The Annals of Applied Probability
2016, Vol. 26, No. 6, 3381–3414
DOI: 10.1214/16-AAP1178

VISCOSITY SOLUTIONS OF FULLY NONLINEAR ELLIPTIC PATH DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS

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This paper extends the recent work on path-dependent PDEs to elliptic equations with Dirichlet boundary conditions. We propose a notion of viscosity solution in the same spirit as [*Ann. Probab.* **44** (2016) 1212–1253, Part 1; Ekren, Touzi and Zhang (2016), Part 2], relying on the theory of optimal stopping under nonlinear expectation. We prove a comparison result implying the uniqueness of viscosity solution, and the existence follows from a Perron-type construction using path-frozen PDEs. We also provide an application to a time homogeneous stochastic control problem motivated by an application in finance.

1. Introduction. In this paper, we develop a theory of viscosity solutions of elliptic PDEs on the continuous path space, by extending the recent literature on path-dependent PDEs (PPDE) to this context.

Nonlinear PPDEs appear in various applications, for example, non-Markovian stochastic control problems are naturally related to path-dependent Hamilton–Jacobi–Bellman equations (see [10]), and non-Markovian stochastic differential games are related to path-dependent Isaacs equations (see [22]). PPDEs are also intimately related to the backward stochastic differential equations introduced by Pardoux and Peng [21], and their extension to the second order in [3, 25]. We refer to the survey paper [24] as an introduction to this new topic. We also refer to the recent applications in [12] to establish a representation of the solution of a class of PPDEs in terms of branching diffusions, and to [16] for the small noise large deviation results of path-dependent diffusions.

In the existing literature, the authors are all focus on developing the well-posedness theory for parabolic PPDEs. In this paper, we explore the notion of an elliptic PPDE. An elliptic PPDE on the continuous path space Ω is of the form:

$$\begin{aligned} &G(\cdot, u, \partial_\omega u, \partial^2_{\omega\omega} u)(\omega) = 0, \qquad \omega \in \mathcal{Q} \subset \Omega, \quad \text{and} \\ &u(\omega) = \xi(\omega), \qquad \omega \in \partial \mathcal{Q}. \end{aligned} \tag{1.1}$$

Received October 2014; revised November 2015.

[1] Supported by grant from RDM-IdF.

MSC2010 subject classifications. 35D40, 35K10, 60H10, 60H30.

Key words and phrases. Viscosity solutions, optimal stopping, path-dependent PDEs, comparison principle, Perron's approach.

Our notions of the derivatives ∂_ω and $\partial^2_{\omega\omega}$ are inspired by the calculus developed in Dupire [7] as well as in Cont and Fournie [4]. Let

$$\Omega^e := \{\omega \in \Omega : \omega = \omega_{t\wedge\cdot} \text{ for some } t \in \mathbb{R}^+\} \quad \text{and} \quad u : \Omega^e \to \mathbb{R},$$

that is, Ω^e is the subspace of all the paths with flat tails. Denote by $\{u_t\}_{t\in\mathbb{R}^+}$ the process $u_t(\omega) := u(\omega_{t\wedge\cdot})$. According to [4, 7], one may define the horizontal and vertical derivatives for the process

$$\begin{aligned} \partial_t u_t(\omega) &:= \lim_{h\to 0} \frac{u_{t+h}(\omega_{t\wedge\cdot}) - u_t(\omega)}{h} \quad \text{and} \\ \partial_\omega u_t(\omega) &:= \lim_{h\to 0} \frac{u_t(\omega) - u_t(\omega_\cdot + h1_{[t,\infty)})}{h}. \end{aligned} \tag{1.2}$$

Also, in [4, 7] the authors proved that a *smooth* process satisfies the functional Itô formula

$$du_t = \partial_t u\, dt + \partial_\omega u\, d\omega_t + \tfrac{1}{2}\partial^2_{\omega\omega} u\, d\langle\omega\rangle_t, \qquad \mathbb{P}\text{-a.s. for all continuous semimartingale measures } \mathbb{P}. \tag{1.3}$$

Note that in the definition (1.2) one requires to extend the process u to the set of càdlàg paths. Although this technical difficulty is addressed and solved in [4], it was observed by Ekren, Touzi and Zhang [8] that it is more convenient to define the derivatives by the Itô decomposition (1.3), namely, we call the continuous processes Λ, Z, Γ the derivatives of the process u if

$$du_t = \Lambda_t\, dt + Z_t\, d\omega_t + \tfrac{1}{2}\Gamma_t\, d\langle\omega\rangle_t, \qquad \mathbb{P}\text{-a.s. for all continuous semimartingale measures } \mathbb{P}.$$

In this paper, we follow this idea to define the path derivatives (see Definition 2.6 below). We next restrict our solution space so that all potential solutions u of elliptic PPDE (1.1) agree with the time-independence property, that is, $\partial_t u = 0$. A function $u : \Omega^e \to \mathbb{R}$ is called to be time-invariant, if

$$u(\omega) = u(\omega_{\ell(\cdot)}) \qquad \text{for all } \omega \text{ and all increasing bijection } \ell : \mathbb{R}^+ \to \mathbb{R}^+,$$

that is, the value of a time-invariant function u is unchanged by any time scaling of path. It follows from the definition of the horizontal derivative in (1.2) that $\partial_t u = 0$. Therefore, the time-invariance implies the time-independence, and in this paper we will prove the well-posedness of time-invariant solutions to PPDE (1.1).

It is noteworthy that the elliptic PPDE (1.1) can reduce to be an elliptic PDE (on the real space). Assume that the nonlinearity G in (1.1) has no dependence on ω, $u : \Omega^e \to \mathbb{R}$ is a smooth solution to (1.1) and that there is a function $v : \mathbb{R}^d \to \mathbb{R}$ such that $u(\omega) = v(\omega_\infty)$ for all $\omega \in \Omega^e$. It follows that the path derivatives reduce to the normal derivatives in the real space, that is, $\partial_\omega u(\omega) = \partial_x v(\omega_\infty)$, $\partial^2_{\omega\omega} u(\omega) = \partial^2_{xx} v(\omega_\infty)$. Then the function v satisfies the corresponding elliptic PDE

$$-G(v, \partial_x v, \partial^2_{xx} v) = 0. \tag{1.4}$$

There is an enormously rich literature studying the elliptic PDE (1.4). In particular, it is known that the solutions to the Dirichlet problem of the equation (1.4) are not always classical (i.e., smooth enough). For example, Nadirashvili and Vladut constructed in [17] a singular solution to an equation $-G(\partial^2_{xx}v) = 0$, where G satisfies the uniform ellipticity condition. A type of weak solutions, viscosity solutions, was introduced by Crandall and Lions [5] to study the equations like the one (1.4), and turns out to be very useful. Since the PDE (1.4) is a special case of the PPDE (1.1), we are motivated to develop a theory of viscosity solutions to elliptic PPDEs.

In this paper, we give a definition of viscosity solutions in the context of elliptic PPDE, and then prove the existence and uniqueness of bounded, uniformly continuous and time-invariant viscosity solutions to the PPDE (1.1) under certain conditions. We try to keep the structure of the paper close to that of Ekren, Touzi and Zhang [11], in which the authors studied the viscosity solutions to parabolic PPDEs. As in [11], our main idea is to construct a viscosity solution to (1.1) by an approximation of piecewise smooth solutions provided by the path-frozen PDEs. Further, we prove the viscosity solution we construct is the unique one through a partial comparison result (i.e., the comparison between a viscosity subsolution and a piecewise smooth supersolution). There are new difficulties in the elliptic context, for example, we need to handle the boundary of Dirichlet problem (in particular, the discontinuity of the hitting time of the boundary H_Q), and we are not allowed to apply certain changes of variables (e.g., $\tilde{u}_t := e^{rt}u_t$), which are quite convenient in the parabolic context. In particular, our argument to verify the uniform continuity of the constructed viscosity solution is new, and quite different from the argument in [11]. Since the path-frozen PDEs do not conserve the uniform continuity of the data of the problem, in [11] the authors require additional uniform continuity assumptions (see their Assumption 3.5) to ensure the uniform continuity of the constructed viscosity solution. Curiously, we observe in the elliptic case that the solutions $\theta^{\omega,\varepsilon}$ to the path-frozen PDEs are "almost" (with an error ε) uniform continuous in the parameter ω, that is,

$$\left|\theta^{\omega^1,\varepsilon} - \theta^{\omega^2,\varepsilon}\right| \le \varepsilon + \rho(2\varepsilon) + C_\varepsilon \rho\big(d^e\big(\omega^1, \omega^2\big)\big), \qquad \text{for some modulus of continuity } \rho$$

[see (5.10) below for the more accurate result], and this intermediate result leads to the uniform continuity of the constructed viscosity solution without any extra assumptions. By comparing to the parabolic context, we think the above property is *intrinsically elliptic*.

We also provide an application of elliptic PPDE to the problem of superhedging a time invariant derivative security under uncertain volatility model. This is a classical time homogeneous stochastic control problem motivated by the application in financial mathematics.

The rest of paper is organized as follows. Section 2 introduces the main notation, as well as the notion of time-invariance, and recalls the result of optimal stopping under nondominated measures. Section 3 defines the viscosity solution of the elliptic PPDEs. Section 4 presents the main results of this paper. In Section 5, we prove the comparison result which implies the uniqueness of viscosity solutions. In Section 6, we verify that a function constructed by a Perron-type approach is an viscosity solution, so the existence follows. We present in Section 7 an application of elliptic PPDE in the field of financial mathematics. Finally, we complete some proofs in the Appendix.

2. Preliminary. Let $\Omega := \{\omega \in C(\mathbb{R}^+, \mathbb{R}^d) : \omega_0 = 0\}$ be the set of continuous paths starting from the origin, B be the canonical process, $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ be the filtration generated by B, $\mathcal{T}$ be the set of all $\mathbb{F}$-stopping times, and $\mathbb{P}_0$ be the Wiener measure.

Denote the L_∞-norm on the continuous path space Ω by $\|\omega\|_\infty := \sup_{s\le\infty} |\omega_s|$. Introduce the concatenation of the continuous paths

$$(\omega \otimes_t \omega')(s) := \omega_s 1_{[0,t)}(s) + (\omega_t + \omega'_{s-t}) 1_{[t,\infty)}(s) \quad \text{for } \omega, \omega' \in \Omega \text{ and } s, t \in \mathbb{R}^+. \tag{2.1}$$

Given a random variable $\xi : \Omega \to \mathbb{R}$ and a process $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$, we define the shifted random variable and the shifted process

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \qquad X^{t,\omega}(s, \omega') := X(t + s, \omega \otimes_t \omega').$$

For a $\tau \in \mathcal{T}$, we often write $\xi^{\tau,\omega}$ (resp. $X^{\tau,\omega}$) instead of $\xi^{\tau(\omega),\omega}$ (resp., $X^{\tau(\omega),\omega}$) for simplicity.

In this paper, we focus on a subset of Ω denoted by Ω^e, which will be considered as the solution space of elliptic PPDEs. Define

$$\Omega^e := \{\omega \in \Omega : \omega = \omega_{t\wedge\cdot} \text{ for some } t \ge 0\}, \quad \text{that is, the set of all paths with flat tails.}$$

We denote the starting of the flat fail of a path $\omega \in \Omega^e$ by

$$\bar{t}(\omega) := \min\{t : \omega = \omega_{t\wedge\cdot}\} \qquad \text{for all } \omega \in \Omega^e.$$

Recall the definition of the concatenation in (2.1). For $\omega \in \Omega^e$, $\omega' \in \Omega$ and $\xi : \Omega \to \mathbb{R}$, we define

$$(\omega \bar{\otimes} \omega')(s) := (\omega \otimes_{\bar{t}(\omega)} \omega')(s) \quad \text{and} \quad \xi^{\omega}(\omega') := \xi^{\bar{t}(\omega),\omega}(\omega') = \xi(\omega \bar{\otimes} \omega').$$

In our arguments, we will be interested in the subsets in Ω^e of some particular form. Denote by

$\mathcal{R}$ the set of all open, bounded and convex subsets of $\mathbb{R}^d$ containing 0.

We are interested in the subsets in Ω^e corresponding to $D \in \mathcal{R}$:

$$\mathcal{D} := \{\omega \in \Omega^e : \omega_t \in D \text{ for all } t \geq 0\}. \tag{2.2}$$

By defining the stopping time

$$\mathrm{H}_D := \inf\{t \geq 0 : \omega_t \notin D\}, \quad \text{and the set} \quad \mathcal{H} := \{\mathrm{H}_D : D \in \mathcal{R}\},$$

we may further define the boundary and the cloture of $\mathcal{D}$:

$$\partial\mathcal{D} := \{\omega \in \Omega^e : \bar{t}(\omega) = H_D(\omega)\}, \qquad \mathrm{cl}(\mathcal{D}) := \mathcal{D} \cup \partial\mathcal{D}.$$

Elliptic equations are devoted to model time-invariant phenomena, and in the path space the time-invariance property can be formulated mathematically as follows.

DEFINITION 2.1. Define the distance on Ω^e:

$$d^e(\omega, \omega') := \inf_{\ell \in \mathcal{I}} \sup_{t \in \mathbb{R}^+} |\omega_{\ell(t)} - \omega'_t| \qquad \text{for } \omega, \omega' \in \Omega^e,$$

where $\mathcal{I}$ is the set of all increasing bijections from $\mathbb{R}^+$ to $\mathbb{R}^+$. We say ω is equivalent to ω', if $d^e(\omega, \omega') = 0$. A function u on Ω^e is time-invariant, if u is well defined on the equivalent class, that is,

$$u(\omega) = u(\omega') \qquad \text{whenever } d^e(\omega, \omega') = 0.$$

For a subset $\mathcal{D} \subset \Omega^e$, $C(\mathcal{D})$ denotes the set of all functions $\varphi : \mathcal{D} \to \mathbb{R}$ continuous with respect to $d^e(\cdot, \cdot)$. The notation $C(\mathcal{D}; \mathbb{R}^d)$, $C(\mathcal{D}; \mathbb{S}^d)$ ($\mathbb{S}^d$ denotes the set of $d \times d$ symmetric matrices) are also used when we need to emphasize the space in which the functions take values.

Finally, we say $u \in \mathrm{BUC}(\mathcal{D})$ if $u : \mathcal{D} \to \mathbb{R}$ is bounded and uniformly continuous with respect to $d^e(\cdot, \cdot)$, that is, there exists a modulus of continuity ρ such that

$$|u(\omega^1) - u(\omega^2)| \leq \rho(d^e(\omega^1, \omega^2)) \qquad \text{for all } \omega^1, \omega^2 \in \mathcal{D}. \tag{2.3}$$

REMARK 2.2. For any modulus of continuity ρ, the concave envelop $\hat{\rho} := \mathrm{conc}[\rho]$ is still a modulus of continuity for the same function. Thus, without loss of generality, we may assume that moduli of continuity are concave.

EXAMPLE 2.3. Let us show an example of two equivalent paths of which the L_∞-distance is large. Let $(t_i, x_i) \in \mathbb{R}^+ \times \mathbb{R}^d$ for each $1 \leq i \leq n$. We denote by

$$\omega := \mathrm{Lin}\{(0, 0), (t_1, x_1), \ldots, (t_n, x_n)\} \tag{2.4}$$

the linear interpolation of the points with a flat tail extending to $t = \infty$ ($\omega_t = x_n$, for $t \geq t_n$). Then by defining another path

$$\omega' := \mathrm{Lin}\{(0, 0), (t'_1, x_1), \ldots, (t'_n, x_n)\},$$

we clearly have $d^e(\omega, \omega') = 0$ regardless of the choice of $\{t'_i\}_{1 \leq i \leq n}$. However, the L_∞-distance $\|\omega - \omega'\|_\infty$ can reach $\max_{1 \leq i, j \leq n} |x_i - x_j|$ by choosing a particular sequence $\{t'_i\}_{1 \leq i \leq n}$.

EXAMPLE 2.4. We show some examples of time-invariant functions:

- Markovian case: Assume that there exists $\bar{u} : \mathbb{R}^d \to \mathbb{R}$ such that $u(\omega) = \bar{u}(\omega_{\bar{t}(\omega)})$. Since $|\omega^1_{\bar{t}(\omega^1)} - \omega^2_{\bar{t}(\omega^2)}| \le d^e(\omega^1, \omega^2)$ for all $\omega^1, \omega^2 \in \Omega^e$, u is time-invariant.
- Maximum dependent case: Assume that there exists $\bar{u} : \mathbb{R} \to \mathbb{R}$ such that $u(\omega) = \bar{u}(\|\omega\|_\infty)$. Note that $\|\omega\|_\infty = d^e(\omega, 0)$ and $d^e(\omega^1, 0) - d^e(\omega^2, 0) \le d^e(\omega^1, \omega^2)$. Thus, $\|\omega^1\|_\infty = \|\omega^2\|_\infty$ whenever $d^e(\omega^1, \omega^2) = 0$. Consequently, u is time-invariant.

