Virtual Reference Feedback Tuning (VRFT): a new direct approach to the design of feedback controllers

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Abstract

In this paper we discuss a new method for the data-based design of feedback controllers in a linear setting. The main features of the method are that it is a direct method (no model identification of the plant is needed) and that it can be applied using a single set of data generated by the plant with no need for specific experiments nor iterations. It will be shown that the method searches for the global optimum of the design criterion and that, in the significant case of restricted complexity controller design, the achieved controller is a sensible approximation (under some reasonable hypotheses) of the restricted complexity global optimal controller.

As an extra contribution it is also presented a controller validation test aiming at ascertain the closed-loop stability before that the designed controller is applied to the plant. A numerical example ends the paper.

1 Introduction

The control problem addressed in this paper can be formally stated as follows. Consider a linear time-invariant discrete-time SISO plant $P(z)$. $P(z)$ is unknown and only a finite set of Input/Output data $\{u(t), y(t)\}_{t=1,..,N}$ (possibly corrupted by noise) is available. Given a set of parameterized controllers $C(z;\theta) = \beta^T(z)\theta$ ($\beta(z) = [\beta_1(z) \quad \beta_2(z) \ldots \beta_n(z)]^T$ is a vector of linear discrete-time transfer functions and $\theta = [\vartheta_1 \quad \vartheta_2 \ldots \vartheta_n]^T \in \mathbf{R}^n$ is the n-dimensional vector of parameters), and a reference model $M(z)$, the problem consists in finding the parameter vector, say $\bar{\theta}$, which minimizes the following model-reference performance index

$$J_{MR}(\theta) = \left\| \left(\frac{P(z)C(z;\theta)}{1+P(z)C(z;\theta)} - M(z) \right) W(z) \right\|_2^2 \quad (1)$$

where $W(z)$ is a user-chosen weighting function.

In this paper we approach the above control problem through a new direct technique called *Virtual Reference Feedback Tuning* (VRFT). The roots of this technique are found in [3, 7, 8], where the "Virtual Reference" framework was first introduced.

A design technique is called "direct" when the I/O data collected on the plant are used to directly tune a parameterized controller, without passing through a plant-identification step. Direct techniques are conceptually more natural than indirect ones (where the controller is designed on the basis of an estimated model of the plant), since they directly target the final goal of tuning the parameters of a given class of controllers. However, despite the appeal of direct methods, very few genuine direct techniques have been proposed in the literature ([3, 4, 10]). In particular (to the best of our knowledge), the only genuine "direct" data-based technique which can be compared with VRFT is a method called *Iterative Feedback Tuning* (*IFT*), recently developed and proposed by Hjalmarsson and coauthors ([4]). Even if IFT and VRFT belong to the same class of design methods, their peculiar features are quite different:

- IFT is based upon a *gradient-descent approach* and it is therefore an *iterative* technique. It typically converges to the *local minimum* closest to the initial condition (henceforth it is a "local" optimization technique). However, it can be shown that, if the initialization vector falls in the basin of attraction of the global minimum of (1), IFT provides an unbiased estimate of the *optimal* parameter vector of the controller. IFT can possibly call for a large number of experiments and, in any case, it requires to perform experiments on the true plant with specific inputs.
- VRFT is a *"one-shot"* method which searches for the *global minimum* of the performance index (1), with no need for iterations nor initialization. It makes use of any set of input-output data (i.e. it does not require specific experiments). However, the VRFT technique is only *near-optimal*, in the sense that, in general, it provides a controller which is close, but not equal, to the one minimizing (1).

Interestingly enough, the application realms of these direct data-based techniques are quite complementary.
Throughout the paper, the frequency-domain interpretation of the performance indices will be used extensively as a main analysis tool, since it allows a deep understanding of the design techniques. It is interesting to note that this analysis framework is mainly borrowed by the recent literature on *"iterative control"* (see e.g. [11, 14]), a design approach which has many similarities with IFT and VRFT, even if it must be classified as an "indirect" control system design technique.
The structure of the paper is as follows. In Section 2 the "Virtual Reference" framework is introduced. Starting from the basic idea of Section 2, the VRFT technique is developed by addressing two main issues: the problem of pre-filtering the data (Section 3), and the problem of dealing with noise (Section 4). This paper delivers an extra contributions in terms of a controller validation test aiming at ascertain the closed-loop stability before that the controller is actually implemented. This is discussed in Section 5. A numerical example ends the paper.

