Open access • Journal Article • DOI:10.1115/1.3157618

Viscoelastic Analysis of Adhesively Bonded Joints — Source link

Feridun Delale, Fazil Erdogan

Institutions: Lehigh University

Published on: 01 Jun 1981 - Journal of Applied Mechanics (American Society of Mechanical Engineers)

Topics: Adhesive bonding, Shear stress, Viscoelasticity, Plane stress and Stress concentration

NASA Contractor Report 159294

NASA-CR-159294
1980 0018201

VISCOELASTIC ANALYSIS OF ADHESIVELY
BONDED JOINTS

F. Delale and F. Erdogan

LEHIGH UNIVERSITY
Bethlehem, Pennsylvania 18015

NASA Grant NGR 39-007-011
March 1980

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

VISCOELASTIC ANALYSIS OF ADHESIVELY BONDED JOINTS*

by

F. Delale and F. Erdogan
Lehigh University, Bethlehem, PA 18015

Abstract

In this paper an adhesively bonded lap joint is analyzed by assuming that the adherends are elastic and the adhesive is linearly viscoelastic. After formulating the general problem a specific example for two identical adherends bonded through a three parameter viscoelastic solid adhesive is considered. The standard Laplace transform technique is used to solve the problem. The stress distribution in the adhesive layer is calculated for three different external loads namely, membrane loading, bending, and transverse shear loading. The results indicate that the peak value of the normal stress in the adhesive is not only consistently higher than the corresponding shear stress but also decays slower.

(*) This work was supported by NASA-Langley under the Grant NGR 39-007-011 and by NSF under the Grant ENG 78-09737.

1. Introduction

In its simplest form an adhesively bonded structure consists of three components of different mechanical properties, namely the adhesive and the two adherends. Because of the nonhomogeneous nature and of the geometrical complexity of the medium, even for the linearly elastic materials the exact analytical treatment of the problem regarding the stress analysis of the structure is, in general, hopelessly complicated. The existing analytical studies are, therefore, based on certain simplifying assumptions with regard to the modeling of the adhesive and the adherends. The adherends are usually modeled as an isotropic or orthotropic membrane (e.g., [1]), plate (e.g., [2,3]), or elastic continuum (e.g., [4,5]). The primary physical consideration used in the selection of a particular model is generally the ratio of the thickness of the adherend to the lateral dimensions of bond region. For example, for adherends with a very small relative thickness the bending stiffness may be neglected whereas if the thickness of the adherend is not small even the plate assumption may be erroneous. As for the adhesives, generally the thickness variation of the stresses is neglected and the adhesive layer is modeled as a linear shear or a tension-shear spring.

In most applications of structural adhesives the operating temperature is such that the adhesive remains in its initial glassy stage through the entire loading period and hence it is not necessary to consider the time-dependent behavior of the stress-strain relations in performing the stress analysis of the joint. However, in certain applications, the temperature and the load duration may be such that the rheological behavior of the adhesive may no longer be negligible. In this paper the adhesively bonded joint problem is considered by assuming that the adhesive is a linear viscoelastic material.

2. Formulation of the Problem

In formulating the adhesively bonded joint problems unless the thickness of the adherends is at least two orders of magnitude smaller than the length characterizing the bond region the generalized plane stress or the membrane assumption does not seem to be very realistic.

On the other hand in an adhesive joint between relatively thin adherends, even if it were possible to formulate the problem by assuming the adherends as elastic continua, the numerical analysis involve such severe convergence problems that the accuracy of the results may be highly questionable [4]. In such problems the plate assumption in modeling the adherends appears to be a fairly good compromise. Thus, in this paper the problem will be formulated under the following primary assumptions: (a) the adherends are treated as linear elastic plates and the transverse shear effects are taken into account; and (b) the adhesive is considered as a viscoelastic solid in which the in-plate strain as well as out-of-plane strain and shear strain are assumed to be nonzero. The secondary assumptions under which the specific problem is formulated and solved simplify the analysis quite considerably but do not affect the character of the solution. These assumptions are: (a) the problem is one of plane strain, that is, the bonded joint is very "wide" and undergoes cylindrical bending; (b) the adherends have the same thickness and are made of the same material; and (c) the structure is a single lap joint. The elastic version of the problem neglecting the transverse shear effects in the adherends was considered in [2]. The solution of, again, the elastic problem for different adherends with a somewhat simpler adhesive model may be found in [3]*.

The geometry of the problem under consideration is shown in Figure 1(a). From the equilibrium of the plate elements for the adherends 1 and 2 the following differential equations may be obtained:

$$\frac{\partial N_{1x}}{\partial x} = \tau, \quad \frac{\partial Q_{1x}}{\partial x} = \sigma - \frac{h_o}{2}\frac{\partial \tau}{\partial x}, \quad \frac{\partial M_{1x}}{\partial x} = Q_{1x} - \frac{h_1}{2}\tau, \qquad (1\ a\text{-}c)$$

$$\frac{\partial N_{2x}}{\partial x} = -\tau, \quad \frac{\partial Q_{2x}}{\partial x} = -\sigma - \frac{h_o}{2}\frac{\partial \tau}{\partial x}, \quad \frac{\partial M_{2x}}{\partial x} = Q_{2x} - \frac{h_2}{2}\tau, \qquad (2\ a\text{-}c)$$

where N_{ix}, Q_{ix}, M_{ix} are respectively the membrane, transverse shear, and moment resultants, the index $i = 1,2$ referring to the adherends 1

(*) Needless to say, the problem has been very widely studied. Some references to further analytical work and to finite element type solutions may be found in [3].

and 2, h_1, h_2, and h_o are the thicknesses of the adherends and the adhesive as shown, and $\sigma(x,t)$ and $\tau(x,t)$ are the interface normal and shear stresses. In modeling the adhesive it is assumed that the stress components $\sigma_y(x,y,t) = \sigma(x,t)$ and $\tau_{xy}(x,y,t) = \tau(x,t)$ in the adhesive layer are independent of the y coordinate.

Assuming cylindrical bending, $\varepsilon_{1z} = 0$, $\varepsilon_{2z} = 0$. The stress resultant-displacement relations may then be expressed as

$$\frac{\partial u_1}{\partial x} = C_1 N_{1x} \ , \quad \frac{\partial \beta_{1x}}{\partial x} = D_1 M_{1x} \ , \quad \frac{\partial v_1}{\partial x} + \beta_{1x} = \frac{Q_{1x}}{B_1} \ , \qquad (3\ \text{a-c})$$

$$\frac{\partial u_2}{\partial x} = C_2 N_{2x} \ , \quad \frac{\partial \beta_{2x}}{\partial x} = D_2 M_{2x} \ , \quad \frac{\partial v_2}{\partial x} + \beta_{2x} = \frac{Q_{2x}}{B_2} \ , \qquad (4\ \text{a-c})$$

where

$$C_i = \frac{1-\nu_i^2}{E_i h_i} \ , \quad D_i = \frac{12(1-\nu_i^2)}{E_i h_i^3} \ , \quad B_i = \frac{5}{6}\mu_i h_i \ , \quad (i=1,2) \qquad (5\ \text{a-c})$$

E_i, μ_i, ν_i, $(i=1,2)$ are the elastic constants, $u_i(x,t)$ and $v_i(x,t)$, $(i=1,2)$ are x and y-components of the displacement vector and β_{ix}, $(i=1,2)$ is the rotation of the normal to the midplane of the adherends.

It may be seen that as stated the problem has 14 unknown functions, namely, σ, τ, u_i, v_i, β_{ix}, N_{ix}, Q_{ix}, M_{ix}, $(i=1,2)$. Equations (1-4) provide 12 relations. The remaining two relations necessary to complete the formulation of the problem are obtained from the continuity conditions for the displacements in the bond region. To do this the mechanics of the adhesive layer, specifically its constitutive relations need to be considered.