Here is some useful notation:

- $O_L := \{x \in \mathbb{R}^d : |x| < L\}$, and $\overline{O}_L := \{x \in \mathbb{R}^d : |x| \le L\}$.
- $[aI_d, bI_d] := \{\gamma \in \mathbb{S}_d : aI_d \le \gamma \le bI_d\}$.
- $\mathbb{H}^0(E)$ denotes the set of all $\mathbb{F}$-progressively measurable processes taking values in the set E, and in particular $\mathbb{H}^0_L := \mathbb{H}^0([\sqrt{2/L}I_d, \sqrt{2L}I_d])$ for $L > 0$.
- Denote the quadratic variation of the path ω by $\langle\omega\rangle_t := |\omega_t|^2 - 2\int_0^t \omega_s \, d\omega_s$, where $\int_0^\cdot \omega_s \, d\omega_s$ is the pathwise stochastic integral defined in Karandikar [13].
- Given $\gamma, \eta \in \mathbb{S}^d$, we define $\gamma : \eta := \mathrm{Trace}[\gamma\eta]$;
- Given a function $\varphi : \Omega \to \mathbb{R}^d$, we may define the corresponding process

$$\varphi_t(\omega) := \varphi(\omega_{t\wedge\cdot}). \tag{2.5}$$

We next introduce the *smooth functions* on the space Ω^e. First, for every constant $L > 0$, we denote by $\mathcal{P}^L$ the collection of all continuous semimartingale measures $\mathbb{P}$ on Ω whose drift and diffusion belong to $\mathbb{H}^0(\overline{O}_L)$ and $\mathbb{H}^0_L$, respectively. More precisely, let $\tilde{\Omega} := \Omega \times \Omega \times \Omega$ be an enlarged canonical space and $\tilde{B} := (B, A, M)$ be the canonical process. A probability measure $\mathbb{P} \in \mathcal{P}^L$ if there exists an extension $\mathbb{Q}^{\alpha,\beta}$ of $\mathbb{P}$ on $\tilde{\Omega}$ such that

$$\begin{aligned} &B = A + M, \qquad A \text{ is absolutely continuous, } M \text{ is a martingale,} \\ &\|\alpha^{\mathbb{P}}\|_\infty \le L, \qquad \beta^{\mathbb{P}} \in \mathbb{H}^0_L \qquad \text{where } \alpha^{\mathbb{P}}_t := \frac{dA_t}{dt}, \beta^{\mathbb{P}}_t := \sqrt{\frac{d\langle M\rangle_t}{dt}}, \\ &\mathbb{Q}^{\alpha,\beta}\text{-a.s.} \end{aligned} \tag{2.6}$$

REMARK 2.5. The definition of $\mathcal{P}^L$ is slightly different from the one in [11], since we urge that the coefficient of diffusion $\beta^{\mathbb{P}} \ge \sqrt{\frac{2}{L}} I_d$.

Further, denote $\mathcal{P}^\infty := \bigcup_{L>0} \mathcal{P}^L$.

DEFINITION 2.6 (Smooth time-invariant processes). Let $D \in \mathcal{R}$, and recall $\mathcal{D} \subset \Omega^e$ defined in (2.2). We say $\varphi \in C^2(\mathcal{D})$, if $\varphi \in C(\mathcal{D})$ and there exist $Z \in C(\mathcal{D}; \mathbb{R}^d)$, $\Gamma \in C(\mathcal{D}; \mathbb{S}^d)$ such that

$$d\varphi_t = Z_t \cdot dB_t + \tfrac{1}{2}\Gamma_t : \langle B\rangle_t \qquad \text{for } t \le \mathrm{H}_D, \mathcal{P}^\infty\text{-q.s.}$$

[φ_t is defined in (2.5)], where $\mathcal{P}^\infty$-q.s. means $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}^\infty$. By a direct localization argument, we see that the above Z and Γ, if they exist, are unique. Denote $\partial_\omega u := Z$ and $\partial^2_{\omega\omega} u := \Gamma$.

REMARK 2.7. In the Markovian case mentioned in Example 2.4, if the function $\bar{u} : \mathbb{R}^d \to \mathbb{R}$ is in $C^2(D)$, then it follows from Itô's formula that $u \in C^2(\mathcal{D})$.

REMARK 2.8. In the path-dependent case, Dupire [7] defined derivatives, $\partial_t u$ and $\partial_\omega u$, for process $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$. In particular, the t-derivative is defined as

$$\partial_t u(s, \omega) := \lim_{h \to 0^+} \frac{u(s+h, \omega_{s\wedge\cdot}) - u(s, \omega)}{h}.$$

Also, Dupire and other authors, for example, [4], proved the functional Itô formula for the processes regular in Dupire's sense:

$$du_s = \partial_t u_s \, ds + \partial_\omega u_s \cdot dB_s + \tfrac{1}{2} \partial^2_{\omega\omega} u_s : \langle B \rangle_s, \qquad \mathcal{P}^\infty\text{-q.s.}$$

Note that in the time-invariant case it always holds that $\partial_t u = 0$. Consequently, the processes with Dupire's derivatives in $C(\mathcal{D})$ are also smooth according to our definition.

We next introduce the notation of nonlinear expectations. For a family of probabilities $\mathcal{P}$, a measurable set $A \in \mathcal{F}_\infty$, a random variable ξ, we define the capacity $\mathcal{C}$, the sublinear expectation $\overline{\mathcal{E}}$ and the superlinear expectation $\underline{\mathcal{E}}$:

$$\mathcal{C}^{\mathcal{P}}[A] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[A], \qquad \overline{\mathcal{E}}^{\mathcal{P}}[\xi] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi], \qquad \underline{\mathcal{E}}^{\mathcal{P}}[\xi] := \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi].$$

We also define the optimal stopping operator (in other words, the Snell envelop) $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$:

$$\overline{\mathcal{S}}^{\mathcal{P}}_t[X](\omega) := \sup_{\tau \in \mathcal{T}} \overline{\mathcal{E}}^{\mathcal{P}}[X^{t,\omega}_\tau], \qquad \underline{\mathcal{S}}^{\mathcal{P}}_t[X](\omega) := \inf_{\tau \in \mathcal{T}} \underline{\mathcal{E}}^{\mathcal{P}}[X^{t,\omega}_\tau],$$

with the barrier process X.

Recall the family of probabilities $\mathcal{P}^L$ defined above. For simplicity, we denote

$$\mathcal{C}^L := \mathcal{C}^{\mathcal{P}^L}, \qquad \overline{\mathcal{E}}^L := \overline{\mathcal{E}}^{\mathcal{P}^L}, \qquad \underline{\mathcal{E}}^L := \underline{\mathcal{E}}^{\mathcal{P}^L}, \qquad \overline{\mathcal{S}}^L := \overline{\mathcal{S}}^{\mathcal{P}^L}, \qquad \underline{\mathcal{S}}^L := \underline{\mathcal{S}}^{\mathcal{P}^L}.$$

The existing literature gives the following results.

LEMMA 2.9 (Tower property, Nutz and van Handel [20]). *For a bounded random variable ξ, we have*

$$\overline{\mathcal{E}}^L[\xi] = \overline{\mathcal{E}}^L[\overline{\mathcal{E}}^L[\xi^{\tau(\cdot),\cdot}]] \qquad \textit{for all } \tau \in \mathcal{T}.$$

LEMMA 2.10 (Snell envelop characterization, Ekren, Touzi and Zhang [9]). *Let $T \in \mathbb{R}^+$, $\mathrm{H}_D \in \mathcal{H}$ and $X \in \mathrm{BUC}(\mathcal{D})$. Denote $\mathrm{H} := \mathrm{H}_D \wedge T$. Define the Snell envelope and the corresponding first hitting time of the obstacles*:

$$Y := \overline{\mathcal{S}}^L[X_{\mathrm{H}\wedge\cdot}], \qquad \tau^* := \inf\{t \geq 0 : Y_t = X_t\}.$$

Then $Y \geq X$, $Y_{\tau^*} = X_{\tau^*}$ and τ^* is an optimal stopping time, that is, $Y_0 = \overline{\mathcal{E}}^L[X_{\tau^*}]$.

It is also important to have the following result, of which the proof can be found in the Appendix.

PROPOSITION 2.11. *Let $D \in \mathcal{R}$, and denote*

$$D^x := \{y : x + y \in D\} \qquad \text{for } x \in D. \tag{2.7}$$

Assume that O is also in $\mathcal{R}$. Define a sequence of stopping times $\{\mathrm{H}_n\}_{n\in\mathbb{N}}$:

$$\mathrm{H}_0 = 0, \qquad \mathrm{H}_n := \inf\{s \geq \mathrm{H}_{n-1} : B_s - B_{\mathrm{H}_{n-1}} \notin O\}, \qquad n \geq 1. \tag{2.8}$$

Then we have:

(i) $\lim_{n\to\infty} \mathcal{C}^L[\mathrm{H}_n < T] = 0$ *for all* $T \in \mathbb{R}^+$,
(ii) $\overline{\mathcal{E}}^L[\mathrm{H}_D] < \infty$,
(iii) $\lim_{T\to\infty} \sup_{x\in D} \mathcal{C}^L[\mathrm{H}_{D^x} > T] = 0$,
(iv) $\lim_{n\to\infty} \sup_{x\in D} \mathcal{C}^L[\mathrm{H}_n < \mathrm{H}_{D^x}] = 0$.

3. Fully nonlinear elliptic PPDEs.

3.1. *Definition of viscosity solutions of uniformly elliptic PPDEs.* Let $Q \in \mathcal{R}$ and consider $\mathcal{Q}$ ($:= \{\omega \in \Omega^e : \omega_t \in Q$ for all $t \geq 0\}$) as the domain of Dirichlet problem of the PPDE:

$$\begin{aligned} &\mathcal{L}u(\omega) := -G(\omega, u, \partial_\omega u, \partial^2_{\omega\omega} u) = 0 \qquad \text{for } \omega \in \mathcal{Q}, \\ &u = \xi \qquad \text{on } \partial\mathcal{Q}, \end{aligned} \tag{3.1}$$

with nonlinearity G and boundary condition by ξ.

ASSUMPTION 3.1. The nonlinearity $G : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ satisfies:

(i) $|G(\cdot, 0, 0, 0)| \leq C_0$;
(ii) G is uniformly elliptic, that is, there exists $L_0 > 0$ such that for all (ω, y, z)

$$G(\omega, y, z, \gamma_1) - G(\omega, y, z, \gamma_2) \geq \frac{1}{L_0} I_d : (\gamma_1 - \gamma_2) \qquad \text{for all } \gamma_1 \geq \gamma_2.$$

(iii) G is uniformly continuous on Ω^e with respect to $d^e(\cdot, \cdot)$, and is uniformly Lipschitz continuous in (y, z, γ) with a Lipschitz constant L_0;

(iv) G is uniformly decreasing in y, that is, there exists a function $\lambda : \mathbb{R} \to \mathbb{R}$ strictly increasing and continuous, $\lambda(0) = 0$, and

$$G(\omega, y_1, z, \gamma) - G(\omega, y_2, z, \gamma) \geq \lambda(y_2 - y_1),$$
$$\text{for all } y_2 \geq y_1, (\omega, z, \gamma) \in \Omega^e \times \mathbb{R}^d \times \mathbb{S}^d.$$

For any time-invariant function u on Ω^e and $\omega \in \mathcal{Q}$, we define the set of test functions:

$$\underline{\mathcal{A}}^{\mathcal{P}} u(\omega) := \{\varphi : \varphi \in C^2(\mathcal{O}_\varepsilon) \text{ and } (\varphi - u^\omega)_0 = \underline{\mathcal{S}}_0^{\mathcal{P}}[(\varphi - u^\omega)_{\mathrm{H}_\varepsilon \wedge \cdot}] \text{ for some } \varepsilon > 0\},$$
$$\overline{\mathcal{A}}^{\mathcal{P}} u(\omega) := \{\varphi : \varphi \in C^2(\mathcal{O}_\varepsilon) \text{ and } (\varphi - u^\omega)_0 = \overline{\mathcal{S}}_0^{\mathcal{P}}[(\varphi - u^\omega)_{\mathrm{H}_\varepsilon \wedge \cdot}] \text{ for some } \varepsilon > 0\},$$
$$\text{with } \mathrm{H}_\varepsilon := \mathrm{H}_{O_\varepsilon} \wedge \varepsilon.$$

We call H_ε a localization of test function φ. In particular, we denote $\overline{\mathcal{A}}^L := \overline{\mathcal{A}}^{\mathcal{P}^L}$, $\underline{\mathcal{A}}^L := \underline{\mathcal{A}}^{\mathcal{P}^L}$, as we choose $\mathcal{P}^L$ as the family of probabilities. Now, we define the viscosity solutions to the elliptic PPDE (3.1).

DEFINITION 3.2. Let $\{u_t\}_{t \in \mathbb{R}^+}$ be a time-invariant progressively measurable process.

(i) u is a $\mathcal{P}$-viscosity subsolution (resp., supersolution) of PPDE (3.1), if we have for all $\omega \in \mathcal{Q}$ and $\varphi \in \underline{\mathcal{A}}^{\mathcal{P}} u(\omega)$ [resp., $\varphi \in \overline{\mathcal{A}}^{\mathcal{P}} u(\omega)$]:

$$-G(\omega, u(\omega), \partial_\omega \varphi_0, \partial^2_{\omega\omega} \varphi_0) \leq (\text{resp. } \geq) \, 0.$$

(ii) u is a $\mathcal{P}$-viscosity solution of PPDE (3.1), if u is both a $\mathcal{P}$-viscosity subsolution and a $\mathcal{P}$-viscosity supersolution of PPDE (3.1).

By very similar arguments as in the proof of Theorem 3.16 and Theorem 5.1 in [10], we may easily prove the following.

THEOREM 3.3 (Consistency with classical solution). *Let Assumption 3.1 hold true and $L > 0$. Given a function $u \in C^2(\mathcal{Q})$, then u is a $\mathcal{P}^L$-viscosity supersolution* (*resp*., *subsolution*, *solution*) *to PPDE* (3.1) *if and only if u is a classical supersolution* (*resp*., *subsolution*, *solution*).

THEOREM 3.4 (Stability). *Let $L > 0$, G satisfy Assumption 3.1, and $u \in$ BUC($\mathcal{Q}$). Assume that*:

(i) *for any $\varepsilon > 0$, there exist G^ε and $u^\varepsilon \in$ BUC($\mathcal{Q}$) such that G^ε satisfies Assumption 3.1 and u^ε is a $\mathcal{P}^L$-viscosity subsolution* (*resp*., *supersolution*) *of PPDE* (3.1) *with generator G^ε*;

(ii) *as* $\varepsilon \to 0$, $(G^\varepsilon, u^\varepsilon)$ *converge to* (G, u) *locally uniformly in the following sense*: *for any* $(\omega, y, z, \gamma) \in \Omega^e \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, *there exits* $\delta > 0$ *such that*

$$\lim_{\varepsilon\to 0} \sup_{(\tilde{\omega},\tilde{y},\tilde{z},\tilde{\gamma})\in O_\delta(\omega,y,z,\gamma)} \left[\left|(G^\varepsilon - G)^\omega(\tilde{\omega}, \tilde{y}, \tilde{z}, \tilde{\gamma})\right| + \left|(u^\varepsilon - u)^\omega(\tilde{\omega})\right|\right] = 0,$$

where we abuse the notation O_δ *to denote the* δ*-ball in the corresponding space.*

Then u *is a* $\mathcal{P}^L$*-viscosity solution* (*resp.*, *supersolution*) *of PPDE* (3.1) *with generator* G.