2 The Virtual Reference framework

The Virtual Reference framework permits one to recast the problem of designing a model-reference feedback controller into a standard system-identification problem. The rationale behind the Virtual Reference framework can be phrased as follows.
Suppose that a controller $C(z;\theta)$ results in a closed-loop system whose transfer function is $M(z)$. Then, if the closed-loop system is fed by *any* reference signal $r(t)$, its output equals $M(z)r(t)$. Hence, a necessary condition such that the closed-loop system has the same transfer function as the reference model is that the output of the two systems is the same for a *given* $\bar{r}(t)$. An usual approach in model reference control to impose the latter (necessary) condition is to first select $\bar{r}(t)$ and then choose $C(z;\theta)$ such that the condition is in fact satisfied. However, for a general selection of $\bar{r}(t)$, the above task may turn out to be complex. The basic idea behind the virtual reference approach consists in performing a wise selection of $\bar{r}(t)$ as explained next.
Given the measured $y(t)$ (i.e. the actual signal measured at the output of the plant), consider a reference $\bar{r}(t)$ such that $M(z)\bar{r}(t) = y(t)$. Such a reference is called "virtual" because it does not exist in reality and in fact it was not used in the generation of $y(t)$. Notice that $y(t)$ is the desired output of the closed-loop system when the reference signal is $\bar{r}(t)$. Next, compute the corresponding *output error* $e(t) = \bar{r}(t) - y(t)$. Even though plant $P(z)$ is not known, we know that when $P(z)$ is fed by $u(t)$ (the actually measured input signal), it generates $y(t)$ as an output. Therefore, a good controller (at least in the condition when the reference signal is the virtual reference $\bar{r}(t)$) is one that generates $u(t)$ when fed by $e(t)$. The idea is then to search for such a controller. What is important to note here is that since both signals $u(t)$ and $e(t)$ are known, this task reduces to the *identification problem* of describing the existing dynamical relationship between these two signals. We also note that all the signals involved in the identification procedure can be pre-processed with a filter $L(z)$, whose choice and shaping is left to the designer.
The above idea can be implemented in a simple algorithm, which represents the "bulk" of the VRFT method. It can be formally stated in the following three steps:

1. Pre-filter the measured I/O data $\{u(t), y(t)\}_{t=1,..,N}$ with a suitable filter $L(z)$:
$$y_L(z) = L(z)y(t), \quad u_L(z) = L(z)u(t). \qquad (2)$$

2. Find a reference input $\bar{r}_L(t)$ such that the output of the reference model $M(z)$ is exactly $y_L(z)$ when fed with $\bar{r}_L(t)$, namely $y_L(t) = M(z)\bar{r}_L(t)$.

3. Select the controller parameter vector, say $\hat{\theta}_N$, that minimizes the following performance index $J_{VR}^N(\theta)$:
$$\begin{aligned} J_{VR}^N(\theta) &= \frac{1}{N}\sum_{t=1}^{N}(u_L(t) - C(z;\theta)e_L(t))^2 \qquad (3)\\ e_L(t) &= \bar{r}_L(t) - y_L(t). \end{aligned}$$

Specifically, if the controller has the form $C(z;\theta) = \beta^T(z)\theta$, the performance index (3) can be given the form
$$\begin{aligned} J_{VR}^N(\theta) &= \frac{1}{N}\sum_{t=1}^{N}(u_L(t) - \varphi_L^T(t)\theta)^2 \qquad (4)\\ \varphi_L(t) &= \beta(z)e_L(t); \end{aligned}$$

since (4) is quadratic in θ, the parameter vector $\hat{\theta}_N$ which minimizes (4) is an explicit function of the data:
$$\hat{\theta}_N = \left[\sum_{t=1}^{N}\varphi_L(t)\varphi_L(t)^T\right]^{-1}\sum_{t=1}^{N}\varphi_L(t)u_L(t). \qquad (5)$$

As it will be shown in forthcoming sections, the Virtual Reference approach provides an approximate solution to the design problem stated in the Introduction (the data-based minimization of $J_{MR}(\theta)$). This is appealing since the Virtual Reference approach is based upon a performance index $J_{VR}^N(\theta)$, characterized by the following attractive features:

- it is quadratic in θ;

- it can be numerically computed directly from the measured I/O data (it does not requires the a-priori estimation of a mathematical model of the plant). Moreover, the I/O data are not required to be collected in specific operating conditions.

Hence, using $J_{VR}^N(\theta)$, the original performance index $J_{MR}(\theta)$ can be easily "directly" minimized, at the expense of a little sub-optimality.
The rest of the paper is devoted to the development of a technique (which we call VRFT) which exploits the Virtual Reference idea so as to give rise to an effective and useful design algorithm. This can be done by addressing the following two main open issues: selection of a pre-filter $L(z)$ in order to achieve a nearly optimal performace and care for the presence of noise.