Referring to Figure 1b the average strains in the adhesive may be expressed as

$$\gamma_{xy} = \left(u_1 - \frac{h_1}{2}\beta_{1x} - u_2 - \frac{h_2}{2}\beta_{2x}\right)/h_o \ ,$$

$$\varepsilon_y = (v_1 - v_2)/h_o \ ,$$

$$\varepsilon_x = \left(\frac{\partial u_1}{\partial x} - \frac{h_1}{2}\frac{\partial \beta_{1x}}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{h_2}{2}\frac{\partial \beta_{2x}}{\partial x}\right)/2 \qquad (6\ \text{a-c})$$

Noting that all the remaining strain components in the adhesive are zero and defining

$$e = (\varepsilon_x + \varepsilon_y)/3, \tag{7}$$

the strain tensor for the adhesive may be decomposed as follows:

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & 0 \\ \gamma_{xy}/2 & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} + \begin{bmatrix} \varepsilon_x - e & \gamma_{xy}/2 & 0 \\ \gamma_{xy}/2 & \varepsilon_y - e & 0 \\ 0 & 0 & -e \end{bmatrix} \tag{8}$$

Similarly, noting that $\sigma_y = \sigma$, $\tau_{xy} = \tau$, the stress tensor for the adhesive may be decomposed as

$$\begin{bmatrix} \sigma_x & \tau & 0 \\ \tau & \sigma & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} + \begin{bmatrix} \sigma_x - s & \tau & 0 \\ \tau & \sigma - s & 0 \\ 0 & 0 & \sigma_z - s \end{bmatrix} \tag{9}$$

where, the hydrostatic component of the stress tensor s is defined by:

$$s = (\sigma_x + \sigma + \sigma_z)/3 \tag{10}$$

The constitutive equations of linear isotropic viscoelastic materials may be expressed in terms of either hereditary integrals by using creep compliance or relaxation functions, or differential operators* [6-8]. In this paper the latter approach is adopted and it is assumed that

$$P_1(s_{ij}) = Q_1(e_{ij}), \quad (i,j) = 1,2,3, \tag{11}$$

$$P_2(s) = Q_2(e) \tag{12}$$

(*) The two formulations are, of course, related through Laplace transforms. For example, the creep compliance J(t) is the inverse Laplace transform of $\bar{P}(s)/s\bar{Q}(s)$ where P and Q are the related differential operators operating on σ and ε, respectively and s is the transform variable.

where s_{ij} and e_{ij} $(i,j) = 1,2,3$, are the deviatoric components of stress and strain tensors, respectively, as given by (8) and (9), s and e are defined by (10) and (7), and P_1, Q_1, P_2, and Q_2 are differential operators of the form $\sum_0^n a_k(t)\, \partial^k/\partial t^k$, the coefficients a_k being generally functions of temperature. More explicitly, from (7)-(12) it may be seen that

$$P_1(2\sigma_x - \sigma - \sigma_z) = Q_1(2\varepsilon_x - \varepsilon_y) \ , \tag{13}$$

$$P_1(2\sigma - \sigma_x - \sigma_z) = Q_1(2\varepsilon_y - \varepsilon_x) \ , \tag{14}$$

$$P_1(2\sigma_z - \sigma_x - \sigma) = - Q_1(\varepsilon_x + \varepsilon_y) \ , \tag{15}$$

$$P_1(\tau) = \frac{1}{2} Q_1(\gamma_{xy}) \ , \tag{16}$$

$$P_2(\sigma_x + \sigma + \sigma_z) = Q_2(\varepsilon_x + \varepsilon_y) \ . \tag{17}$$

Since $\sum_1^3 s_{ii} = 0$ and $\sum_1^3 e_{ii} = 0$, equations (13-15) are not linearly independent. Equation (14) may be obtained by adding (13) and (15) and will, therefore, be ignored in the remainder of the analysis.

Practical experience indicates that under a hydrostatic stress state most viscoelastic materials behave elastically. Hence, it may be assumed that

$$P_2 \equiv 1, \ Q_2 \equiv 3K \ , \tag{18}$$

or

$$\sigma_x + \sigma + \sigma_z = 3K(\varepsilon_x + \varepsilon_y) \tag{19}$$

where K is the bulk modulus of the adhesive. Eliminating σ_x and σ_z from (13), (15) and (19) and using (6 a-c), the constitutive equations may now be written as

$$3P_1\{K[\frac{1}{2}\frac{\partial u_1}{\partial x} - \frac{h_1}{4}\frac{\partial \beta_{1x}}{\partial x} + \frac{1}{2}\frac{\partial u_2}{\partial x} + \frac{h_2}{4}\frac{\partial \beta_{2x}}{\partial x} + \frac{v_1 - v_2}{h_o}] - \sigma\}$$

$$= Q_1\{\frac{1}{2}\frac{\partial u_1}{\partial x} - \frac{h_1}{4}\frac{\partial \beta_{1x}}{\partial x} + \frac{1}{2}\frac{\partial u_2}{\partial x} + \frac{h_2}{4}\frac{\partial \beta_{2x}}{\partial x} - \frac{2}{h_o}(v_1 - v_2)\} \tag{20}$$

$$P_1(\tau) = \frac{1}{2} Q_1\{(u_1 - \frac{h_1}{2}\beta_{1x} - u_2 - \frac{h_2}{2}\beta_{2x})/h_o\} \tag{21}$$

Equations (20) and (21) with (1-4) provide the system of 14 relations necessary to solve for the unknown functions σ, τ, u_i, v_i, β_{ix}, N_{ix}, Q_{ix} and M_{ix}, $(i=1,2)$.

3. Example

As an example we consider a single lap joint which consists of two identical adherends bonded through an adhesive layer which may be represented by a three-parameter viscoelastic solid (Figure 1c). For the adherends we have

$$h_1 = h_2 = h, \; C_1 = C_2 = C = \frac{1-\nu^2}{Eh},$$

$$D_1 = D_2 = D = \frac{12(1-\nu^2)}{Eh^3}, \; B_1 = B_2 = B = \frac{5}{6}\mu h. \tag{22}$$

For the adhesive, referring to Figure 1c it may be shown that

$$P_1 = 1 + a_1 \frac{\partial}{\partial t}, \; Q_1 = b_o + b_1 \frac{\partial}{\partial t} \tag{23 a,b}$$

where

$$a_1 = \frac{\lambda_2}{k_1+k_2}, \; b_o = \frac{k_1 k_2}{k_1+k_2}, \; b_1 = \frac{\lambda_2 k_1}{k_1+k_2} . \tag{24}$$

For a nondecreasing strain under sustained load the following inequality must be satisfied:

$$b_1 > a_1 b_o . \tag{25}$$

Generally, the coefficients a_1, b_o and b_1 are functions of temperature, hence implicitly functions of time, if the temperature does not remain reasonably constant during the period of loading. In the example considered, it is assumed that these coefficients are constant.

Through a relatively straightforward elimination, the governing equations (1-4), (20) and (21) can be reduced to a pair of differential equations in the unknown functions $\sigma(x,t)$ and $\tau(x,t)$. By carrying out this elimination, using (22) and the operators defined by (23) we obtain

$$\frac{\partial^2\tau}{\partial x^2} + a_1 \frac{\partial^3\tau}{\partial x^2\partial t} - \frac{b_o}{2h_o}\left[2C + \frac{hD}{2}(h + h_o)\right]\tau$$

$$- \frac{b_1}{2h_o}\left[2C + \frac{hD}{2}(h+h_o)\right]\frac{\partial\tau}{\partial t} = - \frac{b_o hD}{4h_o} Q_o(t) - \frac{b_1 hD}{4h_o}\frac{dQ_o}{dt} , \tag{26}$$

$$3\frac{\partial^4\sigma}{\partial x^4} + 3a_1 \frac{\partial^5\sigma}{\partial x^4\partial t} + \left[\frac{h}{2}D(3K-b_o) - \frac{2}{h_o B}(3K+2b_o)\right]\frac{\partial^2\sigma}{\partial x^2}$$

$$+ \left[\frac{h}{2}D(3Ka_1-b_1) - \frac{2}{h_o B}(3Ka_1 + 2b_1)\right]\frac{\partial^3\sigma}{\partial x^2\partial t}$$

$$+ \frac{2}{h_o} D(3K+2b_o)\sigma + \frac{2}{h_o} D(3Ka_1+2b_1)\frac{\partial\sigma}{\partial t} = 0. \tag{27}$$

Assuming that no external transverse shear load is applied to the composite plate in $-\ell < x < \ell$ and noting that $\tau(x,t)$ is the average shear stress acting on the adhesive, referring to Figure 1a the equilibrium of transverse shear resultants gives

$$Q_{1x}(x,t) + Q_{2x}(x,t) + h_o\tau = Q_o(t) , \quad (-\ell<x<\ell). \tag{28}$$

Equation (28) has been used in deriving (26).

