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VISCOELASTIC RAYLEIGH WAVES*

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Abstract. A general analysis is given of the propagation of surface waves over a half-space of homogeneous isotropic linearly-viscoelastic material. Particular emphasis is placed on the properties of the particle paths. The detailed examination of particular models shows that (in contrast with elastic materials) (i) more than one surface wave may be possible; (ii) the waves may be either direct or retrograde at the surface; (iii) the motion may change sense at many or no levels below the surface; (iv) the wave speed may be greater than the body-wave speeds.

1. Introduction. It is known [1, 2] that a Rayleigh wave may always propagate over the surface of a semi-infinite, isotropic, homogeneous, linearly-elastic body if the strain-energy density of the material is positive definite. Moreover, as had been conjectured by Rayleigh [1], only one root of the secular equation corresponds to a displacement field that is admissible in the sense that it decays exponentially with distance from the surface [3]. Similar results have been proved for anisotropic elastic materials [4, 5, 6]. Thus for elastic materials there is one, and only one, possible Rayleigh wave. The analysis presented here will show that the situation is very different for viscoelastic materials.

Plane waves in viscoelastic materials have been discussed by many authors, including Hunter [7], Lockett [8] and Hayes and Rivlin [9, 10, 11]. The propagation of Rayleigh waves has been examined by Bland [12] and Caloi (as reported in [13]). Caloi was concerned with the perturbation to the elastic solution caused by the presence of small viscous damping in Voigt solids. Bland derived the secular equation for Rayleigh waves using the correspondence principle. He then raised the question of the number of admissible roots of the secular equation with the comment [12, p. 75]: "It has not yet been shown that for any viscoelastic material there is one and only one such root." We show below that in many cases (including that considered by Caloi) there are two admissible surface waves.

In Secs. 2 and 3 we introduce the basic equations governing the motion of a Rayleigh wave propagating over the surface of a semi-infinite, homogeneous, isotropic, linearly-viscoelastic body. The cubic secular equation determining the wave's slowness component along the surface is derived in the usual way. The coefficients of this equation depend on the frequency-dependent complex Lamé moduli λ, μ of the material. Hence the slow-

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ness is complex, in general, indicating attenuation with distance. In Sec. 4 we show that the paths followed by particles in the wave are generally ellipses, and that the orientation of the axes of the ellipse varies with depth, but tends to a constant orientation at great depth. Particular attention is paid to the sense in which the ellipse is described, whether retrograde or direct, and it is shown that this sense can change at only a finite number of levels below the surface.

In Sec. 5 we consider the first of two particular models. It is assumed that the real parts λ^+, μ^+ of the Lamé moduli are equal and that by comparison with them the imaginary parts λ^-, μ^- are very small. This corresponds to the addition of small viscous damping to one of the cases for elastic bodies treated by Rayleigh [1]. The equations are linearized in terms of a small parameter characterizing the ratio of viscous to elastic effects. It is found that the nature of the admissible solutions depends on the ratio of the imaginary parts λ^-/μ^-. For all values of this ratio there is a surface wave whose characteristics are close to those for the corresponding elastic body. We call this the "quasi-elastic wave". Its speed is *less* than that of either the P or S body-waves in the material. For a certain range of λ^-/μ^- there is also present a second solution, which we call a "viscoelastic wave" whose speed is *intermediate* between the P and S wave speeds. For another different range of λ^-/μ^- there is also a viscoelastic surface wave whose speed is *greater* than the P or S wave speeds. This may be significant in the analysis of arrival times in seismic data.

The characteristics of viscoelastic and quasi-elastic waves are quite different. The quasi-elastic wave is retrograde at the top surface and the sense of the motion changes once from retrograde to direct at a given level below the top surface. The elliptic particle paths have their major axes vertical, and the motion decays rapidly with depth. In contrast, the viscoelastic waves are direct at the top surface and there may be many changes in the sense of the motion, or none at all, depending on the ratio λ^-/μ^-. For one range of values of this ratio, the major axis of the elliptical particle paths oscillates back and forth between two limiting orientations with increasing distance from the surface. These limiting orientations slowly move towards one another to give a constant asymptotic orientation at great depth. Whenever the major axis takes on one of the limiting orientations the sense of the motion changes. But for the other range of λ^-/μ^- for which a viscoelastic wave is possible, the orientation of the major axis of the ellipse is essentially constant, inclined at 30° to the surface, and the sense of the motion never changes. In both cases, the rate of decay with depth of the viscoelastic wave is far less than that of the quasi-elastic wave.

These analytic results are confirmed and supplemented by detailed numerical computations. The resulting graphs and tables show the striking contrast between the behaviors of the two types of wave.

The second model, discussed in Sec. 6, is that of an incompressible material, also considered in detail by Rayleigh [1]. In this case we make no assumption about the size of the viscous damping, and we find that the number of possible solutions depends on the ratio μ^-/μ^+. For one range of values only one wave is possible, but outside this range there are two waves. One of these waves is retrograde at the top surface and the other direct. An example is quoted that shows that this may sometimes be the only essential difference between the two waves, apart from their speeds; in all other respects, e.g. rate of decay with depth, attenuation, number of changes of sense, they are remarkably similar.

Finally, in Sec. 7 we indicate that for wide ranges of the moduli it is possible to have more than one wave. So our detailed results would seem to be typical. We intend to pursue this matter in a further study. In general λ and μ depend on the frequency of the wave. Thus the number and nature of the Rayleigh-wave solutions for a given material will generally depend on the frequency.

It should be remarked that the analysis here will apply equally well to the linearized equations governing the behavior close to the undeformed state of a Rivlin–Ericksen material [14] or a material with memory [15].

2. Basic equations. In a homogeneous linearly-viscoelastic material the stress σ_{ij} at time t, referred to rectangular Cartesian coordinates x_i, is given by

$$\sigma_{ij}(t) = c_{ijkm}e_{km}(t) + \int_{-\infty}^{t} f_{ijkm}(t - \tau)e_{km}(\tau)\, d\tau. \tag{2.1}$$

Here $e_{km}(\tau)$ is the infinitesimal strain at time τ, defined in terms of the displacement $u_i(\tau)$ by

$$2e_{km}(\tau) = u_{k,m}(\tau) + u_{m,k}(\tau), \tag{2.2}$$

where k, denotes $\partial/\partial x_k$. c_{ijkm} are material constants and $f_{ijkm}(t - \tau)$ are material functions of $t - \tau$. c_{ijkm} and f_{ijkm} satisfy the symmetry relations

$$c_{ijkm} = c_{jikm} = c_{ijmk}\,, \qquad f_{ijkm} = f_{jikm} = f_{ijmk}\,. \tag{2.3}$$

In the usual complex notation, a damped, elliptically-polarized sinusoidal wave train with real angular frequency ω has the displacement

$$u_i(\tau) = U_i \exp i\omega(s_p x_p - \tau), \tag{2.4}$$

where s_i is the constant complex slowness vector and U_i is the constant complex amplitude vector. From (2.1) and (2.2) we find

$$\sigma_{ij}(t) = i\omega b_{ijkm}U_k s_m \exp i\omega(s_p x_p - t), \tag{2.5}$$

where

$$b_{ijkm}(\omega) = c_{ijkm} + \int_{-\infty}^{t} f_{ijkm}(t - \tau) \exp i\omega(t - \tau)\, d\tau. \tag{2.6}$$

In an isotropic material b_{ijkm} has the form

$$b_{ijkm} = \mu(\delta_{ik}\,\delta_{jm} + \delta_{jk}\,\delta_{im}) + \lambda\,\delta_{km}\,\delta_{ij}\,, \tag{2.7}$$

where λ and μ are complex functions of ω. In order that the energy dissipated over a cycle be non-negative it is necessary that[1]

$$\mu^- \le 0, \qquad 3\lambda^- + 2\mu^- \le 0. \tag{2.8}$$

The real parts λ^+, μ^+ of the complex moduli represent the elastic-like behavior of the material and are assumed to satisfy

$$\mu^+ > 0, \qquad 3\lambda^+ + 2\mu^+ > 0. \tag{2.9}$$

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[1] Throughout the paper superscripts + and − will be used to denote the real and imaginary parts respectively of complex numbers or vectors.