3 Shaping the pre-filter

Consider the performance index $J_{MR}(\theta)$ of the model reference control problem (eqn.(1)) and the one of the virtual reference approach (eqn.(3)): they appear to be different. Yet, in this section it will be shown that their minimum arguments can in fact be made close to each other by a suitable selection of the pre-filter $L(z)$.
The analysis presented in this section is performed under the assumption that the measured signals $u(t)$ and $y(t)$ are not corrupted by noise. The treatment of noise is postponed to Section 4.
To start with, note that, using the definition of 2-norm of a discrete-time linear transfer function, $J_{MR}(\theta)$ can be given the following frequency-domain form:

$$J_{MR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{PC(\theta)}{1+PC(\theta)} - M\right|^2 |W|^2 d\omega, \quad (6)$$

(we drop the argument $e^{j\omega}$ of transfer functions where this is not a source of confusion).
Introduce now the rational function $C_0(z)$ which exactly solves the model-matching problem, namely $C_0(z)$ is such that

$$\frac{P(z)C_0(z)}{1+P(z)C_0(z)} = M(z). \quad (7)$$

Notice that, in general, $C_0(z)$ does not belong to the family of parameterized controllers $\{C(z;\theta)\}$. Even more so, it is not required to be proper. Throughout the paper we will refer to $C_0(z)$ as the "ideal controller". Using $C_0(z)$, after some manipulations the performance index (6) can be rewritten as:

$$J_{MR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|P|^2\frac{|C(\theta)-C_0|^2}{|1+PC(\theta)|^2}\frac{|W|^2}{|1+PC_0|^2}d\omega. \quad (8)$$

Consider now the performance index $J_{VR}^N(\theta)$. If the measured signals $u(t)$ and $y(t)$ can be considered realizations of stationary and ergodic stochastic processes, when the number of available data grows ($N \longrightarrow \infty$), the following holds:

$$J_{VR}^N(\theta) \longrightarrow J_{VR}(\theta) = E[(u_L(t) - C(z;\theta)e_L(t))^2] =$$
$$= E\left[\left(L(z)\left(1 - \frac{1-M(z)}{M(z)}C(z;\theta)P(z)\right)u(t)\right)^2\right].$$

$J_{VR}(\theta)$ is the *asymptotic* counterpart of $J_{VR}^N(\theta)$, namely the performance index to which $J_{VR}^N(\theta)$ tends as the number of available data goes to infinity. Accordingly, as $N \longrightarrow \infty$, the minimum $\hat{\theta}_N$ of $J_{VR}^N(\theta)$ will converge to the minimum of $J_{VR}(\theta)$, say $\hat{\theta}$. In the rest of the paper, for analysis purposes, $J_{VR}(\theta)$ will be used extensively in place of $J_{VR}^N(\theta)$.
Finally, using the Parseval theorem (see e.g.[6]) and the equation (7), $J_{VR}(\theta)$ can be given the following frequency-domain representation:

$$J_{VR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|P|^2|C(\theta)-C_0|^2|1-M|^2\frac{|L|^2}{|M|^2}\Phi_u d\omega, \quad (9)$$

where Φ_u is the power density spectrum of $u(t)$.
By comparing the frequency-domain expression of $J_{MR}(\theta)$ and $J_{VR}(\theta)$ (eqn.(8) and eqn.(9), respectively), the following two observations can be made.
If $C_0(z) \in \{C(z;\theta)\}$, and both $J_{VR}(\theta)$ and $J_{MR}(\theta)$ have a unique minimum, it is always true that $\hat{\theta} = \bar{\theta}$, namely the minimum of $J_{VR}(\theta)$ coincides with the minimum of $J_{MR}(\theta)$, *whatever* $L(z)$, $W(z)$, $M(z)$ *and* $P(z)$ *are.*
Suppose instead that $C_0(z) \notin \{C(z;\theta)\}$: if the following identity holds

$$|L|^2 = \frac{|M|^2|W|^2}{\Phi_u|1+PC(\theta)|^2} \qquad \forall\omega \in [-\pi;\pi] \quad (10)$$

then $J_{VR}(\theta) = J_{MR}(\theta)$. As a consequence, $\hat{\theta} = \bar{\theta}$.
Clearly, the choice of the filter $L(z)$ suggested by equation (10) is not feasible since $P(z)$ is not known.
Here, the following choice of $L(z)$ is instead proposed: select $L(z)$ such that

$$|L|^2 = \frac{|1-M|^2|M|^2|W|^2}{\Phi_u} \qquad \forall\omega \in [-\pi;\pi]. \quad (11)$$

Note first that all quantities in the right-hand-side of equation (11) are known and therefore $L(z)$ can be actually computed (notice that Φ_u may be considered known only in certain situations where the input signal has been selected by the designer, otherwise, Φ_u can be estimated using many different techniques, among which a high-order AR or ARX model [6], or a high-order state-space model [12]). Moreover, by using equation (7), it is readily seen that expression (11) is equivalent to

$$|L|^2 = \frac{|M|^2|W|^2}{\Phi_u|1+PC_0|^2} \qquad \forall\omega \in [-\pi;\pi]$$

