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VISCOSITY SOLUTIONS OF FULLY NONLINEAR PARABOLIC PATH DEPENDENT PDES: PART II

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In our previous paper [Ekren, Touzi and Zhang (2015)], we introduced a notion of viscosity solutions for fully nonlinear path-dependent PDEs, extending the semilinear case of Ekren et al. [*Ann. Probab.* **42** (2014) 204–236], which satisfies a partial comparison result under standard Lipshitz-type assumptions. The main result of this paper provides a full, well-posedness result under an additional assumption, formulated on some partial differential equation, defined locally by freezing the path. Namely, assuming further that such path-frozen standard PDEs satisfy the comparison principle and the Perron approach for existence, we prove that the nonlinear path-dependent PDE has a unique viscosity solution. Uniqueness is implied by a comparison result.

1. Introduction. This paper is the continuation of our accompanying papers [7, 8]. The main objective of this series of three papers is the following, fully nonlinear parabolic path-dependent partial differential equation:

$$\{-\partial_t u - G(\cdot, u, \partial_\omega u, \partial^2_{\omega\omega} u)\}(t,\omega) = 0, \qquad (t,\omega) \in [0,T) \times \Omega. \tag{1.1}$$

Here Ω consists of continuous paths ω on $[0,T]$ starting from the origin, G is a progressively measurable generator and the path derivatives $\partial_t u$, $\partial_\omega u$, $\partial^2_{\omega\omega} u$ are defined through a functional Itô formula, initiated by Dupire [5]; see also Cont and Fournie [3]. Such equations were first proposed by Peng [16], and they provide a convenient language for many problems arising in non-Markovian, or say path dependent framework, with typical examples, including martingales, backward stochastic differential equations, second-order BSDEs and backward stochastic PDEs. In particular, the value functions of stochastic controls and stochastic differential games with both drift and diffusion controls can be characterized as the solution of the corresponding path dependent PDEs. This extends the classical results in Markovian framework to non-Markovian ones. We refer to [8] and [17] for these connections.

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A path dependent PDE can rarely have a classical solution. We thus turn to the notion of viscosity solutions, which had great success in the finite dimensional case. There have been numerous publications on viscosity solutions of PDEs, both in theory and in applications, and we refer to the classical references [4] and [9]. In our infinite dimensional case, the major difficulty is that the underlying state space Ω is not locally compact, and thus many tools from the standard PDE viscosity theory do not apply to the present context. In our earlier paper [6], which studies semilinear path-dependent PDEs, we replace the pointwise extremality in the standard definition of viscosity solution in PDE literature with the corresponding extremality in the context of an optimal stopping problem under a nonlinear expectation $\mathcal{E}$. More precisely, we introduce a set of smooth test processes φ, which are tangent from above or from below, to the processes of interest u in the sense of the following nonlinear optimal stopping problems:

$$\sup_\tau \overline{\mathcal{E}}[(\varphi - u)_\tau], \qquad \inf_\tau \underline{\mathcal{E}}[(\varphi - u)_\tau] \quad \text{where } \overline{\mathcal{E}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}, \underline{\mathcal{E}} := \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}. \tag{1.2}$$

Here τ ranges over a convenient set of stopping times, and $\mathcal{P}$ is an appropriate set of probability measures. The replacement of the pointwise tangency by the tangency in the sense of the last optimal stopping problem is the key ingredient needed to bypass the local compactness of the underlying space in the standard viscosity solution theory (or the Hilbert structure, which allows us to access local compactness by finite realization approximation of the space). Indeed, the Snell envelope characterization of the solution of the optimal stopping problem allows us to find a "point of tangency." Interestingly, the structure of the underlying space does not play any role, and the standard first and second-order conditions of maximality in the standard optimization theory has the following beautiful counterpart in the optimal stopping problem: the supermartingale property (negative drift; notice that drift is related to the second derivative) of the Snell envelope and the martingale property (zero drift) up the optimal stopping time (first hitting of the obstacle/reward process).

In [6], we proved existence and uniqueness of viscosity solutions for semilinear path-dependent PDEs. In particular, the unique viscosity solution is consistent with the solution to the corresponding backward SDE.

In [6], all probability measures in the class $\mathcal{P}$ are equivalent, and consequently $\mathcal{P}$ is dominated by one measure. In our fully nonlinear context, the class $\mathcal{P}$ becomes nondominated, consisting of mutually singular measures induced by certain linearization of the nonlinear generator G. This causes another major difficulty of the project: the dominated convergence theorem fails under $\overline{\mathcal{E}}^{\mathcal{P}}$. To overcome this, we need some strong regularity for the involved processes, and thus we require some rather sophisticated estimates. In particular, the corresponding optimal stopping problem becomes very technical and is established in a separate paper [7].

We remark that the weak compactness of the class $\mathcal{P}$ plays a very important role in these arguments.

In [8], we introduced the appropriate class $\mathcal{P}$ for fully nonlinear path dependent PDEs (1.1) and the corresponding notion of viscosity solutions. We investigated the connection between our new notion and many other equations in the existing literature of stochastic analysis, for example, backward SDEs, second-order BSDEs and backward SPDEs. Moreover, we proved some basic properties of viscosity solutions, including the partial comparison principle; that is, for a viscosity subsolution u^1 and a classical supersolution u^2, if $u^1_T \le u^2_T$, then $u^1(t,\omega) \le u^2(t,\omega)$ for all $(t,\omega) \in [0,T] \times \Omega$.

In this paper we prove our main result, the comparison principle of viscosity solutions. That is, for a viscosity subsolution u^1 and a viscosity supersolution u^2, if $u^1_T \le u^2_T$, then $u^1(t,\omega) \le u^2(t,\omega)$ for all $(t,\omega) \in [0,T] \times \Omega$. Again, due to the lack of local compactness and now also due to our new definition of viscosity solutions, the standard approach in PDE literature, namely the doubling variable technique combined with Ishii's lemma, does not seem to work here. Our strategy is as follows: We start from the above partial comparison established in [8], but we slightly weaken the smooth requirement of the classical (semi-)solutions. Let $\overline{u}$ denote the infimum of the classical supersolution and $\underline{u}$, the supremum of classical subrsolutions, satisfying appropriate terminal conditions. Then the partial comparison implies $u^1 \le \overline{u}$ and $\underline{u} \le u^2$. Thus the comparison will be a direct consequence of the following claim:

$$\overline{u} = \underline{u}. \tag{1.3}$$

Then clearly our focus is (1.3). We first remark that due to the failure of the dominated convergence theorem under our new $\overline{\mathcal{E}}^{\mathcal{P}}$, the approach in [6] does not work here. In this paper, we shall follow the alternative approach proposed in [8], Section 7, which is also devoted to semilinear path-dependent PDEs. However, as explained in [8], Remark 7.7, there are several major difficulties in the fully nonlinear context, and novel ideas are needed.

Note that (1.3) is more or less equivalent to constructing some classical supersolution $\overline{u}^\varepsilon$ and classical subsolution $\underline{u}^\varepsilon$, for any $\varepsilon > 0$, such that $\lim_{\varepsilon\to 0}[\overline{u}^\varepsilon - \underline{u}^\varepsilon] = 0$. Our main tool is the following local path-frozen PDE: for any $(t,\omega) \in [0,T) \times \Omega$,

$$\begin{aligned} &-\partial_t v(s,x) - g^{t,\omega}\big(s, v, Dv, D^2 v\big) = 0, \\ &\qquad s \in [t, t+\varepsilon], x \in \mathbb{R}^d \text{ such that } |x| \le \varepsilon, \\ &\qquad \text{where } g^{t,\omega}(s,y,z,\gamma) := G(s, \omega_{\cdot\wedge t}, y, z, \gamma). \end{aligned} \tag{1.4}$$

Here D and D^2 denote the gradient and Hessian of v with respect to x, respectively, and we emphasize that $g^{t,\omega}$ is a deterministic function, and thus (1.4) is a standard PDE. We shall assume that the above PDE has a unique viscosity solution (in standard sense), which can be approximated by classical subsolutions and classical supersolutions. One sufficient condition is that after certain smooth mollification of $g^{t,\omega}$, the above local PDE with smooth boundary condition has a classical solution. We then use this classical solution to construct the desired $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$.

We remark that this approach is very much like Perron's approach in standard PDE viscosity theory. However, there are two major differences: First, in the standard Perron approach, $\overline{u}$ and $\underline{u}$ are the extremality of viscosity semi-solutions, while here they are the extremality of classical semi-solutions. This requires the smoothness of the above $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$ and thus makes their construction harder. More importantly, the standard Perron approach assumes the comparison principle and uses it to obtain the existence of viscosity solutions, while we use (1.3) to prove the comparison principle. Thus the required techniques are quite different.

Once we have the comparison principle, then following the idea of the standard Perron approach, we see $\overline{u} = \underline{u}$ is indeed the unique viscosity solution of the path-dependent PDE, so we have both existence and uniqueness. Our result covers quite general classes of path-dependent PDEs, including those not accessible in the existing literature of stochastic analysis. One particular application is the existence of the game value for a path-dependent zero sum stochastic differential game, due to our well-posedness result of the path-dependent Bellman–Isaacs equation; see Pham and Zhang [17]. We also refer to Henry-Labordere, Tan and Touzi [12] and Zhang and Zhuo [18] for applications of our results to numerical methods for path-dependent PDEs.

We also note that there is potentially an alternative way to prove the comparison principle. Roughly speaking, given a viscosity subsolution u^1 and a viscosity supersolution u^2, if we could find certain smooth approximations $u^{i,\varepsilon}$, close to u^i, such that $u^{1,\varepsilon}$ is a classical subsolution and $u^{2,\varepsilon}$ is a classical supersolution, then it follows from partial comparison (actually classical comparison) that $u^{1,\varepsilon} \le u^{2,\varepsilon}$, which leads to the desired comparison immediately by passing ε to 0. Indeed, in PDE literature the convex/concave convolution plays this role. However, in the path-dependent setting, we did not succeed in finding appropriate approximations $u^{i,\varepsilon}$ which satisfy the desired semi-solution property. In our current approach, instead of approximating the (semi-)solution directly, we approximate the path-frozen PDE by mollifying its generator $g^{t,\omega}$. The advantage of our approach is that provided the mollified path-frozen PDE has a classical solution, it will be straightforward to check that the constructed $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$ are classical semi-solutions.

The price of our approach, however, is that we need classical solutions of fully nonlinear PDEs. Partially for this purpose, in the present paper we assume that G is uniformly nondegenerate, which is undesirable in viscosity theory, and for path-dependent Bellman–Isaacs equations, we can only deal with the lower dimensional ($d = 1$ or 2) problems. We shall investigate these important problems and explore further possible direct approximations of u^i as mentioned above, in our future research.

The rest of the paper is organized as follows. Section 2 introduces the general framework and recalls the definition of viscosity solutions introduced in our accompanying paper [8]. Section 3 collects all assumptions needed throughout the paper. The main results are stated in Section 4, where we also outline strategy of proof. In particular, the existence and comparison results follow from the partial comparison principle, the consistency of the Perron approach and the viscosity property of the postulated solution of the PPDE. These crucial results are proved in Sections 5, 6 and 7, respectively. Finally, Section 8 provides some sufficient conditions for our main assumption, under which our well-posedness result is established, together with some concluding remarks.

2. Preliminaries.

In this section, we recall the setup and the notation of [8].

2.1. *The canonical spaces.* Let $\Omega := \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = \mathbf{0}\}$ be the set of continuous paths starting from the origin, B, the canonical process, $\mathbb{F}$, the natural filtration generated by B, $\mathbb{P}_0$, the Wiener measure and $\Lambda := [0,T]\times\Omega$. Here and in the sequel, for notational simplicity, we use $\mathbf{0}$ to denote vectors, matrices or paths with appropriate dimensions whose components are all equal to 0. Let $\mathbb{S}^d$ denote the set of $d\times d$ symmetric matrices, and

$$x\cdot x' := \sum_{i=1}^d x_i x_i' \qquad \text{for any } x, x' \in \mathbb{R}^d,$$
$$\gamma : \gamma' := \operatorname{tr}[\gamma\gamma'] \qquad \text{for any } \gamma, \gamma' \in \mathbb{S}^d.$$

We define a semi-norm on Ω and a pseudometric on Λ as follows: for any $(t,\omega), (t',\omega') \in \Lambda$,

$$\|\omega\|_t := \sup_{0\le s\le t} |\omega_s|, \qquad \mathbf{d}_\infty\big((t,\omega),(t',\omega')\big) := |t-t'| + \|\omega_{\cdot\wedge t} - \omega'_{\cdot\wedge t'}\|_T. \tag{2.1}$$

Then $(\Omega, \|\cdot\|_T)$ is a Banach space, and $(\Lambda, \mathbf{d}_\infty)$ is a complete pseudometric space. We shall denote by $\mathbb{L}^0(\mathcal{F}_T)$ and $\mathbb{L}^0(\Lambda)$ the collection of all $\mathcal{F}_T$-measurable random variables and $\mathbb{F}$-progressively measurable processes, respectively. Let $C^0(\Lambda)$ [resp., $\mathrm{UC}(\Lambda)$] be the subset of $\mathbb{L}^0(\Lambda)$ whose elements are continuous (resp., uniformly continuous) in (t,ω) under $\mathbf{d}_\infty$. The corresponding subsets of bounded processes are denoted by $C_b^0(\Lambda)$ and $\mathrm{UC}_b(\Lambda)$. Finally, $\mathbb{L}^0(\Lambda,\mathbb{R}^d)$ denote the space of $\mathbb{R}^d$-valued processes with entries in $\mathbb{L}^0(\Lambda)$, and we define similar notation for the spaces C^0, C_b^0, UC and UC_b.

We next introduce the shifted spaces. Let $0\le s\le t\le T$.

- Let $\Omega^t := \{\omega \in C([t,T],\mathbb{R}^d) : \omega_t = \mathbf{0}\}$ be the shifted canonical space; B^t the shifted canonical process on Ω^t; $\mathbb{F}^t$ the shifted filtration generated by B^t, $\mathbb{P}_0^t$ the Wiener measure on Ω^t, and $\Lambda^t := [t,T]\times\Omega^t$.
- Define $\|\cdot\|_t^s$ on Ω^s and $\mathbf{d}_\infty^s$ on Λ^s in the spirit of (2.1), and the sets $\mathbb{L}^0(\Lambda^t)$ etc. in an obvious way.

– For $\omega \in \Omega^s$ and $\omega' \in \Omega^t$, define the concatenation path $\omega \otimes_t \omega' \in \Omega^s$ by

$$(\omega \otimes_t \omega')(r) := \omega_r \mathbf{1}_{[s,t)}(r) + (\omega_t + \omega'_r)\mathbf{1}_{[t,T]}(r) \qquad \text{for all } r \in [s,T].$$

– Let $\xi \in \mathbb{L}^0(\mathcal{F}_T^s)$ and $X \in \mathbb{L}^0(\Lambda^s)$. For $(t,\omega) \in \Lambda^s$, define $\xi^{t,\omega} \in \mathbb{L}^0(\mathcal{F}_T^t)$ and $X^{t,\omega} \in \mathbb{L}^0(\Lambda^t)$ by

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \qquad X^{t,\omega}(\omega') := X(\omega \otimes_t \omega'), \qquad \text{for all } \omega' \in \Omega^t.$$

It is clear that for any $(t,\omega) \in \Lambda$ and any $u \in C^0(\Lambda)$, we have $u^{t,\omega} \in C^0(\Lambda^t)$. The other spaces introduced before enjoy the same property.

We denote by $\mathcal{T}$ the set of $\mathbb{F}$-stopping times, and by $\mathcal{H} \subset \mathcal{T}$, the subset of those hitting times $\textsc{h}$ of the form

$$\textsc{h} := \inf\{t : B_t \notin O\} \wedge t_0, \tag{2.2}$$

for some $0 < t_0 \le T$, and some open and convex set $O \subset \mathbb{R}^d$ containing $\mathbf{0}$. The set $\mathcal{H}$ will be important for our optimal stopping problem, which is crucial for the comparison and the stability results. We note that $\textsc{h} = t_0$ when $O = \mathbb{R}^d$, and for any $\textsc{h} \in \mathcal{H}$,

$$0 < \textsc{h}_\varepsilon \le \textsc{h} \text{ for } \varepsilon \text{ small enough, where } \textsc{h}_\varepsilon := \inf\{t \ge 0 : |B_t| = \varepsilon\} \wedge \varepsilon. \tag{2.3}$$

Define $\mathcal{T}^t$ and $\mathcal{H}^t$ in the same spirit. For any $\tau \in \mathcal{T}$ (resp., $\textsc{h} \in \mathcal{H}$) and any $(t,\omega) \in \Lambda$ such that $t < \tau(\omega)$ [resp., $t < \textsc{h}(\omega)$], it is clear that $\tau^{t,\omega} \in \mathcal{T}^t$ (resp., $\textsc{h}^{t,\omega} \in \mathcal{H}^t$).

Finally, the following types of regularity will be important in our framework:

DEFINITION 2.1. Let $u \in \mathbb{L}^0(\Lambda)$.

(i) We say u is right continuous in (t,ω) under $\mathbf{d}_\infty$ if for any $(t,\omega) \in \Lambda$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(s,\tilde{\omega}) \in \Lambda^t$ satisfying $\mathbf{d}_\infty((s,\tilde{\omega}),(t,\mathbf{0})) \le \delta$, we have $|u^{t,\omega}(s,\tilde{\omega}) - u(t,\omega)| \le \varepsilon$.

(ii) We say $u \in \underline{\mathcal{U}}$ if u is bounded from above, right continuous in (t,ω) under $\mathbf{d}_\infty$ and there exists a modulus of continuity function ρ such that for any $(t,\omega), (t',\omega') \in \Lambda$,

$$u(t,\omega) - u(t',\omega') \le \rho(\mathbf{d}_\infty((t,\omega),(t',\omega'))) \qquad \text{whenever } t \le t'. \tag{2.4}$$

(iii) We say $u \in \overline{\mathcal{U}}$ if $-u \in \underline{\mathcal{U}}$.

The progressive measurability of u implies that $u(t,\omega) = u(t,\omega_{\cdot\wedge t})$, and it is clear that $\underline{\mathcal{U}} \cap \overline{\mathcal{U}} = \mathrm{UC}_b(\Lambda)$. We also recall from [7] Remark 3.2 that condition (2.4) implies that u has left-limits and positive jumps.

2.2. *Capacity and nonlinear expectation.* For every constant $L > 0$, we denote by $\mathcal{P}_L$ the collection of all continuous semimartingale measures $\mathbb{P}$ on Ω whose drift and diffusion characteristics are bounded by L and $\sqrt{2L}$, respectively. To be precise, let $\tilde{\Omega} := \Omega^3$ be an enlarged canonical space, $\tilde{B} := (B, A, M)$ be the canonical processes and $\tilde{\omega} = (\omega, a, m) \in \tilde{\Omega}$ be the paths. A probability measure $\mathbb{P} \in \mathcal{P}_L$ means that there exists an extension $\mathbb{Q}$ of $\mathbb{P}$ on $\tilde{\Omega}$ such that

$$B = A + M \qquad A \text{ is absolutely continuous}, M \text{ is a martingale},$$

$$|\alpha^{\mathbb{P}}| \le L, \qquad \frac{1}{2}\operatorname{tr}\bigl((\beta^{\mathbb{P}})^2\bigr) \le L \qquad \text{where } \alpha_t^{\mathbb{P}} := \frac{dA_t}{dt}, \beta_t^{\mathbb{P}} := \sqrt{\frac{d\langle M\rangle_t}{dt}} \quad \mathbb{Q}\text{-a.s.} \tag{2.5}$$

Similarly, for any $t \in [0, T)$, we may define $\mathcal{P}_L^t$ on Ω^t and $\mathcal{P}_\infty^t := \bigcup_{L>0} \mathcal{P}_L^t$.

The set $\mathcal{P}_L^t$ induces the following capacity:

$$\mathcal{C}_t^L[A] := \sup_{\mathbb{P}\in\mathcal{P}_L^t} \mathbb{P}[A], \qquad \text{for all } A \in \mathcal{F}_T^t. \tag{2.6}$$

We denote by $\mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t)$ the set of all $\mathcal{F}_T^t$-measurable r.v. ξ with $\sup_{\mathbb{P}\in\mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[|\xi|] < \infty$. The following nonlinear expectation will play a crucial role:

$$\overline{\mathcal{E}}_t^L[\xi] = \sup_{\mathbb{P}\in\mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[\xi] \quad \text{and} \quad \underline{\mathcal{E}}_t^L[\xi] = \inf_{\mathbb{P}\in\mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[\xi] = -\overline{\mathcal{E}}_t^L[-\xi] \quad \text{for all } \xi \in \mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t). \tag{2.7}$$

DEFINITION 2.2. Let $X \in \mathbb{L}^0(\Lambda)$ satisfy $X_t \in \mathbb{L}^1(\mathcal{F}_t, \mathcal{P}_L)$ for all $0 \le t \le T$. We say that X is an $\overline{\mathcal{E}}^L$-supermartingale (resp., submartingale, martingale) if, for any $(t, \omega) \in \Lambda$ and any $\tau \in \mathcal{T}^t$, $\overline{\mathcal{E}}_t^L[X_\tau^{t,\omega}] \le$ (resp., $\ge, =$) $X_t(\omega)$.

We now state the Snell envelope characterization of optimal stopping under the above nonlinear expectation operators. Given a bounded process $X \in \mathbb{L}^0(\Lambda)$, consider the nonlinear optimal stopping problem

$$\overline{\mathcal{S}}_t^L[X](\omega) := \sup_{\tau\in\mathcal{T}^t} \overline{\mathcal{E}}_t^L\bigl[X_\tau^{t,\omega}\bigr] \quad \text{and} \quad \underline{\mathcal{S}}_t^L[X](\omega) := \inf_{\tau\in\mathcal{T}^t} \underline{\mathcal{E}}_t^L\bigl[X_\tau^{t,\omega}\bigr], \quad (t,\omega) \in \Lambda. \tag{2.8}$$

By definition, we have $\overline{\mathcal{S}}^L[X] \ge X$ and $\overline{\mathcal{S}}_T^L[X] = X_T$.

THEOREM 2.3 ([7]). *Let $X \in \underline{\mathcal{U}}$ be bounded, $\mathrm{H} \in \mathcal{H}$ and set $\widehat{X}_t := X_t \mathbf{1}_{\{t<\mathrm{H}\}} + X_{\mathrm{H}-}\mathbf{1}_{\{t\ge\mathrm{H}\}}$. Define*

$$Y := \overline{\mathcal{S}}^L[\widehat{X}] \quad \textit{and} \quad \tau^* := \inf\{t \ge 0 : Y_t = \widehat{X}_t\}.$$

Then $Y_{\tau^*} = \widehat{X}_{\tau^*}$, Y *is an* $\overline{\mathcal{E}}^L$*-supermartingale on* $[0, \mathrm{H}]$, *and an* $\overline{\mathcal{E}}^L$*-martingale on* $[0, \tau^*]$. *Consequently*, τ^* *is an optimal stopping time*.

We remark that the nonlinear Snell envelope Y is continuous in $[0, \mathrm{H})$ and has left limit at H. However, in general Y may have a negative left jump at H.

2.3. *The path derivatives*. We define the path derivatives via the functional Itô formula, initiated by Dupire [5].

