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Viscosity Solutions of Hamilton-Jacobi Equations in Smooth Banach Spaces

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0. Introduction.

In this paper, we investigate existence and uniqueness of viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. We will be concerned with the Cauchy problem

$$\begin{cases} u_t + H(t, x, Du) = 0 & \text{on} \quad (0, T] \times X \\ u(0, x) = u_0(x) & \text{on} \quad X . \end{cases} \tag{0.1}$$

Here X is a Banach space, T is a given positive number, $H: [0, T] \times X \times X^* \to \mathbf{R}$ is a uniformly continuous function, $u_0 : X \to \mathbf{R}$ is a given uniformly continuous, $u : [0, T] \times X \to \mathbf{R}$ is the unknown, $u_t = \partial u / \partial t$, Du is the Fréchet derivative of u and X^* is the dual of X. This notion of solution was introduced by M. G. Crandall and P. L. Lions (see [4] and [5]). They studied basic properties of viscosity solutions of first order Hamilton-Jacobi equations in finite dimensions. In [6], M. G. Crandall, L. C. Evans and P. L. Lions reformulated and simplified this work. Afterwards, many other authors were interested in this subject, including, for instance, H. Ishii [12], [13], [15], G. Barles [1], I. Capuzzo Dolcetta [2], I. Capuzzo Dolcetta and H. Ishii [3], and others. Some years later, M. G. Crandall and P. L. Lions studied the Hamilton-Jacobi equations in infinite dimensions; see [7]. They proved uniqueness and existence of viscosity solutions with the geometrical assumptions that the Banach space has the Radon-Nikodym property and a smooth bump function, and some technical assumptions of uniform continuity on the Hamiltonian. The existence of viscosity solutions were established by use of differential games.

Recently R. Deville, G. Godefroy and V. Zizler [10] proved both existence and uniqueness of bounded viscosity solutions for the stationary problem of the form

$$u + H(Du) = f \quad \text{on} \quad X .$$

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Here X is a Banach space with a smooth bump function, $H: X^* \to \mathbf{R}$ is a uniformly continuous function and $f: X \to \mathbf{R}$ is bounded and uniformly continuous. For the existence proof they used Perron's method introduced by H. Ishii [14], so they needed a comparison assertion for discontinuous viscosity subsolutions and supersolutions.

The main result of this paper is Theorem 2.1 in the section 2 which will be proved indirectly. First, we will give a comparison result in which one of sub- and supersolutions is assumed to be locally uniformly continuous. In this setting we need the formula of subdifferential of the sum of two functions (see Theorem II-1 [9]). In the second step, we estimate the behabiour of sub- and supersolutions near $t=0$. This is done by regularizing the initial data, constructing locally uniformly Lipschitz viscosity sub- and supersolutions and using the comparison result. Finally, we obtain the desired result by using the smooth variational principle of R. Deville, G. Godefroy and V. Zizler [10] and some technics of the nonlinear analysis. In the section 3 we apply the comparison theorem to show the existence of the viscosity solutions of (0.1) and its regularity.

1. Definitions and preliminaries.

Throughout this paper we assume that the Banach space X satisfies the following geometrical assumption:

(A_0) There exists a C^1 Lipschitz continuous function $\beta: X \to \mathbf{R}$ with bounded nonempty support.

Such a function is called a smooth bump function.

For these Banach spaces we have the following variational principle which was proved by R. Deville, G. Godefroy and V. Zizler in [10] and [11].

THEOREM 1.1. *Let X be a Banach space, $f: X \to \mathbf{R} \cup \{+\infty\}$ a lower semicontinuous bounded below function such that $D(f) := \{x \in X ; f(x) < +\infty\} \neq \emptyset$ and $\varepsilon > 0$. Assume that* (A_0) *holds. Then there exists a C^1-function g on X such that*

a) *$f+g$ has a strong minimum at some point $x_0 \in D(f)$,*

b) *$\|g\|_\infty = \sup\{|g(x)| ; x \in X\} < \varepsilon$ and $\|g'\|_\infty = \sup\{\|g'(x)\| ; x \in X\} < \varepsilon$.*

Moreover, we have the following localization property. There exists a constant $c > 0$ (depending only on the space X) such that whenever $y_0 \in X$ satisfies $f(y_0) \leq \inf_X f + c\varepsilon^2$, then the point x_0 can be chosen so that $\|y_0 - x_0\| < \varepsilon$.

Let us recall that a function $F: X \to \mathbf{R}$ attains a strong minimum at $x_0 \in X$ if $F(x_0) = \inf\{F(x) ; x \in X\}$ and every minimizing sequence $(y_n) \in X$ (i.e. (y_n) satisfies $\lim_n F(y_n) = F(x_0)$) converges to x_0.