3.2. *Equivalent definition by semijets.* Following the standard theory of viscosity solutions for PDEs, we may also define viscosity solutions via semijets. Similar to [24] and [23], we introduce the notion of semijets in the context of PPDE. First, denote functions:

$$\psi^{\alpha,\beta}(\omega) = \alpha \cdot \omega_{\bar{t}(\omega)} + \tfrac{1}{2}\beta : \omega_{\bar{t}(\omega)}\omega^\intercal_{\bar{t}(\omega)}.$$

We next define the sub and superjets:

$$\underline{\mathcal{J}}^L u(\omega) := \{(\alpha, \beta) : \psi^{\alpha,\beta} \in \underline{\mathcal{A}}^L u(\omega)\} \quad \text{and}$$
$$\overline{\mathcal{J}}^L u(\omega) := \{(\alpha, \beta) : \psi^{\alpha,\beta} \in \overline{\mathcal{A}}^L u(\omega)\}.$$

PROPOSITION 3.5. *Let* $u \in \mathrm{BUC}(\mathcal{Q})$. *Then* u *is an* $\mathcal{P}^L$*-viscosity subsolution* (*resp.*, *supersolution*) *of PPDE* (3.1), *if and only if for any* $\omega \in \mathcal{Q}$,

$$-G(\omega, u(\omega), \alpha, \beta) \le (\textit{resp. } \ge) \, 0 \qquad \textit{for all } (\alpha, \beta) \in \underline{\mathcal{J}}^L u(\omega) \; \big(\textit{resp. } \overline{\mathcal{J}}^L u(\omega)\big).$$

PROOF. The "only if" part is trivial by the definitions. It remains to prove the "if" part. We only show the result for $\mathcal{P}^L$-viscosity subsolutions, while the result for the supersolution can be proved similarly. Let $\varphi \in \underline{\mathcal{A}}^L u(\omega)$ and $\mathrm{H}_\delta (:= \mathrm{H}_{O_\delta} \wedge \delta)$ be the corresponding localization. Without loss of generality, we may assume that $\omega = \mathbf{0}$ (i.e., $\omega_t = 0$ for all $t \in \mathbb{R}^+$) and $\varphi_0 = u_0$. Define

$$\alpha := \partial_\omega \varphi_0 \quad \text{and} \quad \beta := \partial^2_{\omega\omega}\varphi_0.$$

Let $\varepsilon > 0$. Since the processes $\partial_\omega \varphi$ and $\partial^2_{\omega\omega}\varphi$ are both continuous, there exists $\delta' \le \delta$ such that

$$|\partial_\omega \varphi_t - \alpha| \le \varepsilon \quad \text{and} \quad |\partial^2_{\omega\omega}\varphi_t - \beta| \le \varepsilon, \qquad \text{for } t \le \mathrm{H}_{O_{\delta'}}.$$

Denote $\beta_\varepsilon := \beta + (1 + 2L)\varepsilon$. Then, for all $\tau \in \mathcal{T}$ such that $\tau \le \mathrm{H}_{\delta'}$, we have

$$\begin{aligned} u_0 - \underline{\mathcal{E}}^L\left[(\psi^{\alpha,\beta_\varepsilon} - u)_\tau\right] &= \overline{\mathcal{E}}^L\left[(u - u_0 - \psi^{\alpha,\beta_\varepsilon})_\tau\right] \\ &\le \overline{\mathcal{E}}^L\left[(u - \varphi)_\tau\right] + \overline{\mathcal{E}}^L\left[(\varphi - \varphi_0 - \psi^{\alpha,\beta_\varepsilon})_\tau\right] \\ &\le \overline{\mathcal{E}}^L\left[\int_0^\tau (\partial_\omega\varphi_s - \alpha)\, dB_s + \frac{1}{2}\int_0^\tau (\partial^2_{\omega\omega}\varphi_s - \beta_\varepsilon)\, ds\right] \\ &\le \overline{\mathcal{E}}^L\left[\int_0^\tau \left(L|\partial_\omega\varphi_s - \alpha| + \frac{1}{2}(\partial^2_{\omega\omega}\varphi_s - \beta_\varepsilon)\right) ds\right] \le 0, \end{aligned}$$

where we used the fact that $\varphi \in \underline{\mathcal{A}}^L u(\mathbf{0})$ and the definition of $\mathcal{P}^L$ in (2.6). Consequently, we obtain $(\alpha, \beta_\varepsilon) \in \underline{\mathcal{J}}^L u(\mathbf{0})$, and thus

$$-G\big(0, u(0), \alpha, \beta_\varepsilon\big) \le 0.$$

Finally, thanks to the continuity of G, we obtain the desired result by sending $\varepsilon \to 0$. $\square$

4. Main results. Following Ekren, Touzi and Zhang [11], we introduce the path-frozen PDEs:

$$(E)^\omega_\varepsilon \qquad \mathbf{L}^\omega v := -G\big(\omega, v, \partial_x v, \partial^2_{xx} v\big) = 0 \qquad \text{on } O_\varepsilon(\omega) := O_\varepsilon \cap Q^\omega, \tag{4.1}$$
$$\text{with } Q^\omega := Q^{\omega_{\bar{t}(\omega)}}.$$

[Recall the notation in (2.7).] Note that ω is a parameter rather than a variable in the above PDE. Similar to [11], our well-posedness result relies on the following condition on the PDE $(E)^\omega_\varepsilon$.

ASSUMPTION 4.1. For $\varepsilon > 0$, $\omega \in \mathcal{Q}$ and $h \in C(\partial O_\varepsilon(\omega))$, we have $\overline{v} = \underline{v}$, where

$$\overline{v}(x) := \inf\big\{w(x) : w \in C^2_0\big(O_\varepsilon(\omega)\big), \mathbf{L}^\omega w \ge 0 \text{ on } O_\varepsilon(\omega), w \ge h \text{ on } \partial O_\varepsilon(\omega)\big\},$$
$$\underline{v}(x) := \sup\big\{w(x) : w \in C^2_0\big(O_\varepsilon(\omega)\big), \mathbf{L}^\omega w \le 0 \text{ on } O_\varepsilon(\omega), w \le h \text{ on } \partial O_\varepsilon(\omega)\big\},$$

and $C^2_0(O_\varepsilon(\omega)) := C^2(O_\varepsilon(\omega)) \cap C(\mathrm{cl}(O_\varepsilon(\omega)))$.

In this paper, we call the classical notion of viscosity solution to PDE (see, e.g., [5]) as Crandall–Lions (C–L) viscosity solution, in order to distinguish the one to PPDE.

EXAMPLE 4.2. Assume that $g : \mathbb{S}^d \to \mathbb{R}$ is convex, and that the corresponding uniformly elliptic PDE

$$\mathbf{L}w = -g\big(\partial^2_{xx} w\big) = 0 \qquad \text{on } O, \qquad w = h \qquad \text{on } \partial O$$

has a C–L viscosity solution. Then according to Caffareli and Cabre [2] (Theorem 6.6 on page 54), the C–L viscosity solution has the interior C^2-regularity. In particular, this equation satisfies Assumption 4.1.

The rest of the paper is devoted to prove the following two main results.

THEOREM 4.3 (Comparison result). *Let Assumptions 3.1 and 4.1 hold true, and $u, v \in \mathrm{BUC}(\mathcal{Q})$ be a $\mathcal{P}^L$-viscosity sub and supersolution to the PPDE (3.1) for some $L > 0$, respectively. If $u \le v$ on $\partial \mathcal{Q}$, then we have $u \le v$ on $\mathcal{Q}$.*

THEOREM 4.4 (Well-posedness). *Let Assumptions 3.1 and 4.1 hold true, and $\xi \in \mathrm{BUC}(\partial \mathcal{Q})$. Then the PPDE (3.1) has a unique $\mathcal{P}^L$-viscosity solution in $\mathrm{BUC}(\mathcal{Q})$ for $L \ge L_0$.*

5. Comparison result.

5.1. *Partial comparison.* Similar to [11], we introduce the class of piecewise smooth processes in our time-invariant context.

DEFINITION 5.1. Let $u : \mathcal{Q} \to \mathbb{R}$. We say $u \in \overline{C}^2(\mathcal{Q})$, if u is bounded, process $\{u_t\}_{t\in\mathbb{R}^+}$ is continuous in t, and there exists an increasing sequence of $\mathbb{F}$-stopping times $\{\mathrm{H}_n\}_{n\geq 0}$ ($\mathrm{H}_0 = 0$) such that:

(i) for each $i \geq 0$ and $\omega \in \mathcal{Q}$, $\Delta\mathrm{H}_{i,\omega} := \mathrm{H}_{i+1}^{\mathrm{H}_i,\omega} - \mathrm{H}_i(\omega)$ is a stopping time in $\mathcal{H}$ whenever $\mathrm{H}_i(\omega) < \mathrm{H}_Q(\omega) < \infty$, that is, there is a set $O_{i,\omega} \in \mathcal{R}$ such that $\Delta\mathrm{H}_{i,\omega}(\omega') = \inf\{t : \omega'_t \notin O_{i,\omega}\}$;

(ii) for each $i \geq 0$ and $\omega \in \mathcal{Q}$, we have

$$u^{\omega_{\mathrm{H}_i\wedge\cdot}} \in \mathrm{BUC}(\mathcal{O}_{i,\omega}) \cap C^2(\mathcal{O}_{i,\omega});$$

(iii) $\{i : \mathrm{H}_i(\omega) < \mathrm{H}_Q(\omega)\}$ is finite $\mathcal{P}^\infty$-q.s. and $\lim_{i\to\infty} \mathcal{C}_0^L[\mathrm{H}_i^\omega < \mathrm{H}_Q^\omega] = 0$ for all $\omega \in \mathcal{Q}$ and $L > 0$.

The rest of the subsection is devoted to the proof of the following partial comparison result.

PROPOSITION 5.2. *Let Assumption 3.1 hold true. Let $u \in \overline{C}^2(\mathcal{Q})$, $v \in$ $\mathrm{BUC}(\mathcal{Q})$ be a $\mathcal{P}^L$-viscosity sub and supersolution of PPDE (3.1) for some $L > 0$, respectively. If $u \leq v$ on ∂Q, then $u \leq v$ in $\mathrm{cl}(\mathcal{Q})$. A similar result holds if we exchange the roles of u and v.*

In preparation to the proof of Proposition 5.2, we prove the following lemma.

LEMMA 5.3. *Let $T > 0$, $D \in \mathcal{R}$ and $X \in \mathrm{BUC}(\mathcal{D})$ and nonnegative. Denote $\mathrm{H} := \mathrm{H}_D \wedge T$. Assume that $X_0 > \overline{\mathcal{E}}^L[X_\mathrm{H}]$, then there exist $\omega^* \in \mathcal{D}$ and $t^* := \bar{t}(\omega^*)$ such that*

$$X(\omega^*) = \overline{\mathcal{S}}_{t^*}^L[X_{\mathrm{H}\wedge\cdot}](\omega^*) \quad \textit{and} \quad X(\omega^*) > 0.$$

PROOF. Denote Y as the Snell envelop of $X_{\mathrm{H}\wedge\cdot}$, that is, $Y_t := \overline{\mathcal{S}}_t^L[X_{\mathrm{H}\wedge\cdot}]$. By Lemma 2.10, the stopping time $\tau^* := \inf\{t : X_t = Y_t\}$ defines an optimal stopping rule. So, we have

$$\overline{\mathcal{E}}^L[X_{\tau^*}] = Y_0 \geq X_0 > \overline{\mathcal{E}}^L[X_\mathrm{H}].$$

Hence, $\{\tau^* < \mathrm{H}\} \neq \phi$. Suppose that $X_{\tau^*} = 0$ on $\{\tau^* < \mathrm{H}\}$. Then

$$0 = X_{\tau^*}1_{\{\tau^*<\mathrm{H}\}}(\omega) = Y_{\tau^*}1_{\{\tau^*<\mathrm{H}\}}(\omega) \geq \overline{\mathcal{E}}^L\big[(X_\mathrm{H})^{\tau^*(\omega),\omega}\big]1_{\{\tau^*<\mathrm{H}\}}(\omega) \geq 0.$$

The last inequality is due to the fact $X \geq 0$. Therefore, $X_{\mathrm{H}} 1_{\{\tau^* < \mathrm{H}\}} = 0$. It follows that $X_{\tau^*} = X_{\mathrm{H}}$ on $\{\tau^* < \mathrm{H}\}$. Thus, we conclude that

$$X_0 \leq Y_0 = \overline{\mathcal{E}}^L[X_{\tau^*}] = \overline{\mathcal{E}}^L[X_{\mathrm{H}}] < X_0.$$

This contradiction implies that $\{\tau^* < \mathrm{H}, X_{\tau^*} > 0\} \neq \phi$. Finally, take $\omega \in \{\tau^* < \mathrm{H}, X_{\tau^*} > 0\}$, and then $\omega^* := \omega_{\tau^*(\omega)\wedge\cdot}$ is a path satisfying the requirements. □

PROOF OF PROPOSITION 5.2. Recall the notation H_i, $\Delta\mathrm{H}_{i,\omega}$ and $O_{i,\omega}$ in Definition 5.1. We divide the proof in two steps.

Step 1. We first show that

$$\begin{aligned}(u-v)^+_{\mathrm{H}_i}(\omega) &\leq \overline{\mathcal{E}}^L\big[(u^{\mathrm{H}_i,\omega} - v^{\mathrm{H}_i,\omega})^+_{\Delta\mathrm{H}_{i,\omega}}\big] \\ &= \overline{\mathcal{E}}^L\big[\big((u_{\mathrm{H}_{i+1}} - v_{\mathrm{H}_{i+1}})^+\big)^{\mathrm{H}_i,\omega}\big] \qquad \text{for all } i \geq 0, \omega \in \mathcal{Q}.\end{aligned}$$

Without loss of generality, we set $i = 0$. Assume the contrary, that is,

$$(u-v)^+(\mathbf{0}) - \overline{\mathcal{E}}^L\big[(u-v)^+_{\mathrm{H}_1}\big] > 0.$$

Denote $X := (u-v)^+$. Since $\lim_{T\to\infty} \mathcal{C}^L[\mathrm{H}_1 \geq T] = 0$ (Proposition 2.11) and u, v are both bounded, there exists $T > 0$ such that

$$X_0 - \overline{\mathcal{E}}^L[X_{\mathrm{H}}] > 0 \qquad \text{with } \mathrm{H} := \mathrm{H}_1 \wedge T.$$

Then, by Lemma 5.3, there exists $\omega^* \in \mathcal{O}_{0,\mathbf{0}}$ and $t^* := \bar{t}(\omega^*)$ such that

$$X(\omega^*) = \overline{\mathcal{S}}^L_{t^*}[X_{\mathrm{H}\wedge\cdot}](\omega^*) \quad \text{and} \quad X(\omega^*) > 0. \tag{5.1}$$

Since $u \in \overline{C}^2(\mathcal{Q})$, in particular $u \in C^2(\mathcal{O}_{0,\mathbf{0}})$, we have $\varphi := u^{\omega^*} \in C^2(\mathcal{O}^{\omega^*}_{0,\mathbf{0}})$ (recall that for a set $D \in \mathcal{R}$ and $\omega \in \Omega^e$, we define $D^\omega := D^{\omega_{\bar{t}(\omega)}}$ and correspondingly we have the definition of $\mathcal{D}^\omega$). Together with (5.1), we get $\varphi \in \overline{\mathcal{A}}^L v(\omega^*)$. By the $\mathcal{P}^L$-viscosity supersolution property of v and Assumption 3.1, this implies that

$$\begin{aligned}0 &\leq -G(\cdot, v, \partial_\omega\varphi_0, \partial^2_{\omega\omega}\varphi_0)(\omega^*) \leq -G(\cdot, u, \partial_\omega u, \partial^2_{\omega\omega} u)(\omega^*) - \lambda(X(\omega^*)) \\ &< -G(\cdot, u, \partial_\omega u, \partial^2_{\omega\omega} u)(\omega^*).\end{aligned}$$

This is in contradiction with the classical subsolution property of u.

Step 2. By the result of Step 1 and the tower property of $\overline{\mathcal{E}}^L$ stated in Lemma 2.9, we have

$$\overline{\mathcal{E}}^L\big[(u-v)^+_{\mathrm{H}_i}\big] \leq \overline{\mathcal{E}}^L\big[(u-v)^+_{\mathrm{H}_{i+1}}\big] \qquad \text{for all } i \geq 0.$$

It follows by induction that

$$(u-v)^+(\mathbf{0}) \leq \overline{\mathcal{E}}^L\big[(u-v)^+_{\mathrm{H}_i}\big] \qquad \text{for all } i \geq 1.$$

Then we obtain

$$(u-v)^+(\mathbf{0}) \le \overline{\mathcal{E}}^L\big[(u-v)^+_{\mathrm{H}_Q}\big] + \overline{\mathcal{E}}^L\big[(u-v)^+_{\mathrm{H}_i} - (u-v)^+_{\mathrm{H}_Q}\big].$$

By Proposition 2.11, we have $\lim_{i\to\infty}\mathcal{C}^L[\mathrm{H}_i < \mathrm{H}_Q] = 0$. Since u, v are both bounded, we have

$$(u-v)^+(\mathbf{0}) \le \overline{\mathcal{E}}^L\big[(u-v)^+_{\mathrm{H}_Q}\big] = 0. \qquad \square$$

5.2. *The Perron type construction.* Define the following two functions:

$$\overline{u}(\omega) := \inf\{\psi(\omega) : \psi \in \overline{\mathcal{D}}^\xi_Q(\omega)\}, \qquad \underline{u}(\omega) := \sup\{\psi(\omega) : \psi \in \underline{\mathcal{D}}^\xi_Q(\omega)\}, \tag{5.2}$$

where

$$\overline{\mathcal{D}}^\xi_Q(\omega) := \{\psi \in \overline{C}^2(\mathcal{Q}^\omega) : \mathcal{L}^\omega\psi \ge 0 \text{ on } \mathcal{Q}, \psi \ge \xi^\omega \text{ on } \partial\mathcal{Q}\},$$
$$\underline{\mathcal{D}}^\xi_Q(\omega) := \{\psi \in \overline{C}^2(\mathcal{Q}^\omega) : \mathcal{L}^\omega\psi \le 0 \text{ on } \mathcal{Q}, \psi \le \xi^\omega \text{ on } \partial\mathcal{Q}\}.$$

As a direct corollary of Proposition 5.2, we have the following.