DEFINITION 2.4. We say $u \in C^{1,2}(\Lambda)$ if $u \in C^0(\Lambda)$, and there exist $\partial_t u \in C^0(\Lambda)$, $\partial_\omega u \in C^0(\Lambda, \mathbb{R}^d)$, $\partial^2_{\omega\omega} u \in C^0(\Lambda, \mathbb{S}^d)$ such that for any $\mathbb{P} \in \mathcal{P}^0_\infty$, u is a $\mathbb{P}$-semimartingale satisfying

$$du = \partial_t u \, dt + \partial_\omega u \cdot dB_t + \tfrac{1}{2} \partial^2_{\omega\omega} u : d\langle B\rangle_t, \qquad 0 \le t \le T, \mathbb{P}\text{-a.s.} \tag{2.9}$$

We remark that the above $\partial_t u$, $\partial_\omega u$ and $\partial^2_{\omega\omega} u$, if they exist, are unique, and thus are called the time derivative, first-order and second-order space derivatives of u, respectively. In particular, it holds that

$$\partial_t u(t, \omega) = \lim_{h \downarrow 0} \frac{1}{h}\bigl[u(t+h, \omega_{\cdot \wedge t}) - u(t, \omega)\bigr]. \tag{2.10}$$

We refer to [8], Remark 2.9, and [2], Remarks 2.3, 2.4, for various discussions on these path derivatives, especially on their comparison with Dupire's original definition. See also Remark 4.5 below. We define $C^{1,2}(\Lambda^t)$ similarly. It is clear that, for any (t, ω) and $u \in C^{1,2}(\Lambda)$, we have $u^{t,\omega} \in C^{1,2}(\Lambda^t)$, and $\partial_\omega(u^{t,\omega}) = (\partial_\omega u)^{t,\omega}$, $\partial^2_{\omega\omega}(u^{t,\omega}) = (\partial^2_{\omega\omega} u)^{t,\omega}$.

For technical reasons, we shall extend the space $C^{1,2}(\Lambda)$ slightly as follows.

DEFINITION 2.5. Let $t \in [0, T]$, $u : \Lambda^t \to \mathbb{R}$. We say $u \in \overline{C}^{1,2}(\Lambda^t)$ if there exist an increasing sequence of $\{\mathrm{H}_i, i \ge 1\} \subset \mathcal{T}^t$, a partition $\{E^i_j, j \ge 1\} \subset \mathcal{F}^t_{\mathrm{H}_i}$ of Ω^t for each i, a constant $n_i \ge 1$ for each i, and $\varphi^i_{jk} \in \mathrm{UC}_b(\Lambda)$, $\psi^i_{jk} \in C^{1,2}(\Lambda) \cap \mathrm{UC}_b(\Lambda)$ for each (i, j) and $1 \le k \le n_i$, such that, denoting $\mathrm{H}_0 := t$, $E^0_1 := \Omega^t$:

(i) for each i and ω, $\mathrm{H}^{\mathrm{H}_i,\omega}_{i+1} \in \mathcal{H}^{\mathrm{H}_i(\omega)}$ whenever $\mathrm{H}_i(\omega) < T$, the set $\{i : \mathrm{H}_i(\omega) < T\}$ is finite for each ω and $\lim_{i\to\infty} \mathcal{C}^L_s[\mathrm{H}^{s,\omega}_i < T] = 0$ for any $(s, \omega) \in \Lambda^t$ and $L > 0$;

(ii) for each (i, j), $\omega, \omega' \in E^i_j$ such that $\mathrm{H}_i(\omega) \le \mathrm{H}_i(\omega')$, it holds for all $\tilde\omega \in \Omega$,

$$0 \le \mathrm{H}_{i+1}\bigl(\omega' \otimes_{\mathrm{H}_i(\omega')} \tilde\omega\bigr) - \mathrm{H}_{i+1}(\omega \otimes_{\mathrm{H}_i(\omega)} \tilde\omega) \le \mathrm{H}_i\bigl(\omega'\bigr) - \mathrm{H}_i(\omega); \tag{2.11}$$

here we abuse the notation that $(\omega \otimes_s \tilde\omega)_r := \omega_r \mathbf{1}_{[t,s)}(r) + (\omega_s + \tilde\omega_{r-s})\mathbf{1}_{[s,T]}(r)$;

(iii) for each i, φ^i_{jk}, ψ^i_{jk}, $\partial_t \psi^i_{jk}$, $\partial_\omega \psi^i_{jk}$, $\partial^2_{\omega\omega} \psi^i_{jk}$ are uniformly bounded, and φ^i_{jk}, ψ^i_{jk} are uniformly continuous, uniformly in (j, k) (but may depend on i);

(iv) u is continuous in t on $[0,T]$, and for each i, $\omega\in\Omega$ and $\mathrm{H}_i(\omega)\le s\le \mathrm{H}_{i+1}(\omega)$,

$$u(s,\omega) = \sum_{j\ge 1}\sum_{k=1}^{n_i}\bigl[\varphi^i_{jk}\bigl(\mathrm{H}_i(\omega),\omega\bigr)\psi^i_{jk}\bigl(s-\mathrm{H}_i(\omega),\omega_{\mathrm{H}_i(\omega)+s}-\omega_{\mathrm{H}_i(\omega)}\bigr)\bigr]\mathbf{1}_{E^i_j}. \tag{2.12}$$

The main idea of the above space is that the processes in $\overline{C}^{1,2}(\Lambda^t)$ are piecewise smooth. However, purely for technical reasons, we require rather technical conditions. For example, (2.11) and (2.12) are mainly needed for Proposition 4.2 below. We remark that these technical requirements may vary from time to time. In particular, the space here requires a more specific structure than the corresponding space in [6] and that in [8] Section 7, both dealing with semilinear PPDEs. Nevertheless, by abusing the notation slightly, we still denote it as $\overline{C}^{1,2}(\Lambda^t)$.

Let $u\in\overline{C}^{1,2}(\Lambda^t)$. One may easily check that $u^{s,\omega}\in\overline{C}^{1,2}(\Lambda^s)$ for any $(s,\omega)\in\Lambda^t$. For any $\mathbb{P}\in\mathcal{P}^t_\infty$, it is clear that the process u is a local $\mathbb{P}$-semimartingale on $[t,T]$ and a $\mathbb{P}$-semimartingale on $[t,\mathrm{H}_i]$ for all i, and

$$du_s=\partial_t u_s\,ds+\tfrac{1}{2}\partial^2_{\omega\omega}u_s : d\langle B^t\rangle_s+\partial_\omega u_s\cdot dB^t_s,\qquad t\le s<T,\mathbb{P}\text{-a.s.} \tag{2.13}$$

By setting $\mathrm{H}_1:=T$, $n_0:=1$, $\varphi^0_{11}:=1$ and $\psi^0_{11}:=u$, we see that $C^{1,2}(\Lambda^t)\subset\overline{C}^{1,2}(\Lambda^t)$.

2.4. *Fully nonlinear path dependent PDEs.* Following the accompanying paper [8], we continue our study of the following fully nonlinear parabolic path-dependent partial differential equation (PPDE, for short):

$$\mathcal{L}u(t,\omega):=\bigl\{-\partial_t u-G\bigl(\cdot,u,\partial_\omega u,\partial^2_{\omega\omega}u\bigr)\bigr\}(t,\omega)=0,\qquad (t,\omega)\in\Lambda, \tag{2.14}$$

where the generator $G:\Lambda\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d\to\mathbb{R}$ satisfies the conditions reported in Section 3.

For any $u\in\mathbb{L}^0(\Lambda)$, $(t,\omega)\in[0,T)\times\Omega$ and $L>0$, define

$$\begin{aligned}\underline{\mathcal{A}}^L u(t,\omega)&:=\bigl\{\varphi\in C^{1,2}(\Lambda^t):(\varphi-u^{t,\omega})_t=0=\underline{\mathcal{S}}^L_t\bigl[(\varphi-u^{t,\omega})_{\cdot\wedge\mathrm{H}}\bigr]\\&\qquad\text{for some }\mathrm{H}\in\mathcal{H}^t\bigr\},\\ \overline{\mathcal{A}}^L u(t,\omega)&:=\bigl\{\varphi\in C^{1,2}(\Lambda^t):(\varphi-u^{t,\omega})_t=0=\overline{\mathcal{S}}^L_t\bigl[(\varphi-u^{t,\omega})_{\cdot\wedge\mathrm{H}}\bigr]\\&\qquad\text{for some }\mathrm{H}\in\mathcal{H}^t\bigr\},\end{aligned} \tag{2.15}$$

where $\overline{\mathcal{S}}^L$ and $\underline{\mathcal{S}}^L$ are the nonlinear Snell envelopes defined in (2.8).

DEFINITION 2.6. (i) Let $L > 0$. We say $u \in \underline{\mathcal{U}}$ (resp., $\overline{\mathcal{U}}$) is a viscosity L-subsolution (resp., L-supersolution) of PPDE (2.14) if, for any $(t, \omega) \in [0, T) \times \Omega$ and any $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$ [resp., $\varphi \in \overline{\mathcal{A}}^L u(t, \omega)$],

$$\{-\partial_t \varphi - G^{t,\omega}(\cdot, \varphi, \partial_\omega \varphi, \partial^2_{\omega\omega}\varphi)\}(t, \mathbf{0}) \le (\text{resp.}, \ge)\ 0.$$

(ii) We say $u \in \underline{\mathcal{U}}$ (resp., $\overline{\mathcal{U}}$) is a viscosity subsolution (resp., supersolution) of PPDE (2.14) if u is viscosity L-subsolution (resp., L-supersolution) of PPDE (2.14) for some $L > 0$.

(iii) We say $u \in \mathrm{UC}_b(\Lambda)$ is a viscosity solution of PPDE (2.14) if it is both a viscosity subsolution and a viscosity supersolution.

As pointed out in [8] Remark 3.11(i), without loss of generality in (2.15), we may always set $\mathrm{H} = \mathrm{H}^t_\varepsilon$ for some small $\varepsilon > 0$,

$$\mathrm{H}^t_\varepsilon := \inf\{s > t : |B^t_s| \ge \varepsilon\} \wedge (t + \varepsilon). \tag{2.16}$$

3. Assumptions.

This section collects all of our assumptions on the nonlinearity G, the terminal condition ξ and the underlying path-frozen PDE.

3.1. *Assumptions on the nonlinearity and terminal conditions.*

We first need the conditions on the nonlinearity G as assumed in [8].

ASSUMPTION 3.1. The nonlinearity G satisfies:

(i) for fixed (y, z, γ), $G(\cdot, y, z, \gamma) \in \mathbb{L}^0(\Lambda)$ and $|G(\cdot, 0, \mathbf{0}, \mathbf{0})| \le C_0$;

(ii) G is uniformly Lipschitz continuous in (y, z, γ), with a Lipschitz constant L_0;

(iii) for any (y, z, γ), $G(\cdot, y, z, \gamma)$ is right continuous in (t, ω) under $\mathbf{d}_\infty$;

(iv) G is elliptic, that is, nondecreasing in γ.

Our main well-posedness result requires the following strengthening of (iii) and (iv) above:

ASSUMPTION 3.2. (i) G is uniformly continuous in (t, ω) under $\mathbf{d}_\infty$ with a modulus of continuity function ρ_0.

(ii) For each ω, G is uniformly elliptic. That is, there exits a constant $c_0 > 0$ such that $G(\cdot, \gamma_1) - G(\cdot, \gamma_2) \ge c_0 \operatorname{tr}(\gamma_1 - \gamma_2)$ for any $\gamma_1 \ge \gamma_2$.

Condition (i) is needed for our uniform approximation of G below; in particular it is used (only) in the proof of Lemma 6.4. We should point out though, for the semi-linear PPDE and the path-dependent HJB considered in [8], Section 4, this condition is violated when σ depends on (t, ω). However, this is a technical condition due to our current approach for uniqueness. Condition (ii) is used to ensure the existence of the viscosity solution for the path-frozen PDE (3.3) below. See also Example 4.7.

Our first condition on the terminal condition ξ is the following:

ASSUMPTION 3.3. $\xi \in \mathbb{L}^0(\mathcal{F}_T)$ is bounded and uniformly continuous in ω under $\|\cdot\|_T$, with the same modulus of continuity function ρ_0 as in Assumption 3.2(i).

REMARK 3.4. The continuity of a random variable in terms of ω seems less natural in stochastic analysis literature. However, since by nature we are in the weak formulation setting, such continuity is in fact natural in many applications. This is emphasized in the two following examples:

- Let $V_0 := \mathbb{E}^{\mathbb{P}_0}[g(X^\sigma_\cdot)]$, for some bounded function $g : \Omega \longrightarrow \mathbb{R}$, and some bounded progressively measurable process σ, with

$$dX^\sigma_t = \sigma_t \, dB_t, \qquad \mathbb{P}_0\text{-a.s.}$$

  In the weak formulation, we define $\mathbb{P}^\sigma$ as the probability measure induced by the process X^σ, and we re-write $V_0 := \mathbb{E}^{\mathbb{P}^\sigma}[g(B_\cdot)]$. Thus the uniform continuity requirement reduces to that of the function g.
- Similarly, the stochastic control problem in strong formulation $V_0 := \sup_{\underline{\sigma} \le \sigma \le \overline{\sigma}} \mathbb{E}^{\mathbb{P}_0}[g(X^\sigma_\cdot)]$ for some constants $0 \le \underline{\sigma} \le \overline{\sigma}$, may be expressed in the weak formulation as $V_0 = \sup_{\underline{\sigma} \le \sigma \le \overline{\sigma}} \mathbb{E}^{\mathbb{P}^\sigma}[g(B_\cdot)]$, thus reducing the uniform continuity requirement of the terminal data to that of the function g.

Our next assumption is a purely technical condition needed in our proof of uniqueness. To be precise, it will be used only in the proof of Lemma 6.3 below to ensure the function θ^ε_n constructed there is continuous in its parameter π_n. When we have a representation for the viscosity solution, for example, in the semilinear case in [8], Section 7, we may construct the θ^ε_n explicitly and thus avoid the following assumption:

For all $\varepsilon > 0$, $n \ge 0$ and $0 \le T_0 < T_1 \le T$, denote

$$\begin{aligned}\Pi^\varepsilon_n(T_0, T_1) := \bigl\{\pi_n = (t_i, x_i)_{1 \le i \le n} : T_0 < t_1 < \cdots < t_n < T_1, \\ |x_i| \le \varepsilon \text{ for all } 1 \le i \le n\bigr\}.\end{aligned} \tag{3.1}$$

For all $\pi_n \in \Pi^\varepsilon_n(T_0, T_1)$, we denote by $\omega^{\pi_n} \in \Omega^{T_0}$ the linear interpolation of $(T_0, \mathbf{0})$, $(t_i, \sum_{j=1}^i x_j)_{1 \le i \le n}$, and $(T, \sum_{j=1}^n x_j)$.

ASSUMPTION 3.5. There exist $0 = T_0 < \cdots < T_N = T$ such that for each $i = 0, \dots, N-1$, for any ε small, any n and any $\omega \in \Omega$, $\tilde{\omega} \in \Omega^{T_{i+1}}$, the functions $\pi_n \mapsto \xi(\omega \otimes_{T_i} \omega^{\pi_n} \otimes_{T_{i+1}} \tilde{\omega})$ and $\pi_n \mapsto G(t, \omega \otimes_{T_i} \omega^{\pi_n} \otimes_{T_{i+1}} \tilde{\omega}, y, z, \gamma)$ are uniformly continuous in $\Pi^\varepsilon_n(T_i, T_{i+1})$, uniformly on $t \ge T_{i+1}$, $(y, z, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ and $\tilde{\omega} \in \Omega^{T_{i+1}}$.

We note that the uniform continuity of ξ and G implies that the above mappings are continuous in $\pi_n \in \Pi^\varepsilon_n(T_i, T_{i+1})$, but not necessarily uniformly continuous. In particular, they may not have limits on the boundary of $\Pi_n^\varepsilon(T_i, T_{i+1})$, namely when $t_i = t_{i+1}$ but $x_i \neq x_{i+1}$. We conclude this subsection with a sufficient condition for Assumption 3.5, where for $\omega \in \Omega$, we use the notation $\overline{\omega}_t := \max_{s \le t} \omega_s$ and $\underline{\omega}_t := \min_{s \le t} \omega_s$, defined componentwise.

LEMMA 3.6. *Let* $\xi(\omega) = g(\omega_{T_1}, \dots, \omega_{T_N}, \overline{\omega}_{T_1}, \dots, \overline{\omega}_{T_N}, \underline{\omega}_{T_1}, \dots, \underline{\omega}_{T_N}, \omega)$ *for some* $0 = T_0 < T_1 < \cdots < T_N = T$, *and some bounded uniformly continuous function* $(\theta, \omega) \in \mathbb{R}^{3dN} \times \Omega \longmapsto g(\theta, \omega) \in \mathbb{R}$. *Assume further that for all* θ, i *and* $\omega \in \Omega$, *there exists a modulus of continuity function* ρ *and* $p > 0$ (*which may depend on the above parameters*), *such that*

$$\left|g(\theta, \omega \otimes_{T_i} \omega^1 \otimes_{T_{i+1}} \tilde{\omega}) - g(\theta, \omega \otimes_{T_i} \omega^2 \otimes_{T_{i+1}} \tilde{\omega})\right| \le \rho\left(\int_{T_i}^{T_{i+1}} |\omega_t^1 - \omega_t^2|^p \, dt\right),$$

for all $\omega^1, \omega^2 \in \Omega^{T_i}$, $\tilde{\omega} \in \Omega^{T_{i+1}}$. *Then* ξ *satisfies Assumptions* 3.3 *and* 3.5.

PROOF. Clearly ξ satisfies Assumption 3.3. For $\omega \in \Omega$, $\tilde{\omega} \in \Omega^{T_{i+1}}$ and $\pi_n, \tilde{\pi}_n \in \Pi_n^\varepsilon(T_i, T_{i+1})$, denote $\hat{\omega}^{\pi_n} := \omega \otimes_{T_i} \omega^{\pi_n} \otimes_{T_{i+1}} \tilde{\omega}$ and $\hat{\omega}^{\tilde{\pi}_n} := \omega \otimes_{T_i} \omega^{\tilde{\pi}_n} \otimes_{T_{i+1}} \tilde{\omega}$. Then

$$\left|\xi(\hat{\omega}^{\tilde{\pi}_n}) - \xi(\hat{\omega}^{\tilde{\pi}_n})\right| \le \rho_0\left(\sum_{k=1}^n |\theta_k - \tilde{\theta}_k|\right) + \rho\left(\int_{T_i}^{T_{i+1}} |\omega_t^{\pi_n} - \omega_t^{\tilde{\pi}_n}|^p \, dt\right),$$

where ρ_0 is the modulus of continuity function of g with respect to (θ, ω). Then one can easily check that the $\pi_n \longmapsto \xi(\hat{\omega}^{\pi_n})$ is uniformly continuous in $\Pi_n^\varepsilon(T_i, T_{i+1})$. $\square$

3.2. *Path-frozen PDEs*. Our main tool for proving the comparison principle for viscosity solutions, or, more precisely, for constructing the $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$, mentioned in the Introduction, so as to prove (1.3), is some path-frozen PDE. Define the following deterministic function on $[t, \infty) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$:

$$g^{t,\omega}(s, y, z, \gamma) := G(s \wedge T, \omega_{\cdot \wedge t}, y, z, \gamma), \qquad (t, \omega) \in \Lambda.$$

For any $\varepsilon > 0$ and $\eta \ge 0$, we denote $T_\eta := (1 + \eta)T$, $\varepsilon_\eta := (1 + \eta)\varepsilon$ and

$$\begin{aligned} & O_\varepsilon := \{x \in \mathbb{R}^d : |x| < \varepsilon\}, \qquad \overline{O}_\varepsilon := \{x \in \mathbb{R}^d : |x| \le \varepsilon\}, \\ & \partial O_\varepsilon := \{x \in \mathbb{R}^d : |x| = \varepsilon\}, \\ & Q_t^{\varepsilon,\eta} := [t, T_\eta) \times O_{\varepsilon_\eta}, \qquad \overline{Q}_t^{\varepsilon,\eta} := [t, T_\eta] \times \overline{O}_{\varepsilon_\eta}, \\ & \partial Q_t^{\varepsilon,\eta} := ([t, T_\eta] \times \partial O_{\varepsilon_\eta}) \cup (\{T_\eta\} \times O_{\varepsilon_\eta}), \end{aligned} \tag{3.2}$$

and we further simplify the notation for $\eta = 0$ as

$$Q_t^\varepsilon := Q_t^{\varepsilon,0}, \qquad \overline{Q}_t^\varepsilon := \overline{Q}_t^{\varepsilon,0}, \qquad \partial Q_t^\varepsilon := \partial Q_t^{\varepsilon,0}.$$

Our additional assumption is formulated on the following localized and path-frozen PDE defined for every $(t,\omega)\in\Lambda$:

$$(3.3)\quad (\mathrm{E})^{t,\omega}_{\varepsilon,\eta}\qquad \mathbf{L}^{t,\omega}v := -\partial_t v - g^{t,\omega}(s,v,Dv,D^2v)=0 \qquad \text{on } Q^{\varepsilon,\eta}_t.$$

Notice that for fixed (t,ω), this is a standard deterministic partial differential equation.

LEMMA 3.7. *Under Assumptions 3.1 and 3.2*(ii), *PDE* (3.3) *satisfies the comparison principle for bounded viscosity solutions* (*in standard sense*, *as in* [4]). *Moreover, for any $h\in C^0(\partial Q^{\varepsilon,\eta}_t)$, PDE* (3.3) *with the boundary condition h has a* (*unique*) *bounded viscosity solution v.*

PROOF. The comparison principle follows from standard theory; see, for example, [4]. Moreover, as we will see later, the $\overline{v}$ and $\underline{v}$ defined in (4.11) are viscosity supersolution and subsolution, respectively, of the PDE (3.3) and satisfy $\overline{v}=\underline{v}=h$ on $\partial Q^{\varepsilon,\eta}_t$. Then the existence follows from the standard Perron approach in the spirit of [4], Theorem 4.1. □

We will use the following additional assumption:

ASSUMPTION 3.8. For any $\varepsilon>0$, $\eta\geq 0$, $(t,\omega)\in\Lambda$ and $h\in C^0(\partial Q^{\varepsilon,\eta}_t)$, we have $\overline{v}=v=\underline{v}$, where v is the unique viscosity solution of PDE (3.3) with boundary condition h, and

$$(3.4)\quad \begin{aligned} \overline{v}(s,x) &:= \inf\{w(s,x): w \text{ classical supersolution of } (\mathrm{E})^{t,\omega}_{\varepsilon,\eta} \\ &\qquad\qquad \text{and } w\geq h \text{ on } \partial Q^{\varepsilon,\eta}_t\},\\ \underline{v}(s,x) &:= \sup\{w(s,x): w \text{ classical subsolution of } (\mathrm{E})^{t,\omega}_{\varepsilon,\eta} \\ &\qquad\qquad \text{and } w\leq h \text{ on } \partial Q^{\varepsilon,\eta}_t\}. \end{aligned}$$

We first note that the above sets of w are not empty. Indeed, one can check straightforwardly that for any $\delta>0$ and denoting $\lambda_\delta := \frac{C_0+L_0\|h\|_\infty}{\delta}+L_0$,

$$(3.5)\quad \overline{w}(t,x) := \|h\|_\infty + \delta e^{\lambda_\delta(T_\eta - t)}, \qquad \underline{w}(t,x) := -\|h\|_\infty - \delta e^{\lambda_\delta(T_\eta - t)}$$

satisfy the requirement for $\overline{v}(s,x)$ and $\underline{v}(s,x)$, respectively. We also observe that our definition (3.4) of $\overline{v}$ and $\underline{v}$ is different from the corresponding definition in the standard Perron approach [13], in which the w is a viscosity supersolution or subsolution. It is also different from the recent development of Bayraktar and Sirbu [1], in which the w is a so called stochastic supersolution or subsolution. Loosely speaking, our Assumption 3.8 requires that the viscosity solution of $(\mathrm{E})^{t,\omega}_{\varepsilon,\eta}$ can be approximated by a sequence of classical supersolutions and a sequence of classical subsolutions. We shall discuss further this issue in Section 8 below. In particular, we will provide some sufficient conditions for Assumption 3.8 to hold.

4. Main results. The following theorem is the main result of this paper:

THEOREM 4.1. *Let Assumptions 3.1, 3.2, 3.3, 3.5 and 3.8 hold true*:

(i) *Let $u^1 \in \underline{\mathcal{U}}$ be a viscosity subsolution and $u^2 \in \overline{\mathcal{U}}$ a viscosity supersolution of PPDE* (2.14) *with $u^1(T,\cdot) \le \xi \le u^2(T,\cdot)$. Then $u^1 \le u^2$ on Λ.*

(ii) *PPDE* (2.14) *with terminal condition ξ has a unique viscosity solution $u \in$* $\mathrm{UC}_b(\Lambda)$.