Let $f: X \to \mathbf{R} \cup \{\pm\infty\}$ and $x \in D(f)$. We say that f is subdifferentiable at x if

$$D^- f(x) = \{\varphi'(x) ; \varphi : X \to \mathbf{R} \text{ is } C^1 \text{ and } f - \varphi \text{ has a local minimum at } x\}$$

is non-empty. We say that f is superdifferentiable at x if

$$D^+f(x)=\{\varphi'(x)\,;\ \varphi: X\to\mathbf{R} \text{ is } C^1 \text{ and } f-\varphi \text{ has a local maximum at } x\}$$

is non-empty (for $x\notin D(f)$ we set $D^+f(x)=D^-f(x)=\emptyset$). Let us mention that, if the Banach space satisfies the assumption (A_0), then $p\in D^+f(x)$ if and only if

$$\limsup_{\|h\|\to 0}\frac{f(x+h)-f(x)-\langle p, h\rangle}{\|h\|}\leq 0 \tag{1.1}$$

and $p\in D^-f(x)$ if and only if

$$\liminf_{\|h\|\to 0}\frac{f(x+h)-f(x)-\langle p, h\rangle}{\|h\|}\geq 0\,. \tag{1.2}$$

Moreover for every lower semicontinuous f: $X\to\mathbf{R}\cup\{+\infty\}$, the set of all point $x\in D(f)$ such that $D^-f(x)\neq\emptyset$ is dense in $D(f)$. (See for instance [10] and [11]).

In this paper the Banach space is $X\times\mathbf{R}$ and it is easy to see that if $\beta: X\to\mathbf{R}$ (resp. $h:\mathbf{R}\to\mathbf{R}$) is a C^1 Lipschitz continuous bump function on X (resp. on $\mathbf{R}$), then $B(t, x)=h(t)\beta(x)$ defines a C^1 Lipschitz continuous bump on $X\times\mathbf{R}$.

We recall the definition of viscosity solutions of (0.1):

Definition 1.2. A function $u: [0, T]\times X\to\mathbf{R}$ is a viscosity subsolution of (0.1) if

i) u is upper semicontinuous,

ii) for every $(t, x)\in(0, T]\times X$ and every $(a, p)\in D^+u(t, x)$:

$$a+H(t, x, p)\leq 0\,,$$

iii) $u(0, x)\leq u_0(x)$ for $x\in X$.

The function u is a viscosity supersolution of (0.1) if

iv) u is lower semicontinuous,

v) for every $(t, x)\in(0, T]\times X$ and every $(a, p)\in D^-u(t, x)$:

$$a+H(t, x, p)\geq 0\,,$$

vi) $u(0, x)\geq u_0(x)$ for $x\in X$.

Finally u is a viscosity solution of (0.1) if u is both a viscosity subsolution and a viscosity supersolution of (0.1).

By the previous definition, we see that a viscosity solution of (0.1) is necessarily continuous and that if (A_0) holds, then a classical solution of (0.1) is also a viscosity solution of (0.1).

In all the sequel we assume that the Hamiltonian H satisfies the following:

(A_1) $\quad H$ is uniformly continuous on $[0, T]\times X\times X^*$,

(A_2) $$x \to H(0, x, 0) \text{ is bounded on } X.$$

For a subset S of a Banach space Y, we denote by $BUC(S)$ the space of bounded uniformly continuous functions on S.

2. Uniqueness of viscosity solutions.

In this section we shall prove the following comparison assertion:

THEOREM 2.1. *Let* (A_0) *holds and* H_1, H_2 *be two Hamiltonians satisfying* (A_1) *and* (A_2). *Let* $u_0, v_0 \in BUC(X)$ *and let* u, v *be two real valued functions on* $[0, T] \times X$, *such that* u *is bounded above and* v *is bounded below.*

If u *is a viscosity subsolution of*

$$\begin{cases} u_t + H_1(t, x, Du) = 0 & \text{on} \quad (0, T] \times X \\ u(0, x) = u_0(x) & \text{on} \quad X \end{cases} \tag{2.1}$$

and v *is a viscosity supersolution of*

$$\begin{cases} v_t + H_2(t, x, Dv) = 0 & \text{on} \quad (0, T] \times X \\ v(0, x) = v_0(x) & \text{on} \quad X. \end{cases} \tag{2.2}$$

Then

$$u(t, x) - v(t, x) \le e^t \max\left\{ \sup_X (u_0 - v_0), \sup_{[0,T] \times X \times X^*} (H_2 - H_1) \right\}$$

for $(t, x) \in [0, T] \times X$.

REMARK 2.2. 1) It is clear that the previous theorem gives the uniqueness of viscosity solutions of (0.1).

2) In the next section, we shall see that using Theorem 2.1 and Perron's method, we obtain the existence of a solution and a modulus of continuity of the solution.

