,a] = [c,a] = 1,$$

then $[h,c] = [c,h] = [c^{-1},h^{-1}] = [c^{-1},h] = [c,h^{-1}]$.

Using this remark, it is an easy exercise to check the following characterization:

Corollary 41. *Denote* $a := [h, c]$ *and* $f := [h, e]$*, then*

$$U_{nc}^{fg}(S,\beta) \cong \frac{\mathrm{Free}(h,c,e)}{\left\langle \begin{array}{c} a^2 = [a,c] = [a,h] = 1, \\ f^2 = [f,e] = [f,h] = 1 \end{array} \right\rangle}.$$

REMARK 42. The element a is nontrivial in U_{nc}^{fg}.

Proof. If one adds the relation $e = 1$ then (recall $a := [h, c]$)

$$G := U_{nc}^{fg}/\langle e = 1\rangle \cong \frac{\mathrm{Free}(h,c)}{\langle\, a^2 = [a,c] = [a,h] = 1, \rangle}$$

G can be described as a central extension of $\mathbb{Z}^2 \cong \mathrm{Free}(h,c)/\langle [h,c]\rangle$ over $\mathbb{Z}/2\mathbb{Z} \cong \langle a : a^2 = 1\rangle$, more precisely, consider the set of monomials

$$M := \{h^i c^j a^\epsilon : i, j \in \mathbb{Z}, \epsilon = 0, 1\}$$

then M is a group with multiplication given by

$$(h^i c^j a^\epsilon)(h^k c^l a^\sigma) = h^{i+k} c^{j+l} a^{jk+\epsilon+\sigma Mod\ 2}$$

and clearly $S \cong M$, so $a \neq 1$ in G. □

REMARK 43. A famous quotient of U_{nc}^{fg} is the quaternion group $H = \{\pm 1, \pm i, \pm j, \pm k\}$ where $e \mapsto 1$, $h \mapsto i$, $c \mapsto j$. One can see that relations go to 1, so we have a well-defined group homomorphism, and $a \mapsto -1$.

REMARK 44. If one uses the abelianization of U_{nc}^{fg}, then one gets essentially a (Laurent) polynomial in the variables h, c, e, and clearly the element a is trivial in $(U_{nc}^{fg})_{ab}$, since $a = [h, c]$. But there are examples where the full non-commutative invariant gives a as answer (see next example), so this non-commutative invariant refines the 2-cocycle one with values only in commutative groups.

EXAMPLE 45. If one uses this virtual pair and the universal 2-cocycle, then the invariant for the virtual link v2.3 is (a, a) twice, (f, f) twice, and 4 times $(1, 1)$.

3.3. State sum. If the target group $(A, \cdot)$ is abelian, then one can perform the state-sum for a pair of maps $f, g \colon X \times X \to A$, defining Boltzmann Weights in the same way. For a given coloring, consider the product over all crossings of the corresponding weights, and then sum over all colorings. If one asks for Reidemeister invariance in this construction, the set of equations are:

ss-f1) $f(x, s(x)) = 1$,

ss-f2) $f(x, y) f(S^2(x, y), z) f(S^1(x, y), S^1(S^2(x, y), z))$
$$= f(x, S^1(y, z)) f(S^2(x, S^1(y, z)), S^2(y, z)) f(y, z),$$

ss-g1) $g(x, s_\beta(x)) = 1$,

ss-g2) $g(x, y)\, g(\beta(x, y)) = 1$,

ss-g3) $g(x, y)\, g(\beta^2(x, y), z)\, g(\beta^1(x, y), \beta^1(\beta^2(x, y), z))$
$$= g(x, \beta^1(y, z))\, g(\beta^2(x, \beta^1(y, z)), \beta^2(y, z))\, g(y, z)$$

ss-m) $g(y,z)\,g(x,\beta^1(y,z))f(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))$
$$= g(S^1(x,y),\beta^1(S^2(x,y),z))\,g(S^2(x,y),z)f(x,y).$$

Conditions ss-f1 and ss-f2 are a consequence of f1, f3, f3. Also ss-m follows from m1, m2, m3. We have ss-g1 and ss-g2 are the same as g1 and g2. But g3, g4, and g5 imply (ss-g3)^2, that is, assuming g3 g4 and g5 one can conclude

$$g(x,y)^2 g(\beta^2(x,y),z)^2 g(\beta^1(x,y),\beta^1(\beta^2(x,y),z))^2$$
$$= g(x,\beta^1(y,z))^2 g(\beta^2(x,\beta^1(y,z)),\beta^2(y,z))^2 g(y,z)^2$$

If the abelian group A has no elements of order 2, then a non-commutative 2-cocycle pair is also a commutative 2-cocycle. One can think of the group $(U_{nc}^{fg})_{ab}$ as a nontrivial way of producing cocycles for virtual state-sum invariants, at least when $(U_{nc}^{fg})_{ab}$ has no elements of order 2.

4. Final questions

We end with some open questions:

1. When S = flip, the compatibility condition for (an involutive) β is non-trivial, but nevertheless there are many solutions (see Remark 26). Is there a characterization in "involutive" terms? e.g. in terms of the dot operation (cyclic set structure), or brace, associated to involutive solutions as considered by Rump [12]?
2. Is it possible to classify connected virtual pairs in group theoretical terms, or at least, the indecomposable ones? The situation is well-known for quandles: recall that if G is a group, $s \in \mathrm{Aut}(G)$, $H \subseteq G$ a subgroup such that $s(h) = h$ for all $h \in H$, then
$$Hy \triangleleft Hx := Hs(yx^{-1})x$$
is a quandle structure on the homogeneous set G/H. This family of quandles contain all indecomposable quandles because of the following characterization:
Theorem 3.1 [13] (see also [4] and [9])**.** Let X be an indecomposable quandle of n elements, $x_0 \in X$, $z := - \triangleleft x_0$, $G = \mathrm{Inn}_(X)$, $H = \{g \in G : g(x_0) = x_0\}$, then
   - G is a transitive group of degree n,
   - z is central in H,
   - X is isomorphic to G/H as quandle with s= conjugation by z.

   An analogous characterization for (indecomposable) biquandles is not known, neither for (indecomposable) virtual pairs. For a given virtual pair (X,S,β), it would be interesting to discover properties of the action of the group generated by the permutations $\{S^1(x,-), S^2(-,y), l_x, r_y : x, y \in X\}$.
3. Given a finite virtual pair (X,S,β), it is easy to produce an algorithm computing generators and relations of $U_{nc}^{fg}(X)$, but one needs to do case by case. Is there a way to compute $U_{nc}^{fg}(X)$ in general at least for a family of virtual pairs? e.g. for S=biAlexander switch and β affine?
4. When β=flip and $g \equiv 1$, then the conditions on f are the same as the 2-cocycle condition considered in [6], which is a generalization of the quandle case considered in [3]. Also, in [3], the authors prove that the noncommutative 2-cocycle invariant (in the quandle case, for classical knots/links) is a quantum invariant. It seems that

the fact that this noncommutative invariant is a quantum one may be generalized to the biquandle case (and still classical knots or links), but it is not clear at all how to proceed when there are virtual crossings. It would be interesting to see what should be the "quantum algebraic" categorical data corresponding to virtual pairs and 2-cocycle pairs.

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