In the special case of the Voigt model [12] for a viscoelastic material

$$\lambda^- = -\omega\lambda', \qquad \mu^- = -\omega\mu', \tag{2.10}$$

where λ', μ' are material constants. If $\mu^+ = \lambda^+ = 0$, the Voigt material is a Newtonian fluid.

For a purely elastic material $\lambda^- = \mu^- = 0$.

In the absence of body forces the equations of motion are

$$\sigma_{ij,j} = \rho(\partial^2 u_i/\partial t^2), \tag{2.11}$$

where ρ is the density of the material. Substitution into (2.11) of (2.4), (2.5) and (2.7) yields an equation relating U_i and s_i. One solution is the transverse mode,

$$\mu s_p s_p = \rho, \qquad U_p s_p = 0, \tag{2.12}$$

and the other is the longitudinal mode,

$$(\lambda + 2\mu)s_p s_p = \rho, \qquad U_i = U s_i, \tag{2.13}$$

for some scalar U.

3. Rayleigh waves. We consider the propagation of waves of period π/ω over the free surface $x_3 = 0$ of a half-space of viscoelastic material occupying the region $x_3 \leq 0$. The wave will be a combination of two displacements of the form (2.4), one corresponding to each of two modes (2.12) and (2.13). Thus the displacement will have the form

$$u_i = \sum_{N=1}^{2} U_i^{(N)} \exp i\omega(s_p^{(N)} x_p - t), \tag{3.1}$$

where $U_i^{(1)}$, $s_i^{(1)}$ satisfy (2.12) and $U_i^{(2)}$, $s_i^{(2)}$ satisfy (2.13). At the free surface we have

$$\sigma_{i3} = 0, \quad \text{on} \quad x_3 = 0. \tag{3.2}$$

Since the material is isotropic, we can assume without loss in generality that the motion is confined to the plane $x_2 = 0$. The boundary conditions (3.2) then require the slowness component s_1 to be the same for each mode. Thus in (3.1)

$$s_1^{(1)} = s_1^{(2)} = s_1 \text{ (say)}. \tag{3.3}$$

Then, from (2.12) and (2.13),

$$\begin{aligned} [s_3^{(1)}]^2 &= (\rho/\mu) - s_1^2, & U_3^{(1)} &= -U_1^{(1)} s_1/s_3^{(1)}, \\ [s_3^{(2)}]^2 &= (\rho/[\lambda + 2\mu]) - s_1^2, & U_3^{(2)} &= U_1^{(2)} s_3^{(2)}/s_1, \end{aligned} \tag{3.4}$$

and both s_2 and U_2 are zero for both modes. Substituting (3.4), (2.5) and (2.7) into (3.2) we find that

$$\frac{U_1^{(1)}}{U_1^{(2)}} = \frac{2 s_3^{(2)} s_3^{(1)}}{2 s_1^2 - (\rho/\mu)} = \frac{(\rho/\mu) - 2 s_1^2}{2 s_1^2}. \tag{3.5}$$

Thus s_1 must satisfy the secular equation

$$-4 s_3^{(1)} s_3^{(2)} = s_1^2 (2 - \rho/\mu s_1^2)^2. \tag{3.6}$$

We write

$$c = \rho/\mu s_1^2. \tag{3.7}$$

Then on squaring both sides of (3.6) and using (3.4) we find a cubic for c, ignoring the root $c = 0$:

$$c^3 - 8c^2 + (24 - 16\mu/[\lambda + 2\mu])c - 16(1 - \mu/[\lambda + 2\mu]) = 0. \tag{3.8}$$

This equation determines the value of c and hence s_1. But not all solutions of (3.8) are admissible. Spurious solutions have been introduced through squaring (3.5). Also we require that the solution decay with distance from the surface, so that it is a true surface wave. We further require that the wave propagates along the top surface in the positive x_1-direction and that its amplitude does not grow as it propagates. Thus a root c of (3.8) is admissible if corresponding to it

(i) $s_1^+ > 0, \quad s_1^- \geq 0,$

(ii) $s_3^{(1)-} < 0, \quad s_3^{(2)-} < 0,$ (3.9)

(iii) the secular equation (3.6) is satisfied.

For the elastic case, Hayes and Rivlin [3] have shown that there is only ever *one* admissible root.

Using (3.4) and (3.5), the real part of the displacement (3.1) may now be written in the form

$$u_i^+ = [A_i^+(x_3) \cos \omega(s_1^+x_1 - t) - A_i^-(x_3) \sin \omega(s_1^+x_1 - t)] \exp -\omega s_1^- x_1 , \tag{3.10}$$

where (putting $U_1^{(1)} = 1$, without loss in generality)

$$\begin{aligned} A_1 &= \exp i\omega s_3^{(1)} x_3 + (2/(c - 2)) \exp i\omega s_3^{(2)} x_3 , \\ A_3 &= -(s_1/s_3^{(1)})[\exp i\omega s_3^{(1)} x_3 + ((c - 2)/2) \exp i\omega s_3^{(2)} x_3]. \end{aligned} \tag{3.11}$$

By (2.5) and (2.7) the corresponding real part of the stress σ_{ij} is given by

$$\sigma_{ij}^+ = [B_{ij}^+(x_3) \cos \omega(s_1^+x_1 - t) - B_{ij}^-(x_3) \sin \omega(s_1^+x_1 - t)] \exp -\omega s_1^- x_1 , \tag{3.12}$$

where, using (3.8), the non-zero components of B_{ij} are

$$\begin{aligned} B_{11} &= 2\mu i\omega s_1[\exp i\omega s_3^{(1)} x_3 - \{1 + c + c^4/8(1 - c)(c - 2)\} \exp i\omega s_3^{(2)} x_3], \\ B_{13} &= B_{31} = \frac{\mu i\omega(c - 2)s_1^2}{s_3^{(1)}} [\exp (i\omega s_3^{(1)} x_3) - \exp (i\omega s_3^{(2)} x_3)], \\ B_{33} &= -2\mu i\omega s_1[\exp (i\omega s_3^{(1)} x_3) - \exp (i\omega s_3^{(2)} x_3)]. \end{aligned} \tag{3.13}$$

4. Particle paths. The displacement (3.10) represents the superposition of two infinite trains of elliptically polarized waves, both travelling with the phase speed $1/s_1^+$ over the top surface. Thus every point in the half-space will move in a plane elliptical orbit with period $2\pi/\omega$. At a given point x_i, the elliptical orbit is found from (3.10) to be

$$(A_3^-u_1^+ + A_1^-u_3^+)^2 + (A_3^+u_1^+ - A_1^+u_3^+)^2 = (A_1^+A_3^- - A_3^+A_1^-)^2 \exp (-2\omega s_1 x_1). \tag{4.1}$$

Moreover, the ellipse is described in a retrograde (direct) sense if

$$(\bar{A}_1A_3)^- = A_1^+A_3^- - A_3^+A_1^- > 0, \qquad (< 0). \tag{4.2}$$

Of course, if $(\bar{A}_1A_3)^- = 0$ the ellipse degenerates into a straight line, as can be seen from (4.1).