The differential equations (26) and (27) are uncoupled and may easily be solved by first reducing them to ordinary differential equations through the use of Laplace transforms defined by

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\, dt , \tag{29}$$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\, ds \tag{30}$$

where $F(s)$ is called the Laplace transform of $f(t)$ and the constant c is selected in such a way that all the singularities of $F(s)$ lie to the left of the line of integration $Re(s) = c$. Assuming that the bonded joint is initially stress-free, the functions $\sigma(x,t)$ and $\tau(x,t)$ are zero for $t<0$ and from (26) and (27), we find

$$\frac{d^2F}{dx^2} - \alpha^2 F = \beta \tag{31}$$

$$\frac{d^4G}{dx^4} - 2\,\gamma^2 \frac{d^2G}{dx^2} + \omega^4 G = 0 \tag{32}$$

where $F(x,s)$ and $G(x,s)$ are the Laplace transforms of $\tau(x,t)$ and $\sigma(x,t)$, respectively and

$$\alpha^2 = \frac{[4C + hD(h+h_o)](b_o + b_1 s)}{4h_o(1 + a_1 s)} \quad , \tag{33}$$

$$\beta = - \frac{hDQ_o(b_o + b_1 s)}{4h_o s(1+a_1 s)} \quad , \tag{34}$$

$$\gamma^2 = \frac{1}{6(1+a_1 s)} \{\frac{2}{h_o B} (3K + 2b_o) - \frac{h}{2} D\ (3K - b_o)$$

$$+ s[\frac{2}{h_o B} (3Ka_1 + 2b_1) - \frac{h}{2} D(3Ka_1 - b_1)]\} \ , \tag{35}$$

$$\omega^4 = \frac{2D}{3h_o(1+a_1 s)} [3K + 2b_o + s(3Ka_1 + 2b_1)] \ . \tag{36}$$

In the example it is assumed that the external loads are given by (see Figure 1a)

$$N_o(t) = N_o H(t),\ M_1(t) = M_1 H(t),\ M_2(t) = M_2 H(t),\ Q_o(t) = Q_o H(t)$$

(37 a-d)

where $H(t)$ is the Heaviside function. For example, the nonhomogeneous term β which appears in (31) and which is given by (34) is obtained by using (37d).

The general solution of (31) and (32) may be written as

$$F(x,s) = A_1 \sinh(\alpha x) + A_2 \cosh(\alpha x) - \frac{\beta}{\alpha^2} \quad , \tag{38}$$

$$G(x,s) = A_3 \sinh(\phi_1 x) + A_4 \cosh(\phi_1 x) + A_5 \sinh(\phi_2 x)$$

$$+ A_6 \cosh(\phi_2 x) \tag{39}$$

where

$$\phi_1 = [\gamma^2 + (\gamma^4 - \omega^4)^{\frac{1}{2}}]^{\frac{1}{2}} \ , \ \phi_2 = [\gamma^2 - (\gamma^4 - \omega^4)^{\frac{1}{2}}]^{\frac{1}{2}} \ , \tag{40}$$

and the unknown functions $A_1(s), \ldots, A_6(s)$ are determined from the boundary conditions.

The problem is solved under three separate loading conditions shown in Figure 2.

(a) Membrane loading (Figure 2a).

For this case the boundary conditions for plates 1 and 2 are given as follows:

$$N_{1x}(\ell,t) = 0, \ M_{1x}(\ell,t) = 0, \ Q_{1x}(\ell,t) = 0, \tag{41 a-c}$$

$$N_{1x}(-\ell,t) = N_o H(t), \ M_{1x}(-\ell,t) = -N_o \frac{h+h_o}{2} H(t), \ Q_{1x}(-\ell,t) = 0, \tag{42 a-c}$$

$$N_{2x}(\ell,t) = N_o H(t), \ M_{2x}(\ell,t) = N_o \frac{h+h_o}{2} H(t), \ Q_{2x}(\ell,t) = 0, \tag{43 a-c}$$

$$N_{2x}(-\ell,t) = 0, \ M_{2x}(-\ell,t) = 0, \ Q_{2x}(-\ell,t) = 0 \ , \tag{44 a-c}$$

Considering the symmetry of the problem in geometry and materials, after some lengthy manipulations it can be shown that (41-44) are equivalent to the following conditions:

$$\tau(x,t) = \tau(-x,t), \ \int_{-\ell}^{\ell} \tau(x,t)\, dx = -N_o H(t) \ , \tag{45 a,b}$$

$$\sigma(x,t) = \sigma(-x,t), \ \int_{-\ell}^{\ell} \sigma(x,t)\, dx = 0 \ , \tag{46 a,b}$$

$$\frac{h}{2}D(3K-b_o)\sigma(\ell,t) + \frac{h}{2}D(3Ka_1-b_1)\frac{\partial}{\partial t}\sigma(\ell,t) - \frac{1}{h_o}(3K+2b_o)[\frac{2}{B}\sigma(\ell,t) + \frac{D}{2}(h+h_o)N_oH(t)] - \frac{1}{h_o}(3Ka_1+2b_1)[\frac{2}{B}\frac{\partial}{\partial t}\sigma(\ell,t) + \frac{D}{2}(h+h_o)N_o\delta(t)] + 3\frac{\partial^2}{\partial x^2}\sigma(\ell,t) + 3a_1\frac{\partial^3}{\partial x^2 \partial t}\sigma(\ell,t) = 0 \ . \tag{47}$$

In this problem since $Q_o = 0$, $\beta = 0$ and substituting from (38) and (39) into (45-47) we obtain

$$A_1(s) = 0 \quad , \quad A_2(s) = -\frac{\alpha N_o}{2s \sinh(\alpha\ell)} \quad ,$$

$$A_3(s) = 0 \quad , \quad A_4(s) = -\frac{(h+h_o)N_o\omega^4 \sinh(\phi_2\ell)}{4s\phi_2\Delta_a(s)} \quad ,$$

$$A_5(s) = 0 \quad , \quad A_6(s) = \frac{(h+h_o)N_o\omega^4 \sinh(\phi_1\ell)}{4s\phi_1\Delta_a(s)} \quad , \tag{48}$$

where

$$\Delta_a(s) = \phi_2\cosh(\phi_1\ell)\sinh(\phi_2\ell) - \phi_1\sinh(\phi_1\ell)\cosh(\phi_2\ell). \tag{49}$$

(b) Bending (Figure 2b)

For this problem the boundary conditions are

$$N_{1x}(\ell,t) = 0,\ M_{1x}(\ell,t) = 0,\ Q_{1x}(\ell,t) = 0 \ , \tag{50 a-c}$$

$$N_{1x}(-\ell,t) = 0,\ M_{1x}(-\ell,t) = M_oH(t),\ Q_{1x}(\ell,t) = 0 \ , \tag{51 a-c}$$

$$N_{2x}(\ell,t) = 0,\ M_{2x}(\ell,t) = M_oH(t),\ Q_{2x}(\ell,t) = 0 \ , \tag{52 a-c}$$

$$N_{2x}(-\ell,t) = 0,\ M_{2x}(-\ell,t) = 0,\ Q_{2x}(-\ell,t) = 0 \ , \tag{53 a-c}$$

Again, considering the symmetry of the problem conditions (50-53) may be shown to be equivalent to the following:

$$\tau(x,t) = -\tau(-x,t) \ , \tag{54}$$

$$\frac{\partial}{\partial x}\tau(\ell,t) + a_1 \frac{\partial^2}{\partial x \partial t}\tau(\ell,t) = -\frac{hD}{4h_o}M_o[b_oH(t) + b_1\delta(t)] \ , \tag{55}$$

$$\sigma(x,t) = -\sigma(-x,t) \ , \tag{56}$$

$$\int_{-\ell}^{\ell} \sigma(x,t)x \, dx = M_oH(t) \ , \tag{57}$$

$$(3K-b_o)\frac{hD}{2}\sigma(\ell,t) + (3Ka_1-b_1)\frac{hD}{2}\frac{\partial}{\partial t}\sigma(\ell,t)$$

$$-\frac{1}{h_o}(3K+2b_o)\left[\frac{2}{B}\sigma(\ell,t) + D\,M_oH(t)\right]$$

$$-\frac{1}{h_o}(3Ka_1+2b_1)\left[\frac{2}{B}\frac{\partial}{\partial t}\sigma(\ell,t) + D\,M_o\,\delta(t)\right]$$

$$+3\frac{\partial^2}{\partial x^2}\sigma(\ell,t) + 3a_1\frac{\partial^3}{\partial x^2\partial t}\sigma(\ell,t) = 0. \tag{59}$$