Before proving our theorem, let us mention the following open problem:

PROBLEM. Is it possible to prove a comparison assertion (Theorem 2.1) in an unbounded viscosity solution case?

Now we prove the following:

PROPOSITION 2.3. *Let* (A_0) *hold and let* H *be a Hamiltonian satisfying* (A_1). *Let* $u_0, v_0 \in BUC(X)$ *and* u, v *be two real valued functions on* $[0, T] \times X$, *such that* u *is bounded above and* v *is bounded below. Suppose that one of these functions is locally uniformly continuous.*

If u *is a viscosity subsolution of* (0.1) *and* v *is a viscosity supersolution of*

$$\begin{cases} v_t + H(t, x, Dv) = 0 & on \quad (0, T] \times X \\ v(0, x) = v_0(x) & on \quad X . \end{cases} \tag{2.3}$$

Then

$$\sup_{[0,T]\times X} (u - v) \leq \sup_X (u_0 - v_0) .$$

PROOF. Here the tool used is the formula of subdifferential of the sum of two functions proved recently by R. Deville and the author (see [9]):

THEOREM. *Let X be a Banach space such that* (A_0) *holds. Let u_1, u_2 be two real valued functions defined on $(0, T] \times X$ such that u_1 is lower semicontinuous and u_2 is locally uniformly continuous. Suppose that (t_0, x_0) and $(a, p) \in D^-(u_1 + u_2)(t_0, x_0)$ are given. Then, for every $\varepsilon > 0$, there exist $(t_1, x_1), (t_2, x_2) \in (0, T] \times X$, there exist $(a_1, p_1) \in D^- u_1(t_1, x_1)$ and $(a_2, p_2) \in D^- u_2(t_2, x_2)$ such that*

i) $\|x_i - x_0\| < \varepsilon$ *and* $|t_i - t_0| < \varepsilon$, *for* $i = 1, 2$,

ii) $|u_1(t_1, x_1) - u_1(t_0, x_0)| < \varepsilon$ *and* $|u_2(t_2, x_2) - u_2(t_0, x_0)| < \varepsilon$,

iii) $\|p_1 + p_2 - p\| < \varepsilon$ *and* $|a_1 + a_2 - a| < \varepsilon$.

To prove that $\sup_{[0,T]\times X}(u - v) \leq \sup_X(u_0 - v_0)$, we suppose the contrary, so there exist $(t_0, x_0) \in (0, T] \times X$ and $\delta > 0$ such that

$$u(t_0, x_0) - v(t_0, x_0) - 2\delta > \sup_X (u_0 - v_0) . \tag{2.4}$$

Fix $0 < \varepsilon < \min(\delta, 2\delta/t_0)$. Consider the function Φ defined on $\mathbf{R} \times X$ by

$$\Phi(t, x) = \begin{cases} u(t, x) - v(t, x) - \delta t/t_0 & \text{if} \quad (t, x) \in [0, T] \times X \\ -\infty & \text{otherwise} . \end{cases}$$

The function Φ is upper semicontinuous and bounded above, so by Theorem 1.1 applied to $f = -\Phi$, there exists a C^1-function $g : \mathbf{R} \times X \to \mathbf{R}$ such that:

(i) $\Phi + g$ has a maximum at some point $(s, y) \in [0, T] \times X$,

(ii) $\|g\|_\infty = \sup\{|g(t, x)| ; (t, x) \in \mathbf{R} \times X\} < \varepsilon/2$ and $\|g'\|_\infty = \sup\{\|g'(t, x)\| ; (t, x) \in \mathbf{R} \times X\} < \varepsilon/2$.

By the first condition in (ii) and (2.4) we have $s \neq 0$, and by (i) if we set $A = \delta/t_0 - g_t(s, y)$ and $p = D_x g(s, y)$, we have $(A, p) \in D^+(u(s, y) - v(s, y))$ with $A > 0$. Now, since one of the functions is locally uniformly continuous, by the formula of subdifferential of the sum of two functions, there exist $(t_1, x_1), (t_2, x_2) \in (0, T] \times X$, there exist $(b_1, p_1) \in D^+ u(t_1, x_1)$ and $(b_2, p_2) \in D^- v(t_2, x_2)$ satisfying:

(i′) $|t_i - s| < \varepsilon$ and $\|x_i - y\| < \varepsilon$ for $i = 1, 2$,

(ii′) $|u(t_1, x_1) - u(s, y)| < \varepsilon$ and $|v(t_2, x_2) - v(s, y)| < \varepsilon$,

(iii′) $|b_1 - b_2 - A| < \varepsilon$ and $\|p_1 - p_2 - p\| < \varepsilon$.