For an elastic material it is known that Rayleigh waves are always retrograde at the top surface, but that they change from retrograde to direct at a certain level below the surface. In this section we shall investigate corresponding results for viscoelastic materials.

At the top surface, the condition (4.2) for the wave to be retrograde (direct) may be simplified using (3.11) to give

$$\left[\frac{s_3^{(1)}}{s_1(2-c)}\right]^- < 0, \qquad (> 0). \tag{4.3}$$

Substitution from (3.6) gives a second equivalent condition

$$\left[\frac{s_3^{(2)}}{s_1(2-c)}\right]^- < 0, \qquad (> 0). \tag{4.4}$$

But, by (3.9), $s_3^{(N)-} < 0$, $N = 1, 2$. Hence

$$\begin{aligned} s_3^{(N)-} &= \left[\frac{s_3^{(N)}}{s_1(2-c)}\, s_1(2-c)\right]^- \\ &= \left[\frac{s_3^{(N)}}{s_1(2-c)}\right]^+ [s_1(2-c)]^- + \left[\frac{s_3^{(N)}}{s_1(2-c)}\right]^- [s_1(2-c)]^+ < 0. \end{aligned} \tag{4.5}$$

But, from (3.6),

$$\left[\frac{s_3^{(1)}}{s_1(2-c)}\right]^+ \left[\frac{s_3^{(2)}}{s_1(2-c)}\right]^- + \left[\frac{s_3^{(1)}}{s_1(2-c)}\right]^- \left[\frac{s_3^{(2)}}{s_1(2-c)}\right]^+ = 0. \tag{4.6}$$

From (4.3)–(4.6) it now follows that the wave is retrograde (direct) at the surface if

$$[s_1(2-c)]^+ > 0, \qquad (< 0). \tag{4.7}$$

In addition, since $\mu^+ > 0$, $\mu^- < 0$ by (2.8) and (2.9), we find from (3.7) that

$$c^+(s_1^{+2} - s_1^{-2}) - 2c^- s_1^+ s_1^- > 0, \tag{4.8}$$

$$2c^+ s_1^+ s_1^- + c^-(s_1^{+2} - s_1^{-2}) > 0. \tag{4.9}$$

By eliminating s_1 between (4.7)–(4.9) we obtain the following sets of sufficient conditions: the wave is *retrograde* if either

$$c^- > 0, \qquad 0 < c^+ < 2,$$

or (4.10)

$$c^- < 0, \quad 0 < c^+ < 2, \quad [(c^-)^2 + (c^+)^2](4 - c^+) < 4c^+;$$

the wave is *direct* if either

$$c^- > 0, \quad c^+ > 2, \quad [(c^-)^2 + (c^+)^2](4 - c^+) < 4c^+,$$

or (4.11)

$$c^- < 0, \qquad (c^-)^2 + (c^+)^2 > 4.$$

For values of c not covered by (4.10) or (4.11) appeal must be made to (4.7) to determine the nature of the wave. In addition, no wave at all is possible if

$$c^- > 0, \qquad c^+ < 0. \tag{4.12}$$

We consider now the change in the form of the particle paths with distance from the surface. The angle Θ made by one of the axes of the ellipse (4.1) with the x_1-axis is given by

$$\tan 2\Theta = \frac{2(A_3^- A_1^- + A_3^+ A_1^+)}{(A_1^+)^2 + (A_1^-)^2 - (A_3^+)^2 - (A_3^-)^2} = \frac{A_1\bar{A}_3 + \bar{A}_1 A_3}{A_1\bar{A}_1 - A_3\bar{A}_3}. \tag{4.13}$$

The ellipse degernerates into a straight line whenever

$$(\bar{A}_1 A_3)^- = A_1^+ A_3^- - A_3^+ A_1^- = 0, \tag{4.14}$$

and when this occurs Θ is given by

$$\tan \Theta = A_3^-/A_1^- = A_3^+/A_1^+. \tag{4.15}$$

Since $(\bar{A}_1 A_3)^-$ is continuous, when the particle paths change from retrograde to direct, or vice-versa, the ellipse must be degenerate, by (4.2). Using (3.11) and (4.14), the ellipse is degenerate when

$$\begin{aligned}\left[\frac{s_1(c-2)}{s_3^{(1)}}\right]^- \Big\{(c^+ - 2) \cosh \omega(s_3^{(1)} - s_3^{(2)})^- x_3 \\ + \left[\frac{(\bar{c}-2)(c-2)}{4} + 1\right] \cos \omega(s_3^{(1)} - s_3^{(2)})^+ x_3\Big\} \\ - \left[\frac{s_1(c-2)}{s_3^{(1)}}\right]^+ \Big\{c^- \sinh \omega(s_3^{(1)} - s_3^{(2)})^- x_3 \\ + \left[\frac{(\bar{c}-2)(c-2)}{4} - 1\right] \sin \omega(s_3^{(1)} - s_3^{(2)})^+ x_3\Big\} = 0.\end{aligned} \tag{4.16}$$

This equation has at most a finite number of roots for x_3, since the hyperbolic terms will eventually dominate as $x_3 \to -\infty$. Thus, *the motion changes from retrograde to direct, or vice-versa, at most a finite number of times.* In an elastic material it is known that there is only one such change. In Sec. 5 we shall show that viscoelastic Rayleigh waves may have many such changes.

Returning now to (4.13), we consider the behavior of Θ at great distances from the surface. Define m by

$$\begin{aligned} m &= s_1/s_3^{(1)} \quad \text{if} \quad s_3^{(1)-} > s_3^{(2)-}, \\ &= -s_3^{(2)}/s_1 \quad \text{if} \quad s_3^{(1)-} < s_3^{(2)-}. \end{aligned} \tag{4.17}$$

Equivalently, by (3.6),

$$\begin{aligned} m &= -4s_3^{(2)}/s_1(c-2)^2 \quad \text{if} \quad s_3^{(1)-} > s_3^{(2)-}, \\ &= (c-2)^2 s_1/4s_3^{(1)} \quad \text{if} \quad s_3^{(1)-} < s_3^{(2)-}. \end{aligned} \tag{4.18}$$

Substituting for A_1 and A_3 from (3.11) in (4.13) and taking the limit as $x_3 \to -\infty$, we find immediately that

$$\tan 2\Theta \to \frac{m + \bar{m}}{m\bar{m} - 1}. \tag{4.19}$$

We conclude that in all cases the angle Θ tends to a limiting value, i.e. that *the ellipse tends to a constant orientation at great distance from the surface.*

The sense of the motion at large distance from the surface is also determined by m. The motion is direct (retrograde) if $m^- > 0$ (< 0).