COROLLARY 5.4. *Let $L > 0$ be constant. Under Assumption 3.1, for all $\mathcal{P}^L$-viscosity supersolutions (resp., subsolution) $u \in \mathrm{BUC}(\mathcal{Q})$ such that $u \ge \xi$ (resp., $u \le \xi$) on $\partial\mathcal{Q}$, we have $u \ge \underline{u}$ (resp., $u \le \overline{u}$) on $\mathcal{Q}$.*

In order to prove the comparison result of Theorem 4.3, it remains to show the following result.

PROPOSITION 5.5. *Let $\xi \in \mathrm{BUC}(\partial\mathcal{Q})$. Under Assumptions 3.1 and 4.1, we have $\overline{u} = \underline{u}$.*

The proof of this proposition is reported in Section 5.4, and requires the preparations in Section 5.3.

5.3. *Preliminary: HJB equations.* In this subsection, we recall the relation between HJB equations and stochastic control problems. Recall the constants L_0 and C_0 in Assumption 3.1 and consider two functions:

$$\begin{aligned} \overline{g}(y,z,\gamma) &:= C_0 + L_0|z| + L_0 y^- + \sup_{\beta\in[\sqrt{2/L_0}I_d, \sqrt{2L_0}I_d]} \frac{1}{2}\beta^2 : \gamma, \\ \underline{g}(y,z,\gamma) &:= -C_0 - L_0|z| - L_0 y^+ + \inf_{\beta\in[\sqrt{2/L_0}I_d, \sqrt{2L_0}I_d]} \frac{1}{2}\beta^2 : \gamma. \end{aligned} \tag{5.3}$$

Then for all nonlinearities G satisfying Assumption 3.1, it holds $\underline{g} \le G \le \overline{g}$. Consider the HJB equations

$$\overline{\mathbf{L}}u := -\overline{g}\big(u, \partial_x u, \partial^2_{xx}u\big) = 0 \quad \text{and} \quad \underline{\mathbf{L}}u := -\underline{g}\big(u, \partial_x u, \partial^2_{xx}u\big) = 0.$$

In the next lemma, we will show that the solutions to the PDEs above with the boundary condition h_D have the stochastic representations

$$\begin{aligned}\overline{w}(x) &:= \sup_{b\in\mathbb{H}^0([0,L_0])} \overline{\mathcal{E}}^{L_0}\Big[h_D(B_{\mathrm{H}_D^x})e^{-\int_0^{\mathrm{H}_D^x} b_r\,dr} + C_0\int_0^{\mathrm{H}_D^x} e^{-\int_0^t b_r\,dr}\,dt\Big],\\ \underline{w}(x) &:= \inf_{b\in\mathbb{H}^0([0,L_0])} \underline{\mathcal{E}}^{L_0}\Big[h_D(B_{\mathrm{H}_D^x})e^{-\int_0^{\mathrm{H}_D^x} b_r\,dr} + C_0\int_0^{\mathrm{H}_D^x} e^{-\int_0^t b_r\,dr}\,dt\Big],\end{aligned} \tag{5.4}$$

where we use the new notation

$$\mathrm{H}_D^x := \mathrm{H}_{D^x}$$

so as to shorten the formulas.

LEMMA 5.6. *Let $h_D(x) := \overline{\mathcal{E}}^{L_0}[v(\mathrm{H}_D^x, B_{\mathrm{H}_D^x\wedge\cdot})]$ for some $v \in \mathrm{BUC}(\mathbb{R}^+\times\Omega^e)$. Then $\overline{w}$ and $\underline{w}$ are the unique C–L viscosity solutions in $\mathrm{BUC}(\mathrm{cl}(D))$ to the equations $\overline{\mathbf{L}}u = 0$ and $\underline{\mathbf{L}}u = 0$, respectively, with the boundary condition $u = h_D$ on ∂D.*

PROOF. We claim and will prove in Proposition A.1 in the Appendix that there exists a modulus of continuity ρ such that

$$\overline{\mathcal{E}}^{L_0}\big[|\mathrm{H}_D^{x_1} - \mathrm{H}_D^{x_2}|\big] \le \rho(|x_1 - x_2|). \tag{5.5}$$

Since $v \in \mathrm{BUC}(\mathbb{R}^+\times\Omega^e)$, we obtain that

$$\begin{aligned}|h_D(x_1) - h_D(x_2)| &\le \overline{\mathcal{E}}^{L_0}\big[|v(\mathrm{H}_D^{x_1}, B_{\mathrm{H}_D^{x_1}\wedge\cdot}) - v(\mathrm{H}_D^{x_2}, B_{\mathrm{H}_D^{x_2}\wedge\cdot})|\big]\\ &\le \rho\big(\overline{\mathcal{E}}^{L_0}\big[|\mathrm{H}_D^{x_1} - \mathrm{H}_D^{x_2}|\big] + \overline{\mathcal{E}}^{L_0}\big[\|B_{\mathrm{H}_D^{x_1}\wedge\cdot} - B_{\mathrm{H}_D^{x_2}\wedge\cdot}\|_\infty\big]\big),\end{aligned} \tag{5.6}$$

where we used the concavity of ρ (recall Remark 2.2) and the Jensen's inequality. Recall the definition of $\mathcal{P}^L$ (each $\mathbb{P}\in\mathcal{P}^L$ corresponds to a measure $\mathbb{Q}^{\alpha,\beta}$ in an extended probability space). We have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\|B_{\mathrm{H}_D^{x_1}\wedge\cdot} - B_{\mathrm{H}_D^{x_2}\wedge\cdot}\|_\infty\big] &\le \mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\Big[\Big\|\int_0^{\mathrm{H}_D^{x_1}\wedge\cdot}\alpha_t\,dt - \int_0^{\mathrm{H}_D^{x_2}\wedge\cdot}\alpha_t\,dt\Big\|_\infty\Big]\\ &\quad + \mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\big[\|M_{\mathrm{H}_D^{x_1}\wedge\cdot} - M_{\mathrm{H}_D^{x_2}\wedge\cdot}\|_\infty^2\big]^{\frac{1}{2}}\\ &\le L_0\overline{\mathcal{E}}^{L_0}\big[|\mathrm{H}_D^{x_1} - \mathrm{H}_D^{x_2}|\big] + \big(2L_0\overline{\mathcal{E}}^{L_0}\big[|\mathrm{H}_D^{x_1} - \mathrm{H}_D^{x_2}|\big]\big)^{\frac{1}{2}},\\ &\qquad\qquad \text{for all } \mathbb{P}\in\mathcal{P}^{L_0}.\end{aligned} \tag{5.7}$$

In view of (5.5), we conclude that $h_D \in \mathrm{BUC}(\mathbb{R}^d)$. Further, since h_D is bounded and the control processes b in (5.4) only takes nonnegative values, it follows that for $x_1, x_2 \in D$,

$$|\overline{w}(x_1) - \overline{w}(x_2)| \le \overline{\mathcal{E}}^{L_0}\big[|h_D(B_{\mathrm{H}_D^{x_1}}) - h_D(B_{\mathrm{H}_D^{x_2}})|\big] + C\overline{\mathcal{E}}^{L_0}\big[|\mathrm{H}_D^{x_1} - \mathrm{H}_D^{x_2}|\big].$$

Since $h_D \in \mathrm{BUC}(\mathbb{R}^d)$, by the same arguments in (5.6) and (5.7), we conclude that $\overline{w} \in \mathrm{BUC}(\mathrm{cl}(D))$. Then, by a verification argument, one can easily show that $\overline{w}$ is the unique C–L viscosity solution to $\overline{\mathbf{L}}u = 0$ with the boundary condition h_D on ∂D. Similarly, we may prove the corresponding result for $\underline{w}$. $\square$

5.4. *Proof of* $\overline{u} = \underline{u}$. Recall the two functions $\overline{u}, \underline{u}$ defined in (5.2). In the next lemma, we will use the path-frozen PDEs to construct the functions θ_n^ε, which will be needed to construct the approximations of $\overline{u}$ and $\underline{u}$ defined in (5.2). Recall the notation of linear interpolation in (2.4). Then:

- let $(x_1, x_2, \dots, x_n) \in (\overline{O}_\varepsilon)^n$, $\mathrm{x}_i := \sum_{j=1}^i x_j$ and then denote

$$\pi_n := \mathrm{Lin}\{(0,0), (1, \mathrm{x}_1), \dots, (n, \mathrm{x}_n)\} \tag{5.8}$$

  (in particular, note that $\pi_n \in \Omega^e$);
- denote $\pi_n^x := \mathrm{Lin}\{\pi_n, (n+1, \mathrm{x}_n + x)\}$ for all $x \in \overline{O}_\varepsilon$ (clearly, we have $\pi_n^x \in \Omega^e$), where we slightly abuse the notation: $\mathrm{Lin}\{\pi_n, (n+1, \mathrm{x}_n + x)\} = \mathrm{Lin}\{(0,0), (1, \mathrm{x}_1), \dots, (n, \mathrm{x}_n), (n+1, \mathrm{x}_n + x)\}$;
- define a sequence of stopping times: $\mathrm{H}_0^x := 0$,

$$\begin{aligned} \mathrm{H}_1^x &:= \inf\{t \geq 0 : x + B_t \notin O_\varepsilon\}, \\ \mathrm{H}_{i+1}^x &:= \inf\{t \geq \mathrm{H}_i^x : B_t - B_{\mathrm{H}_i^x} \notin O_\varepsilon\} \qquad \text{for } i \geq 1, \quad \text{and} \\ \mathrm{H}_i^{\omega,\pi_n,x} &:= \mathrm{H}_i^x \wedge \mathrm{H}_{Q^{\omega \bar{\otimes} \pi_n^x}} \end{aligned} \tag{5.9}$$

  [recall that Q^ω is defined in (4.1)];
- given $\omega \in \Omega$, we define

$$\pi_n^m(x, \omega) := \mathrm{Lin}\{\pi_n, (n+1, \mathrm{x}_n + x + \omega_{\mathrm{H}_1^x}), \dots, (n+m, \mathrm{x}_n + x + \omega_{\mathrm{H}_m^x})\} \qquad \text{for all } m \geq 1.$$

The following lemma plays an essential role in our arguments.

LEMMA 5.7. *Let Assumption 3.1 hold, and assume that* $|\xi| \leq C_0$. *Let* $\omega \in \mathcal{Q}$, $|x_i| = \varepsilon$ *for all* $i \geq 1$, π_n *be defined as in* (5.8), *and* $\omega \bar{\otimes} \pi_n^x \in \mathcal{Q}$. *Then*:

(i) *there exist continuous functions* $(\pi_n, x) \mapsto \theta_n^{\omega,\varepsilon}(\pi_n, x)$, *bounded uniformly in* (ε, n), *such that*

$$\theta_n^{\omega,\varepsilon}(\pi_n; \cdot) \textit{ is a C–L viscosity solution of } (E)_\varepsilon^{\omega \bar{\otimes} \pi_n},$$

with boundary conditions

$$\begin{cases} \theta_n^{\omega,\varepsilon}(\pi_n; x) = \xi(\omega \bar{\otimes} \pi_n^x), & |x| < \varepsilon \textit{ and } x \in \partial Q^{\omega \bar{\otimes} \pi_n}, \\ \theta_n^{\omega,\varepsilon}(\pi_n; x) = \theta_{n+1}^{\omega,\varepsilon}(\pi_n^x; 0), & |x| = \varepsilon \textit{ and } x \in Q^{\omega \bar{\otimes} \pi_n}; \end{cases}$$

(ii) *moreover, there is a modulus of continuity* ρ *and a constant* $C_\varepsilon > 0$ *such that for any* $\omega^1, \omega^2 \in \mathcal{Q}$

$$\left|\theta_0^{\omega^1,\varepsilon}(0;0) - \theta_0^{\omega^2,\varepsilon}(0;0)\right| \le \varepsilon + \rho(2\varepsilon) + C_\varepsilon \rho\bigl(d^e(\omega^1,\omega^2)\bigr). \tag{5.10}$$

REMARK 5.8. For the domain $O_\varepsilon(\omega)$ defined in (4.1), a part of its boundary belongs to ∂Q^ω, while the rest belongs to ∂O_ε. On $\partial Q^\omega \cap \partial O_\varepsilon(\omega)$, we should set the solution to be equal to the boundary condition of the PPDE. Otherwise, on $\partial O_\varepsilon \cap \partial O_\varepsilon(\omega)$, the value of the solution should be consistent with that of the next piece of the path-frozen PDEs. The proof of Lemma 5.7 is similar to that of Lemma 6.2 in [11]. However, the stochastic representations and the estimates that we will use are all in the context of the elliptic equations. So it is necessary to present the proof in detail.

In preparation of the proof of Lemma 5.7, we give the following estimate on the C–L viscosity solutions to the path-frozen PDEs. The proof is reported in the Appendix.

LEMMA 5.9. *Fix* $D \in \mathcal{R}$. *Let* $h^i : \partial D \to \mathbb{R}$ *be continuous* ($i = 1, 2$), G *satisfy Assumption* 3.1, *and* v^i *be the C–L viscosity solutions to the following PDEs*:

$$G\bigl(\omega^i, v^i, \partial_x v^i, \partial^2_{xx} v^i\bigr) = 0 \qquad \text{on } D, \qquad v^i = h^i \qquad \text{on } \partial D.$$

Then we have

$$\bigl(v^1 - v^2\bigr)(x) \le \overline{\mathcal{E}}^{L_0}\bigl[\bigl(h^1 - h^2\bigr)^+(x + B_{\mathrm{H}^x_D})\bigr] + C\rho\bigl(d^e\bigl(\omega^1,\omega^2\bigr)\bigr),$$

where ρ *is a modulus of continuity in* ω *of the function* G. *In particular, if* $\omega^1 = \omega^2$, *then we have*

$$\bigl(v^1 - v^2\bigr)(x) \le \overline{\mathcal{E}}^{L_0}\bigl[\bigl(h^1 - h^2\bigr)^+(x + B_{\mathrm{H}^x_D})\bigr].$$

PROOF OF LEMMA 5.7. Since ε is fixed, to simplify the notation, we omit ε in the superscript in the proof. We divide the proof in five steps.