But u is a subsolution of (0.1) and v is a viscosity supersolution of (2.3), so

$$b_1+H(t_1, x_1, p_1)\le 0\,, \qquad b_2+H(t_2, x_2, p_2)\ge 0\,.$$

If we subtract one from the other, we obtain

$$b_1-b_2+H(t_1, x_1, p_1)-H(t_2, x_2, p_2)\le 0$$

which yields

$$A-\varepsilon+H(t_1, x_1, p_1)-H(t_2, x_2, p_2)\le 0\,.$$

Moreover,

$$|t_1-t_2|\le|t_1-s|+|s-t_2|<2\varepsilon\,,$$

$$\|x_1-x_2\|\le\|x_1-y\|+\|y-x_2\|<2\varepsilon\,,$$

$$\|p_1-p_2\|\le\|p_1-p_2-p\|+\|p\|<\varepsilon+\varepsilon/2=3\varepsilon/2\,.$$

Finally, we send ε to zero after using the uniform continuity of the Hamiltonian H on $[0, T]\times X\times X^*$ to obtain, $A\le 0$, which yields a contradiction.

REMARK 2.4. Before beginning the proof of Theorem 2.1, consider the Cauchy problem (0.1) with the assumptions (A_0), (A_1), (A_2) and suppose that the initial condition u_0 is Lipschitz continuous on X. It is easy to see that if K is a Lipschitz constant of u_0, then for all $x\in X$ and for all $p\in D^-u_0(x)\cup D^+u_0(x)$, $\|p\|\le K$. But $x\to H(0, x, 0)$ is bounded and the Hamiltonian H is uniformly continuous, so by (A_1) and (A_2) it is clear that H is bounded on $[0, T]\times X\times\bigcup_{x\in X}(D^-u_0(x)\cup D^+u_0(x))$. Consequently there exists $M>0$ such that

$$\underline{u}(t, x)=Mt+u_0(x) \text{ is a viscosity supersolution of (0.1)}\,,$$

$$\bar{u}(t, x)=-Mt+u_0(x) \text{ is a viscosity subsolution of (0.1)}\,.$$

In the general case where u_0 is just continuous on X, note that the family $(u_0^\varepsilon)_{\varepsilon>0}$ of Lipschitz continuous functions defined by

$$u_0^\varepsilon(x)=\inf_{y\in X}\left\{u_0(y)+\frac{1}{\varepsilon}\|x-y\|\right\} \tag{2.5}$$

converges uniformly on X to u_0 when ε goes to zero.

PROOF OF THEOREM 2.1. We can assume that $u\ge 0$ and $v\ge 0$. Consider the functions $\tilde{u}$, $\tilde{v}$ defined by $\tilde{u}(t, x)=e^{-t}u(t, x)$ and $\tilde{v}(t, x)=e^{-t}v(t, x)$ for $(t, x)\in[0, T]\times X$, for $i=1, 2$.

We define the maps $\tilde{H}_i\colon[0, T]\times X\times\mathbf{R}\times X^*\to\mathbf{R}$ by setting $\tilde{H}_i(t, x, r, p)=r+e^{-t}H_i(t, x, e^tp)$ for $(t, x, r, p)\in[0, T]\times X\times\mathbf{R}\times X^*$.

We shall need the following:

LEMMA 2.5. *u is a viscosity subsolution of* (2.1) (*resp. v is a viscosity supersolution* *of* (2.2)) *if and only if* $\tilde{u}$ *is a viscosity subsolution of*

$$\begin{cases} \tilde{u}(t,x)+\tilde{H}_1(t,x,\tilde{u}_t(t,x),D_x\tilde{u}(t,x))=0 & \textit{for} \quad (t,x)\in(0,T]\times X \\ \tilde{u}(0,x)=u_0(x) & \textit{for} \quad x\in X \end{cases} \tag{2.6}$$

(*resp.* $\tilde{v}$ *is a viscosity supersolution of*

$$\begin{cases} \tilde{v}(t,x)+\tilde{H}_2(t,x,\tilde{v}_t(t,x),D_x\tilde{v}(t,x))=0 & \textit{for} \quad (t,x)\in(0,T]\times X \\ \tilde{v}(0,x)=v_0(x) & \textit{for} \quad x\in X). \end{cases} \tag{2.7}$$

To see this it is enough to prove that for $0<t\le T$, $(a,p)\in D^+u(t,x)$ (resp. $\in D^-v(t,x)$) if and only if $(-e^{-t}u(t,x)+e^{-t}a, e^{-t}p)\in D^+\tilde{u}(t,x)$ (resp. $\in D^-\tilde{v}(t,x)$).

Let $\beta: X\to\mathbf{R}$ (resp. $h:\mathbf{R}\to\mathbf{R}$) be a bump function on X (resp. on $\mathbf{R}$). We can suppose that $\beta\ge 0$, $\beta(0)=\sup_X\beta=1$ and that the support of β is in the unit ball of X. Similarly for h.