4.1. *Strategy of the proof.* There are two key ingredients for the proof of this main result. The first is the following partial comparison, proved in Section 5, which extends the corresponding result in Proposition 5.3 of [8] to the set $\overline{C}^{1,2}(\Lambda)$. The reason for extending $C^{1,2}(\Lambda)$ to $\overline{C}^{1,2}(\Lambda)$ is that typically we can construct the approximations $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$, mentioned in the Introduction, only in the space $\overline{C}^{1,2}(\Lambda)$, and not in $C^{1,2}(\Lambda)$.

PROPOSITION 4.2. *Assume Assumption 3.1 holds true. Let $u^2 \in \overline{\mathcal{U}}$ be a viscosity supersolution of PPDE* (2.14) *and $u^1 \in \overline{C}^{1,2}(\Lambda)$ bounded from above satisfying $\mathcal{L}u^1(t,\omega) \le 0$ for all $(t,\omega) \in \Lambda$ with $t < T$. If $u^1(T,\cdot) \le u^2(T,\cdot)$, then $u^1 \le u^2$ on Λ.*

A similar result holds if we switch the roles of u^1 and u^2.

The second key ingredient follows the spirit of the Perron approach as in [6]. Let

$$\begin{aligned} \overline{u}(t,\omega) &:= \inf\{\psi_t : \psi \in \overline{\mathcal{D}}_T^\xi(t,\omega)\}, \\ \underline{u}(t,\omega) &:= \sup\{\psi_t : \psi \in \underline{\mathcal{D}}_T^\xi(t,\omega)\}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \overline{\mathcal{D}}_T^\xi(t,\omega) := \{\psi \in \overline{C}^{1,2}(\Lambda^t) : &\ \psi^- \text{ bounded}, \\ &(\mathcal{L}\psi)^{t,\omega} \ge 0 \text{ on } [t,T) \times \Omega^t, \psi_T \ge \xi^{t,\omega}\}, \\ \underline{\mathcal{D}}_T^\xi(t,\omega) := \{\psi \in \overline{C}^{1,2}(\Lambda^t) : &\ \psi^+ \text{ bounded}, \\ &(\mathcal{L}\psi)^{t,\omega} \le 0 \text{ on } [t,T) \times \Omega^t, \psi_T \le \xi^{t,\omega}\}. \end{aligned} \tag{4.2}$$

By using the functional Itô formula (2.13), and following the arguments in [8], Theorem 3.16, we obtain a similar result as the partial comparison of Proposition 4.2, implying that

$$\underline{u} \le \overline{u}. \tag{4.3}$$

Moreover, these processes satisfy naturally a partial dynamic programming principle which implies the following viscosity properties.

PROPOSITION 4.3. *Let Assumptions 3.1, 3.2 and 3.3 hold true. Then the processes $\overline{u}$ and $\underline{u}$ are bounded, uniformly continuous viscosity super solutions and subsolutions, respectively, of PPDE (2.14).*

This result will be proved in Section 7. A crucial step for our proof is to show the consistency of the Perron approach in the sense that equality holds in the last inequality, under our additional assumptions.

PROPOSITION 4.4. *Under the conditions of Theorem 4.1, with $N = 1$ in Assumption 3.5, we have $\overline{u} = \underline{u}$.*

The proof of this proposition is reported in Section 6. Given Propositions 4.2, 4.3 and 4.4, Theorem 4.1 follows immediately.

PROOF OF THEOREM 4.1. We prove the theorem in three steps:

Step 1. We first consider the case $N = 1$ in Assumption 3.5. By Proposition 4.2, we have $u^1 \le \overline{u}$ and $\underline{u} \le u^2$. Then Proposition 4.4 implies $u^1 \le u^2$ immediately, which implies (i) and the uniqueness of the viscosity solution. Finally, by Propositions 4.4 and 4.3, $u := \overline{u} = \underline{u}$ is a viscosity solution of (2.14).

Step 2. For general N, it follows from step 1 that the comparison, existence and uniqueness of the viscosity solution holds on $[T_{N-1}, T_N]$. Let u denote the unique viscosity solution on $[T_{N-1}, T_N]$ with terminal condition ξ, constructed by the Perron approach. Now consider PPDE (2.14) on $[T_{N-2}, T_{N-1}]$ with terminal condition $u(T_{N-1}, \cdot)$. We shall prove below that $u(T_{N-1}, \cdot)$ satisfies the requirement of step 1. Then we may extend the comparison, existence and uniqueness of the viscosity solution to the interval $[T_{N-2}, T_N]$. By repeating the arguments backwardly, we complete the proof of Theorem 4.1.

Step 3. It remains to verify Assumptions 3.3 and 3.5 with $N = 1$ for $u(T_{N-1}, \cdot)$ on $[T_{N-2}, T_{N-1}]$. First, by Proposition 4.3 it is clear that $u(T_{N-1}, \cdot)$ is bounded. Given $\omega \in \Omega$, note that PPDE (2.14) on $[T_{N-1}, T_N]$ can be viewed as a PPDE with generator $G^{T_{N-1},\omega}$ and terminal condition $\xi^{T_{N-1},\omega}$. Then, following the arguments in Lemma 7.3(i) below, one can easily show that $u(T_{N-1}, \omega)$ is uniformly continuous in ω, and it follows from Assumption 3.5 that $u(T_{N-1}, \omega \otimes_{T_{N-2}} \omega^{\pi_n})$ is uniformly continuous in $\pi_n \in \Pi_n^\varepsilon(T_{N-2}, T_{N-1})$. □

4.2. *Heuristic analysis on Proposition* 4.4. While highly technical, Proposition 4.2 follows along the same lines as the partial comparison of [8], Proposition 5.3. Proposition 4.3 has a corresponding result in PDE literature, and is proved in the spirit of the stability result of [8], Theorem 5.1. In this subsection, we provide some heuristic discussions on Proposition 4.4, focusing on the case $\overline{u}_0 = \underline{u}_0$, and the rigorous arguments will be carried out in Section 6 below.

We shall follow [8], Section 7, where Proposition 4.4 is proved in a much simpler, semi-linear setting. The idea is to construct $\overline{u}^\varepsilon \in \overline{\mathcal{D}}_T^\xi(0,0)$ and $\underline{u}^\varepsilon \in \underline{\mathcal{D}}_T^\xi(0,0)$

such that $\lim_{\varepsilon\to 0}[\overline{u}^\varepsilon_0 - \underline{u}^\varepsilon_0] = 0$. To be precise, modulus some technical properties, the approximations $\overline{u}^\varepsilon, \underline{u}^\varepsilon$ should satisfy:

- they are piecewise smooth and $\mathcal{L}\overline{u}^\varepsilon \geq 0 \geq \mathcal{L}\underline{u}^\varepsilon$;
- they are continuous in t;
- $\overline{u}^\varepsilon_T$ and $\underline{u}^\varepsilon_T$ are close to ξ.

To achieve this, we shall discretize the path ω so that we can utilize the path-frozen PDE (3.3). We note that such discretization of ω will not induce big errors, thanks to the uniform continuity of the involved processes. Fix $\varepsilon > 0$, and set $\mathrm{H}_0 := 0$,

$$\mathrm{H}_{i+1} := \{t \geq \mathrm{H}_i : |B_t - B_{t_i}| = \varepsilon\} \wedge T.$$

Denote $\hat{\pi}_n := \{(\mathrm{H}_i, B_{\mathrm{H}_i}), 0 \leq i \leq n\}$. Let $\pi_n = \{(t_i, x_i), 0 \leq i \leq n\}$ be a typical value of $\hat{\pi}_n(\omega)$, and $\omega^{\pi_n} \in \Omega$ be the linear interpolation of π_n. The main idea is to construct a sequence of deterministic functions $v^\varepsilon_n(\pi_n; t, x)$ so that we may construct the desired $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$ from a common process $u^\varepsilon_t := v^\varepsilon_n(\hat{\pi}_n; t, B_t - B_{\mathrm{H}_n})$, $\mathrm{H}_n \leq t < \mathrm{H}_{n+1}$. For this purpose, we require v^ε_n, and hence u^ε, satisfying the following three corresponding properties:

• For each π_n, the function $v^\varepsilon_n(\pi_n; \cdot)$ is in $C^{1,2}(Q^\varepsilon_{t_n})$ and is a classical solution of a certain mollified path-frozen PDE,

$$-\partial_t v^\varepsilon_n - g^{\pi_n}_\varepsilon(t, v^\varepsilon_n, Dv^\varepsilon_n, D^2 v^\varepsilon_n) = 0, \tag{4.4}$$

where $g^{\pi_n}_\varepsilon = g^{t_n,\omega^{\pi_n}}_\varepsilon$. Consequently, the process u^ε is approximately a classical solution of PPDE (2.14) on $[\mathrm{H}_n, \mathrm{H}_{n+1}]$, thanks to the fact that $g^{\hat{\pi}_n(\omega)}_\varepsilon(t, \cdot)$ is a good approximation of $G(t, \omega, \cdot)$.

• $v^\varepsilon_n(\hat{\pi}_n; \mathrm{H}_{n+1}, B_{\mathrm{H}_{n+1}} - B_{\mathrm{H}_n}) = v^\varepsilon_{n+1}(\hat{\pi}_{n+1}; \mathrm{H}_{n+1}, \mathbf{0})$ so that u^ε is continuous in t and is more or less in $\overline{C}^{1,2}(\Lambda)$.

• $v^\varepsilon_n(\pi_n; T, x)$ is constructed from ξ, so that u^ε_T is close to ξ.

Now by the uniform continuity of ξ and G, we will see that $\overline{u}^\varepsilon := u^\varepsilon + \rho_0(2\varepsilon)$ and $\underline{u}^\varepsilon := u^\varepsilon - \rho_0(2\varepsilon)$ satisfy the desired classical semi-solution property. Clearly $\overline{u}^\varepsilon - \underline{u}^\varepsilon \leq 2\rho_0(2\varepsilon)$, implying the result.

In [8], Section 7, the functions v^ε_n can be constructed explicitly via approximating backward SDEs. In the present setting, since we do not have a representation for the candidate solution, we cannot construct v^ε_n directly. By some limiting procedure, in Lemma 6.3 below, we shall construct certain deterministic functions θ^ε_n which satisfy all the above three properties, except that θ^ε_n is only a viscosity solution of PDE (4.4). Now to construct smooth v^ε_n from θ^ε_n, we apply Assumption 3.8. In fact, given the viscosity solution θ^ε_n, Assumption 3.8 allows us to construct the classical supersolution $\overline{v}^\varepsilon_n$ and the classical subsolution $\underline{v}^\varepsilon_n$, rather than one single smooth function v^ε_n, such that $\underline{v}^\varepsilon_n \leq \theta^\varepsilon_n \leq \overline{v}^\varepsilon_n$, and $\overline{v}^\varepsilon_n - \underline{v}^\varepsilon_n$ is small. This procedure is carried out in Lemma 6.4 below, and the construction is done piece by piece, forwardly on each random interval $[\mathrm{H}_n, \mathrm{H}_{n+1}]$.

REMARK 4.5. As we see in the above discussion, the processes we will use to prove the comparison takes the form $v(\Pi_n; t, B_t - B_{\mathrm{H}_n})$, $\mathrm{H}_n \le t < \mathrm{H}_{n+1}$, for some deterministic function v, which is smooth in (t, x). Then it suffices to apply the standard Itô formula on v, rather than the functional Itô formula. Indeed, under our assumptions, we can prove rigorously the well-posedness of viscosity solutions, including existence, stability and comparison and uniqueness, without using the functional Itô formula. In other words, technically speaking, we can establish our theory without involving the path derivatives. However, we do feel that the path derivatives and the functional Itô formula are the natural and convenient language in this path-dependent framework. In particular, it is much more natural to talk about classical solutions of PPDEs by using the path derivatives. Moreover, the current proof relies heavily on the discretization of the underlying path ω, with the help of the path-frozen PDEs. This discretization induces the above piecewise Markovian structure. The functional Itô formula allows us to explore in future research other approaches without using such discretization.

4.3. *The bounding equations.* The proof of Proposition 4.4 requires some estimates, which involve the following particular example analyzed in [8]. Recall the constants L_0 C_0, and c_0 from Assumptions 3.1 and 3.2, and consider the operators

$$
\begin{aligned}
\overline{g}_0(z,\gamma) &:= \sup_{|\alpha|\le L_0, \sqrt{2c_0}\le|\beta|\le\sqrt{2L_0}} \left[\alpha\cdot z + \frac{1}{2}\beta^2 : \gamma\right],\\
\overline{g}(y,z,\gamma) &:= \overline{g}_0(z,\gamma) + L_0|y| + C_0,\\
\underline{g}_0(z,\gamma) &:= \inf_{|\alpha|\le L_0, \sqrt{2c_0}\le|\beta|\le\sqrt{2L_0}} \left[\alpha\cdot z + \frac{1}{2}\beta^2 : \gamma\right],\\
\underline{g}(y,z,\gamma) &:= \underline{g}_0(z,\gamma) - L_0|y| - C_0,
\end{aligned}
\tag{4.5}
$$

which clearly satisfy Assumptions 3.1 and 3.2, and

$$
\underline{g} \le G \le \overline{g}. \tag{4.6}
$$

These operators induce the PPDEs

$$
\begin{aligned}
\overline{\mathcal{L}}u &:= -\partial_t u - \overline{g}(u, \partial_\omega u, \partial_{\omega\omega} u) = 0 \quad \text{and}\\
\underline{\mathcal{L}}u &:= -\partial_t u - \underline{g}(u, \partial_\omega u, \partial_{\omega\omega} u) = 0.
\end{aligned}
\tag{4.7}
$$

Let $\mathcal{B}^t_{L_0} := \{b \in \mathbb{L}^0(\Lambda^t) : |b| \le L_0\}$ and

$$
\begin{aligned}
&\mathcal{P}^t_{L_0,c_0} := \{\mathcal{P}^t_{L_0} : |\beta^{\mathbb{P}}| \ge \sqrt{2c_0}\}, \qquad \overline{\mathcal{E}}^{L_0,c_0}_t := \sup_{\mathbb{P}\in\mathcal{P}^t_{L_0,c_0}} \mathbb{E}^{\mathbb{P}},\\
&\underline{\mathcal{E}}^{L_0,c_0}_t := \inf_{\mathbb{P}\in\mathcal{P}^t_{L_0,c_0}} \mathbb{E}^{\mathbb{P}}.
\end{aligned}
\tag{4.8}
$$

Following the arguments in our accompanying paper ([8], Proposition 4), we see that for a bounded, uniformly continuous $\mathcal{F}_T$-measurable r.v. ξ,

$$
\begin{aligned}
\overline{w}(t,\omega) &:= \sup_{b\in\mathcal{B}^t_{L_0}} \overline{\mathcal{E}}^{L_0,c_0}_t\Big[\xi^{t,\omega}e^{\int_t^T b_r\,dr} + C_0\int_t^T e^{\int_t^s b_r\,dr}\,ds\Big],\\
\underline{w}(t,\omega) &:= \inf_{b\in\mathcal{B}^t_{L_0}} \underline{\mathcal{E}}^{L_0,c_0}_t\Big[\xi^{t,\omega}e^{\int_t^T b_r\,dr} - C_0\int_t^T e^{\int_t^s b_r\,dr}\,ds\Big]
\end{aligned}
\tag{4.9}
$$

are viscosity solutions of the PPDE $\overline{\mathcal{L}}\overline{w} := 0$ and $\underline{\mathcal{L}}\underline{w} := 0$, respectively.

By Lemma 3.7, the PDE version of (4.7),

$$
\begin{aligned}
\overline{\mathbf{L}}v &:= -\partial_t v - \overline{g}\big(v, Dv, D^2v\big) = 0 \quad \text{and}\\
\underline{\mathbf{L}}v &:= -\partial_t v - \underline{g}\big(v, Dv, D^2v\big) = 0 \qquad \text{in } Q_t^{\varepsilon,\eta},
\end{aligned}
\tag{4.10}
$$

satisfies the comparison principle. Moreover, we have the following:

LEMMA 4.6. *Under Assumptions 3.1 and 3.2*(ii), *for any $h \in C^0(\partial Q_t^{\varepsilon,\eta})$, the following functions are the unique viscosity solutions of PDEs (4.10) with boundary condition h*:

$$
\begin{aligned}
\overline{v}(t,x) &:= \sup_{b\in\mathcal{B}^t_{L_0}} \overline{\mathcal{E}}^{L_0,c_0}_t\Big[e^{\int_t^{\mathrm{H}} b_r\,dr}h\big(\mathrm{H}, x+B^t_{\mathrm{H}}\big) + C_0\int_t^{\mathrm{H}} e^{\int_t^s b_r\,dr}\,ds\Big],\\
\underline{v}(t,x) &:= \inf_{b\in\mathcal{B}^t_{L_0}} \underline{\mathcal{E}}^{L_0,c_0}_t\Big[e^{\int_t^{\mathrm{H}} b_r\,dr}h\big(\mathrm{H}, x+B^t_{\mathrm{H}}\big) - C_0\int_t^{\mathrm{H}} e^{\int_t^s b_r\,dr}\,ds\Big],
\end{aligned}
\tag{4.11}
$$

where $\mathrm{H} := \mathrm{H}^{t,x} := \{s > t : (s, x + B^t_s) \notin Q_t^{\varepsilon,\eta}\}$.

PROOF. First, by the arguments in [7], one may easily check that $\overline{v}$ and $\underline{v}$ are continuous and satisfy dynamic programing principle for $t < \mathrm{H}$, which implies the viscosity property immediately. Then it remains to check the boundary conditions. For $x \in \overline{O}_{\varepsilon_\eta}$, since $t \le H^{t,x} \le T$ and h is uniformly continuous with certain modulus of continuity function ρ_h, it is clear that

$$
\begin{aligned}
&\big|\overline{v}(t,x) - h(T,x)\big|\\
&\quad\le \sup_{b\in\mathcal{B}^t_{L_0}} \overline{\mathcal{E}}^{L_0}_t\Big[\Big|e^{\int_t^{\mathrm{H}} b_r\,dr}h\big(\mathrm{H}, x+B^t_{\mathrm{H}}\big) + C_0\int_t^{\mathrm{H}} e^{\int_t^s b_r\,dr}\,ds - h(T,x)\Big|\Big]\\
&\quad= \sup_{b\in\mathcal{B}^t_{L_0}} \overline{\mathcal{E}}^{L_0}_t\Big[\Big|\big[e^{\int_t^{\mathrm{H}} b_r\,dr} - 1\big]h\big(\mathrm{H}, x+B^t_{\mathrm{H}}\big) + \big[h\big(\mathrm{H}, x+B^t_{\mathrm{H}}\big) - h(T,x)\big]\\
&\qquad\qquad + C_0\int_t^{\mathrm{H}} e^{\int_t^s b_r\,dr}\,ds\Big|\Big]
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
&\le C\overline{\mathcal{E}}_t^{L_0}\big[H - t + \rho_h\big(T - \mathrm{H} + \big|B_{\mathrm{H}}^t\big|\big)\big] \\
&\le C\overline{\mathcal{E}}_t^{L_0}\big[T - t + \rho_h\big(T - t + \|B^t\|_T\big)\big] \\
&\to 0,
\end{aligned}
$$

as $t \uparrow T$. Furthermore, let $t < T$ and $\mathbf{0} \neq x \in O_{\varepsilon_\eta}$. Note that for any $a > 0$ and $\mathbb{P} \in \mathcal{P}_{L_0,c_0}^t$,

$$
\begin{aligned}
\mathbb{P}\big(\mathrm{H}^{t,x} - t \ge a\big) &\le \mathbb{P}\Big(\sup_{t \le s \le t+a} \frac{x}{|x|} \cdot B_s^t \le \varepsilon_\eta - |x|\Big) \\
&\le \mathbb{P}\Big(\sup_{t \le s \le t+a} \int_t^s \frac{x}{|x|} \cdot \beta_r^{\mathbb{P}}\, dW_r^{\mathbb{P}} \le \varepsilon_\eta - |x| + L_0 a\Big).
\end{aligned}
$$

Let $A_s := \int_t^s \frac{x^T}{|x|}(\beta_r^{\mathbb{P}})^2 \frac{x}{|x|}\, dr$ and $\tau_s := \inf\{r \ge t : A_r \ge s\}$. Then $M_s := \int_t^{\tau_s} \frac{x^T}{|x|} \beta_r^{\mathbb{P}}\, dW_r^{\mathbb{P}}$ is a $\mathbb{P}$-Brownian motion, and $A_s \ge 2c_0(s-t)$. Thus

$$
\begin{aligned}
\mathbb{P}\big(\mathrm{H}^{t,x} - t \ge a\big) &\le \mathbb{P}\Big(\sup_{t \le s \le t+2c_0 a} M_s \le \varepsilon_\eta - |x| + L_0 a\Big) \\
&= \mathbb{P}_0\big(\|B\|_{2c_0 a} \le \varepsilon_\eta - |x| + L_0 a\big) \\
&= \mathbb{P}_0\big(|B_{2c_0 a}| \le \varepsilon_\eta - |x| + L_0 a\big) \\
&= \mathbb{P}_0\Big(|B_1| \le \frac{1}{\sqrt{2c_0 a}}\big[\varepsilon_\eta - |x| + L_0 a\big]\Big) \\
&\le \frac{C}{\sqrt{a}}\big[\varepsilon_\eta - |x| + L_0 a\big].
\end{aligned}
$$

Set $a := \varepsilon_\eta - |x|$, and we get

$$
\mathbb{P}\big(\mathrm{H}^{t,x} - t \ge \varepsilon_\eta - |x|\big) \le C\sqrt{\varepsilon_\eta - |x|}.
$$

Following similar arguments to those in (4.12), one can easily show that for some modulus of continuity function ρ,

$$
\big|\overline{v}(t,x) - h(t,\tilde{x})\big| \le \rho\big(\varepsilon_\eta - |x|\big) \qquad \text{where } \tilde{x} := \frac{|x|}{\varepsilon_\eta} x \in \partial O_{\varepsilon_\eta}.
$$

Then, for $t_0 < T$, $x_0 \in \partial O_{\varepsilon_\eta}$, $t < T$ and $x \in O_{\varepsilon_\eta}$, noting that

$$
|x - \tilde{x}| \le \varepsilon_\eta - |x| = |x_0| - |x| \le |x - x_0|,
$$

we have, as $(t,x) \to (t_0,x_0)$,

$$
\begin{aligned}
\big|\overline{v}(t,x) - h(t_0,x_0)\big| &\le \big|\overline{v}(t,x) - h(t,\tilde{x})\big| + \big|h(t,\tilde{x}) - h(t_0,x_0)\big| \\
&\le \rho\big(\varepsilon_\eta - |x|\big) + \rho_h\big(|t - t_0| + |x_0 - \tilde{x}|\big) \\
&\le \rho\big(|x_0 - x|\big) + \rho_h\big(|t - t_0| + 2|x_0 - x|\big) \to 0.
\end{aligned}
$$

This implies that $\overline{v}$ is continuous on $\overline{Q}^{\varepsilon,\eta}$. Similarly one can prove the result for $\underline{v}$. $\square$

We remark that (4.9) provides representation for viscosity solutions of PPDEs (4.7), even in the degenerate case $c_0 = 0$. However, this is not true for the PDEs (4.10), due to the boundedness of the domain $Q_t^{\varepsilon,\eta}$, which induces the hitting time H and ruins the required regularity, as we will see in next example.

EXAMPLE 4.7. Assume Assumption 3.1 holds, but G is degenerate, and thus $c_0 = 0$. Let $d = 1$, and set $h(s, x) := s$ on $\partial Q_t^{\varepsilon,\eta}$. Then the $\overline{v}$ defined by (4.11) is discontinuous in $[0, T_\eta) \times \partial O_{\varepsilon_\eta} \subset \partial Q_0^{\varepsilon,\eta}$ and thus is not a viscosity solution of the PDE (4.10).