In this problem, too, $\beta = 0$, and substituting from (38) and (39) into (54-59) we obtain

$$A_1(s) = -\frac{hDM_o(b_o+b_1s)}{4h_o\alpha s(1+a_1s)\cosh(\alpha\ell)}, \quad A_2(s) = 0,$$

$$A_3(s) = -\frac{\omega^4 M_o\cosh(\phi_2\ell)}{2\,s\,\phi_2\,\Delta_b(s)}, \quad A_4(s) = 0,$$

$$A_5(s) = \frac{\omega^4 M_o\cosh(\phi_1\ell)}{2\,s\,\phi_1\Delta_b(s)}, \quad A_6(s) = 0$$

$$\Delta_b(s) = \phi_2\sinh(\phi_1\ell)\cosh(\phi_2\ell) - \phi_1\cosh(\phi_1\ell)\sinh(\phi_2\ell). \tag{60}$$

(c) Transverse Shear (Figure 2c)

For the loading given in Figure 2c the boundary conditions may be expressed as follows:

$$N_{1x}(\ell,t) = 0,\ M_{1x}(\ell,t) = 0,\ Q_{1x}(\ell,t) = 0, \tag{61 a-c}$$

$$N_{1x}(-\ell,t) = 0,\ M_{1x}(-\ell,t) = -Q_o\ell H(t),\ Q_{1x}(-\ell,t) = Q_oH(t), \tag{62 a-c}$$

$$N_{2x}(\ell,t) = 0,\ M_{2x}(\ell,t) = Q_o\ell H(t),\ Q_{2x}(\ell,t) = Q_oH(t), \tag{63 a-c}$$

$$N_{2x}(-\ell,t) = 0,\ M_{2x}(-\ell,t) = 0,\ Q_{2x}(-\ell,t) = 0. \tag{64 a-c}$$

These conditions are equivalent to

$$\tau(x,t) = \tau(-x,t),\ \int_{-\ell}^{\ell}\tau(x,t)\,dx = 0, \tag{65 a,b}$$

$$\sigma(x,t) = \sigma(-x,t), \int_{-\ell}^{\ell} \sigma(x,t)dx = - Q_o H(t), \qquad (66\ a,b)$$

$$\frac{hD}{2}(3K-b_o)\sigma(\ell,t) + \frac{hD}{2}(3Ka_1-b_1)\frac{\partial}{\partial t}\sigma(\ell,t) - \frac{1}{h_o}(3K+2b_o)[\frac{2}{B}\sigma(\ell,t)$$
$$+ DQ_o\ell H(t)] - \frac{1}{h_o}(3Ka_1+2b_1)[\frac{2}{B}\frac{\partial}{\partial t}\sigma(\ell,t) + DQ_o\ell\delta(t)]$$
$$+ 3\frac{\partial^2}{\partial x^2}\sigma(\ell,t) + 3\, a_1 \frac{\partial^3}{\partial x^2 \partial t}\sigma(\ell,t) = 0. \qquad (67)$$

In this case β is given by (34) and the functions $A_1(s),\ldots,A_6(s)$ are obtained as follows:

$$A_1(s) = 0,\ A_2(s) = \frac{\beta\ell}{\alpha \sinh(\alpha\ell)},$$

$$A_3(s) = 0,\ A_4(s) = \frac{Q_o[\omega^2\phi_1\cosh(\phi_2\ell) - \omega^4\ell\sinh(\phi_2\ell)]}{2\, s\, \phi_2\, \Delta_c(s)},$$

$$A_5(s) = 0,\ A_6(s) = -\frac{Q_o[\omega^2\phi_2\cosh(\phi_1\ell) - \omega^4\ell\sinh(\phi_1\ell)]}{2\, s\, \phi_1\, \Delta_c(s)},$$

$$\Delta_c(s) = \phi_2\cosh(\phi_1\ell)\sinh(\phi_2\ell) - \phi_1\sinh(\phi_1\ell)\cosh(\phi_2\ell). \qquad (68)$$

4. Solution and Results

After determining the functions $A_i(s)$, $(i=1,\ldots,6)$ the unknown functions $\tau(x,t)$ and $\sigma(x,t)$ may be obtained by substituting from (38) and (39) into the inversion integral (30). In each case the constant c giving the line of integration is determined by analyzing the singular behavior of the functions F(x,s) and G(x,s) in the complex s plane. Because of the existence of a number of branch points in the complex plane the exact inversion of F and G becomes very complicated and, in light of the fact that the inversion integrals can be evaluated in a straightforward manner numerically, does not seem to be worth the effort. Thus, making the following change in variable

$$s = c + iy, \quad -\infty < y < \infty \qquad (69)$$

the functions τ and σ may be expressed as

$$\tau(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x,c+iy)e^{t(c+iy)}dy \ , \tag{70}$$

$$\sigma(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x,c+iy)e^{t(c+iy)}dy \ , \tag{71}$$

It can be shown that the imaginary parts of the integrands in (70) and (71) are odd functions in y and therefore the integrals give real results.

Examining the functions F and G in the complex plane it is found that s=0 is a simple pole and all the remaining singularities lie in the left hand plane. Hence c is a positive constant. To evaluate the integrals in (70) and (71) first they are expressed in $(0,\infty)$ as follows:

$$\int_{-\infty}^{\infty} f(y)dy = \int_{0}^{\infty}[f(y) + f(-y)]dy. \tag{72}$$

Even though there are routine techniques for evaluating infinite integrals, it is generally a good practice to obtain the asymptotic behavior of the integrands for large values of the argument before selecting a particular technique. In the problem under consideration the integrands do not decay exponentially. Consequently, the numerical integration requires more care. One way to insure that no significant accuracy is lost due to the slow decay of integrands is to evaluate the integral in closed form for large values of the argument. For example, in the lap joint under membrane loading N_o (Figure 2a), after analyzing the asymptotic behavior of the function F, the shear stress τ may be expressed as

$$\tau(x,t) = \frac{1}{2\pi}\int_{0}^{A}[F(x,c+iy)e^{t(c+iy)} + F(x,c-iy)e^{t(c-iy)}]dy$$
$$- \frac{N_o}{2\pi}\sqrt{m/a_1}\ e^{ct}\ \frac{\cosh(x\sqrt{m/a_1})}{\sinh(\ell\sqrt{m/a_1})}\int_{A}^{\infty}\frac{\sin(ty)}{y}dy \tag{73}$$

where

$$m = b_1[4C + h(h+h_o)D]/(4h_o) \tag{74}$$

and A is a "large" number. The second integral is known in closed form and the first is evaluated numerically. The proper selection of A requires some trial calculations. In this problem A selected in 20 to 30 range gives good results. It may also be pointed out that the numerical calculations show the results to be insensitive to the choice of the constant c.

In the numerical example considered it is assumed that the adherends are aluminum alloy plates with the following elastic constants and dimensions (Figure 2)

$$E = 10^7 \text{ psi} = 6.895 \times 10^{10} \text{ N/m}^2 \;, \; \nu = 0.3$$

$$h = 0.09 \text{ in} = 0.229 \times 10^{-2}\text{m} \;, \; \ell = 0.5 \text{ in} = 1.27 \times 10^{-2}\text{m}.$$

In the three parameter viscoelastic solid adopted for the adhesive the coefficients which appear in the operators P_1 and Q_1 (see eqs. 23a,b) are related to the constants shown in Figure 1c by (24). To relate these constants to somewhat more conventional material properties consider the response of the model given in Figure 1c to an input $\tau = \tau_o H(t)$ which is found to be

$$\frac{1}{2}\gamma(t) = \frac{\tau_o}{b_1}\left[t_o(1-e^{-t/t_o}) + a_1 e^{-t/t_o}\right], \; t_o = \frac{b_1}{b_o} = \frac{\lambda_2}{k_2} \;, \tag{75}$$

where t_o is called the retardation time. Now defining

$$\mu_o = \frac{\tau_o}{\gamma(0^+)} \;, \; \mu_\infty = \frac{\tau_o}{\gamma(\infty)} \tag{76}$$

from (75) it is seen that

$$2\mu_o = \frac{b_1}{a_1} = k_1 \;, 2\mu_\infty = b_o = \frac{k_1 k_2}{k_1 + k_2} \tag{77}$$

Thus, the moduli μ_o and μ_∞ and the retardation time t_o may be selected as the three parameters representing the viscoelastic solid.