Hence, choice (11) corresponds to substitute $|1+PC(\theta)|^2$ with $|1+PC_0|^2$ in equation (10), which appears to be a sensible selection since we expect that $|1+PC(\theta)|^2 \approx |1+PC_0|^2$ for $\theta = \bar{\theta}$, where $\bar{\theta}$ is the minimum of $J_{MR}(\theta)$.
In Proposition 1 below, we show that choice (11) is in fact optimal in a certain sense. Before stating this result, some notations must be preliminary settled.
Set

$$\Delta C(z) \quad = \quad C_0(z) - \beta^T(z)\bar{\theta}$$

$$\beta^+(z) = [\beta_1(z) \quad \beta_2(z) \quad \ldots \quad \beta_n(z) \quad \Delta C(z)]^T$$
$$\theta^+ = [\vartheta_1 \quad \vartheta_2 \quad \ldots \quad \vartheta_n \quad \vartheta_{n+1}]^T$$

($\bar{\theta}$ being the parameter vector which minimizes $J_{MR}(\theta)$). Then define an extended family of controllers as follows:

$$C^+(z;\theta^+) = \beta^+(z)^T\theta^+.$$

Finally, consider the extended performance index

$$J^+_{MR}(\theta^+) = \left\| \left(\frac{P(z)C^+(z;\theta^+)}{1+P(z)C^+(z;\theta^+)} - M(z) \right) W(z) \right\|_2^2$$

(note that the difference between $J_{MR}(\theta)$ and $J^+_{MR}(\theta^+)$ is that $J^+_{MR}(\theta^+)$ is parameterized by the family of extended controllers $\{C^+(z;\theta^+)\}$). The second order Taylor expansion around its global minimizer $\bar{\theta}^+$ is denoted by $\tilde{J}^+_{MR}(\theta^+)$, namely:

$$J^+_{MR}(\theta^+) = \tilde{J}^+_{MR}(\theta^+) + o(\|\theta^+ - \bar{\theta}^+\|_2^2).$$

Using the above definitions, Proposition 1 can now be stated.

Proposition 1. The parameter vector $\bar{\theta}$ which minimizes the performance index $J_{MR}(\theta)$, and the parameter vector $\hat{\theta}$ which minimizes the performance index $J_{VR}(\theta)$ when $L(z)$ is selected as in (11) fulfill the following relationships:

$$\bar{\theta} = \arg\min_{\theta} J^+_{MR}([\theta^T \quad 0]^T).$$

$$\hat{\theta} = \arg\min_{\theta} \tilde{J}^+_{MR}([\theta^T \quad 0]^T).$$

proof: see [2] □

The above result is interesting since it provides a formal relationship between the parameter vector $\hat{\theta}$ obtained using the Virtual Reference approach (in the special case when $L(z)$ is selected according to (11)), and the "optimal" parameter vector $\bar{\theta}$, which minimizes the original performance index $J_{MR}(\theta)$. Based on this result, we conclude that if the transfer function $\Delta C(z)$ plays a marginal role in determining $C_0(z)$, namely the family of controllers $\{C(z;\theta)\}$ is only slightly under-parameterized given a certain reference model, then $C(z;\hat{\theta})$ is a good approximation to $C(z;\bar{\theta})$ since $J^+_{MR}(\theta^+)$ is well approximated in a neighborhood of its minimum by its second order expansion $\tilde{J}^+_{MR}(\theta^+)$.

4 Dealing with noisy data

In this section we discuss the case in which the plant output $y(t)$ is affected by additive noise $\xi(t)$, namely

$$\tilde{y}(t) = P(z)u(t) + \xi(t) = y(t) + \xi(t),$$

$\xi(\cdot)$ being an unknown stationary process. We make the hypothesis that the processes $u(\cdot)$ and $\xi(\cdot)$ are *uncorrelated*. Note that, in practice, this means that the data are collected when the plant is working in open-loop configuration. This assumption has been introduced for simplicity but the extension to the closed-loop setting is straightforward (see [2] for details).

If the Virtual Reference algorithm (eqns.(2)-(5)) is applied to the data-set $\{u(t), \tilde{y}(t)\}_{t=1,..,N}$, one obtains a parameter vector, say $\tilde{\theta}_N$, whose limit when $N \to \infty$ differs from $\hat{\theta}$ estimated using the noise-free data $\{u(t), y(t)\}_{t=1,..,N}$ (we say that $\tilde{\theta}_N$ is "biased").

We propose a simple procedures aiming at eliminating the influence of the noise on the estimated controller which is nothing but an instrumental variable method. This procedure guarantees the unbiasedness of the estimated parameter vector (namely the parameter vector estimated from noisy data asymptotically coincides with the parameter vector estimated from noise-free data), but requires two sets of independent measurements, characterized by the same input signal.

Procedure 1.

(i) Collect two independent sets of I/O measurements using the same input signal, namely $\{u(t), \tilde{y}^{(1)}(t)\}_{t=1,..,N}$ and $\{u(t), \tilde{y}^{(2)}(t)\}_{t=1,..,N}$, where $\tilde{y}^{(1)}(t) = y(t) + \xi^{(1)}(t)$, and $\tilde{y}^{(2)}(t) = y(t) + \xi^{(2)}(t)$. (The additive noises $\xi^{(1)}(\cdot)$ and $\xi^{(2)}(\cdot)$ in the two experiments are assumed to be *uncorrelated*).