5. Small viscous terms. We now examine in detail a particular viscoelastic material for which $\lambda^+ = \mu^+$ and λ^- and μ^- are small compared with μ^+. We take

$$\lambda = \mu^+(1 - i\epsilon a), \qquad \mu = \mu^+(1 - i\epsilon b), \tag{5.1}$$

where $\epsilon > 0$ is small. From (2.8)

$$b \geq 0, \qquad 3a + 2b \geq 0. \tag{5.2}$$

For this model there is essentially only one parameter, a/b, which determines the nature of the solutions. We shall show that for certain ranges of this parameter more than one surface wave is possible.

The elastic case with $\lambda = \mu$ is discussed by Rayleigh [1]. We linearize the equations about this state, ignoring terms of order ϵ^2 in comparison with terms of order ϵ. The roots of the cubic (3.8) are found to be

$$\begin{aligned} c_1 &= 4 + 2i\epsilon(a - b), \\ c_2 &= 2 + (2/\sqrt{3}) - i\epsilon(a - b)(3\sqrt{3} + 5)/3\sqrt{3}, \\ c_3 &= 2 - (2/\sqrt{3}) - i\epsilon(a - b)(3\sqrt{3} - 5)/3\sqrt{3}. \end{aligned} \tag{5.3}$$

Root c_1. For this root the value of s_1 satisfying (3.9) is found from (3.7) to be

$$s_1 = (\rho/4\mu^+)^{1/2}\{1 + i\epsilon(3b - a)/4\}, \tag{5.4}$$

and this is admissible if $3b > a$. Corresponding to this, from (3.4), we have

$$\begin{aligned} s_3^{(1)} &= \pm\{3\rho/4\mu^+\}^{1/2}\{1 + i\epsilon(a + 5b)/12\}, \\ s_3^{(2)} &= \pm\{\rho/12\mu^+\}^{1/2}\{1 + i\epsilon(17a - 11b)/12\}. \end{aligned} \tag{5.5}$$

But by (3.9), $s_3^{(1)-} < 0$, $s_3^{(2)-} < 0$. Thus the signs chosen in (5.5) for $s_3^{(1)}$ and $s_3^{(2)}$ depend on the signs of $(a + 5b)$ and $(17a - 11b)$. Hence we find

$$\begin{aligned} s_3^{(1)}/s_1 &= -\text{sgn}\,(5b + a)\sqrt{3}\,\{1 + i\epsilon(a - b)/3\}, \\ s_3^{(2)}/s_1 &= \text{sgn}\,(11b - 17a)\{1 + 5i\epsilon(a - b)/3\}/\sqrt{3}. \end{aligned} \tag{5.6}$$

The boundary conditions (3.6) are now satisfied if $(11b - 17a)(5b + a) > 0$. Together with (5.2) and the condition $3b > a$ found above, this implies that root c_1 corresponds to a Rayleigh wave if

$$-2/3 \leq a/b < 11/17 = 0.65. \tag{5.7}$$

By (4.11), the particles on the top surface move in an elliptical direct path.

Root c_2. In the same way as for root c_1 we find

$$\begin{aligned} s_1 &= \{\rho/\mu^+(2 + 2/\sqrt{3})\}^{1/2}\{1 + i\epsilon[(\sqrt{3} + 5/3)a \\ &\qquad\qquad + (\sqrt{3} + 1/3)b]/4(\sqrt{3} + 1)\}, \\ s_3^{(1)}/s_1 &= \text{sgn}\,([9 + 5\sqrt{3}]a - [39 + 23\sqrt{3}]b) \\ &\qquad\qquad \cdot\{1 - i\epsilon(a - b)(1 + \sqrt{3})/3\}(1 + 2/\sqrt{3})^{1/2}, \\ s_3^{(2)}/s_1 &= \text{sgn}\,([11 + 7\sqrt{3}]a - [5 + 13\sqrt{3}]b) \\ &\qquad\qquad \cdot\{1 - i\epsilon(a - b)(9 + 5\sqrt{3})/6\}\left(\frac{2 - \sqrt{3}}{3\sqrt{3}}\right)^{1/2}. \end{aligned} \tag{5.8}$$

The secular equation (3.6) is satisfied if the signs chosen in (5.8) are opposite. This gives the following condition for c_2 to be an admissible root:

$$1.19 = \frac{5 + 13\sqrt{3}}{11 + 7\sqrt{3}} < \frac{a}{b} < \frac{39 + 23\sqrt{3}}{9 + 5\sqrt{3}} = 4.46. \tag{5.9}$$

By (4.11) the wave is elliptical direct at the top surface.

Root c_3 ; *quasi-elastic wave.* For a purely elastic material this is the only acceptable root and always corresponds to an admissible Rayliegh wave [1]. Not surprisingly, the same is true for small viscous terms, and we call this wave the "quasi-elastic wave". As above, we find

$$\begin{aligned} s_1 &= \{\rho/\mu^+(2 - 2/\sqrt{3})\}^{1/2}\{1 + i\epsilon[(\sqrt{3} - 5/3)a \\ &\qquad\qquad + (\sqrt{3} - 1/3)b]/4(\sqrt{3} - 1)\}, \\ s_3^{(1)}/s_1 &= -\{(2/\sqrt{3}) - 1\}^{1/2}\{i - \epsilon(a - b)(\sqrt{3} - 1)/6\}, \\ s_3^{(2)}/s_1 &= -\{(2/\sqrt{3}) + 1\}^{1/2}\{i + \epsilon(a - b)(9 - 5\sqrt{3})/6\}/\sqrt{3}. \end{aligned} \tag{5.10}$$

The secular equation (3.6) is always satisfied. As in the purely elastic case, the wave is always retrograde at the surface.

Thus for this particular model, if $0.65 \le a/b \le 1.19$ or $4.46 \le a/b$ only one wave, the quasi-elastic wave, is possible. Otherwise two Rayleigh waves are possible, one of them always being the quasi-elastic wave. We call the second wave the "viscoelastic wave". This second wave is possible even for *infinitesimal* viscous damping, provided a/b lies in the appropriate range.