PROOF. It is clear that

$$\overline{v}(t, x) = \overline{\mathcal{E}}_t^{L_0}\left[e^{L_0(\mathrm{H}-t)}\mathrm{H} + C_0 \int_t^{\mathrm{H}} e^{L_0(s-t)}\,ds\right],$$

where the integrand is increasing in H which takes values on $[t, T_\eta]$. Then, by taking the $\mathbb{P}$ corresponding to $\alpha = \beta = 0$, we have $\mathrm{H} = T_\eta$, $\mathbb{P}$-a.s. and thus

$$\overline{v}(t, x) = e^{L_0(T_\eta - t)}T_\eta + C_0 \int_t^{T_\eta} e^{L_0(s-t)}\,ds, \qquad (t, x) \in Q_0^{\varepsilon,\eta}.$$

However, we have $\overline{v}(t, x) = t$ on $\partial Q_0^{\varepsilon,\eta}$, so $\overline{v}$ is discontinuous in $[0, T_\eta) \times \partial O_{\varepsilon_\eta}$. $\square$

4.4. *A change of variables formula.* We conclude this section with a change of variables formula, which is interesting in its own right. We have previously observed in [8], Remark 3.15, that the classical change of variables formula is not known to hold true for our notion of viscosity solutions under Assumption 3.1. We now show that it holds true under the additional Assumption 3.8.

Let $u \in C_b^{1,2}(\Lambda)$ and $\Phi \in C^{1,2}([0, T] \times \mathbb{R})$. Assume Φ is strictly increasing in x, and let Ψ denote its inverse function. Note that Ψ is increasing in x and $\Psi_x > 0$. Define

$$\tilde{u}(t, \omega) := \Phi\big(t, u(t, \omega)\big) \qquad \text{and thus } u(t, \omega) = \Psi\big(t, \tilde{u}(t, \omega)\big). \tag{4.13}$$

Then direct calculation shows that

$$\begin{aligned} \mathcal{L}u(t, \omega) &= \Psi_x\big(t, \tilde{u}(t, \omega)\big)\tilde{\mathcal{L}}\tilde{u}(t, \omega) \quad \text{and} \\ \tilde{\mathcal{L}}\tilde{u} &:= -\partial_t \tilde{u} - \tilde{G}\big(t, \omega, \tilde{u}, \partial_\omega \tilde{u}, \partial^2_{\omega\omega}\tilde{u}\big), \end{aligned} \tag{4.14}$$

where

$$\tilde{G}(t, \omega, y, z, \gamma) := \frac{\Psi_t(t, y) + G(t, \omega, \Psi(t, y), \Psi_x(t, y)z, \Psi_{xx}(t, y)z^2 + \Psi_x(t, y)\gamma)}{\Psi_x(t, y)}.$$

Then the following result is obvious:

PROPOSITION 4.8. *Under the above assumptions on* Ψ, u *is classical solution* (*resp*., *supersolution*, *subsolution*) *of* $\mathcal{L}u = 0$ *if and only if* $\tilde{u} := \Phi(t, u)$ *is a classical solution* (*resp*., *supersolution*, *subsolution*) *of* $\tilde{\mathcal{L}}\tilde{u} = 0$.

Moreover, we have the following:

THEOREM 4.9. *Assume both* (G, ξ) *and* $(\tilde{G}, \Phi(T, \xi))$ *satisfy the conditions of Theorem* 4.1. *Then* u *is the viscosity solution of PPDE* (2.14) *with terminal condition* ξ *if and only if* $\tilde{u} := \Phi(t, u)$ *is the viscosity solution of PPDE* (4.14) *with terminal condition* $\tilde{\xi} := \Phi(T, \xi)$.

PROOF. One may easily check that $\overline{w} = \Phi(t, \overline{u})$, $\underline{w} = \Phi(t, \underline{u})$, where

$$\overline{w}(t, \omega) := \inf\{\psi_t : \psi \in \overline{C}^{1,2}(\Lambda^t), \psi^- \text{ bounded}, \tilde{\mathcal{L}}\psi \geq 0, \psi_T \geq \Phi(T, \xi^{t,\omega})\};$$

$$\underline{w}(t, \omega) := \sup\{\psi_t : \psi \in \overline{C}^{1,2}(\Lambda^t), \psi^+ \text{ bounded}, \tilde{\mathcal{L}}\psi \leq 0, \psi_T \leq \Phi(T, \xi^{t,\omega})\}.$$

Then the result follows immediately from Proposition 4.4 and the arguments in the proof of Theorem 4.1. □

We observe that the above operator $\tilde{G}$ is quadratic in the z-variable, so we need somewhat stronger conditions to ensure the well-posedness.

5. Partial comparison of viscosity solutions.

In this section, we prove Proposition 4.2. The proof is crucially based on the optimal stopping problem reported in Theorem 2.3.

We first prove a lemma. Recall the partition $\{E^i_j, j \geq 1\} \subset \mathcal{F}_{\mathrm{H}_i}$, the constant n_i and the uniform continuous mappings φ^i_{jk} and ψ^i_{jk} in (2.12) corresponding to $u^1 \in \overline{C}^{1,2}(\Lambda)$. For $\delta > 0$, let $0 = t_0 < t_1 < \cdots < t_N = T$ such that $t_{k+1} - t_k \leq \delta$ for $k = 0, \ldots, N-1$, and define $t_{N+1} := T + \delta$.

LEMMA 5.1. *For all* $i, j \geq 1$, *there is a partition* $(\tilde{E}^i_{j,k})_{k\geq 1} \subset \mathcal{F}_{\mathrm{H}_i}$ *of* E^i_j *and a sequence* $(p_k)_{k\geq 1}$ *taking values* $0, \ldots, N$, *such that*

$$\mathrm{H}_i \in [t_{p_k}, t_{p_k+1}) \quad \text{on } \tilde{E}^i_{j,k}, \qquad \sup_{\omega,\omega' \in \tilde{E}^i_{j,k}} \|\omega_{\cdot\wedge \mathrm{H}_i(\omega)} - \omega'_{\cdot\wedge \mathrm{H}_i(\omega')}\| \leq \delta \quad \text{and}$$

$$\min_{\omega \in \tilde{E}^i_{j,k}} \mathrm{H}_i(\omega) = \mathrm{H}_i(\omega^i_{j,k}) =: \tilde{t}^i_{j,k} \qquad \text{for some } \omega^i_{j,k} \in \tilde{E}^i_{j,k}.$$

PROOF. Since i, j are fixed, we simply denote $E := E^i_j$ and $\mathrm{H} := \mathrm{H}_i$. Denote $E_k := E \cap \{t_k \leq \mathrm{H} < t_{k+1}\}$, $k \leq n$. Then $\{E_k\}_k \subset \mathcal{F}_{\mathrm{H}}$ forms a partition of E.

Since Ω is separable, there exists a finer partition $\{E_{k,l}\}_{k,l} \subset \mathcal{F}_{\text{H}}$ such that, for any $\omega, \omega' \in E_{k,l}$, $\|\omega_{\cdot\wedge \text{H}(\omega)} - \omega'_{\cdot\wedge \text{H}(\omega')}\| \le \delta$.

Next, for each $E_{k,l}$, there is a sequence $\omega^{k,l,m} \in E_{k,l}$ such that $t_{k,l,m} := \text{H}(\omega^{k,l,m}) \downarrow \inf_{\omega\in E_{k,l}} \text{H}(\omega)$. Denote $t_{k,l,0} := t_{k+1}$. Define $E_{k,l,m} := E_{k,l} \cap \{t_{k,l,m+1} \le \text{H} < t_{k,l,m}\} \in \mathcal{F}_{\text{H}_i}$, and renumerate them as $(\tilde{E}_k)_{k\ge 1}$. We then verify directly that $(\tilde{E}_k)_{k\ge 1}$ defines a partition of E satisfying the required conditions. $\square$

PROOF OF PROPOSITION 4.2. We only prove $u_0^1 \le u_0^2$. The inequality for general t can be proved similarly. Assume u^2 is a viscosity L-supersolution and $u^1 \in \overline{C}^{1,2}(\Lambda)$ with corresponding hitting times H_i, $i \ge 0$. By Proposition 3.14 of [8], we may assume without loss of generality that

$$G(t,\omega,y_1,z,\gamma) - G(t,\omega,y_2,z,\gamma) \ge y_2 - y_1 \qquad \text{for all } y_1 \le y_2. \tag{5.1}$$

We now prove the proposition in three steps. Throughout the proof, denote

$$\hat{u} := u^1 - u^2.$$

Since u^1 is bounded from above and u^2 bounded from below, we see that $\hat{u}^+$ is bounded.

Step 1. We first show that for all $i \ge 0$ and $\omega \in \Omega$,

$$\hat{u}^+_{\text{H}_i}(\omega) \le \overline{\mathcal{E}}^L_{\text{H}_i(\omega)}\big[\big(\hat{u}^+_{\text{H}_{i+1}-}\big)^{\text{H}_i,\omega}\big]. \tag{5.2}$$

Since $(u^1)^{t,\omega} \in \overline{C}^{1,2}(\Lambda^t)$, clearly it suffices to consider $i = 0$. Assume on the contrary that

$$2Tc := \hat{u}^+_0(\mathbf{0}) - \overline{\mathcal{E}}^L_0\big[\hat{u}^+_{\text{H}_1-}\big] > 0. \tag{5.3}$$

Recall (2.12). Notice that $E_1^0 = \Omega$ and that $\varphi^0_{1k}(0,\mathbf{0})$ are constants, and we may assume without loss of generality that $n_0 = 1$ and

$$u_t^1 = \psi(t,B), \qquad 0 \le t \le \text{H}_1,$$

where $\psi \in C^{1,2}(\Lambda) \cap \text{UC}_b(\Lambda)$ with bounded derivatives. Denote

$$X_t := \big(\psi_t - u_t^2\big)^+ + ct, \qquad 0 \le t \le T.$$

Since u^2 is bounded from below, by the definition of $\underline{\mathcal{U}}$, one may easily check that

$$X \text{ is a bounded process in } \underline{\mathcal{U}}, \text{ and } X_t := \hat{u}_t^+ + ct, 0 \le t \le \text{H}_1.$$

Define

$$\widehat{X} := X\mathbf{1}_{[0,\text{H}_1)} + X_{\text{H}_1-}\mathbf{1}_{[\text{H}_1,T]}; \qquad Y := \overline{\mathcal{S}}^L[\widehat{X}], \qquad \tau^* := \inf\{t \ge 0 : Y_t = \widehat{X}_t\}.$$

Applying Theorem 2.3 and by (5.3), we have

$$\overline{\mathcal{E}}^L_0[\widehat{X}_{\tau^*}] = Y_0 \ge X_0 = \hat{u}^+_0(\mathbf{0}) = 2Tc + \overline{\mathcal{E}}^L_0\big[\hat{u}^+_{\text{H}_1-}\big] \ge Tc + \overline{\mathcal{E}}^L_0[\widehat{X}_{\text{H}_1}].$$

Then there exists $\omega^* \in \Omega$ such that $t^* := \tau^*(\omega^*) < \mathrm{H}_1(\omega^*)$. Next, by the $\overline{\mathcal{E}}^L$-supermartingale property of Y of Theorem 2.3, we have

$$\hat{u}^+(t^*, \omega^*) + ct^* = X_{t^*}(\omega^*) = Y_{t^*}(\omega^*) \geq \overline{\mathcal{E}}^L_{t^*}\big[X^{t^*,\omega^*}_{\mathrm{H}_1^{t^*,\omega^*}-}\big] \geq \overline{\mathcal{E}}^L_{t^*}\big[c\mathrm{H}_1^{t^*,\omega^*}\big] > ct^*,$$

implying that $0 < \hat{u}^+(t^*, \omega^*) = \hat{u}(t^*, \omega^*)$. Since $u^2 \in \overline{\mathcal{U}}$, by (2.3) there exists $\mathrm{H} \in \mathcal{H}^{t^*}$ such that

$$\mathrm{H} < \mathrm{H}_1^{t^*,\omega^*} \quad \text{and} \quad \hat{u}_t^{t^*,\omega^*} > 0 \qquad \text{for all } t \in [t^*, \mathrm{H}]. \tag{5.4}$$

Then $X_t^{t^*,\omega^*} = \varphi_t - (u^2)_t^{t^*,\omega^*}$ for all $t \in [t^*, \mathrm{H}]$, where $\varphi(t,\omega) := \psi^{t^*,\omega^*}(t,\omega) + ct$. Observe that $\varphi \in C^{1,2}(\Lambda^{t^*})$. Using again the $\overline{\mathcal{E}}^L$-supermartingale property of Y of Theorem 2.3, we see that for all $\tau \in \mathcal{T}^{t^*}$,

$$\begin{aligned}
\big(\varphi - (u^2)^{t^*,\omega^*}\big)_{t^*} &= X_{t^*}(\omega^*) = Y_{t^*}(\omega^*) \geq \overline{\mathcal{E}}^L_{t^*}\big[Y^{t^*,\omega^*}_{\tau\wedge \mathrm{H}}\big] \geq \overline{\mathcal{E}}^L_{t^*}\big[X^{t^*,\omega^*}_{\tau\wedge \mathrm{H}}\big] \\
&= \overline{\mathcal{E}}^L_{t^*}\big[\big(\varphi - (u^2)^{t^*,\omega^*}\big)_{\tau\wedge \mathrm{H}}\big].
\end{aligned}$$

That is, $\varphi \in \overline{\mathcal{A}}^L u^2(t^*, \omega^*)$, and by the viscosity L-supersolution property of u^2,

$$\begin{aligned}
0 &\leq \big\{-\partial_t\varphi - G\big(\cdot, u^2, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi\big)\big\}(t^*, \omega^*) \\
&= -c - \big\{\partial_t u^1 + G\big(\cdot, u^2, \partial_\omega u^1, \partial^2_{\omega\omega} u^1\big)\big\}(t^*, \omega^*) \\
&\leq -c - \big\{\partial_t u^1 + G\big(\cdot, u^1, \partial_\omega u^1, \partial^2_{\omega\omega} u^1\big)\big\}(t^*, \omega^*),
\end{aligned}$$

where the last inequality follows from (5.4) and (5.1). Since $c > 0$, this is in contradiction with the subsolution property of u^1 and thus completes the proof of (5.2).

REMARK 5.2. The rest of the proof is only needed in the case where $u^1 \in \overline{C}^{1,2}(\Lambda) \setminus C^{1,2}(\Lambda)$. Indeed, if $u^1 \in C^{1,2}(\Lambda)$, then $H_1 = T$, and it follows from step 1 that $\hat{u}_0^+ \leq \overline{\mathcal{E}}^L_0[\hat{u}^+_{T-}] \leq \overline{\mathcal{E}}^L_0[\hat{u}^+_T] = 0$, and then $u^1_0 \leq u^2_0$. In fact, this is the partial comparison principle proved in [8], Proposition 5.3.

Step 2. We continue by using the following result which will be proved in step 3:

$$\begin{aligned}
&\text{For } i \geq 1,\ \mathbb{P} \in \mathcal{P}_L \text{ and } \mathcal{P}_L(\mathbb{P}, \mathrm{H}_i) := \{\mathbb{P}' \in \mathcal{P}_L : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_{\mathrm{H}_i}\}, \text{ we have} \\
&\Delta_i := \hat{u}^+_{\mathrm{H}_i-} - \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}_L(\mathbb{P}, \mathrm{H}_i)} \mathbb{E}^{\mathbb{P}'}\big[\hat{u}^+_{\mathrm{H}_{i+1}-} \big| \mathcal{F}_{\mathrm{H}_i}\big] \leq 0, \qquad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{5.5}$$

Then by standard arguments, we have

$$\mathbb{E}^{\mathbb{P}}\big[\hat{u}^+_{\mathrm{H}_i-}\big] \leq \sup_{\mathbb{P}' \in \mathcal{P}_L(\mathbb{P}, \mathrm{H}_i)} \mathbb{E}^{\mathbb{P}'}\big[\hat{u}^+_{\mathrm{H}_{i+1}-}\big] \leq \overline{\mathcal{E}}^L_0\big[\hat{u}^+_{\mathrm{H}_{i+1}-}\big].$$

Since $\mathbb{P} \in \mathcal{P}_L$ is arbitrary, this leads to $\overline{\mathcal{E}}^L_0[\hat{u}^+_{\mathrm{H}_i-}] \leq \overline{\mathcal{E}}^L_0[\hat{u}^+_{\mathrm{H}_{i+1}-}]$, and by induction, $\hat{u}_0^+ \leq \overline{\mathcal{E}}^L_0[\hat{u}^+_{\mathrm{H}_i-}]$, for all i. Notice that $\hat{u}^+$ is bounded, $\lim_{i\to\infty} \mathcal{C}^L_0[\mathrm{H}_i < T] = 0$ by

Definition 2.5(i) and $u^2_{T-} \geq u^2_T$ by the definition of $\overline{\mathcal{U}}$. Then, sending $i \to \infty$, we obtain $\hat{u}^+_0 \leq \overline{\mathcal{E}}^L_0[\hat{u}^+_{T-}] \leq \overline{\mathcal{E}}^L_0[\hat{u}^+_T] = 0$, which completes the proof of $u^1_0 \leq u^2_0$.

Step 3. It remains to prove (5.5). Clearly it suffices to prove it on each E^i_j. As in the proof of Lemma 5.1, we omit the dependence on the fixed pair (i, j), thus writing $E := E^i_j$, $n = n_i$, $\mathrm{H} := \mathrm{H}_i$, $\mathrm{H}_1 := \mathrm{H}_{i+1}$, $\varphi_k := \varphi^i_{j,k}$, $\psi_k := \psi^i_{j,k}$, $\Delta := \Delta_i$, and let C denote the common bound of φ_k, ψ_k and ρ, the common modulus of continuity function of φ_k, ψ_k, $1 \leq k \leq n$. We also denote $\tilde{E}_k := \tilde{E}^i_{j,k}$, $\omega^k := \omega^i_{j,k}$ and $\tilde{t}_k := \tilde{t}^i_{j,k}$, as defined in Lemma 5.1.

Fix an arbitrary $\mathbb{P} \in \mathcal{P}_L$ and $\varepsilon > 0$. Since $u^2 \in \overline{\mathcal{U}}$, we have $u^2_{\mathrm{H}-} \geq u^2_{\mathrm{H}}$. Then, for each k, it follows from (5.2) that

$$\hat{u}^+_{\mathrm{H}-}(\omega^k) \leq \hat{u}^+_{\mathrm{H}}(\omega^k) \leq \mathbb{E}^{\mathbb{P}_k}\big[(\hat{u}^+_{\mathrm{H}_1-})^{\tilde{t}_k,\omega^k}\big] + \varepsilon \qquad \text{for some } \mathbb{P}_k \in \mathcal{P}^{\tilde{t}_k}_L.$$

Define $\hat{\mathbb{P}} \in \mathcal{P}_L(\mathbb{P}, \mathrm{H})$ such that for $\mathbb{P}$-a.e. $\omega \in \tilde{E}_k$, the $\hat{\mathbb{P}}^{\mathrm{H}(\omega),\omega}$-distribution of $B^{\mathrm{H}(\omega)}$ is equal to the $\mathbb{P}_k$-distribution of $B^{\tilde{t}_k}$, where $\hat{\mathbb{P}}^{\mathrm{H}(\omega),\omega}$ denotes the r.c.p.d. Then $\mathbb{P}$-a.s. on $\tilde{E}_k$,

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{P}}}\big[\hat{u}^+_{\mathrm{H}_1-}|\mathcal{F}_{\mathrm{H}}\big](\omega) \\
&\quad = \mathbb{E}^{\hat{\mathbb{P}}^{\mathrm{H}(\omega),\omega}}\big[\hat{u}^+\big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} B^{\mathrm{H}(\omega)}_{\cdot}\big)-, \omega \otimes_{\mathrm{H}(\omega)} B^{\mathrm{H}(\omega)}_{\cdot}\big)\big] \\
&\quad = \mathbb{E}^{\mathbb{P}_k}\big[\hat{u}^+\big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)-, \omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)\big],
\end{aligned}$$

where $\tilde{B}^{\tilde{t}_k}_s := B^{\tilde{t}_k}_{s-\mathrm{H}(\omega)+\tilde{t}_k}$, $s \geq \mathrm{H}(\omega)$. Recalling that $\hat{u}^+$ is bounded, $\mathbb{P}$-a.s. this provides

$$\begin{aligned}
\Delta(\omega) &\leq \hat{u}^+_{\mathrm{H}-}(\omega) - \mathbb{E}^{\hat{\mathbb{P}}}\big[\hat{u}^+_{\mathrm{H}_1-}|\mathcal{F}_{\mathrm{H}}\big](\omega) \\
&\leq \varepsilon + \sum_{k\geq 1} \mathbf{1}_{\tilde{E}_k}(\omega)\big(\hat{u}^+_{\mathrm{H}-}(\omega) - \hat{u}^+_{\mathrm{H}-}(\omega^k)\big) \\
&\quad + \sum_{k\geq 1} \mathbf{1}_{\tilde{E}_k}(\omega)\mathbb{E}^{\mathbb{P}_k}\big[(\hat{u}^+_{\mathrm{H}_1-})^{\tilde{t}_k,\omega^k} - \hat{u}^+\big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)-, \omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)\big] \\
&\leq \varepsilon + \sum_{k\geq 1} \mathbf{1}_{\tilde{E}_k}(\omega)\big(\hat{u}_{\mathrm{H}-}(\omega) - \hat{u}_{\mathrm{H}-}(\omega^k)\big)^+ \\
&\quad + \sum_{k\geq 1} \mathbf{1}_{\tilde{E}_k}(\omega)\mathbb{E}^{\mathbb{P}_k}\big[\big((\hat{u}_{\mathrm{H}_1-})^{\tilde{t}_k,\omega^k} - \hat{u}\big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)-, \omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)\big)^+ \\
&\qquad \wedge C\big].
\end{aligned} \tag{5.6}$$

We now estimate the above error for fixed $\omega \in \tilde{E}_k$:

(1) To estimate the terms of the first sum, we recall that $\mathbf{d}_\infty((\mathrm{H}(\omega), \omega), (\tilde{t}_k, \omega^k)) \leq 2\delta$ on $\tilde{E}_k$, by Lemma 5.1. Then since u^1 is continuous, it follows from

(2.12) that on $\tilde{E}_k$,

$$\begin{aligned} u^1_{\mathrm{H}_i-}(\omega) - u^1_{\mathrm{H}_i-}(\omega^j) &= u^1_{\mathrm{H}_i}(\omega) - u^1_{\mathrm{H}_i}(\omega^j) \\ &= \sum_{l=1}^n \big[\varphi_l\big(\mathrm{H}(\omega),\omega\big) - \varphi_l\big(\tilde{t}_k,\omega^k\big)\big]\psi_l(0,\mathbf{0}) \\ &\le Cn\rho(2\delta). \end{aligned}$$

Moreover, denoting by ρ_2 the modulus of continuity of $-u^2 \in \underline{\mathcal{U}}$ in (2.4), we see that

$$\begin{aligned} & u^2_{\mathrm{H}-}(\omega^k) - u^2_{\mathrm{H}-}(\omega) \\ &\quad = u^2(\tilde{t}_k-,\omega^k) - u^2(\tilde{t}_k-,\omega) + u^2(\tilde{t}_k-,\omega) - u^2\big(\mathrm{H}(\omega)-,\omega\big) \\ &\quad \le \rho_2(\delta) + \sup_{\mathrm{H}(\omega)-\delta\le t\le \mathrm{H}(\omega)} \big[u^2(t-,\omega) - u^2\big(\mathrm{H}(\omega)-,\omega\big)\big]. \end{aligned}$$

By the last two estimates, we see that the first sum in (5.6)

$$\sum_{k\ge 1} \mathbf{1}_{\tilde{E}_k}(\omega)\big(\hat{u}_{\mathrm{H}-}(\omega) - \hat{u}_{\mathrm{H}-}(\omega^k)\big)^+ \longrightarrow 0 \qquad \text{as } \delta \searrow 0. \tag{5.7}$$