For the particular epoxy used as the adhesive the properties at t=0 are assumed to be

$$h_o = 0.004 \text{ in.} = 1.016 \times 10^{-4}$$

$$E_o = 5.797 \times 10^5 \text{ psi} = 39.968 \times 10^8 \text{ N/m}^2$$

$$\mu_o = 2.225 \times 10^5 \text{ psi} = 15.341 \times 10^8 \text{ N/m}^2$$

The bulk modulus K is assumed to be constant and may, therefore, be calculated in terms of E_o and the shear modulus μ_o as

$$K = \frac{E_o \mu_o}{3(3\mu_o - E_o)} \tag{78}$$

In the example it is also assumed that

$$\mu_\infty = \mu_o/3 \ , \ t_o = 4 \text{ hrs.}$$

If it is assumed that the adhesive layer is linearly elastic having the constants E_a and ν_a, with the adhesive model used in this paper the solution may be obtained in a straightforward manner. For example, in the case of membrane loading described by (41-44) the adhesive stresses are found to be

$$\tau_e(x) = -\frac{N_o \alpha_e \cosh(\alpha_e x)}{2 \sinh(\alpha_e \ell)} \ , \tag{79}$$

$$\alpha_e^2 = \frac{E_a}{4(1+\nu_a)h_o} \left[4C + hD(h + h_o)\right] \ , \tag{80}$$

$$\sigma_e(x) = B_4 \cosh(m_1 x) + B_6 \cosh(m_2 x) \ , \tag{81}$$

$$B_4 = -\varepsilon_1^4 N_o (h+h_o) \sinh(m_2 \ell)/(4m_2 \Delta) \ ,$$

$$B_6 = \varepsilon_1^4 N_o (h+h_o) \sinh(m_1 \ell)/(4m_1 \Delta) \ ,$$

$$\Delta = m_2 \cosh(m_1 \ell) \sinh(m_2 \ell) - m_1 \sinh(m_1 \ell) \cosh(m_2 \ell) \ ,$$

$$\varepsilon_1^4 = \frac{2DE_a(1-\nu_a)}{h_o(1-\nu_a-2\nu_a^2)} \ ,$$

$$m_1 = [\gamma_1^2 + (\gamma_1^4 - \varepsilon_1^4)^{\frac{1}{2}}]^{\frac{1}{2}}; \ m_2 = [\gamma_1^2 - (\gamma_1^4 - \varepsilon_1^4)^{\frac{1}{2}}]^{\frac{1}{2}} \ ,$$

$$\gamma_1^2 = \frac{(1-\nu_a)E_a}{(1-\nu_a-2\nu_a^2)} \left[\frac{1}{Bh_o} - \frac{hD\nu_a}{4(1-\nu_a)} \right] . \tag{82}$$

On the other hand, in the case of viscoelastic adhesive the elastic response for t=0+ and t=∞ may also be determined by using the limit theorems for the inversion of Laplace transforms. For example, again for the case of membrane loading, from (38) and (48) the shear stress in the adhesive may be obtained as

$$\tau(x,+0) = -\frac{N_o \alpha_o \cosh(\alpha_o x)}{2\sinh(\alpha_o \ell)} , \tag{83}$$

$$\alpha_o^2 = \frac{b_1}{4h_o a_1}[4C + hD(h+h_o)] , \tag{84}$$

and

$$\tau(x,\infty) = -\frac{N_o \alpha_\infty \cosh(\alpha_\infty x)}{2\sinh(\alpha_\infty \ell)} , \tag{85}$$

$$\alpha_\infty^2 = \frac{b_o}{4h_o}[4C + hD(h+h_o)] . \tag{86}$$

Note that at t=+0 $\mu_a = E_a/2(1+\nu_a) = \mu_o$ and $E_a = E_o$, and from (77), (80), and (84) it follows that $\alpha_o = \alpha_e$. Hence, the initial response given by (83) is the expected elastic solution given by (79). Similarly, at t=∞ $\mu_a = \mu_\infty$, and (77), (80) and (86) shows that $\alpha_\infty = \alpha_e$, and hence $\tau(x,\infty) = \tau_e(x)$. Also, it can be shown that $\sigma(x,\infty)$ corresponds to the elastic solution obtained by using $\mu_a = \mu_\infty$ and the bulk modulus of the adhesive which is assumed to be a time-independent constant.

For the three types of loading shown in Figure 2 the calculated results for $\tau(x,t)$ and $\sigma(x,t)$ are shown in Tables 1-6. To visualize the variation of the stresses in time and along bond region some sample results are also given in Figures 3-5. Figure 3 and 4 show the distribution of shear and tensile stresses in the bond region in a single lap joint under membrane loading for some fixed values of time. As expected, there is a certain redistribution of stresses with increasing time. This may also be seen in Figure 5 where the variation of the maximum values of τ and σ is given. From Figures 3-5 and Tables 1-6 it may be observed that the peak values of the tensile stress σ in the adhesive are not only higher than the corresponding shear values but also decay slower. The values τ and σ given in Tables 1 and 2 for t=0 and t=∞ are obtained from the elastic solutions (79) and (81) by using the bulk modulus K which is assumed to be independent of time and the corresponding μ_o and μ_∞.

References

1. F. Erdogan and M. Ratwani, "Stress distribution in bonded joints", Journal of Composite Materials, Vol. 5, pp. 378-393, 1971.

2. M. Goland and E. Reissner, "The stresses in cemented joints", J. Appl. Mech. Vol. 1, Trans. ASME, pp. A17-A27, 1944.

3. U. Yuceoglu and D.P. Updike, "Stress analysis of bonded plates and joints", Journal of Engng. Mech. Div. ASCE, pp. 37-56, 1980.

4. F. Erdogan and M.B. Civelek, "Contact problem for an elastic reinforcement bonded to an elastic plate", J. Appl. Mech., Vol. 41, Trans. ASME, pp. 1014-1018, 1974.

5. F. Erdogan, "Fracture problems in composite materials", J. Engng. Fracture Mechanics, Vol. 4, pp. 811-840, 1972.

6. W. Flügge, Viscoelasticity, Springer-Verlag, 1975.

7. A.C. Pipkin, Lectures on Viscoelasticity Theory, Springer-Verlag, 1972.

8. W.N. Findley, J.S. Lai, and K. Onaran, Creep and Relaxation of Nonlinear Viscoelastic Materials, North-Holland Publ. Co., 1976.

Table 1. Variation of $\tau(x,t)/(N_o/\ell)$ for the case of tension (t in hours)

x/ℓ \ t	0	0.01	0.1	0.5	1.0	2.0	4.0	∞
0	-2.24×10^{-4}	-2.26×10^{-4}	-2.73×10^{-4}	-5.29×10^{-4}	-9.12×10^{-4}	-1.84×10^{-3}	-4.02×10^{-3}	-0.012
0.1	-3.67×10^{-4}	-3.71×10^{-4}	-4.42×10^{-4}	-8.15×10^{-4}	-1.35×10^{-3}	-2.61×10^{-3}	-5.40×10^{-3}	-0.015
0.2	-9.81×10^{-4}	-9.95×10^{-4}	-1.16×10^{-3}	-1.98×10^{-3}	-3.10×10^{-3}	-5.53×10^{-3}	-0.010	-0.023
0.3	-2.85×10^{-3}	-2.89×10^{-3}	-3.29×10^{-3}	-5.24×10^{-3}	-7.78×10^{-3}	-0.013	-0.022	-0.041
0.4	-8.36×10^{-3}	-8.48×10^{-3}	-9.45×10^{-3}	-0.014	-0.020	-0.030	-0.048	-0.075
0.5	-0.026	-0.025	-0.027	-0.038	-0.050	-0.071	-0.101	-0.139
0.6	-0.072	-0.073	-0.078	-0.100	-0.125	-0.164	-0.213	-0.258
0.7	-0.212	-0.213	-0.224	-0.266	-0.309	-0.373	-0.441	-0.481
0.8	-0.624	-0.624	-0.641	-0.702	-0.759	-0.833	-0.891	-0.896
0.9	-1.834	-1.828	-1.838	-1.843	-1.839	-1.816	-0.757	-1.670
1.0	-5.391	-5.351	-5.265	-4.812	-4.382	-3.838	-3.366	-3.112