Let us fix a positive number δ. From Remark 2.4, for ε small enough, we have

$$u_0(x)\le u_0^\varepsilon(x)+\delta \quad \text{and} \quad v_0(x)\ge v_0^\varepsilon(x)-\delta \qquad \text{for} \quad x\in X \tag{2.8}$$

where u_0^ε and v_0^ε are given by the formula (2.5). So, by (2.8) u is also a viscosity subsolution of

$$\begin{cases} u_t+H_1(t,x,Du)=0 & \text{on} \quad (0,T]\times X \\ u(0,x)=u_0^\varepsilon(x)+\delta & \text{on} \quad X \end{cases} \tag{2.9}$$

and v is a viscosity supersolution of

$$\begin{cases} v_t+H_2(t,x,Du)=0 & \text{on} \quad (0,T]\times X \\ v(0,x)=v_0^\varepsilon(x)-\delta & \text{on} \quad X. \end{cases} \tag{2.10}$$

But u_0^ε and v_0^ε are Lipschitz continuous on X, hence there exists a positive number η_0 such that

$$\frac{1}{\eta}t+u_0^\varepsilon(x)+\delta \text{ is a viscosity supersolution of (2.9)},$$

$$-\frac{1}{\eta}t+v_0^\varepsilon(x)-\delta \text{ is a viscosity subsolution of (2.10)}$$

for $\eta\le\eta_0$. From Proposition 2.3 it follows that

$$u\le\frac{1}{\eta}t+u_0^\varepsilon(x)+\delta \quad \text{and} \quad v\ge-\frac{1}{\eta}t+v_0^\varepsilon(x)-\delta \qquad \text{for} \quad (t,x)\in[0,T]\times X. \tag{2.11}$$

Also, we have $\tilde{u}(t,x)\le\sup u$ and $\tilde{v}(t,x)\ge\inf v$ for $(t,x)\in[0,T]\times X$. Let us fix a positive number $\eta\le\eta_0$ and $\lambda>\sup u-\inf v+v(0,0)-u(0,0)+3\varepsilon$.

Consider the real valued function ω defined on $\mathbf{R}^2\times X^2$ by

$$\omega(t,s,x,y)=\begin{cases}\tilde{u}(t,x)-\tilde{v}(s,y)+\lambda h\left(\frac{t-s}{\eta^2}\right)\beta\left(\frac{x-y}{\varepsilon}\right) & \text{if } t,s\in[0,T]\\ -\infty & \text{otherwise}.\end{cases}\tag{2.12}$$

The Banach space $X^2\times\mathbf{R}^2$ satisfies the assumption (A_0) (for example the function $B(t,s,x,y)=h(t)h(s)\beta(x)\beta(y)$ is a bump function on $X^2\times\mathbf{R}^2$).

The function ω is upper semicontinuous and bounded above, so by the smooth variational principle Theorem 1.1 applied to $f=-\omega$, there exists a C^1-function g on $\mathbf{R}^2\times X^2$ such that

a) $\omega-g$ attains its strong maximum at some point $(t_0,s_0,x_0,y_0)\in[0,T]^2\times X^2$,

b) $\|g\|_\infty=\sup\{|g(t,s,x,y)|;\ (t,s,x,y)\in[0,T]^2\times X^2\}<\varepsilon$ and $\|g'\|_\infty=\sup\{|g'(t,s,x,y)|;\ (t,s,x,y)\in[0,T]^2\times X^2\}<\varepsilon$.

We claim that $|t_0-s_0|\le\eta^2$ and $\|x_0-y_0\|\le\varepsilon$. Indeed, otherwise $h((t_0-s_0)/\eta^2)\beta((x_0-y_0)/\varepsilon)=0$ and by a)

$$\tilde{u}(0,0)-\tilde{v}(0,0)+\lambda h(0)\beta(0)\le\tilde{u}(t_0,x_0)-\tilde{v}(s_0,y_0)+2\varepsilon .$$

But $\tilde{u}(t_0,x_0)\le\sup u$, $\tilde{v}(t_0,x_0)\ge\inf v$ and $h(0)=\beta(0)=1$, so

$$\lambda\le\sup u-\inf v+v(0,0)-u(0,0)+2\varepsilon .$$

This gives a contradiction with the choice of λ.