Step 1. We first prove (i) in the case of $G := \overline{g}$, where $\overline{g}$ is defined in (5.3). For any N, denote

$$\overline{\theta}^\omega_{N,N}(\pi_N; 0) := \overline{\mathcal{E}}^{L_0}\bigl[(\xi_{\mathrm{H}_Q})^{\omega \bar{\otimes} \pi_N}\bigr].$$

We define $\overline{\theta}^\omega_{N,n}(\pi_n; \cdot)$ as the C–L viscosity solution of the following PDE:

$$\begin{aligned} &-\overline{g}\bigl(\theta, \partial_x\theta, \partial^2_{xx}\theta\bigr) = 0 \qquad \text{on } O_\varepsilon(\omega \bar{\otimes} \pi_n), \\ &\theta(x) = \overline{\theta}^\omega_{N,n+1}\bigl(\pi^x_n; 0\bigr) \qquad \text{on } \partial O_\varepsilon(\omega \bar{\otimes} \pi_n) \qquad \text{for all } n \le N - 1. \end{aligned} \tag{5.11}$$

In order to shorten the formulas below, we denote the path

$$\Pi_N(\omega,\pi_n^x,B):=\omega\bar{\otimes}\pi_n^{N^\omega-n}(x,B)\bar{\otimes}(B_{\mathrm{H}_{Q^{\omega\bar{\otimes}\pi_n^x}}\wedge\cdot})^{\mathrm{H}^x_{N^\omega-n}}$$
$$\text{with } N^\omega:=\max\{n\le i\le N:\mathrm{H}^x_{i-n}<\mathrm{H}_{Q^{\omega\bar{\otimes}\pi_n^x}}\}.$$

By Lemma 5.6 and simple induction, we have the stochastic representation of $\overline{\theta}^\omega_{N,n}(\pi_n;\cdot)$:

$$\overline{\theta}^\omega_{N,n}(\pi_n;x)=\sup_{b\in\mathbb{H}^0([0,L_0])}\overline{\mathcal{E}}^{L_0}\Big[e^{-\int_0^{\mathrm{H}^{\omega,\pi_n,x}_{N-n}}b_r\,dr}\xi(\Pi_N(\omega,\pi_n^x,B))$$
$$+C_0\int_0^{\mathrm{H}^{\omega,\pi_n,x}_{N-n}}e^{-\int_0^s b_r\,dr}\,ds\Big]\qquad\text{for } n\le N-1.$$

Lemma 5.6 also implies that

$$\overline{\theta}^\varepsilon_{N,n}(\pi_n;x)\text{ is continuous in both variables }(\pi_n,x),\tag{5.12}$$

and clearly, they are uniformly bounded. We next define

$$\overline{\theta}^\omega_n(\pi_n;x):=\sup_{b\in\mathbb{H}^0([0,L_0])}\overline{\mathcal{E}}^{L_0}\Big[e^{-\int_0^{\mathrm{H}_{Q^{\omega\bar{\otimes}\pi_n^x}}}b_r\,dr}\varlimsup_{N\to\infty}\xi(\Pi_N(\omega,\pi_n^x,B))$$
$$+C_0\int_0^{\mathrm{H}_{Q^{\omega\bar{\otimes}\pi_n^x}}}e^{-\int_0^s b_r\,dr}\,ds\Big].$$

Then it follows that

$$\big|\overline{\theta}^\omega_n(\pi_n;x)-\overline{\theta}^\omega_{N,n}(\pi_n;x)\big|\le C\mathcal{C}^{L_0}\big[\mathrm{H}^x_{N-n}<\mathrm{H}_{Q^{\omega\bar{\otimes}\pi_n^x}}\big]\to0,\qquad N\to\infty.$$

By Proposition 2.11, the convergence is uniform in (π_n,x). Together with (5.12), it implies that $\overline{\theta}^\omega_n(\pi_n;x)$ is uniformly bounded and continuous in (π_n,x). Moreover, by the stability of C–L viscosity solutions we see that $\overline{\theta}^\omega_n(\pi_n;\cdot)$ is the C–L viscosity solution of PDE (5.11) in $O_\varepsilon(\omega\bar{\otimes}\pi_n)$, with the boundary condition

$$\begin{cases}\bar{\theta}^\omega_n(\pi_n;x)=\xi(\omega\bar{\otimes}\pi_n^x), & |x|<\varepsilon\text{ and } x\in\partial Q^{\omega\bar{\otimes}\pi_n},\\ \bar{\theta}^\omega_n(\pi_n;x)=\bar{\theta}^\omega_{n+1}(\pi_n^x;0), & |x|=\varepsilon\text{ and } x\in Q^{\omega\bar{\otimes}\pi_n}.\end{cases}$$

Hence, we have showed the desired result in the case $G=\overline{g}$. Similarly, we may show that $\underline{\theta}^\omega_n$ defined below is the C–L viscosity solution to the path-frozen PDE when the nonlinearity is $\underline{g}$:

$$\underline{\theta}^\omega_n(\pi_n;x):=\inf_{b\in\mathbb{H}^0([0,L_0])}\underline{\mathcal{E}}^{L_0}\Big[e^{-\int_0^{\mathrm{H}_{Q^{\omega\bar{\otimes}\pi_n^x}}}b_r\,dr}\varlimsup_{N\to\infty}\xi(\Pi_N(\omega,\pi_n^x,B))$$
$$+C_0\int_0^{\mathrm{H}_{Q^{\omega\bar{\otimes}\pi_n^x}}}e^{-\int_0^s b_r\,dr}\,ds\Big].$$

Step 2. We next prove (ii) in the case of $G = \overline{g}$. Considering $\pi_n^x \in \mathcal{Q}^{\omega^1} \cap \mathcal{Q}^{\omega^2}$, we have the following estimate:

$$\begin{aligned}
&\big|\overline{\theta}^{\omega^1}_{N,n}(\pi_n; x) - \overline{\theta}^{\omega^2}_{N,n}(\pi_n; x)\big| \\
&\quad \le C\overline{\mathcal{E}}^{L_0}\big[\big|\mathrm{H}^{\omega^1,\pi_n,x}_{N-n} - \mathrm{H}^{\omega^2,\pi_n,x}_{N-n}\big|\big] \\
&\qquad + C\overline{\mathcal{E}}^{L_0}\big[\big|\xi\big(\Pi_N(\omega^1, \pi_n^x, B)\big) - \xi\big(\Pi_N(\omega^2, \pi_n^x, B)\big)\big|\big].
\end{aligned}$$

We observe that

$$\begin{aligned}
&\big|\mathrm{H}^{\omega^1,\pi_n,x}_{N-n} - \mathrm{H}^{\omega^2,\pi_n,x}_{N-n}\big| \le \big|\mathrm{H}_{\mathcal{Q}^{\omega^1}\bar{\otimes}\pi_n^x} - \mathrm{H}_{\mathcal{Q}^{\omega^2}\bar{\otimes}\pi_n^x}\big|, \\
&d^e\big(\Pi_N(\omega^1, \pi_n^x, B), \Pi_N(\omega^2, \pi_n^x, B)\big) \\
&\qquad \le d^e(\omega^1, \omega^2) + \|B_{\mathrm{H}_{\mathcal{Q}^{\omega^1}\bar{\otimes}\pi_n^x}\wedge\cdot} - B_{\mathrm{H}_{\mathcal{Q}^{\omega^2}\bar{\otimes}\pi_n^x}\wedge\cdot}\|_\infty + 2\varepsilon.
\end{aligned}$$

As in Lemma 5.6, one may show that

$$\big|\overline{\theta}^{\omega^1}_{N,n} - \overline{\theta}^{\omega^2}_{N,n}\big| \le \rho\big(d^e(\omega^1, \omega^2) + 2\varepsilon\big) \le \rho\big(d^e(\omega^1, \omega^2)\big) + \rho(2\varepsilon),$$

in particular, ρ is independent of N and ε. By sending $N \to \infty$, we obtain that

$$\big|\overline{\theta}^{\omega^1}_{n} - \overline{\theta}^{\omega^2}_{n}\big| \le \rho\big(d^e(\omega^1, \omega^2)\big) + \rho(2\varepsilon).$$

A similar argument provides the same estimate for $\underline{\theta}^{\omega}_n$:

$$\big|\underline{\theta}^{\omega^1}_{n} - \underline{\theta}^{\omega^2}_{n}\big| \le \rho\big(d^e(\omega^1, \omega^2)\big) + \rho(2\varepsilon). \tag{5.13}$$

Step 3. We now prove (i) for general G. Given the construction of Step 1, we define

$$\overline{\theta}^{\omega,m}_m(\pi_m; x) := \overline{\theta}^{\omega}_m(\pi_m; x), \qquad \underline{\theta}^{\omega,m}_m(\pi_m; x) := \underline{\theta}^{\omega}_m(\pi_m; x), \qquad m \ge 1.$$

For $n \le m-1$, we define $\overline{\theta}^{\omega,m}_n$ and $\underline{\theta}^{\omega,m}_n$ as the unique C–L viscosity solution of the path-frozen PDE $(E)^{\omega\bar{\otimes}\pi_n}_\varepsilon$ with the boundary conditions

$$\begin{aligned}
&\overline{\theta}^{\omega,m}_n(\pi_n; x) = \overline{\theta}^{\omega,m}_{n+1}(\pi_n^x; 0), \\
&\underline{\theta}^{\omega,m}_n(\pi_n; x) = \underline{\theta}^{\omega,m}_{n+1}(\pi_n^x; 0) \qquad \text{for } x \in \partial O_\varepsilon(\omega\bar{\otimes}\pi_n).
\end{aligned}$$

Since $\underline{g} \le G \le \overline{g}$, it is obvious that $\overline{\theta}^{\varepsilon,m}_m$ and $\underline{\theta}^{\varepsilon,m}_m$ are respectively C–L viscosity supersolution and subsolution to the path-frozen PDE $(E)^{\omega\bar{\otimes}\pi_m}_\varepsilon$. By the comparison result for C–L viscosity solutions of PDEs, we obtain that

$$\begin{aligned}
\overline{\theta}^{\omega,m}_m(\pi_m; \cdot) &\ge \overline{\theta}^{\omega,m+1}_m(\pi_m; \cdot) \ge \underline{\theta}^{\omega,m+1}_m(\pi_m; \cdot) \\
&\ge \underline{\theta}^{\omega,m}_m(\pi_m; \cdot) \qquad \text{on } O_\varepsilon(\omega\bar{\otimes}\pi_m).
\end{aligned}$$

Further, it follows from the comparison again that

$$\begin{aligned}\overline{\theta}_n^{\omega,m}(\pi_n;\cdot) &\ge \overline{\theta}_n^{\omega,m+1}(\pi_n;\cdot) \ge \underline{\theta}_n^{\omega,m+1}(\pi_n;\cdot)\\ &\ge \underline{\theta}_n^{\omega,m}(\pi_n;\cdot) \qquad \text{on } O_\varepsilon(\omega\bar{\otimes}\pi_n) \text{ for all } n\le m.\end{aligned}\tag{5.14}$$

Denote $\delta\theta_n^{\omega,m} := \overline{\theta}_n^{\omega,m} - \underline{\theta}_n^{\omega,m}$. Applying Lemma 5.9 repeatedly and using the tower property of $\overline{\mathcal{E}}^{L_0}$ stated in Lemma 2.9, we obtain that

$$\left|\delta\theta_n^{\omega,m}(\pi_n;x)\right| \le \overline{\mathcal{E}}^{L_0}\big[\left|\delta\theta_m^{\omega,m}\big(\pi_n^{m-n}(x,B);0\big)\right|1_{\{\mathrm{H}^x_{m-n}<\mathrm{H}_{\mathcal{Q}^\omega\bar{\otimes}\pi_n^x}\}}\big]$$

[we also used the fact that $\delta\theta_m^{\omega,m}(\omega';0)=0$ as $\omega'\in\partial\mathcal{Q}^\omega$]. Then, by Proposition 2.11, we have

$$\left|\delta\theta_n^{\omega,m}(\pi_n;x)\right| \le C\mathcal{C}^{L_0}\big[\mathrm{H}^x_{m-n}<\mathrm{H}_{\mathcal{Q}^\omega\bar{\otimes}\pi_n^x}\big]\to 0 \qquad \text{as } m\to\infty.$$

Together with (5.14), this implies the existence of θ_n^ω such that

$$\overline{\theta}_n^{\omega,m}\downarrow\theta_n^\omega, \qquad \underline{\theta}_n^{\omega,m}\uparrow\theta_n^\omega \qquad \text{as } m\to\infty.\tag{5.15}$$

Clearly, θ_n^ω is uniformly bounded and continuous (because it is both lower and upper semicontinuous). Finally, it follows from the stability of C–L viscosity solutions that θ_n^ω satisfies the statement of (i).

Step 4. We next prove (ii) for a general nonlinearity G. For the simplicity of notation, we denote the stopping times

$$\mathrm{H}^i := \mathrm{H}_{\mathcal{Q}^{\omega^i}\bar{\otimes}\pi_n^x} \qquad \text{for } i=1,2, \qquad \mathrm{H}^{1,2} := \mathrm{H}^1\wedge\mathrm{H}^2.$$

First, considering $\overline{\theta}_n^{\omega,m}$ defined in Step 3, we claim that for $\pi_n^x\in\mathcal{Q}^{\omega^1}\cap\mathcal{Q}^{\omega^2}$

$$\begin{aligned}&\big(\overline{\theta}_n^{\omega^1,m}-\underline{\theta}_n^{\omega^2,m}\big)(\pi_n;x)\\ &\quad\le \overline{\mathcal{E}}^{L_0}\big[\big(\overline{\theta}_m^{\omega^1}-\underline{\theta}_m^{\omega^2}\big)\big(\pi_n^{m-n}(x,B);0\big)1_{\{\mathrm{H}^x_{m-n}\le\mathrm{H}^{1,2}\}}\\ &\qquad +\big(\rho\big(d^e\big(\omega^1,\omega^2\big)\big)+\rho(2\varepsilon)\big)1_{\{\mathrm{H}^x_{m-n}>\mathrm{H}^{1,2}\}}\big]+C(m-n)\rho\big(d^e\big(\omega^1,\omega^2\big)\big).\end{aligned}\tag{5.16}$$

This claim will be proved in Step 5. Since $\overline{\theta}_m^{\omega^1}, \underline{\theta}_m^{\omega^2}$ are both bounded, it follows from (5.16) that

$$\begin{aligned}&\big(\overline{\theta}_n^{\omega^1,m}-\underline{\theta}_n^{\omega^2,m}\big)(\pi_n;x)\\ &\quad\le C\mathcal{C}^L\big[\mathrm{H}^x_{m-n}<\mathrm{H}^{1,2}\big]+C(m-n+1)\rho\big(d^e\big(\omega^1,\omega^2\big)\big)+\rho(2\varepsilon).\end{aligned}$$

Recalling (5.15), we obtain that

$$\begin{aligned}&\big(\theta_n^{\omega^1}-\theta_n^{\omega^2}\big)(\pi_n;x)\\ &\quad\le C\mathcal{C}^L\big[\mathrm{H}^x_{m-n}<\mathrm{H}^{1,2}\big]+C(m-n+1)\rho\big(d^e\big(\omega^1,\omega^2\big)\big)+\rho(2\varepsilon).\end{aligned}$$

Since $\lim_{m\to\infty}\mathcal{C}^L[\mathrm{H}^x_{m-n}<\mathrm{H}^{1,2}]=0$, there is a constant C_ε such that

$$(\theta_n^{\omega^1}-\theta_n^{\omega^2})(\pi_n;x)\le\varepsilon+C_\varepsilon\rho\big(d^e(\omega^1,\omega^2)\big)+\rho(2\varepsilon).$$

By exchanging the roles of ω^1 and ω^2, we have

$$\big|(\theta_n^{\omega^1}-\theta_n^{\omega^2})(\pi_n;x)\big|\le\varepsilon+\rho(2\varepsilon)+C_\varepsilon\rho\big(d^e(\omega^1,\omega^2)\big).$$

Step 5. We now prove Claim (5.16). Suppose that $m\ge n+1$. We first show that

$$\begin{aligned}&(\overline{\theta}_n^{\omega^1,m}-\underline{\theta}_n^{\omega^2,m})(\pi_n;x)\\&\quad\le\overline{\mathcal{E}}^{L_0}\big[(\overline{\theta}_{n+1}^{\omega^1,m}-\underline{\theta}_{n+1}^{\omega^2,m})\big(\pi_n^1(x,B);0\big)1_{\{\mathrm{H}_1^x\le\mathrm{H}^{1,2}\}}\\&\qquad+\big(\rho\big(d^e(\omega^1,\omega^2)\big)+\rho(2\varepsilon)\big)1_{\{\mathrm{H}_1^x>\mathrm{H}^{1,2}\}}\big]+C\rho\big(d^e(\omega^1,\omega^2)\big).\end{aligned}\tag{5.17}$$

Then (5.16) follows from simple induction. Recall that $\overline{\theta}_n^{\omega^1,m}$ (resp., $\underline{\theta}_n^{\omega^2,m}$) is a solution to the PDE with generator $G(\omega^1,\cdot)$ [resp., $G(\omega^2,\cdot)$]. Now we study those two PDEs on the domain

$$O_\varepsilon\cap Q^{\omega^1}\cap Q^{\omega^2}.$$

The boundary of this set can be divided into three parts which belong to ∂O_ε, ∂Q^{ω^1} and ∂Q^{ω^2}, respectively. We denote them by Bd_1, Bd_2 and Bd_3.

(i) On Bd_1, we have $\mathrm{H}_1^x\le\mathrm{H}^{1,2}$, and thus

$$\overline{\theta}_n^{\omega^1,m}(\pi_n;x)=\overline{\theta}_{n+1}^{\omega^1,m}(\pi_n^x;0)\quad\text{and}\quad\underline{\theta}_n^{\omega^2,m}(\pi_n;x)=\underline{\theta}_{n+1}^{\omega^2,m}(\pi_n^x,0).$$

(ii) On Bd_2, we have $\mathrm{H}^1<\mathrm{H}_1^x$, so we have $\overline{\theta}_n^{\omega^1,m}(\pi_n;x)=\xi(\omega^1\bar{\otimes}\pi_n^x)=\underline{\theta}_n^{\omega^1,n}(\pi_n;x)$.