(ii) Compute

$$\tilde{\varphi}^{(1)}_L(t) = \beta(z)\tilde{e}^{(1)}_L(t) = \beta(z)(M(z)^{-1} - 1)L(z)\tilde{y}^{(1)}(t)$$

$$\tilde{\varphi}^{(2)}_L(t) = \beta(z)\tilde{e}^{(2)}_L(t) = \beta(z)(M(z)^{-1} - 1)L(z)\tilde{y}^{(2)}(t)$$

(iii) Compute $\hat{\theta}^{P1}_N$ according to the relation:

$$\hat{\theta}^{P1}_N = \left[\sum_{t=1}^{N} \tilde{\varphi}^{(1)}_L(t)\tilde{\varphi}^{(2)T}_L(t)\right]^{-1} \sum_{t=1}^{N} \tilde{\varphi}^{(1)}_L(t)u_L(t).$$

Remark. (i) Procedure 1 can be simply extended to a closed-loop setting as shown in [2]. (ii) In the case in which only one set of data is available the second experiment can be replaced by a simulation of an accurate (high-order) model of the plant (which can be estimated from the available data). In general, in this way the unbiasedness of the estimated parameter vector is not guaranteed, however, the residual bias is expected to be very small.

5 A controller validation test

In this section we address the issue of ascertain the stability of the closed-loop control system formed by $P(z)$ with the loop closed by $C(z)$ before the controller $C(z)$ is actually implemented. This issue is of course of interest in its own right and it finds a specific application in the context of the VRFT procedure presented in this paper.

The same precise problem has been addressed in [1]. However, our method has a major departure from [1] in that it is directly based on the computation of the probability that $C(z)$ destabilizes $P(z)$ and we make no use of ellipsoidal confidence regions as in [1]. Similarly to [1], we start by identifying $P(z)$ by means of a prediction error identification method.

Let us consider a set of linear models $\{P(z;\eta)\}$ parameterized by the vector η. We assume that the class $\{P(z;\eta)\}$ is rich enough so that $P(z) = P(z;\eta_0)$ for some η_0 (this assumption is the same as in [1]). The true η_0 is not known and an estimate of it is carried out with a Prediction Error identification algorithm: the estimated parameter is named as $\hat{\eta}$. It is well known that, under general conditions, $\hat{\eta}$ is asymptotically normally distributed around η_0 (see [6] Chapter 9):

$$\hat{\eta} \sim \mathcal{N}(\eta_0, \Sigma_{\hat{\eta}}). \tag{12}$$

Therefore, the error between $P(z)$ and $P(z;\hat{\eta})$ is a variance error only, moreover the variance matrix $\Sigma_{\hat{\eta}}$ can be estimated from data.

In [1] an uncertainty set is constructed around the identified model. This uncertainty set is chosen to be the ellipsoidal region in the parameter space of a pre-specified level of confidence centered at $\hat{\eta}$. Then, the criterion for controller validation is stated as: if $C(z)$ stabilizes all the models in the uncertainty set, then it is validated.

In our method, uncertainty is directly used as it is generated by data without any artificial use of confidence regions. We consider convenient to adopt a Bayesian setting. We assume that, prior to the identification experiment, η_0 is distributed according to the distribution $p(\eta_0)$:

$$\eta_0 \sim p(\eta_0). \tag{13}$$

This prior distribution should express the information on η_0 which is available prior that $\hat{\eta}$ has been estimated. For the moment let us assume that $p(\eta_0)$ is given, later on we will discuss its actual choice. In this setting the distribution of $\hat{\eta}$ given in (12) makes sense as the conditional distribution of $\hat{\eta}$ given η_0. Therefore we can write:

$$p(\hat{\eta}|\eta_0) = \mathcal{N}(\eta_0, \Sigma_{\hat{\eta}}). \tag{14}$$

Then, from (13) and (14) we can find, by means of the Bayes theorem, the conditional distribution of η_0 given $\hat{\eta}$. Such distribution is given by:

$$p(\eta_0|\hat{\eta}) = p(\hat{\eta}|\eta_0)\frac{p(\eta_0)}{p(\hat{\eta})} \tag{15}$$

and its expression is given in the following proposition.