By a) for $(t,s,x,y)\in[0,T]^2\times X^2$, we have

$$\begin{aligned}&\tilde{u}(t,x)-\tilde{v}(s,y)+\lambda h\left(\frac{t-s}{\eta^2}\right)\beta\left(\frac{x-y}{\varepsilon}\right)-g(t,s,x,y)\\&\qquad\le\tilde{u}(t_0,x_0)-\tilde{v}(s_0,y_0)+\lambda h\left(\frac{t_0-s_0}{\eta^2}\right)\beta\left(\frac{x_0-y_0}{\varepsilon}\right)-g(t_0,s_0,x_0,y_0).\end{aligned}\tag{2.13}$$

This gives

$$\begin{aligned}\tilde{u}(t,x)-\tilde{v}(s,y)\le\tilde{u}(t_0,x_0)-\tilde{v}(s_0,y_0)-\lambda h\left(\frac{t-s}{\eta^2}\right)\beta\left(\frac{x-y}{\varepsilon}\right)\\+\lambda h\left(\frac{t_0-s_0}{\eta^2}\right)\beta\left(\frac{x_0-y_0}{\varepsilon}\right)+2\varepsilon .\end{aligned}\tag{2.14}$$

We argue by distinguishing the following four cases:

First case. $t_0=s_0=0$. By (2.14) and the choice of β and h, for $(t,x)\in[0,T]\times X$, we have

$$\begin{aligned}e^{-t}(u(t,x)-v(t,x))&\le u_0(x_0)-v_0(y_0)+2\varepsilon\\&\le u_0(x_0)-v_0(x_0)+v_0(x_0)-v_0(y_0)+2\varepsilon\\&\le\sup(u_0-v_0)+v_0(x_0)-v_0(y_0)+2\varepsilon .\end{aligned}$$

Hence,

$$u(t, x)-v(t, x)\leq e^t \sup(u_0-v_0)+e^T(|v_0(x_0)-v_0(y_0)|+2\varepsilon) . \tag{2.15}$$

Second case. $t_0\neq 0$ and $s_0\neq 0$. From (2.13), if we set

$$a=-\frac{\lambda}{\eta^2}h'\left(\frac{t_0-s_0}{\eta^2}\right)\beta\left(\frac{x_0-y_0}{\varepsilon}\right)+g_t(t_0, s_0, x_0, y_0) ,$$

$$p=\frac{\lambda}{\varepsilon}h\left(\frac{t_0-s_0}{\eta^2}\right)\beta'\left(\frac{x_0-y_0}{\varepsilon}\right)+g_x(t_0, s_0, x_0, y_0) ,$$

we have $(a, p)\in D^+\tilde{u}(t_0, x_0)$. Similarly if we fix $(t, x)=(t_0, x_0)$ in (2.13), we obtain $(b, q)\in D^-\tilde{v}(s_0, y_0)$ with

$$b=-\frac{\lambda}{\eta^2}h'\left(\frac{t_0-s_0}{\eta^2}\right)\beta\left(\frac{x_0-y_0}{\varepsilon}\right)-g_s(t_0, s_0, x_0, y_0) ,$$

$$q=-\frac{\lambda}{\varepsilon}h\left(\frac{t_0-s_0}{\eta^2}\right)\beta'\left(\frac{x_0-y_0}{\varepsilon}\right)-g_y(t_0, s_0, x_0, y_0) .$$

But $\tilde{u}$ is a viscosity subsolution of (2.6) and $\tilde{v}$ is a viscosity supersolution of (2.7), so

$$\tilde{u}(t_0, x_0)+\tilde{H}_1(t_0, a, x_0, p)\leq 0 , \quad \tilde{v}(s_0, y_0)+\tilde{H}_2(s_0, b, y_0, q)\geq 0 ,$$

i.e.,

$$\tilde{u}(t_0, x_0)+a+e^{-t_0}H_1(t_0, x_0, e^{t_0}p)\leq 0 ,$$

$$\tilde{v}(s_0, y_0)+b+e^{-s_0}H_2(s_0, y_0, e^{s_0}q)\geq 0 .$$

If we subtract the two inequalities, we obtain

$$\tilde{u}(t_0, x_0)-\tilde{v}(s_0, y_0)+a-b+e^{-t_0}H_1(t_0, x_0, e^{t_0}p)-e^{-s_0}H_2(s_0, y_0, e^{s_0}q)\leq 0 .$$

Thus,

$$\begin{aligned}\tilde{u}(t_0, x_0)-\tilde{v}(s_0, y_0)&\leq b-a+e^{-s_0}H_2(s_0, y_0, e^{s_0}q)-e^{-t_0}H_1(t_0, x_0, e^{t_0}p)\\&=b-a+e^{-s_0}H_2(s_0, y_0, e^{s_0}q)-e^{-s_0}H_1(s_0, y_0, e^{s_0}q)\\&\quad+e^{-s_0}H_1(s_0, y_0, e^{s_0}q)-e^{-t_0}H_1(t_0, x_0, e^{t_0}p)\\&\leq b-a+e^{-s_0}\sup(H_2-H_1)+e^{-s_0}H_1(s_0, y_0, e^{s_0}q)\\&\quad-e^{-t_0}H_1(t_0, x_0, e^{t_0}p) .\end{aligned}$$