Numerical results have been computed for the two cases

$$\epsilon a = 0.02, \qquad \epsilon b = 0.04, \tag{5.11}$$

and

$$\epsilon a = 0.06, \qquad \epsilon b = 0.04. \tag{5.12}$$

Taking the corresponding values of λ and μ given by (5.1) and normalizing by putting $\rho = \mu^+$, we solved Eq. (3.8) on a Honeywell H316 computer using the Cardan formula [16]. For each of the roots, s_1^2 and $(s_3^{(1)})^2$ were calculated from (3.7) and (3.4) and s_1 and $s_3^{(1)}$ taken to satisfy (3.9). $s_3^{(2)}$ was then calculated from (3.6). The root was considered admissible if the resulting value for $s_3^{(2)}$ satisfied (3.9). In agreement with the analytic results given above, for case (5.11) the admissible roots and slownesses are

$$\begin{aligned} c_1 &= 4.0032 - 0.0397i, & s_1 &= 0.499 + 0.012i, \\ s_3^{(1)} &= -0.866 - 0.016i, & s_3^{(2)} &= 0.289 - 0.002i; \end{aligned} \tag{5.13}$$

$$\begin{aligned} c_3 &= 0.8453 + 0.0008i, & s_1 &= 1.087 + 0.021i, \\ s_3^{(1)} &= 0.007 - 0.428i, & s_3^{(2)} &= 0.019 - 0.921i. \end{aligned} \tag{5.14}$$

For the case (5.12), the admissible roots and slownesses are

$$\begin{aligned} c_2 &= 3.1547 - 0.0392i, & s_1 &= 0.563 + 0.015i, \\ s_3^{(1)} &= -0.826 - 0.014i, & s_3^{(2)} &= 0.128 - 0.004i; \end{aligned} \tag{5.15}$$

$$\begin{aligned} c_3 &= 0.8453 - 0.0007i, & s_1 &= 1.087 + 0.022i, \\ s_3^{(1)} &= 0.010 - 0.427i, & s_3^{(2)} &= 0.018 - 0.921i. \end{aligned} \tag{5.16}$$

Having established the existence of two Rayleigh waves under certain conditions, we now discuss their characteristics in more detail. We consider each of the roots in turn. Distance from the surface is measured in terms of x_3/Λ, where Λ is the wavelength of the wave along the top surface given by

$$\Lambda = 2\pi/s_1^{+}\omega. \tag{5.17}$$

Quasi-elastic root c_3. Using (3.11), (5.3) and (5.10) it is found that close to the surface (i.e. for $|\epsilon x_3/\Lambda| \ll 1$) the values of A_1 and A_3 corresponding to c_3 are given (up to terms of order ϵ) by

$$\begin{aligned} A_1 &= \exp\{(2/\sqrt{3}-1)^{1/2}2\pi x_3/\Lambda\} - \sqrt{3}\exp\{(2/\sqrt{3}+1)^{1/2}\,2\pi x_3/(\sqrt{3}\,\Lambda)\}, \\ A_3 &= -i\exp\{(2/\sqrt{3}-1)^{-1/2}[\exp\{(2/\sqrt{3}-1)^{1/2}\,2\pi x_3/\Lambda\} \\ &\qquad - (1/\sqrt{3})\exp\{(2/\sqrt{3}+1)^{1/2}\,2\pi x_3/\sqrt{3}\,\Lambda\}]. \end{aligned} \tag{5.18}$$

A_1 is zero once, at $x_3 = 0.19\Lambda$, and at this level the motion changes from the retrograde motion at the surface to direct motion. This is the only change in the sense of the motion.

A_1 is real and A_3 imaginary, with $|A_3| > |A_1|$. By (4.13), $\Theta = 0$. Thus near the surface, the axes of the ellipse are always aligned with the coordinate axes, the major axis along the x_3-axis. This same orientation of the ellipse persists as $x_3 \to -\infty$, by (4.19) and (5.10). The numerical calculations for cases (5.11) and (5.12) confirm that, to order ϵ, the major axis of the ellipse is aligned with the x_3-axis for all values of x_3 (see Fig. 5).

In Fig. 1 the maximum amplitudes of the displacements and stresses corresponding to (5.14) are plotted as functions of distance from the surface. The displacements and stresses are normalized respectively by the values of $|u_3|$ and $|\sigma_{11}|$ at the top surface.

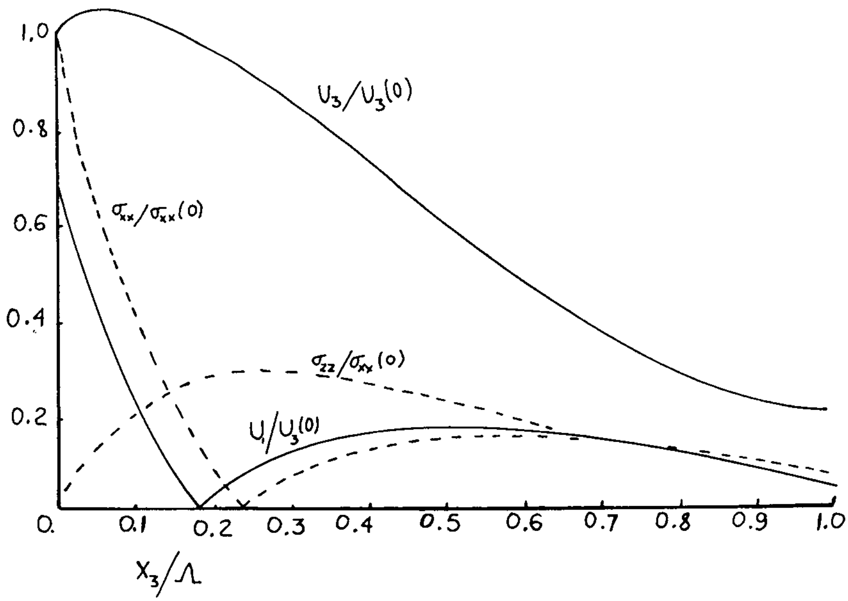

FIG. 1. Amplitudes of the displacements and stresses corresponding to the quasi-elastic wave (5.14) for a material with small viscous damping.

Thus in terms of the solutions (3.10) and (3.12) the graphs give

$$\frac{|u_i(x_3)|}{|u_3(0)|} = \frac{|A_i|}{|A_3(0)|}, \qquad \frac{|\sigma_{ij}(x_3)|}{|\sigma_{11}(0)|} = \frac{|B_{ij}|}{|B_{11}|}. \tag{5.19}$$

The amplitude of u_3 grows to 1.05 times its surface value in the first 0.075Λ and then decays monotonically, being 0.19 times its surface value at a depth of one wavelength. u_1 drops rapidly to almost zero at 0.19Λ. Below about 0.5Λ, the displacements and stresses both decay monotonically.

These graphs are very close to those found for a purely elastic material with Poisson's ratio 1/4 [17]. Thus in all respects, the behavior of the wave corresponding to the quasi-elastic root c_3 is little different from that of the elastic Rayleigh wave.

Viscoelastic root c_1. For $|\epsilon x_3/\Lambda| \ll 1$, A_1 and A_3 are found from (3.11), (5.3) and (5.6) (up to order ϵ):

$$A_1 = \sqrt{3}\, A_3 = \exp\{-i\sqrt{3}\, 2\pi x_3/\Lambda\} + \exp\{i2\pi x_3/(\sqrt{3}\,\Lambda)\}. \tag{5.20}$$

It follows from (4.14) and (4.15) that to this order of accuracy the ellipse is always degenerate near the surface, making an angle of 30° with the x_1-axis. Moreover, the same orientation is found from (4.17) and (5.6) as $x_3 \to -\infty$. The detailed numerical calculations for case (5.13) show that there are indeed only slight deviations of the axis of the ellipse from the 30° orientation. However, the ellipse is never degenerate. Thus the particle paths are not retrograde at any level, there being no change from the direct motion at the surface.

