Viscosity solutions of optimal stopping problems

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Abstract

We prove that the value function of an optimal stopping problem is the unique viscosity solution of the associated variational inequalities. We illustrate by an example how this can be used to solve optimal stopping problems where the high contact (smooth fit) principle does not necessarily hold.

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1 Introduction

There is a well-known connection between optimal stopping and variational inequalities. This connection can (roughly) be described as follows:
Let $\{X_t\}_{t\geq 0}$ be an Ito diffusion in $\boldsymbol{R}^n$, i.e. X_t is the unique strong solution of an Ito stochastic differential equation of the form

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t; \quad t \geq 0 \\ X_0 &= x \in \boldsymbol{R}^n \end{aligned}$$

where $b : \boldsymbol{R}^n \to \boldsymbol{R}^n, \sigma : \boldsymbol{R}^n \to \boldsymbol{R}^{n\times m}$ are given Lipschitz continuous functions (see e.g.[6] Theorem 5.5). Here $(\Omega, \mathcal{F}, \mathcal{F}_t, W_t)$ is m-dimensional Brownian motion (Wiener process).
Suppose $\Phi(x); x \in \boldsymbol{R}^n$ is the value function of an optimal stopping problem for X_t, i.e.

$$\Phi(x) = \sup_{\tau\leq T} E^x \left[\int_0^\tau f(X_t)dt + g(X_\tau)\right] \tag{1}$$

where E^x denotes the expectation with respect to the probability law Q^x of X_t given that $X_0 = x$ and f and g are given functions satisfying certain conditions (see below). The supremum is taken over all $\mathcal{F}_t$-stopping times $\tau \leq T$ where

$$T := \inf\{t > 0; X_t \notin S\}$$

is the first exit time of X_t from a given domain $S \subset \boldsymbol{R}^n$. Let

$$L = \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j,i=1}^n (\sigma\sigma^T)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

be the second order partial differential operator which coincides with the generator A of X_t on the space C_0^2 of twice continuously differentiable functions from $\boldsymbol{R}^n$ into $\boldsymbol{R}$ with compact support. Then if Φ is smooth enough – and some additional conditions are satisfied – we can identify Φ with the (unique) solution of the following variational inequalities (see e.g [6], Theorem 10.18):

$$\Phi \geq g \quad \text{in } S \tag{2}$$

$$L\Phi + f \leq 0 \quad \text{in } S \tag{3}$$

$$L\Phi + f = 0 \quad \text{in } D := \{x \in S; \Phi(x) > g(x)\} \tag{4}$$

$$\Phi = g \quad \text{on } \partial S \text{ (the boundary of S).} \tag{5}$$

The domain D is called the *continuation region* and – again under some additional conditions – an *optimal stopping time* τ^* (i.e. a stopping time τ

for which we achieve the supremum in (1)) is given by the first exit time of X_t from D :

$$\tau^* = \tau_D := \inf\{t > 0; X_t \notin D\}. \tag{6}$$

A more compact way of writing (2) – (5) is

$$\min\{-L\Phi(x) - f(x), \Phi(x) - g(x)\} = 0 \text{ for all } x \in S, \tag{7}$$

$$\Phi(x) = g(x) \text{ for all } x \in \partial S. \tag{8}$$

The connection between (1) and (7) – (8) is very useful. However, it has the drawback that the value function Φ of (1) is not always smooth enough for the expression $L\Phi$ to be defined (for example, Φ is not always twice differentiable). Therefore it is natural to ask if there is a suitable weak formulation of (7) – (8) such that Φ is its unique solution even when it is not smooth. Such a weak solution concept, called *viscosity solution*, was developed by Crandall and Lions [3] for the Hamilton-Jacobi-Bellman equations (HJB) connected to stochastic control problems. Subsequently the concept has been extended to a larger class of equations, including the variational inequalities above. See the expositions in [2] and [5].

This paper is motivated by the papers [4] and [1]. In [4] it is proved that the value function of a target recognition problem for piecewise deterministic processes is the unique viscosity solution of corresponding variational inequalities. In [1] weak, but sufficient, HJB-quasivariational inequalities are established for a class of combined stochastic control/impulse control problems. The existence (and uniqueness) of solutions in Sobolev spaces of such HJB-quasivariational inequalities was established in [7], under some conditions.

The purpose of this paper is to prove that – under mild conditions – the value function Φ of the optimal stopping problem (1) is the unique viscosity solution of (7) – (8). As far as we know this is the first time that – in this generality – the viscosity solution has been given such a stochastic interpretation. See Theorem 2.1. Moreover, we emphasize that our methods enable us to prove uniqueness (which is well-known for *bounded* sets S) for a large class of unbounded sets S, thereby covering most cases of interest for application to optimal stopping (see Theorem 3.4). Finally, in Section 4 we give an example to illustrate the use of our results.