(2) Recall from Lemma 5.1 that $0 \le \mathrm{H}(\omega) - \tilde{t}_k \le \delta$. Then (2.11) leads to

$$0 \le \big[\mathrm{H}_1\big(\omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_k}_\cdot\big) - \tilde{t}_k\big] - \big[\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_\cdot\big) - \mathrm{H}(\omega)\big] \le \mathrm{H}(\omega) - \tilde{t}_k \le \delta,$$

and therefore, denoting $\eta_\delta(\omega) := \delta + \sup\{|\omega_s - \omega_t| : 0 \le t \le T, t \le s \le (t+\delta)\wedge T\}$,

$$\begin{aligned} & \mathbf{d}_\infty\big(\big(\mathrm{H}_1\big(\omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_k}_\cdot\big) - \tilde{t}_k, \tilde{B}^{\tilde{t}_k}\big), \big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_\cdot\big) - \mathrm{H}(\omega), \tilde{B}^{\tilde{t}_k}\big)\big) \\ &\quad \le \eta_\delta\big(\tilde{B}^{\tilde{t}_k}\big) \le \eta_\delta\big(B^{\tilde{t}_k}\big). \end{aligned} \tag{5.8}$$

Then, by using (2.12) again, we see that

$$\begin{aligned} & (u^1)^{\tilde{t}_k,\omega^k}_{\mathrm{H}_1^{\tilde{t}_k,\omega^k}-} - u^1\big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_\cdot\big)-, \omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_\cdot\big) \\ &\quad = u^1\big(\mathrm{H}_1\big(\omega^k \otimes_{\tilde{t}_k} B^{\tilde{t}_k}_\cdot\big), \omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_k}_\cdot\big) - u^1\big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}_1(\omega)} \tilde{B}^{\tilde{t}_1}_\cdot\big), \omega \otimes_{\mathrm{H}_1(\omega)} \tilde{B}^{\tilde{t}_k}_\cdot\big) \\ &\quad = \sum_{l=1}^n \big[\varphi_l\big(\tilde{t}_k,\omega^k\big)\psi_l\big(\mathrm{H}_1\big(\omega \otimes_{\mathrm{H}(\omega)} \tilde{B}^{\tilde{t}_k}_\cdot\big) - \mathrm{H}(\omega), \tilde{B}^{\tilde{t}_k}\big) \\ &\qquad\qquad - \varphi_l\big(\mathrm{H}(\omega),\omega\big)\psi_l\big(\mathrm{H}_1\big(\omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_k}_\cdot\big) - \tilde{t}_k, \tilde{B}^{\tilde{t}_k}\big)\big] \\ &\quad \le Cn\big[\rho(2\delta) + \rho\big(\eta_\delta\big(B^{\tilde{t}_k}\big)\big)\big]. \end{aligned} \tag{5.9}$$

We now similarly estimate the corresponding term with u^2. Since $\tilde{t}_k \le \text{H}(\omega)$, by (2.4) and (5.9) we have

$$
\begin{aligned}
& u^2\big(\text{H}_1\big(\omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)-, \omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big) - \big(u^2_{\text{H}_1-}\big)^{\tilde{t}_k,\omega^k} \\
&\quad = (-u^2)\big(\text{H}_1\big(\omega^k \otimes_{\tilde{t}_k} B^{\tilde{t}_k}_{\cdot}\big)-, \omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_k}_{\cdot}\big) - (-u^2)\big(\text{H}_1\big(\omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)-, \\
&\qquad\qquad \omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big) \\
&\quad \le \rho\big(\mathbf{d}_\infty\big(\big(\text{H}_1\big(\omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}\big), \omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}\big), \big(\text{H}_1\big(\omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_k}\big), \omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_j}\big)\big)\big) \\
&\quad \le \rho\big(\mathbf{d}_\infty\big(\big(\text{H}(\omega), \omega\big), \big(\tilde{t}_k, \omega^k\big)\big) \\
&\qquad\qquad + \mathbf{d}_\infty\big(\big(\text{H}_1\big(\omega^k \otimes_{\tilde{t}_k} \tilde{B}^{\tilde{t}_k}\big) - \tilde{t}_k, \tilde{B}^{\tilde{t}_k}\big), \big(\text{H}_1\big(\omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}\big) - \text{H}(\omega), \tilde{B}^{\tilde{t}_k}\big)\big)\big) \\
&\quad \le \rho\big(2\delta + \eta_\delta\big(B^{\tilde{t}_k}\big)\big).
\end{aligned}
$$

Combining the above with (5.9), this implies that the second summation in (5.6) satisfies

$$
\begin{aligned}
& \sum_{k\ge1} \mathbf{1}_{\tilde{E}_k}(\omega)\mathbb{E}^{\mathbb{P}_k}\big[\big(\big(\hat{u}_{\text{H}_1-}\big)^{\tilde{t}_k,\omega^k} - \hat{u}\big(\text{H}_1\big(\omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)-, \omega \otimes_{\text{H}(\omega)} \tilde{B}^{\tilde{t}_k}_{\cdot}\big)\big)^+ \wedge C\big] \\
&\qquad \le \sum_{k\ge1} \mathbb{E}^{\mathbb{P}_k}\big[\big(Cn(\rho+\rho_2)\big(2\delta + \eta_\delta\big(B^{\tilde{t}_k}\big)\big)\big) \wedge C\big]\mathbf{1}_{\tilde{E}_k}(\omega) \\
&\qquad \le Cn\overline{\mathcal{E}}^L_0\big[(\rho+\rho_2)\big(2\delta + \eta_\delta(B)\big) \wedge C\big].
\end{aligned}
$$

One can easily check that $\lim_{\delta\to0} \overline{\mathcal{E}}^L_0[(\rho+\rho_2)(2\delta+\eta_\delta(B)) \wedge C] = 0$. Then by sending $\delta \to 0$ and $\varepsilon \to 0$ in (5.6), we complete the proof of (5.5). $\square$

6. Consistency of the Perron approach. This section is dedicated to the proof of Proposition 4.4. We follow the strategy outlined in Section 4.2, which is based on the idea in [8], Proposition 7.5. However, as pointed out in [8], Remark 7.7, due to fully nonlinearity, the arguments here are much more involved. We shall divide the proof into several lemmas. As in the previous section, we may assume without loss of generality that G satisfies the monotonity (5.1).

We start with some estimates for viscosity solutions of PDE (3.3).

LEMMA 6.1. *Let Assumptions 3.1 and 3.2*(ii) *hold true. Let $h^i : \partial Q^\varepsilon_t \to \mathbb{R}$ be continuous and v^i be the viscosity solution of the PDE* $(\text{E})^{t,\omega}_{\varepsilon,0}$ *with boundary condition h^i, $i = 1, 2$. Then, denoting $\delta v := v^1 - v^2$, $\delta h := h^1 - h^2$, on Q^ε_t we have*

$$
\begin{aligned}
& \delta v(s,x) \le \overline{\mathcal{E}}^{L_0,c_0}_s\big[(\delta h)^+\big(\text{H}, x + B^s_{\text{H}}\big)\big], \\
& \qquad\qquad \textit{where } \text{H} := T \wedge \inf\big\{r \ge s : \big|x + B^s_r\big| = \varepsilon\big\}.
\end{aligned}
\tag{6.1}
$$

PROOF. Let w denote the right-hand side of (6.1). Following the arguments in Lemma 4.6, it is clear that w is the unique viscosity solution of PDE with boundary condition $(\delta h)^+$,

$$-\partial_t w - \overline{g}_0\big(Dw, D^2 w\big) = 0 \qquad \text{on } \mathcal{O}^\varepsilon_t. \tag{6.2}$$

Let K be a smooth nonnegative kernel with unit total mass. For all $\eta > 0$, we define the mollification $w^\eta := w * K^\eta$ of w. Then w^η is smooth, and it follows from a convexity argument of Krylov [14] that w^η is a classical supersolution of

$$\begin{aligned} -\partial_t w^\eta - \overline{g}_0\big(Dw^\eta, D^2 w^\eta\big) \geq 0 \qquad &\text{on } \mathcal{O}^\varepsilon_t, \\ w^\eta = (\delta h)^+ * K^\eta \qquad &\text{on } \partial\mathcal{O}^\varepsilon_t. \end{aligned} \tag{6.3}$$

We claim that

$$\begin{aligned} &\tilde{w}^\eta + v^2 \ \text{ supersolution of the PDE } (\mathrm{E})^{t,\omega}_{\varepsilon,0}, \\ &\text{where } \tilde{w}^\eta := w^\eta + \big\|w^\eta - (\delta h)^+\big\|_{\mathbb{L}^\infty(\partial Q^\varepsilon_t)}. \end{aligned} \tag{6.4}$$

Then, noting that $\tilde{w}^\eta + v^2 = w^\eta + h^2 + \|w^\eta - (\delta h)^+\|_{\mathbb{L}^\infty(\partial Q^\varepsilon_t)} \geq h^1 = v^1$ on ∂Q^ε_t, we deduce from the comparison result of Lemma 3.7 that $\tilde{w}^\eta + v^2 \geq v^1$ on $\overline{Q}^\varepsilon_t$. Sending $\eta \searrow 0$, this implies that $w + v^2 \geq v^1$, which is the required result.

It remains to prove that $\tilde{w}^\eta + v^2$ is a supersolution of the PDE $(\mathrm{E})^{t,\omega}_{\varepsilon,0}$. Let $(t_0, x_0) \in \mathcal{O}^\varepsilon_t$, $\phi \in C^{1,2}(\mathcal{O}^\varepsilon_t)$ be such that $0 = (\phi - \tilde{w}^\eta - v^2)(t_0, x_0) = \max(\phi - \tilde{w}^\eta - v^2)$. Then it follows from the viscosity supersolution property of v^2 that $\mathbf{L}^{t,\omega}(\phi - \tilde{w}^\eta)(t_0, x_0) \geq 0$. Hence, at the point (t_0, x_0), by (5.1) and (6.3), we have

$$\begin{aligned} \mathbf{L}^{t,\omega}\phi &\geq \mathbf{L}^{t,\omega}\phi - \mathbf{L}^{t,\omega}(\phi - \tilde{w}^\eta) \\ &= -\partial_t w^\eta - g^{t,\omega}\big(\cdot, \phi, D\phi, D^2\phi\big) \\ &\quad + g^{t,\omega}\big(\cdot, \phi - \tilde{w}^\eta, D(\phi - w^\eta), D^2(\phi - w^\eta)\big) \\ &\geq -\partial_t w^\eta - g^{t,\omega}\big(\cdot, \phi, D\phi, D^2\phi\big) + g^{t,\omega}\big(\cdot, \phi, D(\phi - w^\eta), D^2(\phi - w^\eta)\big) \\ &\geq \overline{g}_0\big(Dw^\eta, D^2 w^\eta\big) - \alpha \cdot Dw^\eta - \gamma : D^2 w^\eta \geq 0, \end{aligned}$$

where $|\alpha| \leq L_0$ and $|\gamma| \leq L_0$, thanks to Assumption 3.1. This proves (6.4). □

6.1. *Viscosity solutions of a discretized path-frozen PDE.* Denote $\Pi^\varepsilon_n := \Pi^\varepsilon_n(0, T)$ in (3.1), and by $\overline{\Pi}^\varepsilon_n$ its closure. Under Assumption 3.5 (with $N = 1$), clearly one may extend the mapping $\pi_n \in \Pi^\varepsilon_n \longmapsto \xi(\omega^{\pi_n})$ continuously to the compact set $\overline{\Pi}^\varepsilon_n$, and we shall still denote it as $\xi(\omega^{\pi_n})$ for all $\pi_n \in \overline{\Pi}^\varepsilon_n$.

We first construct some stopping times, in light of Definition 2.5. For $\pi_n \in \Pi^\varepsilon_n$ and $(t, x) \in Q^\varepsilon_{t_n}$, define the sequence $\mathrm{H}^{\varepsilon,\pi_n,t,x}_m := \mathrm{H}_m$ as follows: First, $\mathrm{H}_0 := t$, and

$$\begin{aligned} \mathrm{H}_1 &:= \inf\big\{s \geq t : |x + B^t_s| = \varepsilon\big\} \wedge T, \\ \mathrm{H}_{m+1} &:= \big\{s > \mathrm{H}_m : \big|B^t_s - B^t_{\mathrm{H}_m}\big| = \varepsilon\big\} \wedge T, \qquad m \geq 1; \end{aligned} \tag{6.5}$$

$$\pi_n^m(t,x,B^t) := \big(\pi_n, \big(\mathrm{H}_1, x+B^t_{\mathrm{H}_1}\big), \big(\mathrm{H}_2, B^t_{\mathrm{H}_2}-B^t_{\mathrm{H}_1}\big), \dots, \big(\mathrm{H}_m, B^t_{\mathrm{H}_m}-B^t_{\mathrm{H}_{m-1}}\big)\big).$$

It is clear that $\pi_n^m(t,x,B^t)\in\Pi^\varepsilon_{n+m}$ whenever $\mathrm{H}_m<T$.

LEMMA 6.2. $\{\mathrm{H}_m^{\varepsilon,\pi_n,t,x}, m\ge 0\}$ *satisfies the requirements of Definition* 2.5(i)–(ii), *with* $E^m_j=\Omega^t$ *in* (ii).

PROOF. For notational simplicity, we omit the superscripts ${}^{\varepsilon,\pi_n,t,x}$. It is clear that $\mathrm{H}_{m+1}^{\mathrm{H}_m,\omega}\in\mathcal{H}^{\mathrm{H}_m(\omega)}$ whenever $\mathrm{H}_m(\omega)<T$. Next, if $\mathrm{H}_m(\omega)<T$ for all m, then $|B^t_{\mathrm{H}_{m+1}}-B^t_{\mathrm{H}_m}|(\omega)=\varepsilon$ for all m. This contradicts the fact that ω is (left) continuous at $\lim_{m\to\infty}\mathrm{H}_m(\omega)$, and thus $\mathrm{H}_m(\omega)=T$ when m is large enough. Moreover, for each m,

$$\{\mathrm{H}_m<T\}\subset\big\{\big|B^t_{\mathrm{H}_{i+1}}-B^t_{\mathrm{H}_i}\big|=\varepsilon, i=1,\dots,m-1\big\}$$
$$\subset\left\{\sum_{i=1}^{m-1}\big|B^t_{\mathrm{H}_{i+1}}-B^t_{\mathrm{H}_i}\big|^2\ge(m-1)\varepsilon^2\right\}.$$

Then, for any $L>0$,

$$\begin{aligned}\mathcal{C}^L_t[\mathrm{H}_m<T]&\le\frac{1}{(m-1)\varepsilon^2}\overline{\mathcal{E}}^L_t\left[\sum_{i=1}^{m-1}\big|B^t_{\mathrm{H}_{i+1}}-B^t_{\mathrm{H}_i}\big|^2\right]\\&\le\frac{CL^2}{(m-1)\varepsilon^2}\to0\qquad\text{as } m\to\infty.\end{aligned}\tag{6.6}$$

Similarly one can show that $\lim_{m\to\infty}\mathcal{C}^L_s[\mathrm{H}_m^{s,\omega}<T]=0$ for any $(s,\omega)\in\Lambda^t$. Finally, for $\omega,\tilde\omega\in\Omega$ and using the notation in Definition 2.5(ii), we have

$$\begin{aligned}\mathrm{H}_{m+1}(\omega\otimes_{\mathrm{H}_m(\omega)}\tilde\omega)&=T\wedge\inf\big\{t\ge\mathrm{H}_i(\omega):|\tilde\omega_{t-\mathrm{H}_m(\omega)}|=\varepsilon\big\}\\&=T\wedge\big[\mathrm{H}_m(\omega)+\tilde{\mathrm{H}}(\tilde\omega)\big],\end{aligned}$$

where $\tilde{\mathrm{H}}(\tilde\omega):=\inf\{t:|\tilde\omega_t|=\varepsilon\}$ is independent of ω. Then, given $\mathrm{H}_n(\omega)\le\mathrm{H}_n(\omega')$, (2.11) follows immediately. □

We next prove the existence of the functions θ^ε_n, as mentioned in Section 4.2, which allows us to construct classical super and subsolutions in Lemma 6.4 below.

LEMMA 6.3. *Let Assumptions* 3.1, 3.2(ii), 3.3 *and* 3.5 *with* $N=1$ *hold true. Then there exists a sequence of continuous functions* $\theta^\varepsilon_n:(\pi_n,(t,x))\in\overline{\Pi}^\varepsilon_{n+1}\mapsto\mathbb{R}$, *bounded uniformly in* (ε,n), *such that*

$$\begin{aligned}&\theta^\varepsilon_n(\pi_n;\cdot)\ \textit{is a viscosity solution of}\ (\mathrm{E})^{t_n,\omega^{\pi_n}}_{\varepsilon,0};\\&\theta^\varepsilon_n(\pi_n;t,x)=\xi\big(\omega^{\pi_n,(t,x)}\big)\qquad\textit{if}\ t=T,\\&\theta^\varepsilon_n(\pi_n;t,x)=\theta^\varepsilon_{n+1}\big(\pi_n,(t,x);t,0\big)\qquad\textit{if}\ |x|=\varepsilon.\end{aligned}\tag{6.7}$$

PROOF. *Step* 1. We first prove the lemma in the cases $G = \overline{g}$ and $G = \underline{g}$, as introduced in (4.5). For any n, denote

$$\overline{\theta}^{\varepsilon}_{n,n}(\pi_n; t_n, \mathbf{0}) := \xi(\omega^{\pi_n}),$$

which is continuous for $\pi_n \in \overline{\Pi}^{\varepsilon}_n$, thanks to Assumption 3.5 (with $N = 1$). For $m := n-1, \dots, 0$, let $\theta := \overline{\theta}^{\varepsilon}_{n,m}(\pi_m; \cdot)$ be the unique viscosity solution of the PDE

$$\begin{aligned} \overline{\mathbf{L}}\theta &:= -\partial_t\theta - \overline{g}(\theta, D\theta, D^2\theta) = 0 \qquad \text{in } Q^{\varepsilon}_{t_m}, \\ \theta(t,x) &= \overline{\theta}^{\varepsilon}_{n,m+1}(\pi_m, (t,x); t, \mathbf{0}) \qquad \text{on } \partial Q^{\varepsilon}_{t_m}. \end{aligned} \tag{6.8}$$

Applying Lemma 6.1 repeatedly and recalling Assumption 3.5 (with $N = 1$) again, we see that $\overline{\theta}^{\varepsilon}_{n,m}(\pi_m; t, x)$ are uniformly bounded and continuous in all variables (π_m, t, x). Now for any $\pi_m \in \overline{\Pi}^{\varepsilon}_m$ and $(t,x) \in \overline{Q}^{\varepsilon}_{t_m}$, define

$$\overline{\theta}^{\varepsilon}_m(\pi_m, t, x) := \sup_{b \in \mathcal{B}^t_{L_0}} \overline{\mathcal{E}}^{L_0}_t \Big[e^{\int_t^T b_r\,dr} \varlimsup_{n\to\infty} \xi(\omega^{\pi^{n-m}_m(t,x,B^t)}) + C_0 \int_t^T e^{\int_t^s b_r\,dr}\,ds \Big].$$

Then, by (6.6),

$$\big|\overline{\theta}^{\varepsilon}_m(\pi_m, t, x) - \overline{\theta}^{\varepsilon}_{n,m}(\pi_m, t, x)\big| \le C\mathcal{C}^{L_0}_{t_m}[\mathrm{H}_{n-m} < T] \le \frac{C}{(n-m-1)\varepsilon^2} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

This implies that $\overline{\theta}^{\varepsilon}_m(\pi_m; t, x)$ are uniformly bounded, uniform in (ε, m) and are continuous in all variables (π_m, t, x). Moreover, by stability of the viscosity solutions, we see that

$$\begin{aligned} &\overline{\theta}^{\varepsilon}_m(\pi_m; \cdot) \qquad \text{is the viscosity solution of PDE (6.8) in } Q^{\varepsilon}_{t_m} \\ &\qquad\qquad\qquad \text{with the boundary condition} \\ &\overline{\theta}^{\varepsilon}_m(\pi_m; T, x) = \xi(\omega^{\pi_m, (T,x)}), \qquad |x| \le \varepsilon, \\ &\overline{\theta}^{\varepsilon}_m(\pi_m; t, x) = \overline{\theta}^{\varepsilon}_{m+1}(\pi_m, (t,x); t, 0), \qquad |x| = \varepsilon. \end{aligned}$$

Similarly we may define from $\underline{g}$ the following $\underline{\theta}^{\varepsilon}_m$ satisfying the corresponding properties:

$$\underline{\theta}^{\varepsilon}_m(\pi_m, t, x) := \inf_{b \in \mathcal{B}^t_{L_0}} \underline{\mathcal{E}}^{L_0}_t \Big[e^{\int_t^T b_r\,dr} \varlimsup_{n\to\infty} \xi(\omega^{\pi^{n-m}_m(t,x,B^t)}) - C_0 \int_t^T e^{\int_t^s b_r\,dr}\,ds \Big].$$

Step 2. We now prove the lemma for G. Given the construction of step 1, define

$$\overline{\theta}^{\varepsilon,m}_m(\pi_m; t, x) := \overline{\theta}^{\varepsilon}_m(\pi_m; t, x), \qquad \underline{\theta}^{\varepsilon,m}_m(\pi_m; t, x) := \underline{\theta}^{\varepsilon}_m(\pi_m; t, x); \qquad m \ge 1.$$

For $i = m-1, \dots, 0$, by Lemma 3.7 we may define $\overline{\theta}_i^{\varepsilon,m}$ and $\underline{\theta}_i^{\varepsilon,m}$ as the unique viscosity solution of the PDE $(\mathrm{E})_{\varepsilon,0}^{t_i,\omega^{\pi_i}}$ with boundary conditions $\overline{\theta}_i^{\varepsilon,m} = \overline{\theta}_{i+1}^{\varepsilon,m}$ and $\underline{\theta}_i^{\varepsilon,m} = \underline{\theta}_{i+1}^{\varepsilon,m}$ on $\partial Q_{t_i}^{\varepsilon}$. Note that for $(t,x) \in \partial Q_{t_m}^{\varepsilon}$,

$$\overline{\theta}_m^{\varepsilon,m}(\pi_m; t, x) = \overline{\theta}_{m+1}^{\varepsilon,m+1}(\pi_m^{t,x}; t, 0), \qquad \underline{\theta}_m^{\varepsilon,m}(\pi_m; t, x) = \underline{\theta}_{m+1}^{\varepsilon,m+1}(\pi_m^{t,x}; t, 0).$$

Since $\underline{g} \le g^{t,\omega} \le \overline{g}$, it follows from the comparison result of the PDEs defined by the operators $\overline{g}$ and $\underline{g}$ that

$$\overline{\theta}_m^{\varepsilon,m}(\pi_m; \cdot) \ge \overline{\theta}_m^{\varepsilon,m+1}(\pi_m; \cdot) \ge \underline{\theta}_m^{\varepsilon,m+1}(\pi_m; \cdot) \ge \underline{\theta}_m^{\varepsilon,m}(\pi_m; \cdot) \qquad \text{in } Q_{t_m}^{\varepsilon}.$$

Then, by an immediate backward induction, the comparison result of Lemma 3.7 implies

$$\begin{aligned} &\overline{\theta}_i^{\varepsilon,m}(\pi_i; \cdot) \ge \overline{\theta}_i^{\varepsilon,m+1}(\pi_i; \cdot) \ge \underline{\theta}_i^{\varepsilon,m+1}(\pi_i; \cdot) \ge \underline{\theta}_i^{\varepsilon,m}(\pi_i; \cdot) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } Q_{t_i}^{\varepsilon}, \text{ for all } i \le m. \end{aligned} \tag{6.9}$$

Denote $\delta\theta_i^{\varepsilon,m} := \overline{\theta}_i^{\varepsilon,m} - \underline{\theta}_i^{\varepsilon,m}$. For any π_i and any $(t,x) \in Q_{t_i}^{\varepsilon}$, recall the notation in (6.5). Applying Lemma 6.1 repeatedly, and following similar but much easier arguments as those in Lemma 5.5, we see that

$$\left|\delta\theta_i^{\varepsilon,m}(\pi_i; t, x)\right| \le \overline{\mathcal{E}}_t^{L_0}\left[\left|\delta\theta_m^{\varepsilon,m}\left(\pi_i^{m-i}(t, x, B^t); \mathrm{H}_{m-i}, 0\right)\right|\right].$$

Note that $\delta\theta_i^{\varepsilon,m}(\pi_i; t, x) = 0$ when $t = T$. Then, by (6.6) again,

$$\left|\delta\theta_i^{\varepsilon,m}(\pi_i; t, x)\right| \le C\mathcal{C}_t^{L_0}[\mathrm{H}_{m-i} < T] \le \frac{C}{(m-i-1)\varepsilon^2} \to 0 \qquad \text{as } m \to \infty.$$

Together with (6.9), this implies the existence of θ_i^{ε} such that $\overline{\theta}_i^{\varepsilon,m} \searrow \theta_i^{\varepsilon}$, $\underline{\theta}_i^{\varepsilon,m} \nearrow \theta_i^{\varepsilon}$, as $m \to \infty$. Clearly θ_i^{ε} are uniformly bounded and continuous. Finally, it follows from the stability of the viscosity solutions that θ_i^{ε} satisfies (6.7). □

6.2. *Approximating classical super and subsolutions of the PPDE.* We now apply Assumption 3.8 to θ_n^{ε} to construct smooth approximations of $\overline{u}$ and $\underline{u}$, namely the $\overline{u}^{\varepsilon}$ and $\underline{u}^{\varepsilon}$ mentioned in Section 4.2. Define $\mathrm{H}_i^{\varepsilon} := \mathrm{H}_i^{\varepsilon,(0,\mathbf{0}),(0,\mathbf{0})}$, that is,

$$\mathrm{H}_0^{\varepsilon} := 0 \quad \text{and} \quad \mathrm{H}_{n+1}^{\varepsilon} := T \wedge \inf\left\{t \ge \mathrm{H}_n^{\varepsilon} : |B_t - B_{\mathrm{H}_n^{\varepsilon}}| = \varepsilon\right\} \qquad \text{for all } n \ge 0.$$

Let $\hat{\pi}_n$ denote the sequence $(\mathrm{H}_i^{\varepsilon}, B_{\mathrm{H}_i^{\varepsilon}})_{1 \le i \le n}$, and $\omega^{\varepsilon} := \lim_{n\to\infty} \omega^{\hat{\pi}_n}$. It is clear that

$$\|\omega - \omega^{\varepsilon}\|_T \le 2\varepsilon \quad \text{and} \quad \|\omega^{\hat{\pi}_n}_{\cdot \wedge \mathrm{H}_n} - \omega\|_{\mathrm{H}_{n+1}} \le 2\varepsilon \qquad \text{for all } n, \omega. \tag{6.10}$$

Recall the common modulus of continuity function ρ_0 of G in Assumption 3.2, and let θ_n^{ε} be given as in Lemma 6.3. We then approximate θ_0^{ε} by a piecewise smooth processes in $\overline{C}^{1,2}(\Lambda)$.