The displacements and stresses corresponding to (5.15) are plotted for the case (5.13) in Figs. 2 and 3. Both displacements oscillate with the same periodicity, the peaks

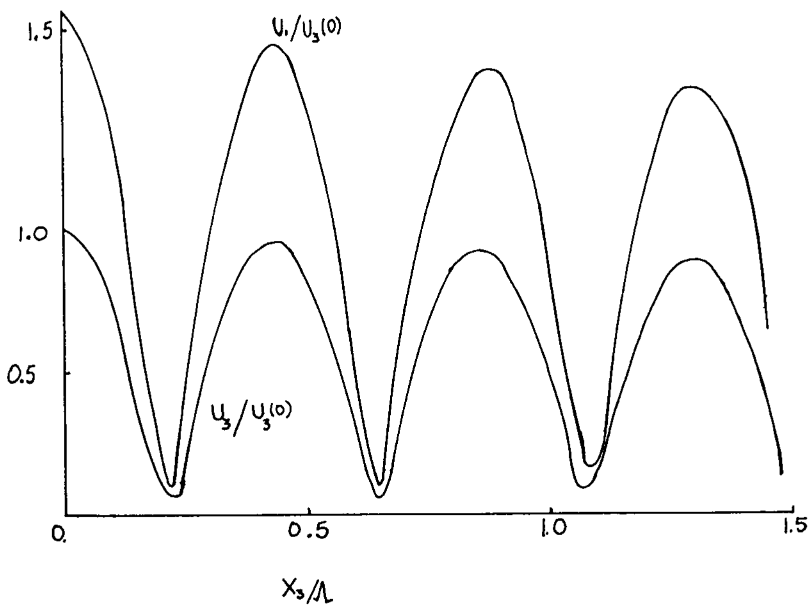

FIG. 2. Amplitudes of the displacements corresponding to the viscoelastic wave (5.13) for a material with small viscous damping.

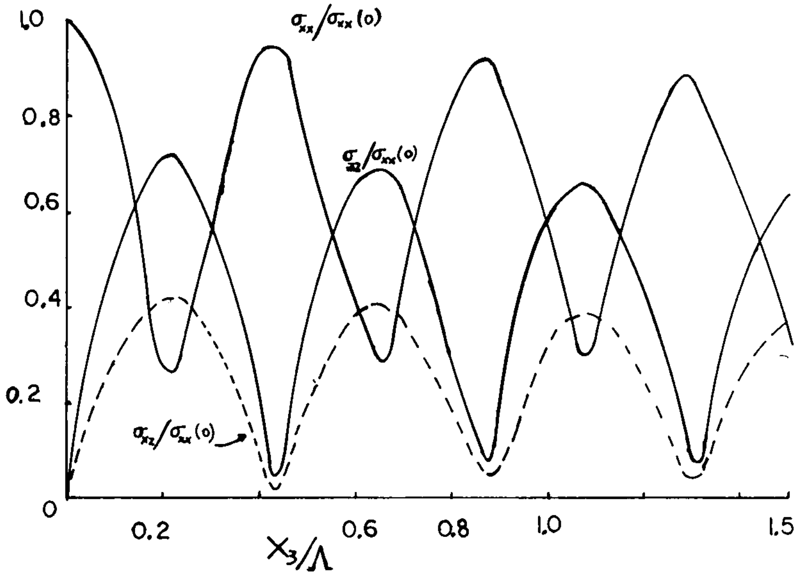

FIG. 3. Amplitudes of the stresses corresponding to the viscoelastic wave (5.13) for a material with small viscous damping.

diminishing slowly. The behavior of the stresses is similar, with σ_{11} out of phase with σ_{33} and σ_{13}. The graphs show that for the viscoelastic roots the decay of the distrubance with distance from the surface, being governed by s_3^- which is $O(\epsilon)$, is far slower than for the quasi-elastic root for which s_3^- is $O(1)$.

Viscoelastic root c_2. The most interesting behavior of the particle paths is exhibited for this root. Assuming a/b satisfies (5.9), the values of A_1 and A_3 for $|\epsilon x_3/\Lambda| \ll 1$ are found (up to order ϵ) from (3.11), (5.3) and (5.8):

$$\begin{aligned} A_1 &= \exp\{-i(1 + 2/\sqrt{3})^{1/2}\, 2\pi x_3/\Lambda\} \\ &\qquad + \sqrt{3}\exp\{i[(2 - \sqrt{3})/3\sqrt{3}]^{1/2}\, 2\pi x_3/\Lambda\}, \\ A_3 &= (1 + 2/\sqrt{3})^{-1/2}[\exp\{-i(1 + 2/\sqrt{3})^{1/2}\, 2\pi x_3/\Lambda\} \\ &\qquad + (1/\sqrt{3})\exp\{i[(2 - \sqrt{3})/3\sqrt{3}]^{1/2}\, 2\pi x_3/\Lambda\}]. \end{aligned} \tag{5.21}$$

By (4.14) the ellipse is degenerate when

$$\frac{x_3}{\Lambda} = -\frac{\beta n}{2}, \qquad \beta^{-1} = (1 + 2/\sqrt{3})^{1/2} + \{(2 - \sqrt{3})/3\sqrt{3}\}^{1/2}, \tag{5.22}$$

where n is any non-negative integer. At these points the corresponding value of Θ is found from (4.15) to be $(-1)^n\Theta_0$, where

$$\Theta_0 = \tan^{-1}\{3 + 2\sqrt{3}\}^{-1/2} = 21°27'. \tag{5.23}$$

From (4.13) it is now found that

$$\tan 2\Theta = \tan 2\Theta_0 \left\{\frac{\sqrt{3} + 2\cos(2\pi x_3/\beta\Lambda)}{2 + \sqrt{3}\cos(2\pi x_3/\beta\Lambda)}\right\}. \tag{5.24}$$

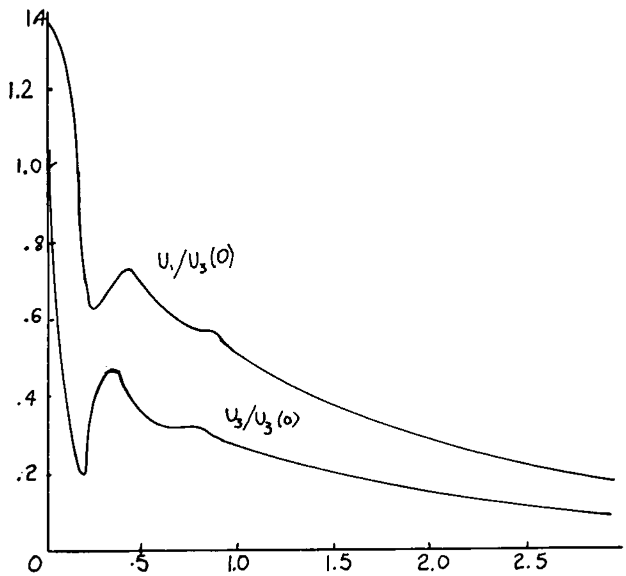

FIG. 4. Amplitudes of the displacements for the direct wave (6.10) for an incompressible material.