Proposition 2. If $p(\eta_0) = \mathcal{N}(m_{\eta_0}, \Sigma_{\eta_0})$ and $p(\hat{\eta}|\eta_0)$ is given by (14), then

$$p(\eta_o|\hat{\eta}) = \mathcal{N}(m_{\eta_0|\hat{\eta}}, \Sigma_{\eta_0|\hat{\eta}}) \tag{16}$$

where $\Sigma_{\eta_0|\hat{\eta}} = \left(\Sigma_{\hat{\eta}}^{-1} + \Sigma_{\eta_0}^{-1}\right)^{-1}$ and $m_{\eta_0|\hat{\eta}} = \Sigma_{\eta_0|\hat{\eta}}\left(\Sigma_{\hat{\eta}}^{-1}\hat{\eta} + \Sigma_{\eta_0}^{-1} m_{\eta_0}\right)$. □

In some cases, one could find difficult to assign any prior distribution for η_0. In this event, with the aim of expressing the absence of any a-priori information, it is a common procedure (see [9] pag. 20-21 and the works cited therein) to take the posterior distribution in the limit as the prior distribution spreads over the parameter space. In our case, we have:

$$\bar{p}(\eta_0|\hat{\eta}) = \lim_{\Sigma_{\eta_0}^{-1} \to 0} p(\eta_0|\hat{\eta}) = \mathcal{N}(\hat{\eta}, \Sigma_{\hat{\eta}}). \tag{17}$$

Therefore, (16) can be used in case a prior $p(\eta_0)$ is given, while (17) can be adopted in the case when no prior information is available.

The posterior probability that $C(z)$ destabilizes $P(z)$ is given by:

$$\mathbf{p_d} = \int \mathcal{I}(\eta_0) p(\eta_0|\hat{\eta}) d\eta_0 \tag{18}$$

where $\mathcal{I}(\eta_0)$ is an indicator function defined as

$$\mathcal{I}(\eta_0) = \begin{cases} 1 & \text{if } C(z) \text{ destabilizes } P(z;\eta_0) \\ 0 & \text{otherwise} \end{cases}$$

Suppose for a moment that we are able to compute $\mathbf{p_d}$. Then, we can compare its value with a previously chosen (small) maximal threshold $\bar{\mathbf{p}}_\mathbf{d}$ and validate $C(z)$ if $\mathbf{p_d} < \bar{\mathbf{p}}_\mathbf{d}$. On the other hand, the actual calculation of $\mathbf{p_d}$ may be very difficult. As a matter of fact η_0 is typically of high dimension and a standard numerical computation could be excessive.

Computing $\mathbf{p_d}$

Notice that $\mathbf{p_d} = E[\mathcal{I}(\eta_0)|\hat{\eta}]$. Let us introduce some terminology.

Given a sample $\eta_0^m = \{\eta_{01} \quad \eta_{02} \cdots \eta_{0m}\}$ of parameter vectors independently extracted according to $p(\eta_0|\hat{\eta})$, a *sample estimate* of $E[\mathcal{I}(\eta_0)|\hat{\eta}]$ is:

$$\hat{\mathbf{p}}_\mathbf{d} = \frac{1}{m}\sum_{i=1}^{m} \mathcal{I}(\eta_{0i}). \tag{19}$$

We say that: (i) $\hat{\mathbf{p}}_\mathbf{d}$ has *accuracy* ϵ if $|\hat{\mathbf{p}}_\mathbf{d} - \mathbf{p_d}| < \epsilon$. (ii) $\hat{\mathbf{p}}_\mathbf{d}$ has *accuracy* ϵ with *confidence* δ if the probability of drawing a sample η_0^m such that $|\hat{\mathbf{p}}_\mathbf{d} - \mathbf{p_d}| > \epsilon$ is smaller than δ. Notice that, since the distribution of probability $p(\eta_0|\hat{\eta})$ is a known normal distribution, we are able to extract the sample η_0^m and then employ (19) to obtain $\hat{\mathbf{p}}_\mathbf{d}$. An interesting fact is that the number m of random extractions can be easily computed for given values of ϵ and δ through the following result (see [13]).

Hoeffding's inequality. Let x be a random variable valued in $[0,1]$ with distribution p. Given a sample $x^m = \{x_1 \quad x_2 \cdots x_m\}$ of points independently extracted according to p, let

$$\hat{E}[x] = \frac{1}{m}\sum_{i=1}^{m} x_i$$

be the corresponding sample estimate of $E[x]$ based on x^m. Then, $p^m\{x^m : |\hat{E}[x] - E[x]| > \epsilon\} \leq 2e^{-2m\epsilon^2}$ □

The Hoeffding's inequality establishes a link between the size m of the sample η_0^m and the accuracy and confidence of $\hat{\mathbf{p}}_\mathbf{d}$. As a matter of fact, after some simple calculations, we can state that: if we choose m such that

$$m > \frac{1}{2\epsilon^2} \ln \frac{2}{\delta} \tag{20}$$

then the corresponding sampling estimate $\hat{\mathbf{p}}_\mathbf{d}$ is guaranteed to have accuracy ϵ and confidence δ. At this point, $\hat{\mathbf{p}}_\mathbf{d}$ can be actually used as an estimate of $\mathbf{p}_\mathbf{d}$ since its dependability is known and can be fixed a-priori.