(iii) On Bd_3, we have $\mathrm{H}^2<\mathrm{H}_1^x$, so we have $\underline{\theta}_n^{\omega^2,m}(\pi_n;x)=\xi(\omega^2\bar{\otimes}\pi_n^x)=\overline{\theta}_n^{\omega^2,n}(\pi_n;x)$.

Then it follows from Lemma 5.9 that

$$\begin{aligned}&(\overline{\theta}_n^{\omega^1,m}-\underline{\theta}_n^{\omega^2,m})(\pi_n;x)\\&\quad\le\overline{\mathcal{E}}^{L_0}\big[(\overline{\theta}_{n+1}^{\omega^1,m}-\underline{\theta}_{n+1}^{\omega^2,m})\big(\pi_n^1(x,B);0\big)1_{\{\mathrm{H}_1^x\le\mathrm{H}^{1,2}\}}\\&\qquad+\big(\underline{\theta}_n^{\omega^1,n}(\pi_n;x+B_{\mathrm{H}^1})-\underline{\theta}_n^{\omega^2,m}(\pi_n;x+B_{\mathrm{H}^1})\big)1_{\{\mathrm{H}^1<\mathrm{H}_1^x\le\mathrm{H}^2\}}\\&\qquad+\big(\overline{\theta}_n^{\omega^1,m}(\pi_n;x+B_{\mathrm{H}^2})-\overline{\theta}_n^{\omega^2,n}(\pi_n;x+B_{\mathrm{H}^2})\big)1_{\{\mathrm{H}^2<\mathrm{H}_1^x\le\mathrm{H}^1\}}\big]\\&\qquad+C\rho\big(d^e(\omega^1,\omega^2)\big).\end{aligned}\tag{5.18}$$

We next estimate

$$\Delta := \underline{\theta}_n^{\omega^1,n}(\pi_n; x + B_{\mathrm{H}^1}) - \underline{\theta}_n^{\omega^2,m}(\pi_n; x + B_{\mathrm{H}^1}).$$

As in Step 3, the comparison result of C–L viscosity solution implies that

$$\underline{\theta}_n^{\omega^2,m}(\pi_n; x + B_{\mathrm{H}^1}) \geq \underline{\theta}_n^{\omega^2,n}(\pi_n; x + B_{\mathrm{H}^1}).$$

It follows from (5.13) that

$$\Delta \leq \underline{\theta}_n^{\omega^1,n}(\pi_n; x + B_{\mathrm{H}^1}) - \underline{\theta}_n^{\omega^2,n}(\pi_n; x + B_{\mathrm{H}^1}) \leq \rho\big(d^e(\omega^1, \omega^2)\big) + \rho(2\varepsilon).$$

Similarly, we can obtain the same estimate for $\overline{\theta}_n^{\omega^1,m}(\pi_n; x + B_{\mathrm{H}^2}) - \overline{\theta}_n^{\omega^2,n}(\pi_n; x + B_{\mathrm{H}^2})$. Together with (5.18), we obtain (5.17). $\square$

The previous lemma shows the existence of C–L viscosity solution to the path-frozen PDEs. Further, we will use Assumption 4.1 to construct piecewise smooth super and subsolutions to the PPDE. Recall the stopping times defined in (5.9), and denote

$$\theta_n^\varepsilon := \theta_n^{\mathbf{0},\varepsilon}, \qquad \mathrm{H}_n := \mathrm{H}_n^0 \wedge \mathrm{H}_Q \quad \text{and} \quad \hat{\pi}_n := \mathrm{Lin}\big\{\big(\mathrm{H}_i(\omega), \omega_{\mathrm{H}_i(\omega)}\big); 0 \leq i \leq n\big\}.$$

LEMMA 5.10. *There exists $\psi^\varepsilon \in \overline{C}^2(\mathcal{Q})$ such that*

$$\psi^\varepsilon(\mathbf{0}) = \theta_0^\varepsilon(\mathbf{0}) + \varepsilon, \qquad \psi^\varepsilon \geq \xi \qquad \text{on } \partial\mathcal{Q},$$

$$-G\big(\hat{\pi}_n, \psi^\varepsilon(\omega), \partial_\omega\psi^\varepsilon(\omega), \partial^2_{\omega\omega}\psi^\varepsilon(\omega)\big) \geq 0 \qquad \text{when } \mathrm{H}_n(\omega) \leq \bar{t}(\omega) < \mathrm{H}_{n+1}(\omega), \quad \text{for all } n \in \mathbb{N},$$

where $\partial_\omega\psi^\varepsilon$, $\partial^2_{\omega\omega}\psi^\varepsilon$ are the derivatives of φ^ε on the corresponding intervals.

PROOF. For simplicity, in the proof, we omit the superscript ε. First, since PDE $(E)_\varepsilon^0$ satisfies Assumption 4.1 and $G(\omega, y, z, \gamma)$ is decreasing in y, there exists a function $v_0 \in C_0^2(O_\varepsilon(\mathbf{0}))$ such that

$$v_0(0) = \theta_0(0) + \frac{\varepsilon}{2}, \qquad \mathbf{L}^0 v_0 \geq 0 \qquad \text{on } O_\varepsilon(\mathbf{0}) \quad \text{and} \quad v_0 \geq \theta_0 \qquad \text{on } \partial O_\varepsilon(\mathbf{0}).$$

Denote $v_0(\mathbf{0}; \cdot) := v_0(\cdot)$. Similarly, applying Assumption 4.1 to PDE $(E)_\varepsilon^{\hat{\pi}_n}$ ($n \geq 1$), we can find a function $v_n(\hat{\pi}_n; \cdot) \in C_0^2(O_\varepsilon(\hat{\pi}_n))$ such that

$$v_n(\hat{\pi}_n; 0) = v_{n-1}(\hat{\pi}_{n-1}; \omega_{\mathrm{H}_n(\omega)} - \omega_{\mathrm{H}_{n-1}(\omega)}) + 2^{-n-1}\varepsilon,$$

$$\mathbf{L}^{\hat{\pi}_n} v_n(\hat{\pi}_n; \cdot) \geq 0 \qquad \text{on } O_\varepsilon(\hat{\pi}_n), \qquad v_n(\hat{\pi}_n; \cdot) \geq \theta_n(\hat{\pi}_n; \cdot) \qquad \text{on } \partial O_\varepsilon(\hat{\pi}_n).$$

We now give the definition of the required function $\psi : \mathcal{Q} \to \mathbb{R}$:

$$\psi(\omega) := \sum_{n=0}^{\infty} \big(v_n(\hat{\pi}_n; \omega_{\bar{t}(\omega)} - \omega_{\mathrm{H}_n(\omega)}) + \varepsilon - 2^{-n-1}\varepsilon\big) 1_{\{\mathrm{H}_n(\omega) \leq \bar{t}(\omega) < \mathrm{H}_{n+1}(\omega)\}}.$$

Clearly, we have $\psi \in \overline{C}^2(\mathcal{Q})$. Consider a path ω such that $\mathrm{H}_n(\omega) \le \bar{t}(\omega) < \mathrm{H}_{n+1}(\omega)$. Since $\psi(\omega) \ge v_n(\hat{\pi}_n; \omega_{\bar{t}(\omega)} - \omega_{\mathrm{H}_n(\omega)})$, it follows from the monotonicity of G

$$-G(\hat{\pi}_n, \psi(\omega), \partial_\omega \psi(\omega), \partial^2_{\omega\omega}\psi(\omega)) \ge \mathbf{L}^{\hat{\pi}_n} v_n(\hat{\pi}_n; \omega_{\bar{t}(\omega)} - \omega_{\mathrm{H}_n(\omega)}) \ge 0.$$

Finally, we may easily check that $\psi(0) - \theta_0(0) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and that $\psi \ge \xi$ on $\partial \mathcal{Q}$. □

Now we have done all the necessary constructions and are ready to show the main result of the section.

PROOF OF PROPOSITION 5.5. For any $\varepsilon > 0$, let ψ^ε be as in Lemma 5.10, and $\overline{\psi}^\varepsilon := \psi^\varepsilon + \rho(2\varepsilon) + \lambda^{-1}(\rho(2\varepsilon))$, where ρ is the common modulus of continuity of ξ and G, and λ^{-1} is the inverse of the function in Assumption 3.1. Then clearly $\overline{\psi}^\varepsilon \in \overline{C}^2(\mathcal{Q})$ and bounded. Also,

$$\overline{\psi}^\varepsilon(\omega) - \xi(\omega) \ge \psi^\varepsilon(\omega) + \rho(2\varepsilon) - \xi(\omega) \ge \xi(\omega^\varepsilon) - \xi(\omega) + \rho(2\varepsilon) \ge 0 \qquad \text{on } \partial\mathcal{Q}.$$

Moreover, when $\bar{t}(\omega) \in [H_n(\omega), H_{n+1}(\omega))$, we have that

$$\begin{aligned}
\mathcal{L}\overline{\psi}^\varepsilon(\omega) &= -G(\omega, \overline{\psi}^\varepsilon, \partial_\omega \psi^\varepsilon, \partial^2_{\omega\omega}\psi^\varepsilon) \\
&\ge -G(\hat{\pi}_n, \psi^\varepsilon + \lambda^{-1}(\rho(2\varepsilon)), \partial_\omega \psi^\varepsilon, \partial^2_{\omega\omega}\psi^\varepsilon) - \rho(2\varepsilon) \\
&\ge -G(\hat{\pi}_n, \psi^\varepsilon, \partial_\omega \psi^\varepsilon, \partial^2_{\omega\omega}\psi^\varepsilon) \ge 0.
\end{aligned}$$

Then by the definition of $\overline{u}$ we see that

$$\begin{aligned}
\overline{u}(0) \le \overline{\psi}^\varepsilon(0) &= \psi^\varepsilon + \rho(2\varepsilon) + \lambda^{-1}(\rho(2\varepsilon)) \\
&\le \theta_0^\varepsilon(0) + \varepsilon + \rho(2\varepsilon) + \lambda^{-1}(\rho(2\varepsilon)).
\end{aligned} \tag{5.19}$$

Similarly, $\underline{u}(0) \ge \theta_0^\varepsilon(0) - \varepsilon - \rho(2\varepsilon) - \lambda^{-1}(\rho(2\varepsilon))$. That implies that

$$\overline{u}(0) - \underline{u}(0) \le 2\varepsilon + 2\rho(2\varepsilon) + 2\lambda^{-1}(\rho(2\varepsilon)).$$

Since ε is arbitrary, this shows that $\overline{u}(0) = \underline{u}(0)$. Similarly, we can show that $\overline{u}(\omega) = \underline{u}(\omega)$ for all $\omega \in \mathcal{Q}$. □

6. Existence.

In this section, we verify that

$$u := \overline{u} = \underline{u} \tag{6.1}$$

is the unique $\mathcal{P}^L$-viscosity solution in $\mathrm{BUC}(\mathcal{Q})$ to the PPDE (3.1) for $L \ge L_0$. We will prove that $u \in \mathrm{BUC}(\mathcal{Q})$ in Section 6.1 and u satisfies the viscosity property in Section 6.2.

6.1. *Regularity.* The noncontinuity of the hitting time $\mathrm{H}_Q(\cdot)$ brings difficulty to the proof of the regularity of u. One cannot adapt the method used in [11]. In our approach, we make use of the estimate (5.10) for the solution of the path-frozen PDEs.

PROPOSITION 6.1. *Let Assumption 3.1 hold and $\xi \in \mathrm{BUC}(\partial \mathcal{Q})$. Then $\overline{u}$ is bounded from above and $\underline{u}$ is bounded from below.*

PROOF. Assume that $|\xi| \leq C_0$. Define

$$\psi := \lambda^{-1}(C_0) + C_0.$$

Obviously, $\psi \in \overline{C}^2$. Observe that $\psi_T \geq C_0 \geq \xi$. Also,

$$\mathcal{L}^\omega \psi_s = -G^\omega(\cdot, \psi_s, 0, 0) \geq C_0 - G^\omega(\cdot, 0, 0, 0) \geq 0.$$

It follows that $\psi \in \overline{\mathcal{D}}_Q^\xi(\omega)$, and thus $\overline{u}(\omega) \leq \psi(0) = \lambda^{-1}(C_0) + C_0$. Similarly, one can show that $\underline{u}(\omega) \geq -\lambda^{-1}(C_0) - C_0$. $\square$

PROPOSITION 6.2. *The function u defined in (6.1) is uniformly continuous in $\mathcal{Q}$.*

PROOF. Recall (5.19), that is, for $\omega^1, \omega^2 \in \mathcal{Q}$, it holds that

$$\overline{u}(\omega^1) \leq \theta_0^{\omega^1}(0) + \varepsilon + \rho(2\varepsilon) \quad \text{and} \quad \underline{u}(\omega^2) \geq \theta_0^{\omega^2}(0) - \varepsilon - \rho(2\varepsilon).$$

Hence, it follows from Lemma 5.7 that

$$\begin{aligned} u(\omega^1) - u(\omega^2) &= \overline{u}(\omega^1) - \underline{u}(\omega^2) \\ &\leq \theta_0^{\omega^1}(0) - \theta_0^{\omega^2}(0) + 2\big(\varepsilon + \rho(2\varepsilon)\big) \\ &\leq C_\varepsilon \rho\big(d^e(\omega^1, \omega^2)\big) + 3\big(\varepsilon + \rho(2\varepsilon)\big) \qquad \text{for all } \varepsilon > 0. \end{aligned}$$

By exchanging the roles of ω^1 and ω^2, we obtain $|u(\omega^1) - u(\omega^2)| \leq C_\varepsilon \rho(d^e(\omega^1, \omega^2)) + 3(\varepsilon + \rho(2\varepsilon))$, from which the uniform continuity of u can be easily deduced. $\square$

6.2. *Viscosity property.* After having shown that u is uniformly continuous, we need to verify that it indeed satisfies the viscosity property. The following proof is similar to that of Proposition 4.3 in [11].

PROPOSITION 6.3. *The function u defined in (6.1) is a $\mathcal{P}^L$-viscosity solution to PPDE (3.1) for $L \geq L_0$.*

PROOF. We only prove that $\overline{u}$ is a $\mathcal{P}^L$-viscosity supersolution. The subsolution property can be proved similarly. Without loss of generality, we only show the $\mathcal{P}^{L_0}$-viscosity supersolution property at the point $\mathbf{0}$. Assume the contrary, that is, there exists $\varphi \in \overline{\mathcal{A}}^{L_0}\overline{u}(\mathbf{0})$ such that $-c := \mathcal{L}\varphi(\mathbf{0}) < 0$. For any $\psi \in \overline{\mathcal{D}}^{\xi}_{Q}(\mathbf{0})$ and $\omega \in \mathcal{Q}$, it is clear that $\psi^{\omega} \in \overline{\mathcal{D}}^{\xi}_{Q}(\omega)$ and $\psi(\omega) \geq \overline{u}(\omega)$. Now by the definition of $\overline{u}$, there exists $\psi^n \in \overline{C}^2(\mathcal{Q})$ such that

$$\delta_n := \psi^n(0) - \overline{u}(0) \downarrow 0 \qquad \text{as } n \to \infty, \qquad \mathcal{L}\psi^n(\omega) \geq 0, \qquad \omega \in \mathcal{Q}. \tag{6.2}$$

Let $\mathrm{H}_\varepsilon := \varepsilon \wedge \mathrm{H}_{O_\varepsilon}$ be a localization of test function φ. Since $\varphi \in C^2(\mathcal{O}_\varepsilon)$ and $\overline{u} \in \mathrm{BUC}(\mathcal{Q})$, without loss of generality we may assume that

$$\mathcal{L}\varphi(\omega_{t\wedge\cdot}) \leq -\frac{c}{2} \quad \text{and} \quad |\varphi_t - \varphi_0| + |\overline{u}_t - \overline{u}_0| \leq \frac{c}{6L_0} \qquad \text{for all } t \leq \mathrm{H}_{O_\varepsilon}. \tag{6.3}$$

Since $\varphi \in \overline{\mathcal{A}}^{L_0}\overline{u}(\mathbf{0})$, this implies for all $\mathbb{P} \in \mathcal{P}^{L_0}$ that

$$0 \geq \mathbb{E}^{\mathbb{P}}[(\varphi - \overline{u})_{\mathrm{H}_\varepsilon}] \geq \mathbb{E}^{\mathbb{P}}[(\varphi - \psi^n)_{\mathrm{H}_\varepsilon}]. \tag{6.4}$$