Table 2. Variation of $\sigma(x,t)/(N_0/\ell)$ for the case of tension (t in hours)

x/ℓ \ t	0	0.01	0.1	0.5	1.0	2.0	4.0	∞
0	1.01×10^{-5}	1.18×10^{-5}	2.05×10^{-5}	5.80×10^{-5}	9.57×10^{-5}	1.48×10^{-4}	1.99×10^{-4}	2.29×10^{-4}
0.1	-4.61×10^{-5}	-4.33×10^{-5}	-3.01×10^{-5}	2.77×10^{-5}	8.59×10^{-5}	1.67×10^{-4}	2.46×10^{-4}	2.92×10^{-4}
0.2	-4.45×10^{-4}	-4.38×10^{-4}	-4.12×10^{-4}	-2.90×10^{-4}	-1.69×10^{-4}	-8.28×10^{-7}	1.61×10^{-4}	2.50×10^{-4}
0.3	-2.45×10^{-3}	-2.44×10^{-3}	-2.40×10^{-3}	-2.20×10^{-3}	-2.00×10^{-3}	-1.73×10^{-3}	-1.49×10^{-3}	-1.38×10^{-3}
0.4	-0.011	-0.011	-0.011	-0.011	-0.011	-0.011	-0.011	-0.011
0.5	-0.044	-0.044	-0.044	-0.045	-0.046	-0.048	-0.049	-0.050
0.6	-0.153	-0.152	-0.154	-0.158	-0.163	-0.169	-0.174	-0.177
0.7	-0.457	-0.456	-0.460	-0.469	-0.478	-0.489	-0.498	-0.502
0.8	-1.052	-1.048	-1.052	-1.050	-1.048	-1.045	-1.040	-1.037
0.9	-0.882	-0.876	-0.866	-0.808	-0.755	-0.690	-0.639	-0.621
1.0	9.017	8.971	8.938	8.656	8.397	8.096	7.872	7.801

Table 3. Variation of $\tau(x,t)/(M_o/\ell^2)$ for the case of bending (t in hours)

x/ℓ \ t	0.01	0.1	0.5	1.0	2.0	4.0
0.	0.	0.	0.	0.	0.	0.
0.1	-2.38×10^{-3}	-2.81×10^{-3}	-5.01×10^{-3}	-8.09×10^{-3}	-0.015	-0.029
0.2	-7.82×10^{-3}	-9.07×10^{-3}	-0.015	-0.024	-0.042	-0.077
0.3	-0.023	-0.026	-0.042	-0.062	-0.103	-0.175
0.4	-0.068	-0.076	-0.113	-0.159	-0.245	-0.382
0.5	-0.200	-0.219	-0.303	-0.401	-0.573	-0.818
0.6	-0.587	-0.629	-0.808	-1.005	-1.324	-1.720
0.7	-1.720	-1.803	-2.143	-2.495	-3.010	-3.553
0.8	-5.036	-5.171	-5.658	-6.123	-6.718	-7.185
0.9	-14.74	-14.82	-14.86	-14.83	-14.64	-14.17
1.0	-43.15	-42.46	-38.81	-35.34	-30.95	-27.14

Table 4. Variation of $\sigma(x,t)/(M_o/\ell^2)$ for the case of bending (t in hours)

x/ℓ \ t	0.01	0.1	0.5	1.0	2.0	4.0
0.	0.	0.	0.	0.	0.	0.
0.1	-5.94×10^{-4}	-4.84×10^{-4}	5.50×10^{-6}	5.00×10^{-4}	1.19×10^{-3}	1.88×10^{-3}
0.2	-4.72×10^{-3}	-4.45×10^{-3}	-3.19×10^{-3}	-1.93×10^{-4}	-1.85×10^{-4}	1.51×10^{-3}
0.3	-0.026	-0.026	-0.023	-0.021	-0.019	-0.016
0.4	-0.118	-0.118	-0.117	-0.116	-0.116	-0.115
0.5	-0.465	-0.469	-0.480	-0.491	-0.506	-0.521
0.6	-1.621	-1.638	-1.686	-1.733	-1.796	-1.852
0.7	-4.847	-4.891	-4.991	-5.084	-5.205	-5.301
0.8	-11.15	-11.19	-11.17	-11.15	-11.11	-11.06
0.9	-9.318	-9.209	-8.599	-8.031	-7.341	-6.802
1.0	95.43	95.09	92.09	89.33	86.13	83.75

Table 5. Variation of $\tau(x,t)/(Q_o/\ell)$ for the case of shearing (t in hours)

x/ℓ \ t	0.01	0.1	0.5	1.0	2.0	4.0
0.	4.030	4.030	4.028	4.025	4.017	4.000
0.1	4.029	4.029	4.027	4.021	4.011	3.989
0.2	4.024	4.023	4.016	4.007	3.988	3.949
0.3	4.009	4.006	3.990	3.969	3.928	3.854
0.4	3.964	3.956	3.919	3.873	3.787	3.649
0.5	3.832	3.813	3.729	3.631	3.459	3.214
0.6	3.445	3.404	3.225	3.027	2.709	2.312
0.7	2.312	2.229	1.890	1.537	1.022	0.479
0.8	-1.004	-1.139	-1.626	-2.091	-2.686	-3.153
0.9	-10.71	-10.79	-10.83	-10.80	-10.61	-10.14
1.0	-39.12	-38.43	-34.78	-31.31	-26.92	-23.11

Table 6. Variation of $\sigma(x,t)/(Q_o/\ell)$ for the case of shearing (t in hours)

x/ℓ \ t	0.01	0.1	0.5	1.0	2.0	4.0
0.	1.37×10^{-4}	2.25×10^{-4}	6.02×10^{-4}	9.78×10^{-4}	1.50×10^{-3}	2.00×10^{-3}
0.1	-3.86×10^{-4}	-2.53×10^{-4}	3.33×10^{-4}	9.21×10^{-4}	1.73×10^{-3}	2.53×10^{-3}
0.2	-4.21×10^{-3}	-3.93×10^{-3}	-2.68×10^{-3}	-1.42×10^{-3}	3.10×10^{-4}	1.98×10^{-3}
0.3	-0.024	-0.023	-0.021	-0.019	-0.016	-0.014
0.4	-0.109	-0.109	-0.107	-0.106	-0.105	-0.104
0.5	-0.433	-0.438	-0.445	-0.454	-0.466	-0.479
0.6	-1.521	-1.538	-1.578	-1.619	-1.674	-1.724
0.7	-4.595	-4.635	-4.726	-4.811	-4.921	-5.009
0.8	-10.79	-10.84	-10.83	-10.81	-10.79	-10.75
0.9	-10.47	-10.38	-9.832	-9.322	-8.707	-8.225
1.0	82.09	81.77	79.09	76.62	73.76	71.63