Now from (2.14), for $(t, x)\in[0, T]\times X$, we have

$$\begin{aligned}\tilde{u}(t, x)-\tilde{v}(t, x)&\leq\tilde{u}(t_0, x_0)-\tilde{v}(s_0, y_0)+2\varepsilon\\&\leq b-a+e^{-s_0}\sup(H_2-H_1)+e^{-s_0}H_1(s_0, y_0, e^{s_0}q)\\&\quad-e^{-t_0}H_1(t_0, x_0, e^{t_0}p)+2\varepsilon .\end{aligned}$$

And hence,

$$u(t,x)-v(t,x)\leq e^{t}\{|b-a|+e^{-s_0}\sup(H_2-H_1)+|e^{-s_0}H_1(s_0,y_0,e^{s_0}q)$$
$$-e^{-t_0}H_1(t_0,x_0,e^{t_0}p)|+2\varepsilon\}\,. \tag{2.16}$$

Third case. $t_0\neq 0$ and $s_0=0$. Let $(t,x)\in[0,T]\times X$, the fomula (2.14) gives

$$\tilde{u}(t,x)-\tilde{v}(t,x)\leq\tilde{u}(t_0,x_0)-\tilde{v}(s_0,y_0)+2\varepsilon\,. \tag{2.17}$$

By (2.11) and the fact that $t_0\leq\eta^2$, we obtain

$$u(t,x)-v(t,x)\leq e^{t}\{\eta+u_0^{\varepsilon}(x_0)+\delta-v_0(y_0)+2\varepsilon)$$
$$=e^{t}(\eta+u_0^{\varepsilon}(x_0)-u_0(x_0)+u_0(x_0)-v_0(x_0)+v_0(x_0)-v_0(y_0)+\delta+2\varepsilon\}$$
$$\leq e^{T}(\eta+2\delta)+e^{t}\sup(u_0-v_0)+e^{T}\{|v_0(x_0)-v_0(y_0)|+2\varepsilon\}\,. \tag{2.18}$$

Fourth case. $s_0\neq 0$ and $t_0=0$. Here the fomulas (2.17) and (2.11) with the equality $s_0\leq\eta^2$ give the following: For $(t,x)\in[0,T]\times X$,

$$u(t,x)-v(t,x)\leq e^{t}\{\eta+u_0(x_0)-v_0^{\varepsilon}(y_0)+2\varepsilon\}$$
$$=e^{t}\{\eta+u_0(x_0)-v_0(x_0)+v_0(x_0)-v_0(y_0)+v_0(y_0)-v_0^{\varepsilon}(y_0)+\delta+2\varepsilon\}$$
$$\leq e^{T}(\eta+2\delta)+e^{t}\sup(u_0-v_0)+e^{T}\{|v_0(x_0)-v_0(y_0)|+2\varepsilon\} \tag{2.19}$$

for $(t,x)\in[0,T]\times X$.

Moreover,

$$|t_0-s_0|\leq\eta^2\,,\quad \|x_0-y_0\|\leq\varepsilon\,,$$

$$|a-b|=|g_t(t_0,s_0,x_0,y_0)+g_s(t_0,s_0,x_0,y_0)|\leq\|g'\|_\infty+\|g'\|_\infty<2\varepsilon\,,$$

$$\|p-q\|=\|g_x(t_0,s_0,x_0,y_0)+g_y(t_0,s_0,x_0,y_0)\|\leq\|g'\|_\infty+\|g'\|_\infty<2\varepsilon\,.$$

Finally, since H_1 and v_0 are uniformly continuous, by sending ε, η and δ to zero in (2.15), (2.16), (2.18) and (2.19) we find the required result and the proof is complete.

3. Existence of viscosity solutions.

Recall that if u is a locally bounded function defined on $[0,T]\times X$, the upper semicontinuous envelope u^* of u is defined by

$$u^*(t,x)=\limsup_{r\to 0}\{u(s,y)\,;\,|t-s|+\|y-x\|\leq r\}$$

and the lower semicontinuous envelope u_* of u is defined by

$$u_*(t,x)=\liminf_{r\to 0}\{u(s,y)\,;\,|t-s|+\|y-x\|\leq r\}\,.$$

At first, we consider the problem (0.1) with u_0 being bounded and Lipschitz continuous on X. From Remark 2.4, there exists $M>0$ such that $\underline{u}(t,x)=Mt+u_0(x)$

is a viscosity supersolution and $\bar{u}(t, x) = -Mt + u_0(x)$ is a viscosity subsolution of (0.1). Moreover $\underline{u}(0, x) = \bar{u}(0, x) = u_0(x)$ for $x \in X$. By Perron's method [14], it is clear that there exists $u : [0, T] \times X \to \mathbf{R}$, satisfying $\bar{u} \leq u \leq \underline{u}$ such that the upper semicontinuous envelope u^* of u is a viscosity subsolution of (0.1) and the lower semicontinuous envelope u_* of u is a viscosity supersolution of (0.1). From Theorem 2.1, we obtain $u^* \leq u_*$. And since $u_* \leq u^*$ we conclude that u is a unique bounded viscosity solution of (0.1).