Denote $\mathcal{G}^{\mathbb{P}}\phi := \alpha^{\mathbb{P}} \cdot \partial_\omega\phi + \frac{1}{2}(\beta^{\mathbb{P}})^2 : \partial^2_{\omega\omega}\phi$. Then, since $\varphi \in C^2(\mathcal{O}_\varepsilon)$ and $\psi^n \in \overline{C}^2(\mathcal{Q})$, it follows from (6.2) that

$$\begin{aligned}
\delta_n &\geq \mathbb{E}^{\mathbb{P}}[(\varphi - \psi^n)_{\mathrm{H}_\varepsilon} - (\varphi - \psi^n)_0] \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathrm{H}_\varepsilon} \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)(B_{s\wedge\cdot})\,ds\right] \\
&\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathrm{H}_\varepsilon} \left(\frac{c}{2} - G(\cdot, \varphi, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) + G(\cdot, \psi^n, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n)\right.\right. \\
&\qquad \left.\left. + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\right)(B_{s\wedge\cdot})\,ds\right] \\
&\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathrm{H}_\varepsilon} \left(\frac{c}{2} - G(\cdot, \varphi, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) + G(\cdot, \overline{u}, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n)\right.\right. \\
&\qquad \left.\left. + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\right)(B_{s\wedge\cdot})\,ds\right],
\end{aligned}$$

where the last inequality is due to the monotonicity in y of G. Since $\varphi_0 = \overline{u}_0$ and G is L_0-Lipschitz continuous in y, it follows from (6.3) that

$$\begin{aligned}
\delta_n \geq \mathbb{E}^{\mathbb{P}}\biggl[\int_0^{\mathrm{H}_\varepsilon} \biggl(&\frac{c}{3} - G(\cdot, \overline{u}_0, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) \\
&+ G(\cdot, \overline{u}_0, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\biggr)(B_{s\wedge\cdot})\,ds\biggr].
\end{aligned}$$

We next let $\eta > 0$, and for each n, define $\tau_0^n := 0$ and

$$\begin{aligned}\tau_{j+1}^n(\omega) := {} & \mathrm{H}_\varepsilon(\omega) \wedge \inf\{t \ge \tau_j^n : \rho(d^e(\omega_{t\wedge\cdot}, \omega_{\tau_j^n\wedge\cdot})) + |\partial_\omega\varphi(\omega_{t\wedge\cdot}) - \partial_\omega\varphi(\omega_{\tau_j^n\wedge\cdot})| \\ & + |\partial^2_{\omega\omega}\varphi(\omega_{t\wedge\cdot}) - \partial^2_{\omega\omega}\varphi(\omega_{\tau_j^n\wedge\cdot})| + |\partial_\omega\psi^n(\omega_{t\wedge\cdot}) - \partial_\omega\psi^n(\omega_{\tau_j^n\wedge\cdot})| \\ & + |\partial^2_{\omega\omega}\psi^n(\omega_{t\wedge\cdot}) - \partial^2_{\omega\omega}\psi^n(\omega_{\tau_j^n\wedge\cdot})| \ge \eta\},\end{aligned}$$

where ρ is a modulus of continuity in ω of G. Since $\varphi \in C^2(\mathcal{O}_\varepsilon)$ and $\psi^n \in \overline{C}^2(\mathcal{Q})$, one can easily check that $\tau_j^n \uparrow \mathrm{H}_\varepsilon$, $\mathcal{P}^{L_0}$-q.s. as $j \to \infty$. Thus,

$$\begin{aligned}\delta_n &\ge \left(\frac{c}{3} - C\eta\right)\mathbb{E}^{\mathbb{P}}[\mathrm{H}_\varepsilon] + \sum_{j\ge 0}\mathbb{E}^{\mathbb{P}}(\tau_j^n - \tau_{j+1}^n)\big(-G(\cdot, \overline{u}_0, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) \\ &\quad + G(\cdot, \overline{u}_0, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\big)(B_{\tau_j^n\wedge\cdot}) \\ &= \left(\frac{c}{3} - C\eta\right)\mathbb{E}^{\mathbb{P}}[\mathrm{H}_\varepsilon] + \sum_{j\ge 0}\mathbb{E}^{\mathbb{P}}(\tau_j^n - \tau_{j+1}^n) \\ &\quad \times \left(\alpha_j^n \cdot \partial_\omega(\psi^n - \varphi) + \frac{1}{2}(\beta_j^n)^2 : \partial^2_{\omega\omega}(\psi^n - \varphi) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\right)(B_{\tau_j^n\wedge\cdot}),\end{aligned}$$

for some α_j^n, β_j^n such that $|a_j^n| \le L$ and $\beta_j^n \in \mathbb{H}_L^0$. Note that α_j^n and β_j^n are both $\mathcal{F}_{\tau_j^n}$-measurable. Take $\mathbb{P}_n \in \mathcal{P}^{L_0}$ such that $\alpha_t^{\mathbb{P}_n} = \alpha_j^n$, $\beta_t^{\mathbb{P}_n} = \beta_j^n$ for $t \in [\tau_j^n, \tau_{j+1}^n)$. Then

$$\delta_n \ge \left(\frac{c}{3} - C\eta\right)\mathbb{E}^{\mathbb{P}_n}[\mathrm{H}_\varepsilon].$$

Let $\eta := \frac{c}{6C}$. It follows that $\underline{\mathcal{E}}^{L_0}[\mathrm{H}_\varepsilon] \le \mathbb{E}^{\mathbb{P}_n}[\mathrm{H}_\varepsilon] \le \frac{6}{c}\delta_n$. By letting $n \to \infty$, we get $\underline{\mathcal{E}}^{L_0}[\mathrm{H}_\varepsilon] = 0$, contradiction. □

7. Path-dependent time-invariant stochastic control. In this section, we present an application of fully nonlinear elliptic PPDE. An important question which is most relevant since the recent financial crisis is the risk of model misspecification. The uncertain volatility model (see Avellaneda, Levy and Paras [1], Lyons [15] or Nutz [19]) provides a conservative answer to this problem.

In the present application, the canonical process B represents the price process of some primitive asset, and our objective is the hedging of the derivative security defined by the payoff $\xi(B_\cdot)$ at some maturity H_Q defined as the exiting time from some domain Q.

In contrast with the standard Black–Scholes modeling, we assume that the probability space $(\Omega, \mathcal{F})$ is endowed with a family of probability measures $\mathcal{P}^{\mathrm{UVM}}$. In the uncertain volatility model, the quadratic variation of the canonical process is assumed to lie between two given bounds,

$$\underline{\sigma}^2\, dt \le d\langle B\rangle_t \le \overline{\sigma}^2\, dt, \qquad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}^{\mathrm{UVM}}.$$

Then, by the possible frictionless trading of the underlying asset, it is well known that the nonarbitrage condition is characterized by the existence of an equivalent martingale measure. Consequently, we take

$$\mathcal{P}^{\mathrm{UVM}} := \Big\{\mathbb{P} \in \mathcal{P}^\infty : B \text{ is a continuous } \mathbb{P}\text{-martingale and } \frac{d\langle B\rangle_t}{dt} \in [\underline{\sigma}^2, \overline{\sigma}^2], \mathbb{P}\text{-a.s.}\Big\}.$$

The superhedging problem under model uncertainty was initially formulated by Denis and Martini [6] and Neufeld and Nutz [18], and involves delicate quasi-sure analysis. Their main result expresses the cost of robust superhedging as

$$u_0 := \overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r\mathrm{H}_Q}\xi(B_{\mathrm{H}_Q\wedge\cdot})\big] := \overline{\mathcal{E}}^{\mathcal{P}^{\mathrm{UVM}}}\big[e^{-r\mathrm{H}_Q}\xi(B_{\mathrm{H}_Q\wedge\cdot})\big],$$

where r is the discount rate. Further, define u on Ω^e as

$$u(\omega) := \overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r\mathrm{H}_{Q^\omega}}\xi(\omega \bar{\otimes} B_{\mathrm{H}_{Q^\omega}\wedge\cdot})\big] \qquad \text{for all } \omega \in \mathcal{Q}. \tag{7.1}$$

We are interested in characterizing u as a viscosity solution of the corresponding fully nonlinear elliptic PPDE.

ASSUMPTION 7.1. Assume that

$$\xi \in \mathrm{BUC}(\partial\mathcal{Q}), \qquad \underline{\sigma} > 0, \quad \text{and the discount rate } r \geq 0.$$

PROPOSITION 7.2. *Let L be a constant such that $\frac{1}{L} \leq \underline{\sigma}$ and $L \geq \overline{\sigma}$. Under Assumption 7.1, the function u defined in (7.1) is in* BUC($\mathcal{Q}$) *and is a $\mathcal{P}^L$-viscosity solution to the elliptic path-dependent HJB equation*

$$ru - \sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]} \tfrac{1}{2}\gamma^2\partial^2_{\omega\omega}u = 0 \qquad \text{on } \mathcal{Q}, \quad \text{and} \quad u = \xi \qquad \text{on } \partial\mathcal{Q}.$$

LEMMA 7.3. *The function u defined in* (7.1) *is in* BUC($\mathcal{Q}$).

PROOF. As in Lemma 5.6, the required result follows easily from the fact $\xi \in \mathrm{BUC}(\partial\mathcal{Q})$. □

LEMMA 7.4. *We have $u_0 = \overline{\mathcal{E}}^{\mathrm{UVM}}[e^{-r\tau}u_\tau]$ [recall that $u_t(\omega) := u(\omega_{t\wedge\cdot})$] for all $\tau \leq \mathrm{H}_Q$.*

PROOF. By the definition of u, we have

$$\begin{aligned} e^{-rt}u(\omega_{t\wedge\cdot}) &= e^{-rt}\overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r\mathrm{H}_{Q^{\omega_{t\wedge\cdot}}}}\xi(\omega \otimes_t B_{\mathrm{H}_{Q^{\omega_{t\wedge\cdot}}}\wedge\cdot})\big] \\ &= e^{-rt}\overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r((\mathrm{H}_Q)^{t,\omega}-t)}(\xi_{\mathrm{H}_Q})^{t,\omega}\big] \\ &= \overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r(\mathrm{H}_Q)^{t,\omega}}(\xi_{\mathrm{H}_Q})^{t,\omega}\big]. \end{aligned}$$

Then it follows the tower property (Lemma 2.9) that

$$u_0 = \overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r\mathrm{H}_Q}\xi(B_{\mathrm{H}_Q\wedge\cdot})\big] = \overline{\mathcal{E}}^{\mathrm{UVM}}\big[\overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r(\mathrm{H}_Q)^{\tau,\cdot}}(\xi_{\mathrm{H}_Q})^{\tau,\cdot}\big]\big]$$
$$= \overline{\mathcal{E}}^{\mathrm{UVM}}\big[e^{-r\tau}u_\tau\big]. \qquad \square$$

PROOF OF PROPOSITION 7.2. *Step* 1. We first verify the viscosity supersolution property. Without loss of generality, we only verify it at the point $\mathbf{0}$. Recall the equivalent definition of viscosity solutions in Proposition 3.5. Let $(\alpha,\beta)\in\overline{\mathcal{J}}^L u(\mathbf{0})$, that is, $-u_0 = \max_\tau \overline{\mathcal{E}}^L[(\psi^{\alpha,\beta}-u)_{\mathrm{H}_\varepsilon\wedge\tau}]$, with $\mathrm{H}_\varepsilon := \varepsilon\wedge\mathrm{H}_{O_\varepsilon}$. Then we have for all $\mathbb{P}\in\mathcal{P}^{\mathrm{UVM}}\subset\mathcal{P}^L$ and $h>0$ that

$$0 \ge \mathbb{E}^{\mathbb{P}}\big[\psi^{\alpha,\beta}_{\mathrm{H}_\varepsilon\wedge h} - u_{\mathrm{H}_\varepsilon\wedge h} + u_0\big]$$
$$\ge \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\beta\langle B\rangle_{\mathrm{H}_\varepsilon\wedge h} + \alpha B_{\mathrm{H}_\varepsilon\wedge h}\right] + \mathbb{E}^{\mathbb{P}}\big[\big(e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1\big)u_{\mathrm{H}_\varepsilon\wedge h}\big]$$
$$- \mathbb{E}^{\mathbb{P}}\big[e^{-r(\mathrm{H}_\varepsilon\wedge h)}u_{\mathrm{H}_\varepsilon\wedge h}\big] + u_0.$$

It follows from Lemma 7.4 that $u_0 = \overline{\mathcal{E}}^{\mathrm{UVM}}[e^{-r(\mathrm{H}_\varepsilon\wedge h)}u_{\mathrm{H}_\varepsilon\wedge h}] \ge \mathbb{E}^{\mathbb{P}}[e^{-r(\mathrm{H}_\varepsilon\wedge h)} \times u_{\mathrm{H}_\varepsilon\wedge h}]$. Therefore,

$$0 \ge \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\beta\langle B\rangle_{\mathrm{H}_\varepsilon\wedge h} + \alpha B_{\mathrm{H}_\varepsilon\wedge h}\right] + \mathbb{E}^{\mathbb{P}}\big[\big(e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1\big)u_{\mathrm{H}_\varepsilon\wedge h}\big].$$

Now, we take $\mathbb{P}_\gamma\in\mathcal{P}^{\mathrm{UVM}}$ such that there exists a $\mathbb{P}_\gamma$-Brownian motion W such that $B_t = \gamma W_t$, $\mathbb{P}_\gamma$-a.s. It follows that

$$0 \ge \frac{1}{h}\mathbb{E}^{\mathbb{P}_\gamma}\left[\frac{1}{2}\gamma^2\beta(\mathrm{H}_\varepsilon\wedge h) + \big(e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1\big)u_{\mathrm{H}_\varepsilon\wedge h}\right].$$

Let $h\to 0$, we obtain that $0\ge -ru_0+\frac{1}{2}\gamma^2\beta$. Since $\gamma\in[\underline{\sigma},\overline{\sigma}]$ can be arbitrary, we finally have

$$ru_0 - \sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]}\frac{1}{2}\gamma^2\beta \ge 0.$$

Step 2. Now we verify the viscosity subsolution property. Without loss of generality, we only verity it at the point $\mathbf{0}$. Let $(\alpha,\beta)\in\underline{\mathcal{J}}^L u(\mathbf{0})$, that is, $-u_0 = \min_\tau \underline{\mathcal{E}}^L[(\psi^{\alpha,\beta}-u)_{\mathrm{H}_\varepsilon\wedge\tau}]$, with $\mathrm{H}_\varepsilon := \varepsilon\wedge\mathrm{H}_{O_\varepsilon}$. For any $h>0$, we have

$$0 \le \underline{\mathcal{E}}^L\big[\psi^{\alpha,\beta}_{\mathrm{H}_\varepsilon\wedge h} - u_{\mathrm{H}_\varepsilon\wedge h} + u_0\big].$$

So we have for all $\mathbb{P}\in\mathcal{P}^{\mathrm{UVM}}\subset\mathcal{P}^L$ that

$$0 \le \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\beta\langle B\rangle_{\mathrm{H}_\varepsilon\wedge h}\right] + \mathbb{E}^{\mathbb{P}}\big[\big(e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1\big)u_{\mathrm{H}_\varepsilon\wedge h}\big] - \mathbb{E}^{\mathbb{P}}\big[e^{-r(\mathrm{H}_\varepsilon\wedge h)}u_{\mathrm{H}_\varepsilon\wedge h}\big] + u_0$$
$$\le \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]}\gamma^2\beta(\mathrm{H}_\varepsilon\wedge h) + \big(e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1\big)u_{\mathrm{H}_\varepsilon\wedge h}\right]$$
$$- \mathbb{E}^{\mathbb{P}}\big[e^{-r(\mathrm{H}_\varepsilon\wedge h)}u_{\mathrm{H}_\varepsilon\wedge h}\big] + u_0.$$

Since $u_0 = \overline{\mathcal{E}}^{\mathrm{UVM}}[e^{-r(\mathrm{H}_\varepsilon\wedge h)}u_{\mathrm{H}_\varepsilon\wedge h}]$ (Lemma 7.4), it follows that

$$0 \le \overline{\mathcal{E}}^{\mathrm{UVM}}\Big[\frac{1}{2}\sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]}\gamma^2\beta(\mathrm{H}_\varepsilon\wedge h) + \big(e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1\big)u_{\mathrm{H}_\varepsilon\wedge h}\Big]. \tag{7.2}$$

Since we have

$$\begin{aligned}\left|\frac{e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1}{h}u_{\mathrm{H}_\varepsilon\wedge h} + ru_0\right| &\le \left|\frac{e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1}{h}+r\right||u_{\mathrm{H}_\varepsilon\wedge h}| + r|u_{\mathrm{H}_\varepsilon\wedge h}-u_0| \\ &\le C\left|\frac{e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1}{h}+r\right| + r\rho(\varepsilon),\end{aligned}$$

where ρ is a modulus of continuity of u. By denoting

$$\delta(h) := \sup_{0\le s\le h}\left|\frac{e^{-rs}-1}{s}+r\right|,$$

we have the following estimate:

$$\begin{aligned}&\left|\frac{e^{-r(\mathrm{H}_\varepsilon\wedge h)}-1}{h}u_{\mathrm{H}_\varepsilon\wedge h} + ru_0\right| \\ &\quad\le \big(C\delta(h)+r\rho(\varepsilon)\big)1_{\{\mathrm{H}_\varepsilon>h\}} + \big(C\big(r+\delta(h)\big)+r\rho(\varepsilon)\big)1_{\{\mathrm{H}_\varepsilon\le h\}}.\end{aligned}$$

Together with (7.2), we obtain that

$$\begin{aligned}0 &\le \overline{\mathcal{E}}^{\mathrm{UVM}}\Big[\frac{1}{2}\sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]}\gamma^2\beta\frac{\mathrm{H}_\varepsilon\wedge h}{h} - ru_0\Big] + C\delta(h) + r\rho(\varepsilon) \\ &\quad + \big(C\big(r+\delta(h)\big)+r\rho(\varepsilon)\big)\mathcal{C}^{\mathcal{P}^{\mathrm{UVM}}}[\mathrm{H}_\varepsilon\le h] \\ &\le \frac{1}{2}\sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]}\gamma^2\beta - ru_0 + C\delta(h) + r\rho(\varepsilon) \\ &\quad + \Big(C\big(r+\delta(h)\big)+r\rho(\varepsilon)+\frac{1}{2}\overline{\sigma}^2|\beta|\Big)\mathcal{C}^{\mathcal{P}^{\mathrm{UVM}}}[\mathrm{H}_\varepsilon\le h].\end{aligned}$$

By letting $h\to 0$, we get $ru_0 - r\rho(\varepsilon) - \sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]}\frac{1}{2}\gamma^2\beta \le 0$. Finally, by letting $\varepsilon\to 0$, we obtain

$$ru_0 - \sup_{\gamma\in[\underline{\sigma},\overline{\sigma}]}\frac{1}{2}\gamma^2\beta \le 0. \qquad \square$$

APPENDIX

PROOF OF PROPOSITION 2.11. The first result is easy, and we omit its proof. We decompose the proof in two steps.