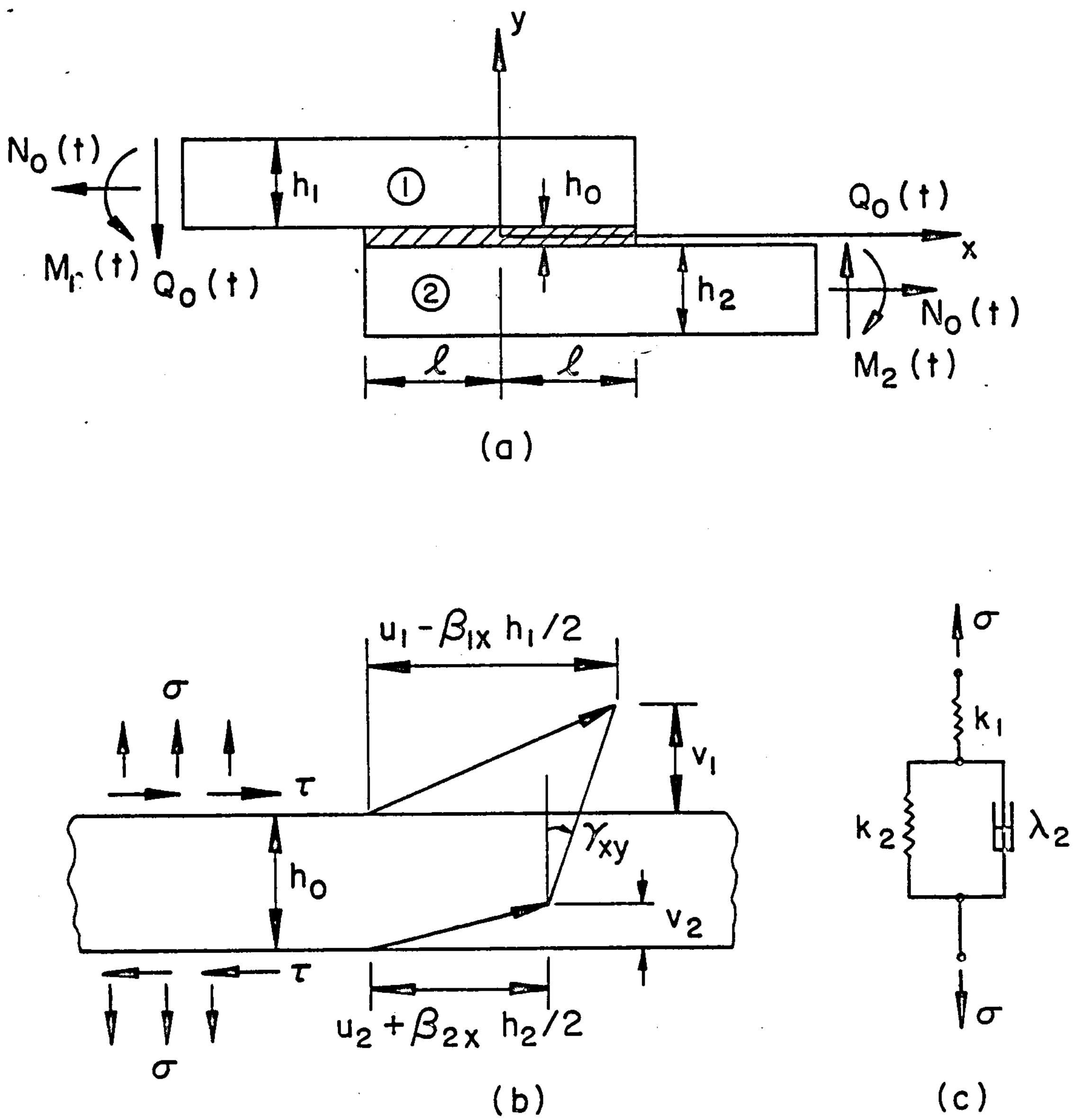

Figure 1. Geometry of the bonded joint and the viscoelastic adhesive model.

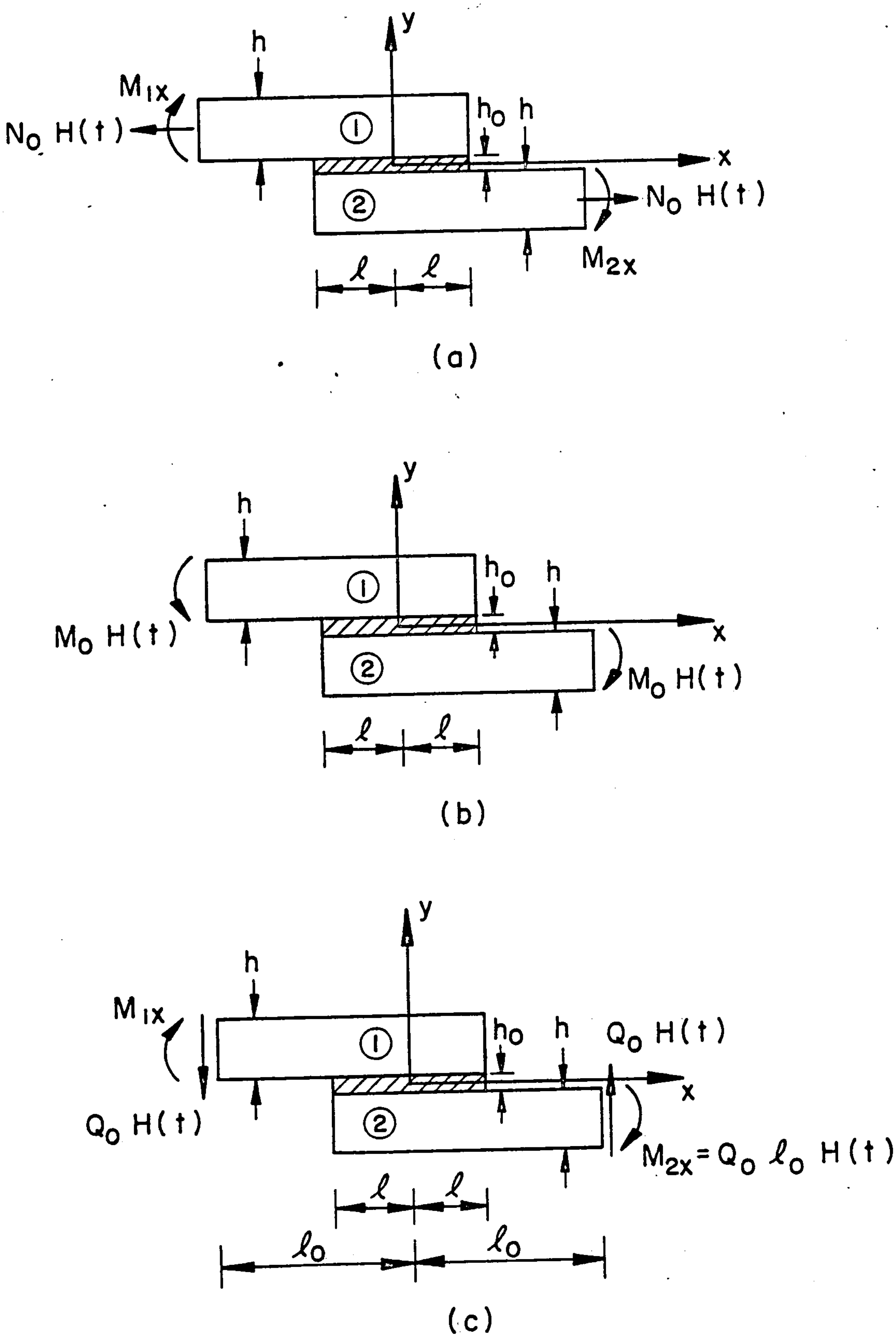

Figure 2. The loading conditions.