2 Optimal Stopping

From now on we will assume that $S \in \boldsymbol{R}^n$ is a fixed domain whose boundary ∂S is *regular* for the process X_t, i.e.

$$T := \inf\{t > 0; X_t \notin S\} = 0 \text{ a.s } Q^x \text{ for all } x \in \partial S. \tag{9}$$

We assume that f and g are continuous functions on S and $\bar{S}$, respectively, satisfying

$$E^x[\int_0^T |f(X_t)|dt] < \infty \text{ for all } x \in \bar{S} \tag{10}$$

and the family

$$\{g(X_\tau); \tau \text{ stopping time}, \tau \leq T\} \tag{11}$$

is uniformly integrable with respect to Q^x for all $x \in \bar{S}$. With Φ defined by (1) we will assume that

$$\Phi \text{ is continuous on } \bar{S}. \tag{12}$$

and, as in (6), we assume that

$$\tau_D := \inf\{t > 0; X_t \notin D\} < \infty \text{ a.s } Q^x \text{ for all } x \in D, \tag{13}$$

where

$$D := \{x \in S; \Phi(x) > g(x)\}$$

is the continuation region, as in (4). Then it is well known that Φ can be expressed by

$$\Phi(x) = E^x[\int_0^{\tau_D} f(X_t)dt + g(X_{\tau_D})],$$

so that $\tau^* = \tau_D$ is an optimal stopping time for the problem (1). Moreover, we have the f-harmonicity property in D:

$$\Phi(x) = E^x[\int_0^\tau f(X_t)dt + \Phi(X_\tau)] \tag{14}$$

for all stopping times $\tau \leq \tau_D$. And we have the f-superharmonicity property in $\bar{S}$:

$$\Phi(x) \geq E^x[\int_0^\tau f(X_t)dt + \Phi(X_\tau)] \tag{15}$$

for all stopping times $\tau \leq T$. For these and other results on optimal stopping we refer to [6], Chapter 10.

We now give the definition of viscosity solutions for a variational inequality.

Definition 2.1. *Let u be a real continuous function on $\bar{S}$ (the closure of S). Consider the following variational inequality (in u):*

$$\min\{-Lu(x) - f(x), u(x) - g(x)\} = 0 \text{ for all } x \in S \tag{16}$$

$$u(x) = g(x) \text{ for all } x \in \partial S. \tag{17}$$

Then viscosity solutions are defined as follows,

(a) *u is a* viscosity subsolution *of (16)-(17) in S if (17) holds and for each $\psi \in C^2(S)$ and each $y_0 \in S$ such that $\psi \geq u$ on S and $\psi(y_0) = u(y_0)$ we have,*

$$\min\{-L\psi(y_0) - f(y_0), \psi(y_0) - g(y_0)\} \leq 0 \tag{18}$$

(b) *u is a* viscosity supersolution *of (16)- (17) in S if (17) holds and for each $\phi \in C^2(S)$ and each $y_0 \in S$ such that $\phi \leq u$ on S and $\phi(y_0) = u(y_0)$ we have,*

$$\min\{-L\phi(y_0) - f(y_0), \phi(y_0) - g(y_0)\} \geq 0 \tag{19}$$

(c) *u is a* viscosity solution *of (16)-(17) in S if it is both a viscosity subsolution and a viscosity supersolution of (16)-(17).*

Remark Note that without loss of generality we may – and will – assume that the functions ψ and ϕ above have compact support.

THEOREM 2.1. *Assume that (9)-(12) hold. Then $\Phi(y)$ defined by (1) is a viscosity solution of the equation*

$$\min\{-L\Phi(y) - f(y), \Phi(y) - g(y)\} = 0 \quad \text{for all} \quad y \in S \tag{20}$$

$$\Phi(y) = g(y) \text{ for all } y \in \partial S \tag{21}$$

Proof. Note that (21) follows immediately from assumption (9). Next we turn to (20). First we treat the viscosity subsolution case. Let $\psi \in C^2$ and $y_0 \in S$ be such that $\psi \geq \Phi$ on S and $\psi(y_0) = \Phi(y_0)$.
If $y_0 \notin D$ then $\Phi(y_0) = g(y_0)$ so $\psi(y_0) - g(y_0) = 0$ and hence

$$\min\{-L\psi(y_0) - f(y_0), \psi(y_0) - g(y_0)\} \leq 0$$

and so (18) holds.