At $x_3/\Lambda = -\beta(2n + 1)/4$,

$$(\bar{A}_1 A_3)^- = \{(\sqrt{3} + 2)/\sqrt{3}\}^{1/2}(2/\sqrt{3})(-1)^n. \tag{5.25}$$

We can see from (5.24) that the axis of the ellipse oscillates back and forth between $\pm\,\Theta_0$ as we move away from the surface $x_3 = 0$. The extreme values $\pm\,\Theta_0$ are taken on at periodic values of x_3, and at these levels the ellipse is degenerate and the sense of the motion changes. These same levels correspond to maxima and minima of $|A_1|$ and $|A_3|$, as is seen from (5.21) and (5.22). Using (5.25) to determine the sense of the motion at intermediate values of x_3, we conclude that when $\Theta = -\,\Theta_0$ the motion changes from retrograde to direct and $|A_1|$, $|A_3|$ have minima; when $\Theta = \Theta_0$ the change is from direct to retrograde and $|A_1|$, $|A_3|$ have maxima.

This approximate analysis predicts degeneracy at $x_3 = 0$, on putting $n = 0$ in (5.22). We know from above that the motion is in fact direct at the surface. Exact calculation shows that the motion changes just below the surface. Moreover, the analysis holds only near the surface; the general treatment of Sec. 4 shows there are only a finite number of points at which the ellipse is degenerate, and that Θ tends to a constant orientation at great depth, found to be 12°47′ from (4.19) and (5.8). Nevertheless, numerical calculations for the case (5.15) show that there are about 60 changes from retrograde to direct or vice-versa within a distance of 17 wavelengths of the surface and still more below this level. The position of these changes is predicted accurately by (5.22) for $n \geq 1$. But the limiting values of Θ are approximately $\pm\,\Theta_0$ only for $n = 0, 1, 2$. It is found numerically that the limiting values of Θ move steadily closer together with depth, the bottom limit changing faster than the top, to coalesce at the asymptotic orientation 12°47′ found above. The values of the limiting orientations down to 5 wavelengths are given in Table 1. In Figs. 6 and 7 the behavior of the ellipses is shown diagramatically.

The behavior of the displacements and stresses for this root is similar to that for root c_1.

TABLE 1. Levels at which the sense of the motion changes (DR—direct to retrograde, RD—retrograde to direct) and the corresponding orientation angle Θ, corresponding to case (5.15).

$-x_3/\Lambda$	$\Theta°$	Sense	$-x_3/\Lambda$	$\Theta°$	Sense
.009	21.45	DR	2.65	−6.55	RD
.295	−19.20	RD	2.96	19.79	DR
.598	21.10	DR	3.24	−4.50	RD
.884	−15.17	RD	3.55	19.48	DR
1.19	20.76	DR	3.83	−2.71	RD
1.47	−11.83	RD	4.14	19.17	DR
1.78	20.43	DR	4.42	−1.17	RD
2.06	−8.97	RD	4.73	18.87	DR
2.37	20.10	DR	5.01	0.18	RD

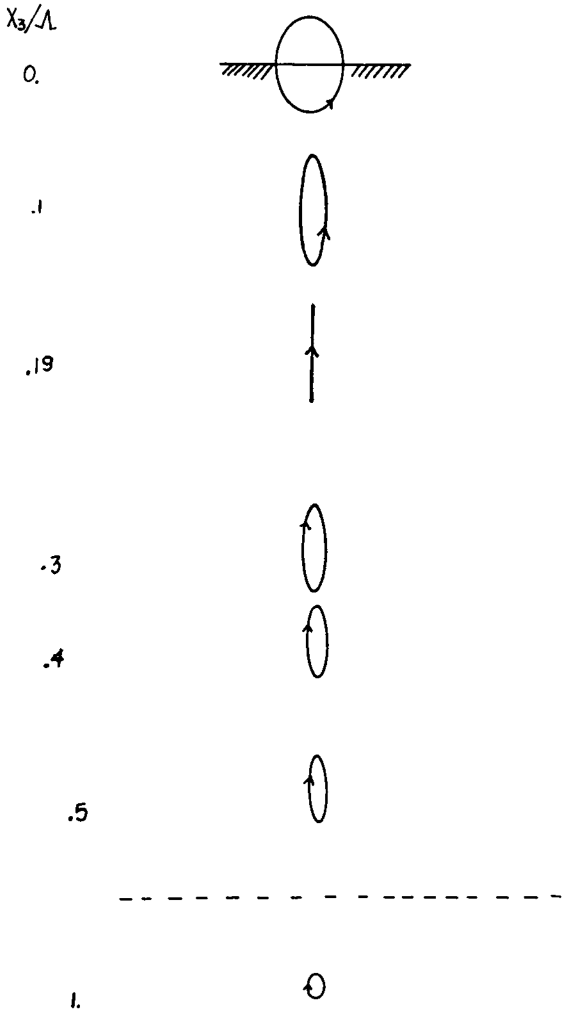

FIG. 5. Variation of the elliptic particle paths with depth corresponding to the quasi-elastic wave (5.14) for a material with small viscous damping.

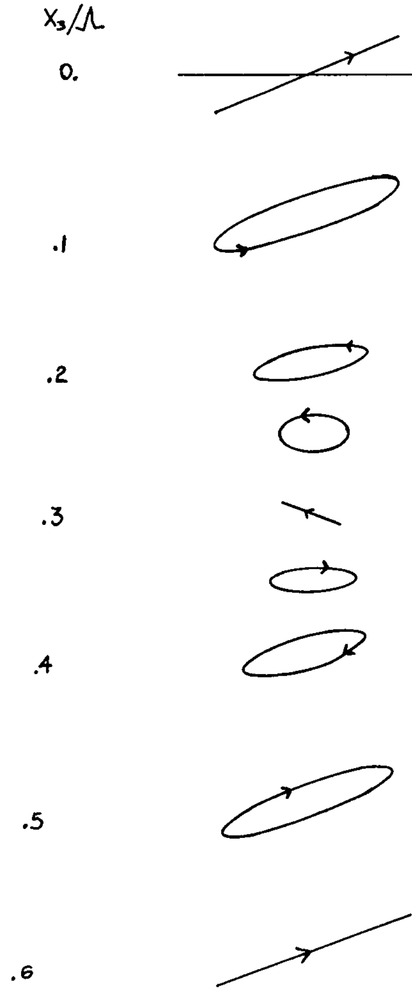

FIG. 6. Variation near the surface of the elliptic particle paths with depth corresponding to the viscoelastic wave (5.15) for a material with small viscous damping.

We note one final difference between the surface waves corresponding to the three roots c_1, c_2 and c_3. Let V_P, V_S be the longitudinal and transverse body wave speeds in the material. Then, neglecting terms of order ϵ, $V_P = (3\mu^+/\rho)^{1/2}$, $V_S = (\mu^+/\rho)^{1/2}$. Denote by V_1, V_2 and V_3 the speeds along the top surface of the surface waves corresponding to the roots c_1, c_2 and c_3. Then, by (5.4), (5.8) and (5.10),

$$V_3 < V_S < V_2 < V_P < V_1 . \tag{5.26}$$

Thus both viscoelastic waves are faster than the transverse S-wave, and one is faster than the longitudinal P-wave. As expected, the quasi-elastic wave is slower than both body waves.

6. Incompressible material. The second case we consider in detail is that of an incompressible material. There is no restriction on the magnitude of the viscous terms.