6 A numerical example

We conclude the paper with an illustrative numerical example. The transfer function $P(z)$ is: $P(z) = z^{-2}B(z)/A(z)$ where $A(z) = 1 - 1.41833z^{-1} + 1.58939z^{-2} - 1.31608z^{-3} + .88642z^{-4}$ and $B(z) = .28261z^{-1} + .50666z^{-2}$. $P(z)$ is the discrete-time model (sampling time $T_s = 0.05s$) of a flexible transmission system which was proposed in [5] as a benchmark for digital control design. The reference model is: $M(z) = z^{-3}(1-\alpha)^2/(1-\alpha z^{-1})^2$ where $\alpha = e^{-T_s \bar{\omega}}$ and $\bar{\omega} = 6$. The magnitudes of $P(z)$ and $M(z)$ are shown in fig.(1). The class of controllers is: $C(\theta, z) = (\vartheta_0 + \vartheta_1 z^{-1} + \vartheta_2 z^{-2} + \vartheta_3 z^{-3} + \vartheta_4 z^{-4})/(1 - z^{-1})$. In order to compute $\hat{\theta}_N$ via the VRFT method, a set of data have been obtained by feeding $P(z)$ with $N = 2000$ samples of a zero-mean Gaussian white noise ($\Phi_u = 1$). In the following, we will present three different VRFT design cases for this specific control problem. Before starting, let us notice that the argument of the design criterion (9) can be written as $|PC(\theta) - PC_0|^2|T|^2$ (where $|T|^2 = |1-M|^2|L|^2|M|^{-2}\Phi_u$) from which it can be clearly seen that such argument is the weighted mismatch between the open-loop transfer function $PC(\theta)$ and the ideal open-loop transfer function PC_0. In the first two cases, we will show how the shape of $|T|^2$ affects the design. The third design case is instead characterized by the presence of noise.

Case 1: $L = 1$.
The designed control system is shown in fig.(2.a). We obtain $J_{MR}(\hat{\theta}_N^1) = 0.0472$. In fig.(2.b) the open-loop transfer function $PC(\hat{\theta}_N^1)$, the "ideal" open-loop transfer function PC_0 and the weighting factor $|T_1|^2 = |1-M|^2|M^{-1}|^2$ are shown. The step response is shown in fig.(5.a). Notice how the designed control system does not reproduce the desired reference-model with high accuracy. The reason is that we deliberately have not made use of the filter $L(z)$. This choice results in a weighting term which emphasizes the high-frequency region (see. fig.(2.b)). This shows that a wise choice of the filter $L(z)$ is crucial for a successful design.

Case 2: $L = M(1-M)$.
In this case the filter $L(z)$ has been chosen according to (11) with $W(z) = 1$. The designed control system is shown in fig.(3.a). In fig.(3.b) the open-loop transfer function $PC(\hat{\theta}_N^2)$, the "ideal" open-loop transfer function PC_0 and the weighting factor $|T_2|^2 = |1-M|^4$ are shown. The step response is shown in fig.(5.b).
The result obtained through the choice of $L(z)$ as suggested in equation (11) can be compared with the optimal solution $\bar{\theta}$. In fig.(4) the solution $\bar{\theta}$ is illustrated. From a comparison of fig.(3) and fig.(4) it is clear that the solution $\hat{\theta}_N^2$ is very close to the global solution $\bar{\theta}$. As a matter of fact the J_{MR} value for $\hat{\theta}_N^2$ and $\bar{\theta}$ is: $J_{MR}(\hat{\theta}_N^2) = 0.0280$, $J_{MR}(\bar{\theta}) = 0.0276$.

Case 3: $L = M(1-M)$, noisy data.
In this case the output signal has been corrupted by a zero mean white disturbance such that the signal to noise ratio is $SNR = \sigma^2_{y_{n.f.}}/\sigma^2_\xi = 10$ (SNR is the ratio between the variance of $y_{n.f.}(t) = P(z)u(t)$ and the variance of the noise signal). The filter $L(z)$ has been chosen as in Case 2. First the controller parameter vector $\hat{\theta}_N^3$ is estimated without paying attention to the presence of noise (i.e. the same procedure as in Case 2 has been used). The step response of the achieved control system is shown in fig.(6.a). Notice how the performance degrades dramatically with respect to the noise-free case (Case 2). The influence of noise can be counteracted by means of Procedure 1 proposed in Section 4. The parameter vector $\hat{\theta}_N^4$ estimated through Procedure 1 gives rise to the step response shown in fig.(6.b).

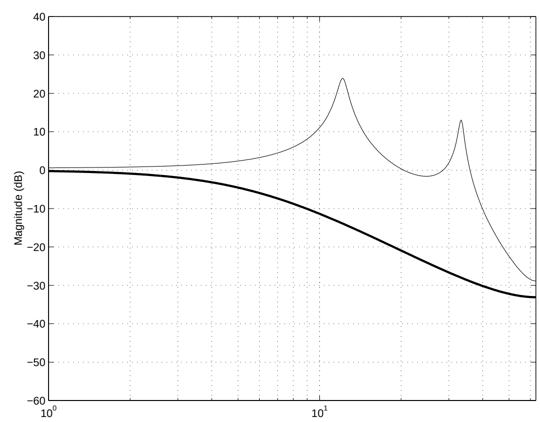


Figure 1: Magnitude plots. The plant (thin line) and the reference-model (bold line).