LEMMA 6.4. *Under the conditions of Theorem 4.1, with $N = 1$ in Assumption 3.5, there exists $\psi^\varepsilon \in \overline{C}^{1,2}(\Lambda)$ bounded from below with corresponding stopping times H_n^ε such that*

$$\text{(6.11)} \qquad \begin{aligned} &\psi^\varepsilon(0, \mathbf{0}) = \theta_0^\varepsilon(0, \mathbf{0}) + \varepsilon + T\rho_0(2\varepsilon), \\ &\psi^\varepsilon(T, \omega) \ge \xi(\omega^\varepsilon), \qquad \mathcal{L}\psi^\varepsilon \ge 0 \qquad \textit{on } [0, T). \end{aligned}$$

PROOF. For notational simplicity, in this proof we omit the superscript ε and denote $\theta_n := \theta_n^\varepsilon$, $\psi = \psi^\varepsilon$ etc. Moreover, we extend the domain of $\theta_n(\pi_n; \cdot)$ to $[t_n, \infty) \times \mathbb{R}^d$,

$$\theta_n(\pi_n; t, x) := \theta(\pi_n; t \wedge T, \mathrm{proj}_{\overline{O}_\varepsilon}(x)),$$

where $\mathrm{proj}_{\overline{O}_\varepsilon}$ is the orthogonal projection on $\overline{O}_\varepsilon$, the closed centered ball with radius ε. We shall construct ψ on each $[\mathrm{H}_n, \mathrm{H}_{n+1})$ forwardly, by induction on n.

Step 1. First, let $\eta > 0$, $\lambda > 0$ be small numbers which will be decided later. Consider PDEs (3.3) and (4.10) on $Q_0^{\varepsilon,\eta}$, and recall the operators $\underline{\mathbf{L}}$ and $\overline{\mathbf{L}}$ at (4.10). Thanks to Lemma 3.7, let $v_0^{\eta,\lambda}$, $\overline{v}_0^{\eta,\lambda}$ and $\underline{v}_0^{\eta,\lambda}$ denote the unique viscosity solutions of PDEs $(\mathrm{E})_{\varepsilon,\eta}^{0,\mathbf{0}}$, $\overline{\mathbf{L}}v = 0$ and $\underline{\mathbf{L}}v = 0$, respectively, with the same boundary condition $\theta_0 + \lambda$ on $\partial Q_0^{\varepsilon,\eta}$.

By comparison, we have $\underline{v}_0^{\eta,\lambda} \le v_0^{\eta,\lambda} \le \overline{v}_0^{\eta,\lambda}$. Then, by using the estimate in Lemma 6.1, one can easily show that there exist $\eta_0(\lambda)$ and $C_0(\lambda)$, which may depend on L_0, λ and the regularity of θ_0, such that, for all $\eta \le \eta_0(\lambda)$,

$$0 \le v_0^{\eta,\lambda} - \theta_0 \le C_0(\lambda) \qquad \text{on } \overline{Q}_0^{\varepsilon,\eta} \setminus Q_0^\varepsilon \text{ with } C_0(\lambda) \searrow 0, \text{ as } \lambda \searrow 0.$$

In particular, the above inequalities hold on ∂Q_0^ε. Then, by the comparison principle, Lemmas 3.7 and 6.1, we have

$$0 \le v_0^{\eta,\lambda} - \theta_0 \le C_0(\lambda) \qquad \text{in } \overline{Q}_0^{\varepsilon,\eta}.$$

Fix λ_0 such that $C_0(\lambda_0) < \frac{\varepsilon}{4}$, and set $\eta_0 := \eta_0(\lambda_0)$. Then $v_0^{\eta_0,\lambda_0} < \theta_0 + \frac{\varepsilon}{4}$. On the other hand, by Assumption 3.8, there exists $v_0 \in C^{1,2}(Q_0^{\varepsilon,\eta_0})$ satisfying

$$\begin{aligned} &v_0(0, \mathbf{0}) \le v_0^{\eta_0,\lambda_0}(0, \mathbf{0}) + \frac{\varepsilon}{4} < \theta_0(0, \mathbf{0}) + \frac{\varepsilon}{2}, \\ &\mathbf{L}^{0,\mathbf{0}} v_0 \ge 0 \qquad \text{in } Q_0^{\varepsilon,\eta_0}, \qquad v_0 \ge v_0^{\eta_0,\lambda_0} \qquad \text{on } \partial Q_0^{\varepsilon,\eta_0}. \end{aligned}$$

By the comparison principle and Lemma 3.7, the last inequality on $\partial Q_0^{\varepsilon,\eta_0}$ implies that

$$v_0 \ge v_0^{\eta_0,\lambda_0} \ge \theta_0 \qquad \text{on } \overline{Q}_0^{\varepsilon,\eta_0}.$$

By modifying v_0 outside of $Q_0^{\varepsilon,\eta_0/2}$ and by the monotonicity (5.1), without loss of generality we may assume $v_0 \in C^{1,2}([0,T],\mathbb{R}^d)$ with bounded derivatives such that

$$v_0(0,\mathbf{0}) = \theta_0(0,\mathbf{0}) + \frac{\varepsilon}{2}, \qquad \mathbf{L}^{0,\mathbf{0}} v_0 \ge 0 \quad \text{in } Q_0^\varepsilon, \qquad v_0 \ge \theta_0 \quad \text{on } \partial Q_0^\varepsilon.$$

We now define

$$\psi(t,\omega) := v_0(t,\omega_t) + \frac{\varepsilon}{2} + \rho_0(2\varepsilon)(T-t), \qquad t \in [0, \mathrm{H}_1]. \tag{6.12}$$

Note that $(t,\omega_t) \in Q_0^\varepsilon$ for $t < \mathrm{H}_1$, $(\mathrm{H}_1, \omega_{\mathrm{H}_1}) \in \partial Q_0^\varepsilon$, and θ_0 is bounded. Then

$$\begin{aligned} &\psi(0,\mathbf{0}) = \theta_0(0,\mathbf{0}) + \varepsilon + T\rho_0(2\varepsilon), \\ &v_0(\mathrm{H}_1,\omega) \ge \theta_0(\mathrm{H}_1, \omega_{\mathrm{H}_1}) = \theta_1(\hat{\pi}_1; \mathrm{H}_1, 0), \qquad \psi \ge -C \text{ on } [0,\mathrm{H}_1]. \end{aligned} \tag{6.13}$$

Moreover, by monotonicity (5.1) again, and by Assumption 3.2 and (6.10),

$$\begin{aligned} \mathcal{L}\psi(t,\omega) &= \rho_0(2\varepsilon) - \partial_t v_0(t,\omega_t) - G\big(t,\omega,\psi, Dv_0(t,\omega_t), D^2 v_0(t,\omega_t)\big) \\ &\ge \rho_0(2\varepsilon) - \partial_t v_0(t,\omega_t) - G\big(t,\omega, v_0(t,\omega_t), Dv_0(t,\omega_t), D^2 v_0(t,\omega_t)\big) \\ &\ge -\partial_t v_0(t,\omega_t) - g^{0,\mathbf{0}}\big(t, v_0(t,\omega_t), Dv_0(t,\omega_t), D^2 v_0(t,\omega_t)\big) \\ &= \mathbf{L}^{0,\mathbf{0}} v_0(t,\omega_t) \ge 0 \qquad \text{for } 0 \le t < \mathrm{H}_1(\omega). \end{aligned} \tag{6.14}$$

Here we use the fact that $\partial_\omega[v_0(t,\omega_t)] = (Dv_0)(t,\omega_t)$; see [8], Remark 2.9(i).

Step 2. Let η, λ, δ be small positive numbers which will be decided later. Set $s_i := (1-\delta)^i T$, $i \ge 0$. Since $\overline{O}_\varepsilon$ is compact, there exists a partition $D_1,\dots,D_n$ such that $|y - \tilde{y}| \le T\delta$ for any $y, \tilde{y} \in D_j$, $j = 1,\dots,n$. For each j, fix a point $y_j \in D_j$. Now for each (i,j), let $v_{ij}^{\eta,\lambda}$ denote the unique viscosity solution of the PDE $(\mathrm{E})_{\varepsilon,\eta}^{s_i,\omega^{(s_i,y_j)}}$ with the boundary condition $v_{ij}^{\eta,\lambda}(t,x) = \theta_1(s_i, y_j; t, x) + \lambda$ on $\partial Q_{s_i}^{\varepsilon,\eta}$. Here $\omega^{(s_i,y_j)}$ denotes the linear interpolation of $(0,\mathbf{0})$, (s_i, y_j), (T, y_j). Then, by the same arguments as in step 1, there exist $\eta_0(\lambda)$ and $C_0(\lambda)$, which may depend on L_0, λ and the regularity of θ_1, but independent of δ and (i,j), such that for all $\eta \le \eta_0(\lambda)$,

$$\begin{aligned} &0 \le v_{ij}^{\eta,\lambda}(t,x) - \theta_1(s_i, y_j; t, x) \le C_0(\lambda) \qquad \text{on } \overline{Q}_{s_i}^{\varepsilon,\eta} \setminus Q_{s_i}^\varepsilon \text{ and} \\ &C_0(\lambda) \searrow 0 \qquad \text{as } \lambda \searrow 0. \end{aligned}$$

Following the arguments in step 1, we may fix λ_0, η_0, independently of δ and (i,j), and there exists $v_{ij} \in C^{1,2}([s_i,T],\mathbb{R}^d)$ with bounded derivatives such that

$$\begin{aligned} &v_{ij}(s_i,\mathbf{0}) = \theta_1(s_i, y_j; s_i, \mathbf{0}) + \frac{\varepsilon}{4}, \qquad \mathbf{L}^{s_i,\omega^{(s_i,y_j)}} v_{ij} \ge 0 \qquad \text{in } Q_{s_i}^\varepsilon, \\ &v_{ij} \ge \theta_1(s_i, y_j; \cdot) \qquad \text{on } \partial Q_{s_i}^\varepsilon. \end{aligned}$$

Denote

$$E^1_{ij} := \{s_{i+1} < \mathrm{H}_1 \le s_i\} \cap \{B_{\mathrm{H}_1} \in D_j\} \in \mathcal{F}_{\mathrm{H}_1}.$$

Here we are using (i, j) instead of j as the index, and clearly E^1_{ij} form a partition of Ω. We then define ψ on $[\mathrm{H}_1, \mathrm{H}_2]$ in the form of (2.12) with $n_1 = 2$,

$$\begin{aligned}\psi_t &:= \sum_{i,j}\Big[v_0(\mathrm{H}_1, B_{\mathrm{H}_1}) + v_{ij}(s_i + t - \mathrm{H}_1, B_t - B_{\mathrm{H}_1}) - v_{ij}(s_i, \mathbf{0}) + \frac{\varepsilon}{2}\Big]\mathbf{1}_{E^1_{ij}} \\ &\quad + \rho_0(2\varepsilon)(T - t), \qquad t \in [\mathrm{H}_1, \mathrm{H}_2].\end{aligned} \tag{6.15}$$

We show that ψ satisfies all the requirements on $[\mathrm{H}_1, \mathrm{H}_2]$ when δ is small enough.

- First, by (6.15), we have

$$\begin{aligned}\psi_{\mathrm{H}_1} &= \sum_{i,j}\Big[v_0(\mathrm{H}_1, B_{\mathrm{H}_1}) + \frac{\varepsilon}{2}\Big]\mathbf{1}_{E^1_{ij}} + \rho_0(2\varepsilon)(T - \mathrm{H}_1) \\ &= v_0(\mathrm{H}_1, B_{\mathrm{H}_1}) + \frac{\varepsilon}{2} + \rho_0(2\varepsilon)(T - \mathrm{H}_1),\end{aligned}$$

which is consistent with (6.12), and thus ψ is continuous at $t = \mathrm{H}_1$.

- We next check, similar to (6.14), that

$$\mathcal{L}\psi(t, \omega) \ge 0, \qquad \mathrm{H}_1 \le t < \mathrm{H}_2. \tag{6.16}$$

Note that $(\mathrm{H}_1, B_{\mathrm{H}_1}) \in \partial Q^\varepsilon_0$ and $0 \le s_i - \mathrm{H}_1 \le s_i - s_{i+1} = \delta s_i \le \delta T$ on E^1_{ij}, then

$$\begin{aligned}v_0(\mathrm{H}_1, B_{\mathrm{H}_1}) - v_{ij}(s_i, \mathbf{0}) + \frac{\varepsilon}{2} &\ge \theta_1(\mathrm{H}_1, B_{\mathrm{H}_1}; \mathrm{H}_1, \mathbf{0}) - \theta_1(s_i, y_j; s_i, \mathbf{0}) + \frac{\varepsilon}{4} \\ &\ge \frac{\varepsilon}{4} - \rho_1(3T\delta) \qquad \text{on } E^1_{ij},\end{aligned}$$

where ρ_1 is the modulus of continuity function of θ_1. In particular, $\rho_1(3T\delta) < \frac{\varepsilon}{4}$ when δ is small enough. Now on E^1_{ij}, denoting $t_1 := \mathrm{H}_1$, $x := \omega_{\mathrm{H}_1}$, $\tilde{t} := s_i - \mathrm{H}_1 + t$, by (5.1), Assumption 3.2(i) and (6.10) again, we have

$$\begin{aligned}\mathcal{L}\psi(t, \omega) &\ge \mathcal{L}\psi(t, \omega) - \mathbf{L}^{s_i, \omega^{(s_i, y_j)}} v_{ij}(\tilde{t}, x) \\ &= \rho_0(2\varepsilon) - G\big(t, \omega, \psi(t, \omega), Dv_{ij}(\tilde{t}, x), D^2 v_{ij}(\tilde{t}, x)\big) \\ &\quad + G\big(\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}, v_{ij}(\tilde{t}, x), Dv_{ij}(\tilde{t}, x), D^2 v_{ij}(\tilde{t}, x)\big) \\ &\ge \frac{\varepsilon}{4} - \rho_1(3T\delta) - G\big(t, \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}, v_{ij}(\tilde{t}, x), Dv_{ij}(\tilde{t}, x), D^2 v_{ij}(\tilde{t}, x)\big) \\ &\quad + G\big(\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}, v_{ij}(\tilde{t}, x), Dv_{ij}(\tilde{t}, x), D^2 v_{ij}(\tilde{t}, x)\big) \\ &\ge \frac{\varepsilon}{4} - \rho_1(3T\delta) - \rho_0\big(\mathbf{d}_\infty\big((t, \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}), (\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i})\big)\big).\end{aligned} \tag{6.17}$$

Without loss of generality, assume $\varepsilon \le T$. Then

$$\begin{aligned}
&\mathbf{d}_\infty\big((t,\omega^{\hat\pi_1}_{\cdot\wedge t_1}),(\tilde t\wedge T,\omega^{(s_i,y_j)}_{\cdot\wedge s_i})\big)\\
&\quad\le |t-\tilde t| + \sup_{0\le s\le T}\left|\frac{s\wedge t_1}{t_1}x - \frac{s\wedge s_i}{s_i}y_j\right|\\
&\quad\le \delta T + \sup_{0\le s\le T}\left|\frac{s\wedge t_1}{t_1}x - \frac{s\wedge t_1}{t_1}y_j\right| + \sup_{0\le s\le T}\left|\frac{s\wedge t_1}{t_1}y_j - \frac{s\wedge s_i}{s_i}y_j\right|\\
&\quad\le 2\delta T + \varepsilon \sup_{0\le s\le T}\left|\frac{s\wedge t_1}{t_1} - \frac{s\wedge s_i}{s_i}\right|\\
&\quad= 2\delta T + \varepsilon\left[1-\frac{t_1}{s_i}\right] \le 2\delta T + \varepsilon\left[1-\frac{s_{i+1}}{s_i}\right] = 3\delta T.
\end{aligned}$$

Then $\mathcal{L}\psi(t,\omega) \ge \frac{\varepsilon}{4} - [\rho_0+\rho_1](3T\delta)$. By choosing δ small enough, we obtain (6.16).

• Finally, we emphasize that the bound of v_{ij} and its derivatives depend only on the properties of θ_1 (and the η_0 which again depends on θ_1), but not on (i,j). Then ψ satisfies Definition 2.5(iii) on $[\mathrm{H}_1,\mathrm{H}_2]$. Moreover, since θ_1 is bounded, by comparison we see that $\psi \ge -C$ on $[\mathrm{H}_1,\mathrm{H}_2]$.

Step 3. Repeating the arguments, we may define ψ on $[\mathrm{H}_n,\mathrm{H}_{n+1}]$ for all n. From the construction and recalling Lemma 6.2, we see that $\psi \in \overline{C}^{1,2}(\Lambda)$ bounded from below, $\psi(0,\mathbf{0}) = \theta_0(0,\mathbf{0}) + \varepsilon + T\rho_0(2\varepsilon)$ and $\mathcal{L}\psi \ge 0$ on $[0,T)$. Finally, since $\mathrm{H}_n = T$ when n is large enough, we see that $\psi(T,\omega) = \psi(\mathrm{H}_n(\omega),\omega) \ge \theta_n(\omega^\varepsilon) = \xi(\omega^\varepsilon)$. □

Now we are ready to prove the main result of this section.

PROOF OF PROPOSITION 4.4. For any $\varepsilon > 0$, let H^ε_n, $n \ge 0$ and ψ^ε be as in Lemma 6.4, and define $\overline{\psi}^\varepsilon := \psi^\varepsilon + \rho_0(2\varepsilon)$. Then clearly $\overline{\psi}^\varepsilon \in \overline{C}^{1,2}(\Lambda)$, $\overline{\psi}^\varepsilon$ is bounded from below, and

$$\overline{\psi}^\varepsilon(T,\omega) - \xi(\omega) = \psi^\varepsilon(T,\omega) + \rho_0(2\varepsilon) - \xi(\omega) \ge \xi(\omega^\varepsilon) - \xi(\omega) + \rho_0(2\varepsilon) \ge 0,$$

where the last inequality follows from (6.10). Moreover, for $t \in [\mathrm{H}_n,\mathrm{H}_{n+1})$, by (5.1) again,

$$\begin{aligned}
\mathcal{L}\overline{\psi}^\varepsilon(t,\omega) &= -\partial_t\psi^\varepsilon(t,\omega) - G\big(t,\omega,\psi^\varepsilon+\rho_0(2\varepsilon),\partial_\omega\psi^\varepsilon,\partial^2_{\omega\omega}\psi^\varepsilon\big)\\
&\ge -\partial_t\psi^\varepsilon(t,\omega) - G\big(t,\omega,\psi^\varepsilon,\partial_\omega\psi^\varepsilon,\partial^2_{\omega\omega}\psi^\varepsilon\big) = \mathcal{L}\psi^\varepsilon(t,\omega) \ge 0.
\end{aligned}$$

Then by the definition of $\overline{u}$ we see that

$$\overline{u}(0,\mathbf{0}) \le \overline{\psi}^\varepsilon(0,\mathbf{0}) = \psi^\varepsilon(0,\mathbf{0}) + \rho_0(2\varepsilon) \le \theta^\varepsilon_0(0,\mathbf{0}) + \varepsilon + (T+1)\rho_0(2\varepsilon).$$

Similarly, $\underline{u}(0,\mathbf{0}) \ge \theta_0^\varepsilon(0,\mathbf{0}) - \varepsilon - (T+1)\rho_0(2\varepsilon)$. This implies that

$$\overline{u}(0,\mathbf{0}) - \underline{u}(0,\mathbf{0}) \le 2\big(\varepsilon + (T+1)\rho_0(2\varepsilon)\big).$$

Since $\varepsilon > 0$ is arbitrary, this shows that $\overline{u}(0,\mathbf{0}) = \underline{u}(0,\mathbf{0})$. Similarly we can show that $\overline{u}(t,\omega) = \underline{u}(t,\omega)$ for all $(t,\omega) \in \Lambda$. □

Fo later use, we conclude this section with a complete well-posedness result for a special PPDE.

COROLLARY 6.5. *Let $G(t,\omega,y,z,\gamma) = \overline{g}(y,z,\gamma)$ satisfy Assumptions 3.1 and 3.2, and assume that ξ satisfies Assumptions 3.3 and 3.5 with $N = 1$. Then $\overline{u} = \underline{u}$ and is the unique viscosity solution of the PPDE (2.14).*

PROOF. We first observe that $\overline{g}$ satisfies Assumption 3.2(i). We shall prove in Proposition 8.2 below that it also satisfies Assumption 3.8. Then it follows from the last proof that $\underline{u} = \overline{u}$. Moreover, the process $\overline{w}$ introduced in (4.9) is a viscosity solution of PPDE (2.14) with terminal condition ξ. Then it follows from the partial comparison of Proposition 4.2 that $\underline{u} \le \overline{w} \le \overline{u}$, hence equality. □

7. Viscosity properties of $\overline{u}$ and $\underline{u}$. This section is devoted to the proof of Proposition 4.3. The idea is similar to the corresponding result in the PDE literature, and in spirit is similar to the stability of the viscosity solutions as in [8], Theorem 5.1. However, we shall first establish the required regularities of $\overline{u}$ and $\underline{u}$.