Step 1. We first prove that $\overline{\mathcal{E}}^L[\mathrm{H}_D] < \infty$. Without loss of generality, we may assume that $D = O_r$. Denote by B^1 the first entry of B. Since

$$\mathrm{H}_{O_r} \le \mathrm{H}_r^1 := \inf\{t \ge 0 : |B_t^1| \ge r\},$$

it is enough to show that $\overline{\mathcal{E}}^L[\mathrm{H}_r^1] < \infty$. Thus, without loss of generality, we may assume that the dimension $d = 1$.

We first consider the following Dirichlet problem of ODE:

$$-L|\partial_x u| - \frac{1}{L}\partial^2_{xx} u - 1 = 0, \qquad u(r) = u(-r) = 0. \tag{A.1}$$

It is easy to verify that equation (A.1) has a classical solution:

$$\begin{aligned} &u(x) = \frac{1}{L^3}\big(e^{L^2 r} - e^{L^2 x}\big) - \frac{1}{L}(R - x) \qquad \text{for } 0 \le x \le r, \quad \text{and} \\ &u(x) = u(-x) \qquad \text{for } -r \le x \le 0. \end{aligned}$$

Further, it is clear that u is concave, so u is also a classical solution to the equation

$$-L|\partial_x u| - \frac{1}{2}\sup_{\frac{2}{L} \le \beta \le 2L} \beta \partial^2_{xx} u - 1 = 0, \qquad u(r) = u(-r) = 0. \tag{A.2}$$

Then by Itô's formula we obtain

$$0 = u(B_{\mathrm{H}_{O_r}}) = u_0 + \int_0^{\mathrm{H}_{O_r}} \partial_x u(B_t)\, dB_t + \frac{1}{2}\int_0^{\mathrm{H}_{O_r}} \partial^2_{xx} u(B_t)\, d\langle B\rangle_t.$$

Recalling the definition of $\mathbb{Q}^{\alpha,\beta}$ in (2.6) and taking the expectation on both sides, we have

$$\begin{aligned} &0 = u_0 + \mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\left[\int_0^{\mathrm{H}_{O_r}} \left(\alpha_t \partial_x u(B_t) + \frac{1}{2}\beta_t^2 \partial^2_{xx} u(B_t)\right) dt\right] \\ &\qquad\qquad \text{for all } \|\alpha\| \le L, \frac{2}{L} \le \beta_\cdot \le 2L. \end{aligned} \tag{A.3}$$

Since u is a solution of equation (A.2), we have

$$\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\left[\int_0^{\mathrm{H}_{O_r}} \left(\alpha_t \partial_x u(B_t) + \frac{1}{2}\beta_t^2 \partial^2_{xx} u(B_t)\right) dt\right] \le -\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}[\mathrm{H}_{O_r}].$$

Hence, $u_0 \ge \overline{\mathcal{E}}^L[\mathrm{H}_{O_r}]$. On the other hand, taking $\alpha^* := L\,\mathrm{sgn}(\partial_x u(B_t))$ and $\beta^* := \sqrt{\frac{2}{L}}$, we obtain from (A.2) and (A.3) that

$$u_0 = \mathbb{E}^{\mathbb{Q}^{\alpha^*,\beta^*}}[\mathrm{H}_{O_r}].$$

So, we have proved that $u_0 = \overline{\mathcal{E}}^L[\mathrm{H}_{O_r}]$. Consequently, $\overline{\mathcal{E}}^L[\mathrm{H}_{O_r}] < \infty$.

Step 2. Note that

$$\mathcal{C}^L[\mathrm{H}_D \geq T] \leq \frac{\overline{\mathcal{E}}^L[\mathrm{H}_D]}{T}.$$

By the result of Step 1, we have $\mathcal{C}^L[\mathrm{H}_D \geq T] \leq \frac{C}{T}$, and then $\lim_{T\to\infty} \mathcal{C}^L[\mathrm{H}_D \geq T] = 0$. Further,

$$\begin{aligned}\mathcal{C}^L[\mathrm{H}_n < \mathrm{H}_D] &\leq \mathcal{C}^L[\mathrm{H}_n < \mathrm{H}_D; \mathrm{H}_D \leq T] + \mathcal{C}^L[\mathrm{H}_n < \mathrm{H}_D; \mathrm{H}_D > T] \\ &\leq \mathcal{C}^L[\mathrm{H}_n < T] + \mathcal{C}^L[\mathrm{H}_D > T].\end{aligned} \tag{A.4}$$

We conclude that $\lim_{n\to\infty} \mathcal{C}^L[\mathrm{H}_n < \mathrm{H}_D] = 0$.

Further, define $\hat{D} := \bigcup_{x\in D} D^x$. Note that $\mathrm{H}_D^x \leq \mathrm{H}_{\hat{D}}$ for all $x \in D$. Hence, we have

$$\sup_{x\in D} \mathcal{C}^L\big[\mathrm{H}_D^x \geq T\big] \leq \mathcal{C}^L[\mathrm{H}_{\hat{D}} \geq T] \to 0.$$

Together with (A.4), we obtain $\lim_{n\to\infty} \sup_{x\in D} \mathcal{C}^L[\mathrm{H}_n < \mathrm{H}_D^x] = 0$. $\square$

PROOF OF LEMMA 5.9. For simplicity, denote

$$g^i := G(\omega^i, \cdot, \cdot, \cdot) \qquad (i = 1, 2), \qquad c_0 := \rho\big(d^e(\omega^1, \omega^2)\big) \qquad \big(\geq |g^1 - g^2|\big),$$
$$\mathbf{L}^i u := -g^i\big(u, \partial_x u, \partial^2_{xx} u\big) \qquad (i = 1, 2), \quad \text{and} \quad \delta h := h^1 - h^2.$$

By standard argument, one can easily verify that function

$$w(x) := \overline{\mathcal{E}}^{L_0}\big[\delta h^+(x + B_{\mathrm{H}_D^x}) + c_0 \mathrm{H}_D^x\big]$$

is a C–L viscosity solution of the nonlinear PDE

$$-c_0 - L_0|\partial_x w| - \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}} I_d \leq \gamma \leq \sqrt{2L_0} I_d} \gamma^2 : \partial^2_{xx} w = 0 \qquad \text{on } D, \quad \text{and}$$
$$w = (\delta h)^+ \qquad \text{on } \partial D.$$

Let K be a smooth nonnegative kernel with unit total mass. For all $\eta > 0$, we define the mollification $w^\eta := w * K^\eta$ of w. Then w^η is smooth, and it follows from a convexity argument as in [14] that w^η is a classic supersolution of

$$\begin{aligned}&-c_0 - L_0\big|\partial_x w^\eta\big| - \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}} I_d \leq \gamma \leq \sqrt{2L_0} I_d} \gamma^2 : \partial^2_{xx} w^\eta \geq 0 \qquad \text{on } D, \quad \text{and} \\ &w^\eta = (\delta h)^+ * K^\eta \qquad \text{on } \partial D.\end{aligned} \tag{A.5}$$

We claim that

$\bar{w}^\eta + v^2$ is a C–L viscosity supersolution to the PDE with generator g^1, where $\bar{w}^\eta := w^\eta + \delta$, with $\delta := \max_{x\in\partial D} |w^\eta(x) - (\delta h)^+(x)|$. Then we note that

$$\bar{w}^\eta + v^2 \ge w^\eta + h^2 + \delta \ge h^1 = v^1 \qquad \text{on } \partial D.$$

By comparison principle for the C–L viscosity solutions of PDEs, we have $\bar{w}^\eta + v^2 \ge v^1$ on $\mathrm{cl}(D)$. Setting $\eta \to 0$, we obtain that $v^1 - v^2 \le w$. The desired result follows.

It remains to prove that $\bar{w}^\eta + v^2$ is a C–L viscosity supersolution of the PDE with generator g^1. Let $x_0 \in D$, $\phi \in C^2(D)$ be such that $0 = (\phi - \bar{w}^\eta - v^2)(x_0) = \max(\phi - \bar{w}^\eta - v^2)$. Then it follows from the viscosity supersolution property of v^2 that $\mathbf{L}^2(\phi - \bar{w}^\eta)(x_0) \ge 0$. Hence, at the point x_0, by (A.5) we have

$$\begin{aligned}
\mathbf{L}^1\phi &\ge \mathbf{L}^1\phi - \mathbf{L}^2(\phi - \bar{w}^\eta)\\
&= -g^1(\phi, \partial_x\phi, \partial^2_{xx}\phi) + g^2(\phi - \bar{w}^\eta, \partial_x(\phi - \bar{w}^\eta), \partial^2_{xx}(\phi - \bar{w}^\eta))\\
&\ge -g^1(\phi, \partial_x\phi, \partial^2_{xx}\phi) + g^2(\phi, \partial_x(\phi - \bar{w}^\eta), \partial^2_{xx}(\phi - \bar{w}^\eta))\\
&\ge -c_0 - L_0|\partial_x w^\eta| - \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}} I_d \le \gamma \le \sqrt{2L_0} I_d} \gamma^2 : \partial^2_{xx} w^\eta\\
&\ge 0,
\end{aligned}$$

where the last inequality is due to (A.5). $\square$

PROPOSITION A.1. *For all $n \ge 1$, there exists a modulus of continuity ρ such that*

$$\overline{\mathcal{E}}^L\big[\big|\mathrm{H}_Q^{x_1} - \mathrm{H}_Q^{x_2}\big|\big] \le \rho(|x_1 - x_2|).$$

PROOF. By the tower property, we have

$$\begin{aligned}
\overline{\mathcal{E}}^L\big[\big|\mathrm{H}_Q^{x_1} - \mathrm{H}_Q^{x_2}\big|\big] &\le \overline{\mathcal{E}}^L\big[\big|\mathrm{H}_Q^{x_1} - \mathrm{H}_Q^{x_2}\big| 1_{\{\mathrm{H}_Q^{x_1} \le \mathrm{H}_Q^{x_2}\}}\big] + \overline{\mathcal{E}}^L\big[\big|\mathrm{H}_Q^{x_1} - \mathrm{H}_Q^{x_2}\big| 1_{\{\mathrm{H}_Q^{x_1} > \mathrm{H}_Q^{x_2}\}}\big]\\
&\le \overline{\mathcal{E}}^L\big[\overline{\mathcal{E}}^L\big[\mathrm{H}_Q^{x_2 + B_{\mathrm{H}_Q^{x_1}}}\big] 1_{\{\mathrm{H}_Q^{x_1} \le \mathrm{H}_Q^{x_2}\}}\big] + \overline{\mathcal{E}}^L\big[\overline{\mathcal{E}}^L\big[\mathrm{H}_Q^{x_1 + B_{\mathrm{H}_Q^{x_2}}}\big] 1_{\{\mathrm{H}_Q^{x_1} > \mathrm{H}_Q^{x_2}\}}\big].
\end{aligned}$$

So, it suffices to show that there exists a modulus of continuity ρ such that

$$\overline{\mathcal{E}}^L\big[\mathrm{H}_Q^{x_2 + \omega'_{\mathrm{H}_Q^{x_1}}}\big] \le \rho(|x_1 - x_2|), \qquad \text{for all } \omega' \text{ such that } \mathrm{H}_Q^{x_1}(\omega') \le \mathrm{H}_Q^{x_2}(\omega').$$

Denote $y_i := x_i + \omega'_{\mathrm{H}_Q^{\omega^1}}$ for $i = 1, 2$. Note that

$$|y_1 - y_2| = |x_1 - x_2|, \qquad y_1 \in \partial Q, y_2 \in Q.$$

In the case of the dimension $d = 1$, we may assume that $Q = [0, h]$ for some $h > 0$. Next, consider the Dirichlet problem of ODE:

$$\text{(A.6)} \qquad -L|\partial_x u| - \frac{1}{2} \sup_{\frac{2}{L} \le \beta \le 2L} \beta \partial^2_{xx} u - 1 = 0 \quad \text{and} \quad u\left(-\frac{h}{2}\right) = u\left(\frac{h}{2}\right) = 0.$$

Then, as in the proof of Proposition 2.11 above, we can prove that equation (A.6) has a classical solution u and

$$\overline{\mathcal{E}}^L[\mathrm{H}_Q^{y_2}] = u\left(\frac{h}{2} - |x_1 - x_2|\right) = u\left(\frac{h}{2} - |x_1 - x_2|\right) - u\left(\frac{h}{2}\right) \le \rho(|x_1 - x_2|),$$

where ρ is the modulus of continuity of u.

In the case $d > 1$, we need the following discussion. Since Q is bounded and convex, there exists a d-dimensional open cube $\widehat{Q}$ such that $Q \subset \widehat{Q}$, $d(y_2, \partial\widehat{Q}) \le |y_1 - y_2| = |x_1 - x_2|$ and there is a unique point $y^* \in \partial\widehat{Q}$ such that $d(y_2, \partial\widehat{Q}) = |y_2 - y^*|$. Since $\mathrm{H}_Q^{y_2} \le \mathrm{H}_{\widehat{Q}}^{y_2}$, it is enough to prove

$$\overline{\mathcal{E}}^L[\mathrm{H}_{\widehat{Q}}^{y_2}] \le \rho(|x_1 - x_2|). \tag{A.7}$$

Denote the unit vector $e^* := \frac{y^* - y_2}{|y^* - y_2|}$. Note that

$$\begin{aligned} &y_2 + |y^* - y_2| e^* \in \partial\widehat{Q} \\ &\text{and there is a constant } \ell > 0 \text{ such that } y_2 - \ell e^* \in \partial\widehat{Q}. \end{aligned} \tag{A.8}$$

Denote a new stopping time

$$\mathrm{H}^* := \inf\{t \ge 0 : B \cdot e^* \notin (-\ell, |y^* - y_2|)\}.$$

Since $\widehat{Q}$ is a cube, it follows from (A.8) that $\mathrm{H}_{\widehat{Q}}^{y_2} \le \mathrm{H}^*$. Since $B \cdot e^*$ takes values in $\mathbb{R}^1$, it follows from the previous result in the case $d = 1$ that

$$\overline{\mathcal{E}}^L[\mathrm{H}^*] \le \rho(|y^* - y_2|) \le \rho(|x_1 - x_2|), \qquad \text{for some modulus of continuity } \rho.$$

Together with the fact $\mathrm{H}_{\widehat{Q}}^{y_2} \le \mathrm{H}^*$, we finally obtain (A.7). □

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