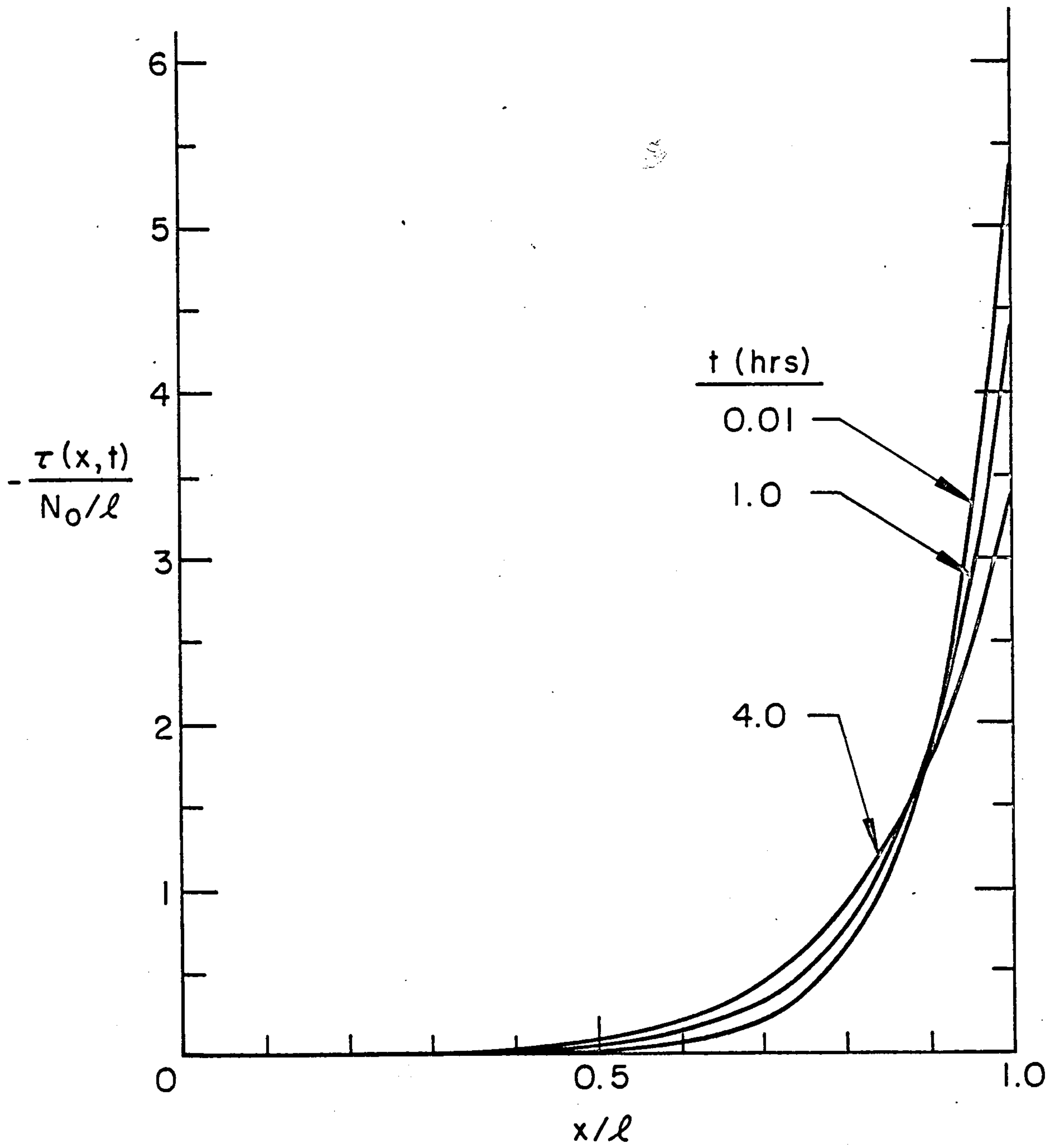

Figure 3. The shear stress $\tau_{xy} = \tau(x,t)$ in the adhesive layer.

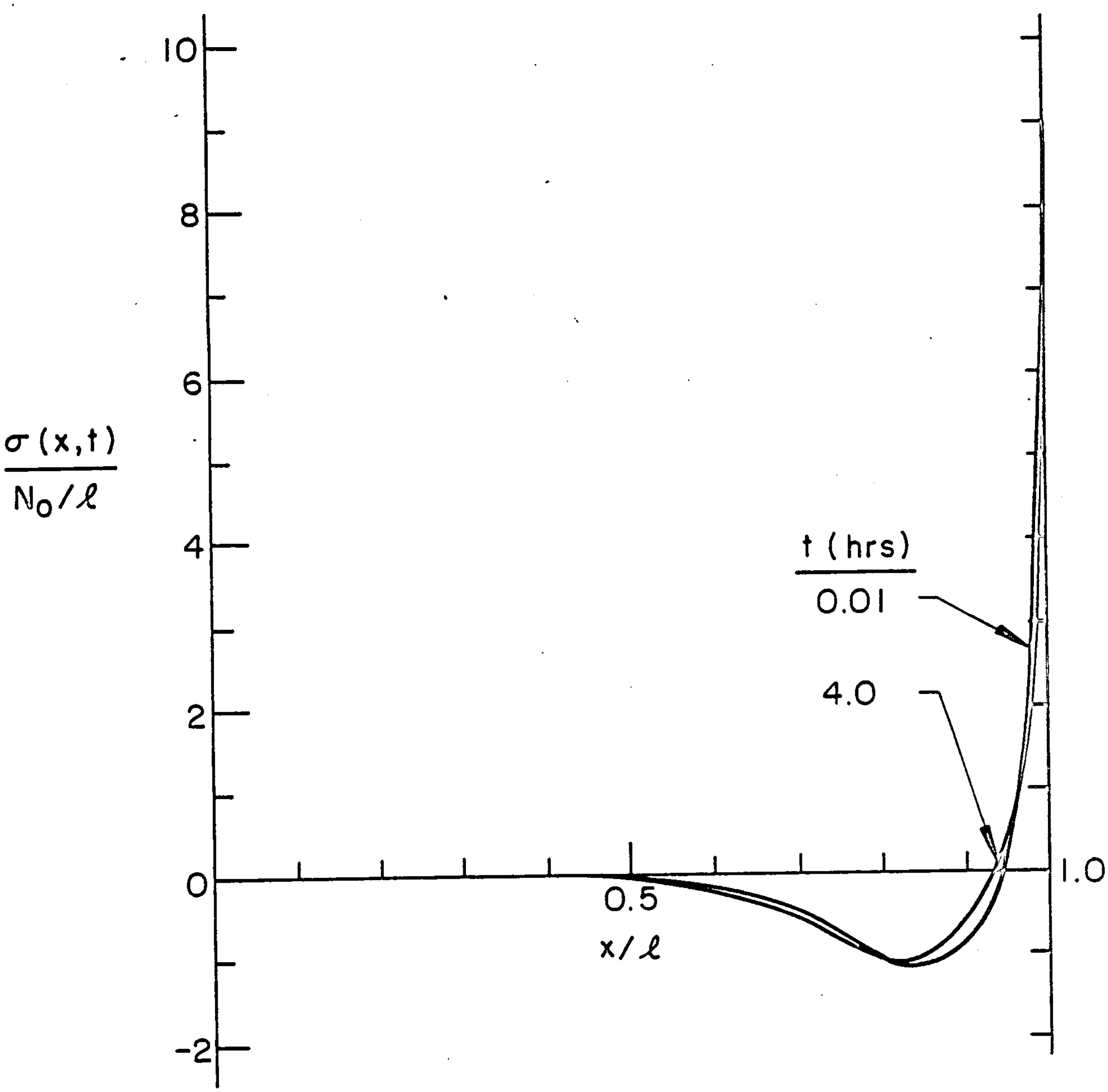

Figure 4. The normal stress $\sigma_y = \sigma(x,t)$ in the adhesive layer.

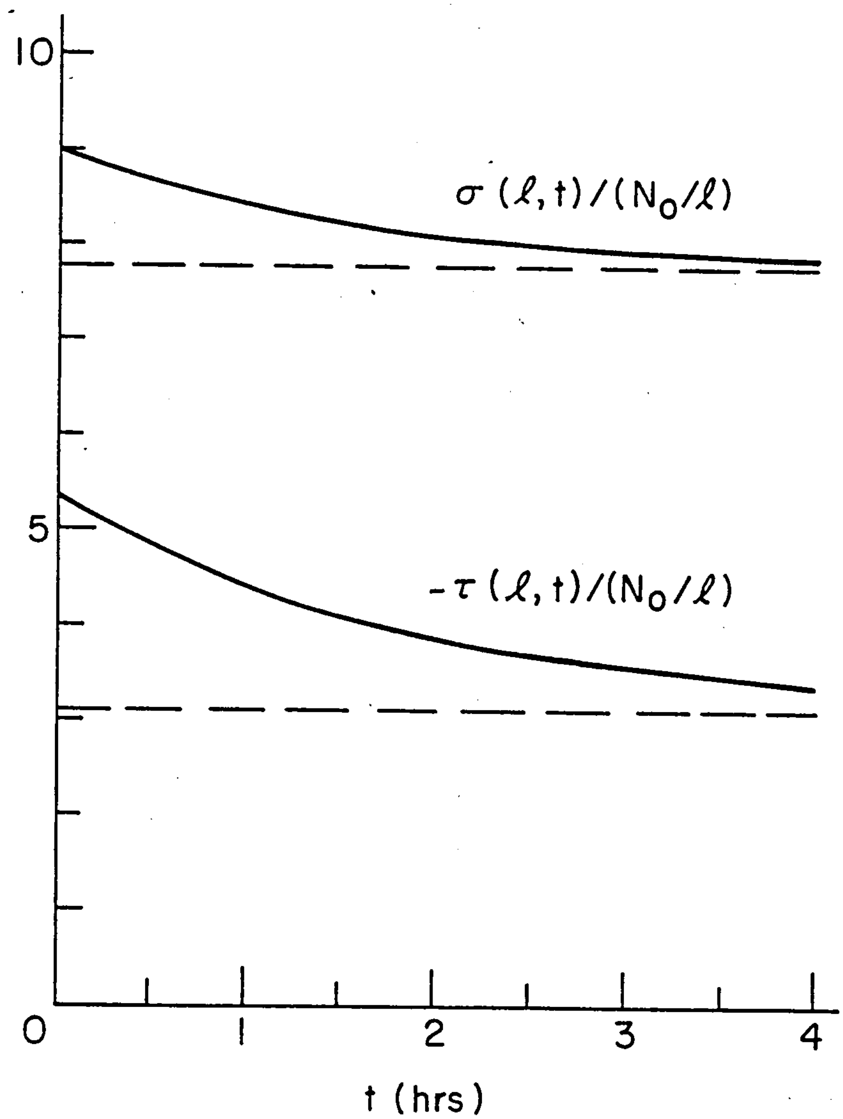

Figure 5. Variation of maximum shear and normal stresses in the adhesive as functions of time.

End of Document