LEMMA 7.1. *Under Assumptions 3.1 and 3.3, the processes $\overline{u}$, $\underline{u}$ are bounded.*

PROOF. We shall only prove the result for $\overline{u}$, the proof for $\underline{u}$ being similar. Fix (t,ω), and set

$$\psi(s,\tilde{\omega}) := C_0(L_0+1)e^{(L_0+1)(T-s)}.$$

Then $\psi \in C^{1,2}(\Lambda^t) \subset \overline{C}^{1,2}(\Lambda^t)$, $\psi \ge 0$, $\psi_T \ge C_0(L_0+1) \ge C_0 \ge \xi^{t,\omega}$, and we compute that

$$\begin{aligned}(\mathcal{L}\psi)_s^{t,\omega} &= (L_0+1)\psi_s - G^{t,\omega}(\cdot,\psi_s,\mathbf{0},\mathbf{0}) \ge \psi_s - G^{t,\omega}(\cdot,\mathbf{0},\mathbf{0},\mathbf{0})\\ &\ge C_0(L_0+1) - C_0 \ge 0.\end{aligned}$$

This implies that $\psi \in \overline{\mathcal{D}}_T^\xi(t,\omega)$, and thus $\overline{u}(t,\omega) \le \psi(t,\mathbf{0})$.

On the other hand, by similar arguments one can show that $-\psi$ is a classical subsolution of PPDE (2.14) satisfying $-\psi_T \le \xi^{t,\omega}$. Then by partial comparison Proposition 4.2, $\overline{u}(t,\omega) \ge -\psi(t,\mathbf{0})$. Hence $|\overline{u}(t,\omega)| \le \psi(t,\mathbf{0}) \le C_0(L_0+1)e^{(L_0+1)T}$. □

We next prove that $\overline{u}$ and $\underline{u}$ satisfy a partial dynamic programming principle.

LEMMA 7.2. *Under Assumptions* 3.1 *and* 3.3, *for* $0 \le t_1 < t_2 \le T$, *we have*

$$\overline{u}(t_1,\omega) \ge \inf\{\psi_{t_1} : \psi \in \overline{\mathcal{D}}_{t_2}^{\overline{u}_{t_2}}(t_1,\omega)\}, \qquad \underline{u}(t_1,\omega) \le \sup\{\psi_{t_1} : \psi \in \underline{\mathcal{D}}_{t_2}^{\underline{u}_{t_2}}(t_1,\omega)\}.$$

PROOF. We only prove the result for $\overline{u}$. For any arbitrary $\psi \in \overline{\mathcal{D}}_T^{\xi}(t_1,\omega)$, notice that $\psi^{t_2,\omega'} \in \overline{C}^{1,2}(\Lambda^{t_2})$ and $\psi_{t_2}(\omega') \ge \overline{u}_{t_2}^{t_1,\omega}(\omega')$ for any $\omega' \in \Omega^{t_1}$. Then $\psi \in \overline{\mathcal{D}}_{t_2}^{\overline{u}_{t_2}}(t_1,\omega)$, and the result follows. □

The next result shows that the functions $\overline{u}, \underline{u}$ are uniformly continuous. We observe that with this regularity in hand, and following standard techniques, we may prove that the equality holds in Lemma 7.2, so that $\overline{u}, \underline{u}$ satisfy a dynamic programming principle. However, this is not needed for the present analysis. Moreover, the result is true in degenerate case $c_0 = 0$ as well.

LEMMA 7.3. *Under Assumptions* 3.1, 3.2(ii) *and* 3.3, *we have* $\overline{u}, \underline{u} \in \mathrm{UC}_b(\Lambda)$.

PROOF. We only prove the result for $\overline{u}$.

(i) We first prove that $\overline{u}$ is uniformly continuous in ω, uniformly in t. For $t \in [0,T]$ and $\omega^1, \omega^2 \in \Omega$, denote $\delta := \|\omega^1 - \omega^2\|_t$. For $\psi^1 \in \overline{\mathcal{D}}_T^{\xi}(t,\omega^1)$, define

$$\psi^2(s,\tilde{\omega}) := \psi^1(s,\tilde{\omega}) + \psi(s) \qquad \text{where } \psi(s) := e^{(L_0+1)(T-s)}[\rho_0(\delta) + \delta].$$

Notice that $e^{-(L_0+1)s} = e^{-(L_0+1)\mathrm{H}_i} e^{-(L_0+1)(s-\mathrm{H}_i)}$, and one can easily check that $\psi^2 \in \overline{C}^{1,2}(\Lambda^t)$ with the same H_i as those of ψ^1. Moreover, ψ^2 is bounded from below, and

$$\begin{aligned}
\psi_T^2 &= \psi_T^1 + \psi_T \ge \xi_T^{t,\omega^1} + \rho_0(\delta) \ge \xi^{t,\omega^2};\\
(\mathcal{L}\psi^2)_s^{t,\omega^2} &\ge (\mathcal{L}\psi^2)_s^{t,\omega^2} - (\mathcal{L}\psi^1)_s^{t,\omega^1}\\
&= (L_0+1)\psi_s - G^{t,\omega^2}(s,\cdot,\psi^2,\partial_\omega\psi^1,\partial^2_{\omega\omega}\psi^1)\\
&\quad + G^{t,\omega^1}(s,\cdot,\psi^1,\partial_\omega\psi^1,\partial^2_{\omega\omega}\psi^1)\\
&\ge (L_0+1)\psi_s - \rho_0(\delta) - L_0\psi_s = \psi_s - \rho_0(\delta) \ge \delta > 0.
\end{aligned}$$

Then $\psi^2 \in \overline{\mathcal{D}}_T^{\xi}(t,\omega^2)$, and therefore $\overline{u}(t,\omega^2) \le \psi^2(t,\mathbf{0})$, implying that

$$\begin{aligned}
\overline{u}(t,\omega^2) - \psi^1(t,\mathbf{0}) &\le \psi^2(t,\mathbf{0}) - \psi^1(t,\mathbf{0}) = e^{(L_0+1)(T-t)}[\rho_0(\delta)+\delta]\\
&\le C[\rho_0(\delta)+\delta].
\end{aligned}$$

Since $\psi^1 \in \overline{\mathcal{D}}_T^{\xi}(t,\omega^1)$ is arbitrary, we obtain $\overline{u}(t,\omega^2) - \overline{u}(t,\omega^1) \le C[\rho_0(\delta)+\delta]$. By symmetry, this shows the required uniform continuity of $\overline{u}$ in ω, uniformly in t.

(ii) We now prove that $-\overline{u}$ satisfies (2.4). Fix $t_1 < t_2 \le T$, and consider the process

$$\underline{w}(t,\omega) := \inf_{b\in\mathcal{B}^t_{L_0}} \underline{\mathcal{E}}^{L_0,c_0}_t\Big[e^{\int_t^{t_2} b_r\,dr}\overline{u}(t_2,\omega\otimes_t B^t) - C_0\int_t^{t_2} e^{\int_t^s b_r\,dr}\,ds\Big], \tag{7.1}$$
$$(t,\omega)\in[0,t_2]\times\Omega.$$

By (4.9), $\underline{w}$ is a viscosity solution of the PPDE

$$\underline{\mathcal{L}w} := -\partial_t\underline{w} - \underline{g}(\underline{w},\partial_\omega\underline{w},\partial^2_{\omega\omega}\underline{w}) = 0, \tag{7.2}$$
$$t\in[0,t_2),\omega\in\Omega,\ \underline{w}(t_2,\omega) = \overline{u}(t_2,\omega).$$

Recalling (4.6) and applying partial comparison principle Proposition 4.2 on PPDE (7.2), we see that $\psi_{t_1} \ge \underline{w}(t_1,\omega)$ for any $\psi\in\overline{\mathcal{D}}^{\overline{u}_{t_2}}_{t_2}(t_1,\omega)$. Then $\overline{u}(t_1,\omega)\ge \underline{w}(t_1,\omega)$, and thus

$$\begin{aligned}
&\overline{u}(t_2,\omega) - \overline{u}(t_1,\omega)\\
&\quad\le \overline{u}(t_2,\omega) - \underline{w}(t_1,\omega)\\
&\quad= \sup_{b\in\mathcal{B}^{t_1}_{L_0}} \overline{\mathcal{E}}^{L_0,c_0}_{t_1}\Big[\overline{u}(t_2,\omega) - e^{\int_{t_1}^{t_2} b_r\,dr}\overline{u}(t_2,\omega\otimes_{t_1} B^{t_1}) + C_0\int_{t_1}^{t_2} e^{\int_{t_1}^s b_r\,dr}\,ds\Big].
\end{aligned}$$

Then it follows from (i) and Lemma 7.1 that

$$\begin{aligned}
&\overline{u}(t_2,\omega) - \overline{u}(t_1,\omega)\\
&\quad\le C(t_2-t_1) + C\overline{\mathcal{E}}^{L_0,c_0}_{t_1}\big[\big|\overline{u}(t_2,\omega) - \overline{u}(t_2,\omega\otimes_{t_1} B^{t_1})\big|\big]\\
&\quad\le C(t_2-t_1) + C\overline{\mathcal{E}}^{L_0,c_0}_{t_1}\big[\rho\big(\mathbf{d}_\infty\big((t_1,\omega),(t_2,\omega)\big) + \|B^{t_1}\|_{t_2}\big)\big],
\end{aligned}$$

where ρ is the modulus of continuity of $\overline{u}(t_2,\cdot)$. Now it is straightforward to check that $-\overline{u}$ satisfies (2.4).

(iii) We finally prove that $\overline{u}$ satisfies (2.4). This, together with Lemma 7.1 and (ii), implies that $\overline{u}\in \mathrm{UC}_b(\Lambda)$. For $t_1<t_2$, $\omega\in\Omega$ and $\psi^2\in\overline{\mathcal{D}}^{\xi}_T(t_2,\omega)$, define

$$\xi_{t_2}(\tilde\omega) := \psi^2(t_2,\mathbf{0}) + e^{L_0(T-t_2)}\rho_0\big(\mathbf{d}_\infty\big((t_1,\omega),(t_2,\omega)\big) + \|\tilde\omega\|_{t_2}\big), \qquad \tilde\omega\in\Omega^{t_2}$$

and

$$\overline{w}(t,\tilde\omega) := \sup_{b\in\mathcal{B}^t_{L_0}} \overline{\mathcal{E}}^{L_0,c_0}_t\Big[e^{\int_t^{t_2} b_r\,dr}\xi_{t_2}(t_2,\tilde\omega\otimes_t B^t) + C_0\int_t^{t_2} e^{\int_t^s b_r\,dr}\,ds\Big], \tag{7.3}$$
$$(t,\tilde\omega)\in[t_1,t_2]\times\Omega^{t_1}.$$

By Lemma 7.1, we may assume without loss of generality that $|\psi^2(t_2,\mathbf{0})| \le C$. Then

$$\begin{aligned}
&\big|\overline{w}(t_1,\mathbf{0}) - \psi^2(t_2,\mathbf{0})\big| \\
&\quad\le C(t_2-t_1) + C\overline{\mathcal{E}}^{L_0,c_0}_{t_1}\big[\rho_0\big(\mathbf{d}_\infty\big((t_1,\omega),(t_2,\omega)\big) + \|B^{t_1}\|_{t_2}\big)\big] \\
&\quad\le C\rho\big(\mathbf{d}_\infty\big((t_1,\omega),(t_2,\omega)\big)\big),
\end{aligned} \tag{7.4}$$

for some modulus of continuity ρ.

By (4.9), the process $\overline{w}$ is a viscosity solution of the PPDE

$$\begin{aligned}
&\overline{\mathcal{L}}\overline{w} := -\partial_t\overline{w} - \overline{g}\big(\overline{w}, \partial_\omega\overline{w}, \partial^2_{\omega\omega}\overline{w}\big) = 0, \\
&\qquad (t,\tilde{\omega}) \in [t_1,t_2) \times \Omega^{t_1} \text{ and } \overline{w}(t_2,\cdot) = \xi_{t_2}.
\end{aligned} \tag{7.5}$$

Notice that ξ_{t_2} satisfies the conditions of Corollary 6.5, and therefore $\overline{w} = \overline{(\overline{w})}$, where $\overline{(\overline{w})}$ is defined for PPDE (7.5) in the spirit of (4.1). Then for any $\varepsilon > 0$, there exists $\psi^0 \in \overline{C}^{1,2}(\Lambda^{t_1})$ bounded from below such that

$$\begin{aligned}
&\psi^0(t_1,\mathbf{0}) \le \overline{w}(t_1,\mathbf{0}) + \varepsilon, \\
&\psi^0(t_2,\tilde{\omega}) \ge \overline{w}(t_2,\tilde{\omega}) \quad \text{and} \\
&-\partial_t\psi^0 - \overline{g}\big(\psi^0, \partial_\omega\psi^0, \partial^2_{\omega\omega}\psi^0\big) \ge 0.
\end{aligned} \tag{7.6}$$

Therefore, for $t \in [t_1,t_2)$, by (4.5) and (4.6), we have

$$\begin{aligned}
\mathcal{L}\psi^0 &= -\partial_t\psi^0 - G\big(\cdot,\psi^0,\partial_\omega\psi^0,\partial^2_{\omega\omega}\psi^0\big) \\
&\ge \overline{g}_0\big(\psi^0,\partial_\omega\psi^0,\partial^2_{\omega\omega}\psi^0\big) - G\big(\cdot,\psi^0,\partial_\omega\psi^0,\partial^2_{\omega\omega}\psi^0\big) \ge 0.
\end{aligned} \tag{7.7}$$

Now define ψ^1 on Λ^{t_1} by

$$\begin{aligned}
\psi^1(t,\tilde{\omega}) &:= \psi^0(t,\tilde{\omega})\mathbf{1}_{[t_1,t_2)}(t) \\
&\qquad + \big[\psi^2(t,\tilde{\omega}^{t_2}) + \big(\psi^0(t_2,\tilde{\omega}) - \psi^2(t_2,\mathbf{0})\big)e^{L_0(t_2-t)}\big]\mathbf{1}_{[t_2,T]}(t),
\end{aligned} \tag{7.8}$$

where $\tilde{\omega}^{t_2}_s := \tilde{\omega}_s - \tilde{\omega}_{t_2}$ for $\tilde{\omega} \in \Omega^{t_1}$ and $s \in [t_2,T]$. Since ψ^0, ψ^2 and $-\psi^2(t_2,\mathbf{0})$ are bounded from below, then so is ψ^1. We shall prove in (iv) below that $\psi^1 \in \overline{C}^{1,2}(\Lambda^{t_1})$. Then it follows from (7.5) and (7.6) that $\psi^0(t_2,\tilde{\omega}) \ge \overline{w}(t_2,\tilde{\omega}) \ge \psi^2(t_2,\mathbf{0})$, and thus $\psi^1(t,\tilde{\omega}) \ge \psi^2(t,\tilde{\omega}^{t_2})$ for $t \ge t_2$. Then, for $t \in [t_2,T]$,

$$\begin{aligned}
\mathcal{L}\psi^1 &= -\partial_t\psi^2 + L_0\big(\psi^1 - \psi^2\big(t,\tilde{\omega}^{t_2}\big)\big) - G\big(\cdot,\psi^1,\partial_\omega\psi^2,\partial^2_{\omega\omega}\psi^2\big) \\
&\ge L_0\big(\psi^1 - \psi^2\big(t,\tilde{\omega}^{t_2}\big)\big) + G\big(\cdot,\psi^2,\partial_\omega\psi^2,\partial^2_{\omega\omega}\psi^2\big) \\
&\quad - G\big(\cdot,\psi^1,\partial_\omega\psi^2,\partial^2_{\omega\omega}\psi^2\big) \\
&\ge 0.
\end{aligned} \tag{7.9}$$

Moreover, by (7.8), (7.6) and (7.5),

$$\psi^1(T,\tilde\omega) \ge \psi^2(T,\tilde\omega^{t_2}) + (\overline{w}(t_2,\tilde\omega) - \psi^2(t_2,\mathbf{0}))e^{L_0(t_2-T)}$$
$$\ge \xi^{t_2,\omega}(\tilde\omega^{t_2}) + \rho_0(\mathbf{d}_\infty((t_1,\omega),(t_2,\omega)) + \|\tilde\omega\|_{t_2}) \ge \xi^{t_1,\omega}(\tilde\omega).$$

This, together with (7.7) and (7.9), implies that $\psi^1 \in \overline{\mathcal{D}}_T^{\xi}(t_1,\omega)$. Then it follows from (7.6) and (7.4) that

$$\overline{u}(t_1,\omega) \le \psi^1(t_1,\mathbf{0}) = \psi^0(t_1,\mathbf{0}) \le \overline{w}(t_1,\mathbf{0}) + \varepsilon$$
$$\le \psi^2(t_2,\mathbf{0}) + C\rho(\mathbf{d}_\infty((t_1,\omega),(t_2,\omega))) + \varepsilon.$$

Since $\psi^2 \in \overline{\mathcal{D}}_T^{\xi}(t_2,\omega)$ and $\varepsilon > 0$ are arbitrary, this provides (2.4).

(iv) It remains to verify that $\psi^1 \in \overline{C}^{1,2}(\Lambda^{t_1})$. Let H_i^0, $E_j^{0,i}$ correspond to ψ^0 and H_i^2, $E_j^{2,i}$ correspond to ψ^2 in Definition 2.5. Define a random index

$$I := \inf\{i : \mathrm{H}_i^0 \ge t_2\}.$$

Set $\mathrm{H}_i^1 := \mathrm{H}_i^0$ for $i < I$ and $\mathrm{H}_i^1(\omega) := \mathrm{H}_{i-I}^2(\omega^{t_2})$ for $i \ge I$. Moreover, set $E_{2j-1}^{1,i} := E_j^{0,i} \cap \{I > i\}$ and $E_{2j}^{1,i} := E_j^{2,i-I} \cap \{I \le i\}$, $j \ge 1$.

Noting that $\mathrm{H}_{i+1}^1 = \mathrm{H}_{i+1}^0 \wedge t_2$ whenever $\mathrm{H}_i^0 < t_2$, it is clear that H_i^1 are $\mathbb{F}$-stopping times and $(\mathrm{H}^1)_{i+1}^{\mathrm{H}_i^1(\omega),\omega} \in \mathcal{H}^{\mathrm{H}_i^1(\omega)}$ whenever $\mathrm{H}_i^1(\omega) < T$. From the construction of $E_j^{1,i}$ one can easily see that $\{E_j^{1,i}, j \ge 1\} \subset \mathcal{F}_{\mathrm{H}_i^1}$ and form a partition of Ω^{t_1}. Moreover, since on each $E_j^{1,i}$, either $\mathrm{H}_i^1 = \mathrm{H}_i^0$ or $\mathrm{H}_i^1 = \mathrm{H}_{i-I}^2$, Definitions 2.5(ii)–(iv) are obvious.

It remains to prove

$$\{i : \mathrm{H}_i^1 < T\} \text{ is finite} \quad \text{and} \quad \lim_{i\to\infty} \mathcal{C}_t^L[(\mathrm{H}_i^1)^{t,\omega} < T] = 0 \qquad \text{for any } (t,\omega) \in \Lambda^{t_1}. \tag{7.10}$$

Notice that, denoting by $[\frac{i}{2}]$ the largest integer below $\frac{i}{2}$,

$$\{\mathrm{H}_i^1 < T\} = \left\{\mathrm{H}_i^1 < T, I > \left[\frac{i}{2}\right]\right\} \cup \left\{\mathrm{H}_i^1 < T, I \le \left[\frac{i}{2}\right]\right\}$$
$$\subset \{\mathrm{H}_{[i/2]}^0 < t_2\} \cup \{\omega \in \Omega^{t_1} : \mathrm{H}_{[i/2]}^2(\omega^{t_2}) < T\}.$$

Then $\{i : \mathrm{H}_i(\omega) < T\}$ is finite for all ω. Furthermore, for any $L > 0$ and $\mathbb{P} \in \mathcal{P}_L^{t_1}$,

$$\mathbb{P}[\mathrm{H}_i^1 < T] \le \mathbb{P}[\mathrm{H}_{[i/2]}^0 < t_2] + \mathbb{P}[\{\omega \in \Omega^{t_1} : \mathrm{H}_{[i/2]}^2(\omega^{t_2}) < T\}]$$
$$\le \mathcal{C}_{t_1}^L[\mathrm{H}_{[i/2]}^0 < T] + \mathbb{E}^{\mathbb{P}}[\mathbb{P}^{t_2,\omega}[\mathrm{H}_{[i/2]}^2 < T]]$$
$$\le \mathcal{C}_{t_1}^L[\mathrm{H}_{[i/2]}^0 < T] + \mathcal{C}_{t_2}^L[\mathrm{H}_{[i/2]}^2 < T],$$

and thus

$$\lim_{i\to\infty}\mathcal{C}^L_{t_1}[\mathrm{H}^1_i < T] \le \lim_{i\to\infty}[\mathcal{C}^L_{t_1}[\mathrm{H}^0_{[i/2]} < T] + \mathcal{C}^L_{t_2}[\mathrm{H}^2_{[i/2]} < T]] = 0.$$

Similarly one can show (7.10) for any $(t,\omega)\in\Lambda^{t_1}$. □

PROOF OF PROPOSITION 4.3. In view of Lemmas 7.1 and 7.3, it remains to prove that $\overline{u}$ and $\underline{u}$ are the viscosity L_0-supersolution and subsolution, respectively, of PPDE (2.14). Without loss of generality, we may assume that the generator G satisfies (5.1), and we prove only that $\overline{u}$ is a viscosity L_0-supersolution at $(0,\mathbf{0})$.