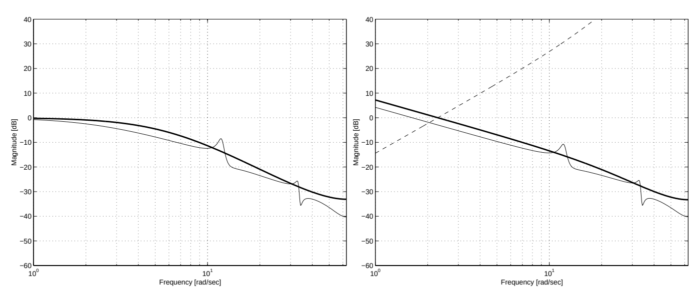
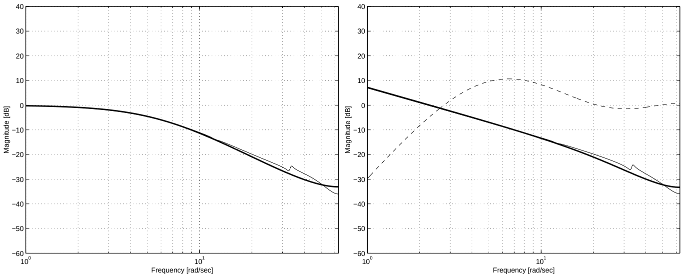
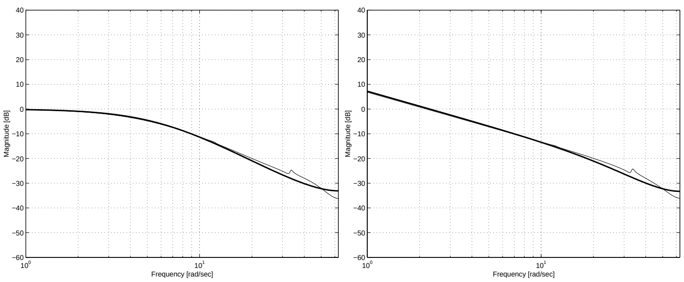


Figure 2: Magnitude plots. **(a)** the control system given by $\hat{\theta}_N^1$ (thin line) and the reference-model (bold line); **(b)** $PC(\hat{\theta}_N^1)$ (thin line), PC_0 (bold line) and $|T_1|^2$ (dashed line).



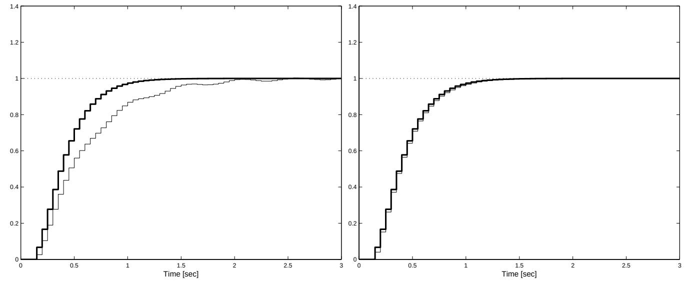
(a) (b)

Figure 3: Magnitude plots. **(a)** the control system given by $\hat{\theta}_N^2$ (thin line) and the reference-model (bold line); **(b)** $PC(\hat{\theta}_N^2)$ (thin line), PC_0 (bold line) and $|T_2|^2$ (dashed line).



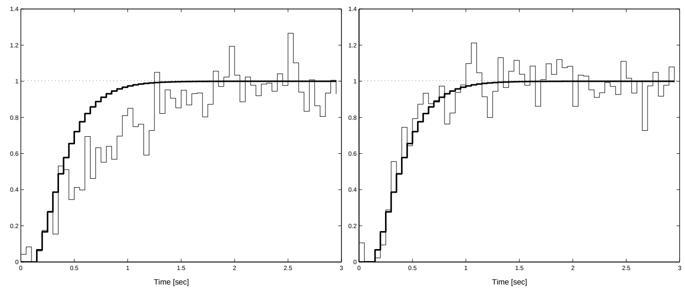
(a) (b)

Figure 4: Magnitude plots. **(a)** the control system given by $\bar{\theta}$ (thin line) and the reference-model (bold line); **(b)** $PC(\bar{\theta})$ (thin line) and PC_0 (bold line).



(a) (b)

Figure 5: Step responses. **(a)** the control system given by $\hat{\theta}_N^1$ (thin line) and the reference-model (bold line); **(b)** the control system given by $\hat{\theta}_N^2$ (thin line) and the reference-model (bold line).



(a) (b)

Figure 6: Step responses. **(a)** the control system given by $\hat{\theta}_N^3$ (thin line) and the reference-model (bold line); **(b)** the control system given by $\hat{\theta}_N^4$ (thin line) and the reference-model (bold line).

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