In the general case where u_0 is bounded and uniformly continuous, we have the following result:

THEOREM 3.1. *Let* (A_0), (A_1) *and* (A_2) *hold. Let* $u_0 \in BUC(X)$. *Then the problem* (0.1) *has a unique bounded viscosity solution. Moreover this solution is uniformly continuous on* $[0, T) \times X$.

PROOF. For $\varepsilon > 0$ consider the problem

$$\begin{cases} u_t + H(t, x, Du) = 0 & \text{on} \quad (0, T] \times X \\ u(0, x) = u_0^\varepsilon(x) & \text{on} \quad X \end{cases} \tag{3.1$_\varepsilon$}$$

where u_0^ε is given by the formula (2.5).

By the previous remark, $(3.1)_\varepsilon$ has a unique bounded viscosity solution u^ε. Now Theorem 2.1 shows that if u^{ε_1} is the viscosity solution of $(3.1)_{\varepsilon_1}$ and u^{ε_2} is the viscosity solution of $(3.1)_{\varepsilon_2}$ with ε_1 and ε_2 two positive numbers, then

$$\|u^{\varepsilon_1} - u^{\varepsilon_2}\|_\infty \leq e^T \|u_0^{\varepsilon_1} - u_0^{\varepsilon_2}\|_\infty . \tag{3.2}$$

Here $\|\cdot\|_\infty$ is the norm of uniform convergence. Since $(u_0^\varepsilon)_{\varepsilon>0}$ converges uniformly on X to u_0 as ε goes to zero, the sequence $(u_0^{1/n})_{n\geq 1}$ is a Cauchy sequence in $(C(X, \mathbf{R}), \|\cdot\|_\infty)$. Consequently, by (3.2), $(u^{1/n})_{n\geq 1}$ is also a Cauchy sequence on $(C([0, T] \times X, \mathbf{R}), \|\cdot\|_\infty)$. Let u be the limit of $u^{1/n}$. By the stability of subdifferentials and superdifferentials Theorem II-5 [8], it is easy to prove that u is a viscosity solution of (0.1).

Now we prove that this solution is uniformly continuous on $[0, T) \times X$. Suppose that (A_0), (A_1) and (A_2) hold. Let us fix $\varepsilon > 0$. By (2.11) and the uniform continuity of the initial data u_0 and the Hamiltonian H, we can find $0 < \delta_1 < T$ such that for all $0 \leq \delta_0 \leq \delta_1$, we have

$$\sup_X |u(\delta, x) - u_0(x)| < e^{-T}\varepsilon/2 , \qquad \sup_X |u_0(x+h) - u_0(x)| < e^{-T}\varepsilon/2$$

and

$$\sup_{[0,T] \times X \times X^*} |H(t+\delta, x+h, p) - H(t, x, p)| < e^{-T}\varepsilon$$

for $0 \leq \delta$, $\|h\| \leq \delta_0$. Let us fix $0 < \delta_0 \leq \delta_1$ and $0 \leq \delta$, $\|h\| \leq \delta_0$. The function u_2 defined on $[0, T - \delta_0] \times X$ by $u_2(t, x) = u(t+\delta, x+h)$ is a viscosity subsolution of

$$\begin{cases} u_t(t, x) + H(t+\delta, x+h, Du(t, x+h)) = 0 & \text{on} \quad (0, T-\delta_0] \times X \\ u(0, x) = u_0(x+h) + e^{-T}\varepsilon/2 & \text{on} \quad X \end{cases}$$

and is a viscosity supersolution of

$$\begin{cases} u_t(t, x) + H(t+\delta, x+h, Du(t, x)) = 0 & \text{on } (0, T-\delta_0] \times X \\ u(0, x) = u_0(x+h) - e^{-T}\varepsilon/2 & \text{on } X. \end{cases}$$

Finally, by Theorem 2.1,

$$|u_2(t, x) - u(t, x)| \le e^t \max\left\{ \sup_X |u_0(x+h) - u_0(x)| + e^{-T}\frac{\varepsilon}{2}, \right.$$

$$\left. \sup_{[0,T]\times X\times X^*} |H(t+\delta, x+h, p) - H(t, x, p)| \right\} < \varepsilon,$$

for $(t, x) \in [0, T-\delta_0] \times X$. So, $|u(t+\delta, x+h) - u(t, x)| \le \varepsilon$ for $(t, x) \in [0, T-\delta_0] \times X$ and $0 \le \delta$, $\|h\| \le \delta_0$. Thus u is uniformly continuous on $[0, T-\delta_0] \times X$ and the proof is complete.

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