Assume to the contrary that there exists $\varphi\in\overline{\mathcal{A}}^{L_0}\overline{u}(0,\mathbf{0})$ such that $-c := \mathcal{L}\varphi(0,\mathbf{0}) < 0$. Following the proof of the partial dynamic programming principle of Lemma 7.2, we observe that for any $\psi\in\overline{\mathcal{D}}^{\xi}_T(0,\mathbf{0})$ and any $(t,\omega)\in\Lambda$, it is clear that $\psi^{t,\omega}\in\overline{\mathcal{D}}^{\xi}_T(t,\omega)$ and then $\psi(t,\omega)\ge\overline{u}(t,\omega)$. By the definition of $\overline{u}$ in (4.1), there exist $\psi^n\in\overline{C}^{1,2}(\Lambda)$ such that

$$\begin{aligned}&\delta_n := \psi^n(0,\mathbf{0}) - \overline{u}(0,\mathbf{0})\downarrow 0 \qquad \text{as } n\to\infty,\\ &(\mathcal{L}\psi^n)_s \ge 0 \quad\text{and}\quad \psi^n_s\ge\overline{u}_s, \qquad s\in[0,T].\end{aligned}\tag{7.11}$$

Let H be the hitting time required in $\overline{\mathcal{A}}^{L_0}\overline{u}(0,\mathbf{0})$, and since $\varphi\in C^{1,2}(\Lambda)$ and $\overline{u}\in \mathrm{UC}_b(\Lambda)\subset\overline{\mathcal{U}}$, without loss of generality, we may assume

$$\mathcal{L}\varphi(t,\omega)\le -\frac{c}{2}\quad\text{and}\quad |\varphi_t-\varphi_0| + \overline{u}_t - \overline{u}_0 \le \frac{c}{6L_0}, \qquad \text{for all } t\le \mathrm{H}.\tag{7.12}$$

We emphasize that the above H is independent of n. Now let $\{\mathrm{H}^n_i, i\ge 1\}$ correspond to $\psi^n\in\overline{C}^{1,2}(\Lambda)$. Since $\varphi\in\overline{\mathcal{A}}^{L_0}\overline{u}(0,\mathbf{0})$, this implies for all $\mathbb{P}\in\mathcal{P}_{L_0}$ and n, i that

$$0\ge\mathbb{E}^{\mathbb{P}}[(\varphi-\overline{u})_{\mathrm{H}\wedge\mathrm{H}^n_i}]\ge\mathbb{E}^{\mathbb{P}}[(\varphi-\psi^n)_{\mathrm{H}\wedge\mathrm{H}^n_i}].\tag{7.13}$$

Recall the processes $\alpha^{\mathbb{P}}$, $\beta^{\mathbb{P}}$ in the definition of $\mathbb{P}\in\mathcal{P}_L$ [see (2.5)], and denote $\mathcal{G}^{\mathbb{P}}\phi := \alpha^{\mathbb{P}}\cdot\partial_\omega\phi + \frac{1}{2}(\beta^{\mathbb{P}})^2 : \partial^2_{\omega\omega}\phi$. Then, applying functional Itô formula in (7.13) and recalling that ψ^n is a semi-martingale on $[0,\mathrm{H}^n_i]$, it follows from (7.11) that

$$\begin{aligned}\delta_n &\ge \mathbb{E}^{\mathbb{P}}[\psi^n_0 - \psi^n_{\mathrm{H}\wedge\mathrm{H}^n_i} + \varphi_{\mathrm{H}\wedge\mathrm{H}^n_i} - \varphi_0]\\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathrm{H}\wedge\mathrm{H}^n_i}(\partial_t + \mathcal{G}^{\mathbb{P}})(\varphi-\psi^n)\,ds\right]\\ &\ge \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathrm{H}\wedge\mathrm{H}^n_i}\left(\frac{c}{2} - G(\cdot,\varphi,\partial_\omega\varphi,\partial^2_{\omega\omega}\varphi) + G(\cdot,\psi^n,\partial_\omega\psi^n,\partial^2_{\omega\omega}\psi^n)\right.\right.\\ &\qquad\qquad\left.\left. + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)\right)ds\right]\end{aligned}$$

$$\geq \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\mathrm{H}\wedge \mathrm{H}_i^n}\Big(\frac{c}{2} - G(\cdot,\varphi,\partial_\omega\varphi,\partial^2_{\omega\omega}\varphi) + G(\cdot,\overline{u},\partial_\omega\psi^n,\partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)\Big)\,ds\Big],$$

where the last inequality follows from (5.1) and the fact that $\overline{u} \leq \psi^n$ by (7.11). Since $\varphi_0 = \overline{u}_0$, by (7.12) and (5.1), we get

$$\delta_n \geq \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\mathrm{H}\wedge \mathrm{H}_i^n}\Big(\frac{c}{3} - G(\cdot,\overline{u}_0,\partial_\omega\varphi,\partial^2_{\omega\omega}\varphi) + G(\cdot,\overline{u}_0,\partial_\omega\psi^n,\partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)\Big)\,ds\Big].$$

Now let $\eta > 0$ be a small number. For each n, define $\tau_0^n := 0$, and

$$\begin{aligned}\tau_{j+1}^n := \mathrm{H}\wedge\inf\big\{t\geq\tau_j^n : {}&\rho_0\big(\mathbf{d}_\infty\big((t,\omega),(\tau_j^n,\omega)\big)\big) + \big|\partial_\omega\varphi(t,\omega)-\partial_\omega\varphi(\tau_j^n,\omega)\big| \\ &+ \big|\partial^2_{\omega\omega}\varphi(t,\omega)-\partial^2_{\omega\omega}\varphi(\tau_j^n,\omega)\big| + \big|\partial_\omega\psi^n(t,\omega)-\partial_\omega\psi^n(\tau_j^n,\omega)\big| \\ &+ \big|\partial^2_{\omega\omega}\psi^n(t,\omega)-\partial^2_{\omega\omega}\psi^n(\tau_j^n,\omega)\big| \geq \eta\big\}.\end{aligned}$$

Recalling Definitions 2.5(iii)–(iv), we see the uniform regularity of ψ^n on $[0,\mathrm{H}_i^n]$ for each i. Then, together with the smoothness of G and φ, one can easily check that $\tau_j^n \uparrow \mathrm{H}$ as $j\to\infty$. Thus

$$\begin{aligned}\delta_n &\geq \Big[\frac{c}{3} - C\eta\Big]\mathbb{E}^{\mathbb{P}}[\mathrm{H}\wedge\mathrm{H}_i^n] \\ &\quad + \sum_{j\geq 0}\mathbb{E}^{\mathbb{P}}\big[(\tau_{j+1}^n\wedge\mathrm{H}_i^n - \tau_j^n\wedge\mathrm{H}_i^n) \\ &\qquad \times \big(G(\cdot,\overline{u}_0,\partial_\omega\psi^n,\partial^2_{\omega\omega}\psi^n) - G(\cdot,\overline{u}_0,\partial_\omega\varphi,\partial^2_{\omega\omega}\varphi) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)\big)_{\tau_j^n}\big] \\ &= \Big[\frac{c}{3} - C\eta\Big]\mathbb{E}^{\mathbb{P}}[\mathrm{H}\wedge\mathrm{H}_i^n] \\ &\quad + \sum_{j\geq 0}\mathbb{E}^{\mathbb{P}}\Big[\big(\tau_{j+1}^n\wedge\mathrm{H}_i^n - \tau_j^n\wedge\mathrm{H}_i^n\big) \\ &\qquad \times\Big(\alpha_{\tau_j^n}\cdot\partial_\omega(\psi^n-\varphi) + \frac{1}{2}\beta^2_{\tau_i^n}:\partial^2_{\omega\omega}(\psi^n-\varphi) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)_{\tau_j^n}\Big)\Big]\end{aligned}$$

for some appropriate $\alpha_{\tau_j^n}, \beta_{\tau_j^n}$. Now choose $\mathbb{P}_n \in \mathcal{P}_{L_0}$ such that $\alpha_t^{\mathbb{P}_n} = \alpha_{\tau_j^n}$, $\beta_t^{\mathbb{P}_n} = \beta_{\tau_j^n}$ for all $\tau_j^n \leq t < \tau_{j+1}^n$. Then $\delta_n \geq [\frac{c}{3} - C\eta]\mathbb{E}^{\mathbb{P}_n}[\mathrm{H}\wedge\mathrm{H}_i^n]$. Set $\eta := \frac{c}{6C}$, send $i\to\infty$ and recall from Definition 2.5 that $\lim_{i\to\infty}\mathcal{C}_0^{L_0}(\mathrm{H}_i^n < T) = 0$. This leads to

$\delta_n \geq \frac{c}{6}\mathbb{E}^{\mathbb{P}_n}[\mathrm{H}] \geq \underline{\mathcal{E}}_0^{L_0}[\mathrm{H}]$, and by sending $n \to \infty$, we obtain $\underline{\mathcal{E}}_0^{L_0}[\mathrm{H}] = 0$. However, since $\mathrm{H} \in \mathcal{H}$, by [8], Lemma 2.4, we have $\underline{\mathcal{E}}^{L_0}[\mathrm{H}] > 0$. This is a contradiction. □

8. On Assumptions 3.8 and 3.2(i).

8.1. *Sufficient conditions for Assumption* 3.8. In this subsection we discuss the validity of our Assumption 3.8 which is clearly related to the classical Perron approach, the key argument for the existence in the theory of viscosity solutions, as shown by Ishii [13]. However, our definition of $\overline{v}$ and $\underline{v}$ involves classical supersolutions and subsolutions, while the classical definition in [13] involves viscosity solutions. We remark that Fleming and Vermes [10, 11] have some studies in this respect. The main issue here is to approximate viscosity solutions by classical supersolutions or subsolutions. This is a difficult problem which requires some restrictions on the nonlinearity. In this section, we provide some sufficient conditions, and we hope to address this issue in a more systematic way in future research.

For ease of presentation, we first simplify the notation in Assumption 3.8. Let

$$\begin{aligned} O &:= \{x \in \mathbb{R}^d : |x| < 1\}, \qquad \overline{O} := \{x \in \mathbb{R}^d : |x| \leq 1\}, \\ \partial O &:= \{x \in \mathbb{R}^d : |x| = 1\}; \\ Q &:= [0, T) \times O, \qquad \overline{Q} := [0, T] \times \overline{O}, \\ \partial Q &:= ([0, T] \times \partial O) \cup (\{T\} \times O). \end{aligned} \tag{8.1}$$

We shall consider the following (deterministic) PDE on Q:

$$\begin{aligned} &\mathbf{L}v := -\partial_t v - g(s, x, v, Dv, D^2 v) = 0 \qquad \text{in } Q \quad \text{and} \\ &v = h \qquad \text{on } \partial Q. \end{aligned} \tag{8.2}$$

We remark that in (3.3) the generator g is independent of x.

ASSUMPTION 8.1. (i) g and h are continuous in (t, x);
(ii) g is uniformly Lipschitz continuous in (y, z, γ) and uniformly elliptic in γ.

As in Lemma 3.7, under the above assumption, we see that PDE (8.2) has a unique viscosity solution v, and the comparison principle holds in the sense of viscosity solutions within the class of bounded functions. Define

$$\begin{aligned} \overline{v}(t, x) &:= \inf\{w(t, x) : w \text{ classical supersolution of PDE (8.2)}\}, \\ \underline{v}(t, x) &:= \sup\{w(t, x) : w \text{ classical subsolution of PDE (8.2)}\}. \end{aligned}$$

By the comparison principle we have $\underline{v} \leq v \leq \overline{v}$.

Denote $\mathbb{S}_+^d := \{\gamma \in \mathbb{S}^d : \gamma \geq \mathbf{0}\}$. The following proposition is the main result of this section:

PROPOSITION 8.2. *Under Assumption 8.1, we have $\overline{v} = \underline{v}$ if g is either convex in γ or the dimension $d \le 2$.*

PROOF. For the case $d \le 2$ we refer to Pham and Zhang [17]. Below, we prove the result only for the case when g is convex in γ. As in (5.1), we assume without loss of generality that

$$g(\cdot, y_1, \cdot) - g(\cdot, y_2, \cdot) \le y_2 - y_1 \qquad \text{for all } y_1 \ge y_2. \tag{8.3}$$

For any $\alpha > 0$, we define $O^\alpha := \{x \in \mathbb{R}^d : |x| < 1 + \alpha\}$, $Q^\delta := [0, (1+\alpha)T) \times O^\alpha$, and similar to (8.1), define their closures and boundaries. Let μ, η be smooth mollifiers on Q and $Q^1 \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, and define for any $\alpha' > 0$,

$$h_\alpha(t,x) := (h * \mu_\alpha)\left(\frac{t}{1+\alpha}, \frac{x}{(1+\alpha)}\right), \qquad (t,x) \in \overline{Q}^\alpha,$$

$$g_0(t,x,y,z,\gamma) := \min_{(t',x') \in Q} \{g(t',x',y,z,\gamma) + 2\rho_0(|t-t'| + |x-x'|)\},$$

$$g_{\alpha'} := (g_0 * \eta_{\alpha'}), \qquad (t,x,y,z,\gamma) \in Q^1 \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d.$$

By the uniform continuity of g, we have $c(\alpha') := \|g - g_{\alpha'}\|_\infty \to 0$ as $\alpha' \searrow 0$. Set

$$\underline{g}_{\alpha'} := g_{\alpha'} - c(\alpha') \quad \text{and} \quad \overline{g}_{\alpha'} := g_{\alpha'} + c(\alpha').$$

By our assumptions on g and h, it follows from Theorem 14.15 of Lieberman [15] that there exist $\underline{v}_{\alpha,\alpha'}, \overline{v}_{\alpha,\alpha'} \in C^{1,2}(Q^\alpha) \cap C(\overline{Q}^\alpha)$ solutions of the equations

$$(\underline{E}_{\alpha,\alpha'}) : -\partial_t v - \underline{g}_{\alpha'}(\cdot, v, Dv, D^2 v) = 0 \qquad \text{in } Q^\alpha \quad \text{and} \quad v = h_\alpha \qquad \text{on } \partial Q^\alpha,$$

$$(\overline{E}_{\alpha,\alpha'}) : -\partial_t v - \overline{g}_{\alpha'}(\cdot, v, Dv, D^2 v) = 0 \qquad \text{in } Q^\alpha \quad \text{and} \quad v = h_\alpha \qquad \text{on } \partial Q^\alpha,$$

respectively. In particular, their restriction to $\overline{Q}$ are in $C^{1,2}(\overline{Q})$. By the comparison principle, $\underline{v}_{\alpha,\alpha'} \le \overline{v}_{\alpha,\alpha'}$. Moreover, it follows from (8.3) that

$$\overline{g}_{\alpha'}(\cdot, y + 2c(\alpha'), \cdot) \le \overline{g}_{\alpha'}(\cdot, y, \cdot) - 2c(\alpha') = \underline{g}_{\alpha'}(\cdot, y, \cdot).$$

This shows that $v_{\alpha,\alpha'} + 2c(\alpha')$ is a classical supersolution of $(\overline{E}_{\alpha,\alpha'})$, and therefore

$$\underline{v}_{\alpha,\alpha'} + 2c(\alpha') \ge \overline{v}_{\alpha,\alpha'} \ge \underline{v}_{\alpha,\alpha'}.$$

Additionally, notice that the solutions $\underline{v}_{\alpha,\alpha'}, \overline{v}_{\alpha,\alpha'}$ are bounded uniformly in α, α' for α, α' small enough. The generators $\underline{g}_{\alpha'}, \overline{g}_{\alpha'}$ have the same uniform ellipticity constants as g, and they verify the hypothesis of Theorem 14.13 of Liebermann [15] uniformly in α'. Therefore $\underline{v}_{\alpha,\alpha'}, \overline{v}_{\alpha,\alpha'}$ are Lipschitz continuous with the same Lipshitz constant for all α, α'. Then, denoting $\overline{h}_{\alpha,\alpha'} := \overline{v}_{\alpha,\alpha'}|_{\partial Q}$ and $\underline{h}_{\alpha,\alpha'} := \underline{v}_{\alpha,\alpha'}|_{\partial Q}$, this implies that

$$c(\alpha,\alpha') := \max\{\|\overline{h}_{\alpha,\alpha'} - h\|_\infty, \|\underline{h}_{\alpha,\alpha'} - h\|_\infty\} \longrightarrow 0$$
$$\text{as } \alpha \to 0, \text{ uniformly in } \alpha'.$$

Now for fixed $\varepsilon > 0$, choose $\alpha_0, \alpha_0' > 0$ so that $c(\alpha_0, \alpha') < \varepsilon/4$ for all $\alpha' > 0$, and $c(\alpha_0') \le \varepsilon/4$. Then $\overline{w}_{\alpha_0,\alpha_0'} := \overline{v}_{\alpha_0,\alpha_0'} + c(\alpha_0, \alpha_0')$ and $\underline{w}_{\alpha_0,\alpha_0'} := \underline{v}_{\alpha_0,\alpha_0'} - c(\alpha_0, \alpha_0')$ are respectively the classical supersolution and subsolution of (8.2) on $\overline{Q}$. Thus $\underline{w}_{\alpha_0,\alpha_0'} \le \underline{v}$ and $\overline{w}_{\alpha_0,\alpha_0'} \ge \overline{v}$. Therefore,

$$\begin{aligned}\overline{v} - \underline{v} &\le \overline{w}_{\alpha_0,\alpha_0'} - \underline{w}_{\alpha_0,\alpha_0'} = \overline{v}_{\alpha_0,\alpha_0'} - \underline{v}_{\alpha_0,\alpha_0'} + 2c(\alpha_0, \alpha_0') \le 2c(\alpha_0') + 2c(\alpha_0, \alpha_0') \\ &\le \varepsilon.\end{aligned}$$

Then it follows from the arbitrariness of ε that $\overline{v} = \underline{v}$. $\square$

8.2. *A weaker version of Assumption* 3.2(i). We remark that, while seemingly reasonable, the uniform continuity of G in (t, ω) is violated even for semilinear PPDEs when the diffusion coefficient σ depends on (t, ω). In this subsection we weaken the uniform regularity in Assumption 3.2 slightly so as to fit into the framework of Pham and Zhang [17], which deals with path-dependent Bellman–Isaacs equations associated to stochastic differential games.

ASSUMPTION 8.3. There exist a modulus of continuity functions ρ_0, $\tilde{\rho}_0$ such that, for any $(t, \omega), (\tilde{t}, \tilde{\omega}) \in \Lambda$ and any (y, z, γ),

$$\begin{aligned}&\big|G(t, \omega, y, z, \gamma) - G(\tilde{t}, \tilde{\omega}, y, z, \gamma)\big| \\ &\quad \le \tilde{\rho}_0\big(|t - \tilde{t}|\big)\big[|z| + |\gamma|\big] + \rho_0\big(\mathbf{d}_\infty\big((t, \omega), (\tilde{t}, \tilde{\omega})\big)\big).\end{aligned}$$

Recall the parameters $\varepsilon, \delta, \eta_0$ and the functions v_{ij} introduced in the proof of Lemma 6.4. Notice that Assumption 3.2 is used only in the proof of Lemma 6.4, more precisely in (6.14) and (6.17). We also note that the smooth functions v_{ij} are typically constructed as the classical solution to some PDE, as in Section 8 and in [17], and thus satisfy certain estimates. Assume the following:

(8.4) There exists a constant $C_{\eta_0} > 0$, which may depend on η_0 (and ε), but is independent of δ, such that $|Dv_{ij}(t, x)| \le C_{\eta_0}$, $|D^2 v_{ij}(t, x)| \le C_{\eta_0}$ for all $(t, x) \in \overline{Q}_0^\varepsilon$.

We claim that Lemma 6.4, hence our main result, Theorem 4.1, still holds true if we replace Assumption 3.2 by (8.4) and Assumption 8.3.

Indeed, in (6.14), note that

$$\begin{aligned}&G\big(t, \omega, \big(v_0, Dv_0, D^2 v_0\big)(t, \omega_t)\big) - g^{0,\mathbf{0}}\big(t, \big(v_0, Dv_0, D^2 v_0\big)(t, \omega_t)\big) \\ &\quad = G\big(t, \omega, \big(v_0, Dv_0, D^2 v_0\big)(t, \omega_t)\big) - G\big(t, \mathbf{0}, \big(v_0, Dv_0, D^2 v_0\big)(t, \omega_t)\big) \le \rho_0(\varepsilon),\end{aligned}$$

thanks to Assumption 8.3. Thus we still have (6.14).

To see (6.17) under our new assumption, we first note that as in (6.17) and by (5.1),

$$\begin{aligned}\mathcal{L}\psi(t, \omega) \ge \rho_0(2\varepsilon) + \frac{\varepsilon}{4} - \rho_1(3T\delta) - G\big(t, \omega, v_{ij}(\tilde{t}, x), Dv_{ij}(\tilde{t}, x), D^2 v_{ij}(\tilde{t}, x)\big) \\ + G\big(\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}, v_{ij}(\tilde{t}, x), Dv_{ij}(\tilde{t}, x), D^2 v_{ij}(\tilde{t}, x)\big).\end{aligned}$$

Now by Assumption 8.3 and (8.4) we have, at $(\tilde{t}, x) \in \overline{Q}_0^\varepsilon$,

$$
\begin{aligned}
&G(t, \omega, v_{ij}, Dv_{ij}, D^2 v_{ij}) - G(\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}, v_{ij}, Dv_{ij}, D^2 v_{ij}) \\
&\quad = G(t, \omega, v_{ij}, Dv_{ij}, D^2 v_{ij}) - G(t, \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}, v_{ij}, Dv_{ij}, D^2 v_{ij}) \\
&\qquad + G(t, \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}, v_{ij}, Dv_{ij}, D^2 v_{ij}) - G(\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}, v_{ij}, Dv_{ij}, D^2 v_{ij}) \\
&\quad \le \rho_0(\|\omega - \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}\|_t) + \tilde{\rho}_0(|t - \tilde{t} \wedge T|)[|Dv_{ij}| + |D^2 v_{ij}|] \\
&\qquad + \rho_0(\mathbf{d}_\infty((t, \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}), (\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}))) \\
&\quad \le \rho_0(2\varepsilon) + C_{\eta_0} \tilde{\rho}_0(T\delta) + \rho_0(\mathbf{d}_\infty((t, \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}), (\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}))).
\end{aligned}
$$

Thus

$$
\mathcal{L}\psi(t, \omega) \ge \frac{\varepsilon}{4} - \rho_1(3T\delta) - C_{\eta_0} \tilde{\rho}_0(T\delta) - \rho_0(\mathbf{d}_\infty((t, \omega^{\hat{\pi}_1}_{\cdot \wedge t_1}), (\tilde{t} \wedge T, \omega^{(s_i, y_j)}_{\cdot \wedge s_i}))).
$$

Substituting this inequality to (6.17), we see that the rest of the proof of Lemma 6.4 remains the same.

8.3. *Concluding remarks*. We now summarize the conditions under which we have the complete wellposedness result.

THEOREM 8.4. *Assume the following hold true*:

- *Assumptions* 3.1 *and* 3.2(ii);
- *Assumptions* 3.3 *and* 3.5 *or*, *more specifically*, *the sufficient conditions of Lemma* 3.6;
- *G is either convex in γ or the dimension $d \le 2$*;
- *Assumption* 3.2(i), *or more generally*, *Assumption* 8.3 *and* (8.4).

Then the results of Theorem 4.1 *hold true*.

We conclude with some final remarks on our assumptions. We first note that the highly technical requirements of the space $\overline{C}^{1,2}(\Lambda)$ are needed only in the proofs, and are not part of our assumptions. Assumptions 3.1 and 3.3 are more or less standard, and are in fact the conditions used in [8]. In particular, due to the failure of the dominated convergence theorem under $\overline{\mathcal{E}}^{\mathcal{P}_L}$, the regularity of the involved processes become crucial, and some assumptions on regularity of data are more or less necessary.

Assumption 3.5 on the additional structure of ξ is purely technical, due to our current approach. Indeed, in situations where we have a representation for the viscosity solution, for example, in the semilinear case, as in [8], Section 7, this assumption is not needed. We believe this assumption can also be removed if we consider path-dependent HJB equations where the function θ_n^ε in Lemma 6.3 can be constructed directly via second-order BSDEs.

The uniform continuity of G in (t,ω) in Assumption 3.2(i) excludes the dependence of the diffusion coefficient σ on (t,ω) for stochastic control or stochastic differential game problems (see [8], Section 4 and [17]) and thus is not desirable. This is due to our approach of approximating PPDEs by path-frozen PDEs. This assumption may not be needed if we do not use this approximation.

The uniform nondegeneracy of G in Assumption 3.2(ii) is of course serious, as in PDE literature.

Finally, Assumption 3.8 is crucial in our current approach. For path-dependent HJB equations, namely when G is convex in γ, we have, more or less, complete results in the uniformly nondegenerate case. However, in the present paper we verify this assumption by the existence of classical solutions of the mollified path-frozen PDE. Unfortunately, for Bellman–Isaacs equations, we are able to obtain classical solutions only when $d \le 2$; see [17]. It will be very interesting to explore more PDE estimates to see if we can verify Assumption 3.8 directly without getting into classical solutions of high-dimensional Bellman–Isaacs equations.

We note that the essential point of our whole argument is to find approximations $\overline{u}^\varepsilon, \underline{u}^\varepsilon \in \overline{C}^{1,2}(\Lambda)$ such that $\mathcal{L}\overline{u}^\varepsilon \ge 0 \ge \mathcal{L}\underline{u}^\varepsilon$. Assumptions 3.2, 3.5 and 3.8 all serve this purpose. There is potentially an alternative way to prove the comparison principle directly. Let u^1 be a viscosity subsolution and u^2 a viscosity supersolution such that $u^1_T \le u^2_T$. Instead of mollifying the PDE to obtain classical solutions, we may try to mollify u^i directly so that the corresponding $u^{i,\varepsilon}$ will be automatically smooth (in some appropriate sense). In fact, in the PDE literature, the convex/concave convolution exactly serves this purpose. However, in this case, the main challenge is that we need to check that $u^{1,\varepsilon}$ is a classical subsolution, and $u^{2,\varepsilon}$ a classical supersolution, which, if true, will imply the comparison immediately. It will be interesting to explore this approach as well in future research.

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