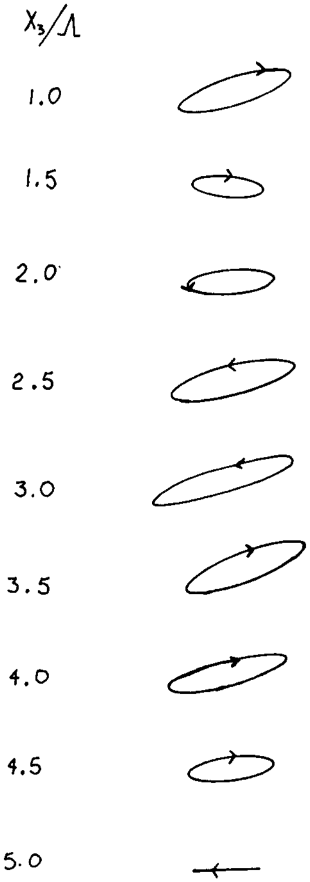

FIG. 7. Variation over five wavelengths of the elliptic particle paths with depth corresponding to the viscoelastic wave (5.15) for a material with small viscous damping.

The constitutive equation (2.1) must be amended to allow for an arbitrary hydrostatic pressure. However, the equations governing the Rayleigh wave (3.4)–(3.8) are found to be correct if the limit $\lambda \to \infty$ is taken. In this case c satisfies the real equation

$$c^3 - 8c^2 + 24c - 16 = 0. \tag{6.1}$$

The roots of this cubic are [1]

$$\begin{aligned} c_1 &= 3.5437 + 2.2303i, \\ c_2 &= 3.5437 - 2.2303i, \\ c_3 &= 0.9126. \end{aligned} \tag{6.2}$$

Root c_1. It is easily shown that root c_1 is never admissible. From (3.4), with $\lambda \to \infty$, and (3.9) we have

$$s_3^{(2)} = -is_1 . \tag{6.3}$$

Thus from (3.6) we have

$$s_3^{(1)} = -is_1(2 - c_1)^2/4 = S_1(1.7215 + 0.6478i). \tag{6.4}$$

Since s_1^+, s_1^- are positive, $s_3^{(1)-} > 0$, contradicting (3.9).

Root c_2. From (3.4),

$$s_3^{(1)} = \pm s_1(1.7215 - 0.6478i), \qquad s_3^{(2)} = -is_1 . \tag{6.5}$$

The boundary conditions are satisfied identically provided the negative sign is chosen for $s_3^{(1)}$. The condition that $s_3^{(1)-} < 0$ then gives

$$1.7215s_1^- - 0.6478s_1^+ > 0. \tag{6.6}$$

But from (3.7)

$$\frac{s_1^+}{s_1^-} - \frac{s_1^-}{s_1^+} = \frac{2(\mu^- c_2^- - \mu^+ c_2^+)}{(\mu^+ c_2^- + \mu^- c_2^+)}. \tag{6.7}$$

Combining (6.6) and (6.7), we find that this root is admissible if

$$\mu^-/\mu^+ < -0.159. \tag{6.8}$$

The motion of the top surface is direct, by (4.11).

Root c_3. This root is the only admissible one for a purely elastic material [1]. We find

$$s_3^{(1)} = -0.2956is_1 , \qquad s_3^{(2)} = -is_1 , \tag{6.9}$$

and the boundary conditions (3.6) are always satisfied. By (4.10) the motion of the top surface is retrograde.

Thus for the incompressible material two Rayleigh waves are possible if $\mu^-/\mu^+ < -0.159$ and one of the waves is retrograde, the other direct, at the top surface. If $\mu^-/\mu^+ > -0.159$, only one wave is possible and this is retrograde. These results apply for any values of μ^+, μ^- not necessarily close to the elastic case. Indeed, as $\mu^+ \to 0$ for the Voigt model (2.10) the material behaves like a Newtonian fluid.

When the material is neither "close" to an elastic solid nor a Newtonian fluid both waves have similar characteristics regarding their decay along the top surface or with depth. For example, if $\mu^+ = -2\mu^-$ then corresponding to root c_2 we get

$$\begin{aligned} s_1 &= \{\rho/\mu^+\}^{1/2}\{0.403 + 0.227i\}, \\ s_3^{(1)} &= -\{\rho/\mu^+\}^{1/2}\{0.840 + 0.129i\}, \end{aligned} \tag{6.10}$$

while for root c_3 we get

$$\begin{aligned} s_1 &= \{\rho/\mu^+\}^{1/2}\{0.964 + 0.227i\}, \\ s_3^{(1)} &= \{\rho/\mu^+\}^{1/2}\{0.067 - 0.285i\}. \end{aligned} \tag{6.11}$$

The essential difference between these two waves lies in the fact that (6.10) is direct and (6.11) retrograde at the surface. The elliptical particle paths of both waves become degenerate at only one value of x_3. For the root c_2 with the values (6.10) the motion changes from direct to retrograde at 0.13 wavelengths from the surface. For c_3 with values (6.11) the change from retrograde to direct occurs at 0.14 wavelengths. Once again (5.26) holds (ignoring V_P and V_1), the speed of transverse S-body waves being $1.09\ (\mu^+/\rho)^{1/2}$.

TABLE 2. Ratios of the complex moduli and number (n) of admissible roots.

λ^+/μ^+	.5	.5	.5	.5	.2	.2	.2	.2
$-\lambda^-/\mu^+$	.05	.05	.6	.6	.2	.2	2.4	2.4
$-\mu^-/\mu^+$	.04	.15	.25	.15	.08	.24	.8	2.0
n	2	1	2	1	1	2	2	1

The variation with depth of the displacements corresponding to (6.10) are shown in Fig. 4. The behavior is midway between the sinusoidal variation of Fig. 2 and the almost monotonic decay of Fig. 1.

7. Conclusions. We have demonstrated the existence of two Rayleigh waves for certain ranges of the moduli in two explicit cases. In Table 2 we show other typical values of the ratios λ^+/μ^+, λ^-/μ^+, μ^-/μ^+ and the corresponding number of admissible roots. This was determined numerically, using the method described in Sec. 5. This table indicates that the results we have obtained above are typical. We intend to investigate further this general question of the number of admissible roots. We note that Chadwick and Windle [18] found two thermoelastic Rayleigh waves in certain cases.

Our principal conclusions are that Rayleigh waves on isotropic viscoelastic materials differ from such waves on elastic materials in the following ways:

(i) more than one wave may be possible, whereas in the elastic case there is only ever one wave;

(ii) waves may be either retrograde or direct at the top surface, whereas elastic Rayleigh waves are always retrograde;

(iii) the motion may change sense from retrograde to direct, or vice-versa, at many or no levels below the surface, rather than just once as in the elastic case;

(iv) a viscoelastic Rayleigh wave may propagate with a speed greater than either the P or S body-waves, whereas in the elastic case the speed is always less than that of P and S waves.

These properties have been demonstrated explicitly for two models. For the case of small viscous terms, the two admissible waves could be classified as a quasi-elastic wave (always present) and a viscoelastic wave (present for certain values of the moduli). Because of the slow decay of its disturbance with depth, the viscoelastic wave does not possess the characteristics of a surface wave to the same degree as the quasi-elastic wave, and may therefore be of less importance. However, this objection cannot be raised for the case of an incompressible material. In Sec. 6 we exhibited two admissible Rayleigh waves with very similar decay properties. One is retrograde and travels slower than the body waves; the other is direct and travels faster than the body wave.

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