Next, suppose $y_0 \in D$. For $\tau \leq \tau_D$ we have by (14)

$$\Phi(y_0) = E^{y_0}[\int_0^\tau f(X_t)dt + \Phi(X_\tau)]$$

Hence, using the relationship between Φ and ψ, we obtain by Dynkin's formula, if τ is also assumed to be bounded,

$$\begin{aligned}
\Phi(y_0) &= E^{y_0}[\int_0^\tau f(X_t)dt + \Phi(X_\tau)] \\
&\leq E^{y_0}[\int_0^\tau f(X_t)dt + \psi(X_\tau)] \\
&= E^{y_0}[\int_0^\tau f(X_t)dt + \psi(y_0) + \int_0^\tau L\psi(X_t)dt] \\
&= E^{y_0}[\int_0^\tau (f(X_t) + L\psi(X_t))dt] + \psi(y_0)
\end{aligned}$$

or

$$E^{y_0}[\int_0^\tau (f(X_t) + L\psi(X_t))dt] \geq 0$$

On dividing by $E^{y_0}[\tau]$ and letting $\tau \to 0$, we get

$$f(y_0) + L\psi(y_0) \geq 0.$$

Thus

$$\min\{-L\psi(y_0) - f(y_0), \psi(y_0) - g(y_0)\} \leq 0.$$

This shows that Φ is a viscosity subsolution.
The viscosity supersolution case is similar: Let $\phi \in C^2(S)$ and $y_0 \in S$ be such that $\phi \leq \Phi$ on S and $\phi(y_0) = \Phi(y_0)$.
For all bounded stopping times $\tau \leq T$ we have by (15)

$$\Phi(y_0) \geq E^{y_0}[\int_0^\tau f(X_t)dt + \Phi(X_\tau)]$$

Hence, using the relationship between Φ and ϕ, we obtain by Dynkin's formula

$$\begin{aligned}
\Phi(y_0) &= E^{y_0}[\int_0^\tau f(X_t)dt + \Phi(X_\tau)] \\
&\geq E^{y_0}[\int_0^\tau f(X_t)dt + \phi(X_\tau)] \\
&= E^{y_0}[\int_0^\tau f(X_t)dt + \phi(y_0) + \int_0^\tau L\phi(X_t)dt] \\
&= E^{y_0}[\int_0^\tau (f(X_t) + L\phi(X_t))dt] + \phi(y_0)
\end{aligned}$$

or

$$E^{y_0}[\int_0^\tau (f(X_t) + L\psi(X_t))dt] \leq 0$$

On dividing by $E^{y_0}[\tau]$ and letting $\tau \to 0$, we get

$$f(y_0) + L\phi(y_0) \leq 0.$$

Thus

$$\min\{-L\phi(y_0) - f(y_0), \phi(y_0) - g(y_0)\} \geq 0,$$

which shows that Φ is a viscosity supersolution. □

3 Uniqueness

We now turn to the question of uniqueness of the viscosity solution of the variational inequality (16) – (17). It is known that if S is bounded, then (16) – (17) has a unique solution. (See e.g Theorem 3.3, [2], which applies to a more general equation than (16) – (17).) But since many important optimal stopping problems involve unbounded sets S (even $S = \boldsymbol{R}^n$), it is of interest to obtain uniqueness also in such cases. However, as the following simple example shows, uniqueness does not hold unless *some* conditions are imposed:

Example 3.1. *Let $S = \boldsymbol{R}$, $f = 0$ and*

$$g(x) = \frac{x^2}{1+x^2}; \quad x \in \boldsymbol{R}.$$

Let $X_t = W_t$, Brownian motion on $\boldsymbol{R}$. Then it is easy to see that

$$\Phi(x) = \sup_\tau E^x[g(W_\tau)] = 1 \text{ for all } x \in \boldsymbol{R}.$$

However, any constant function

$$u(x) = a$$

where $a \geq 1$ is a viscosity solution of the variational inequality (16) – (17) *in this case, viz.*

$$\min\{-\frac{1}{2}u''(y), u(y) - g(y)\} = 0 \text{ for all } y \in \boldsymbol{R}.$$

In order to prove our uniqueness result, we need the following result of independent interest:

Proposition 3.1. *Let U be a domain in $\boldsymbol{R}^n$ such that*

$$\tau_U := \inf\{t > 0; X_t \notin U\} < \infty \text{ a.s } Q^x \text{for all } x \in \overline{U}.$$

Let F and G be continuous functions on U and $\overline{U}$, respectively, satisfying

$$E^x[\int_0^{\tau_U} |F(X_t)|dt] < \infty \text{ for all } x \in \overline{U} \tag{22}$$

and

$$E^x[|G(X_{\tau_U})|] < \infty \text{ for all } x \in \overline{U} \tag{23}$$

a) *(Existence) Define*

$$w(x) = E^x[\int_0^{\tau_U} F(X_t)dt + G(X_{\tau_U})]$$

Then $u = w$ is a viscosity solution of the equation

$$Lu + F = 0 \text{ in } U \tag{24}$$

Moreover, $u = w$ satisfies the condition

(25) *the family*
$\{u(X_\tau); \tau \text{ stopping time }, \tau \leq \tau_U\}$
is Q^x – uniformly integrable, for all $x \in \overline{U}$.

b) *(Uniqueness) Conversely, if u is a viscosity solution of (24) with the boundary values*

$$u = G \text{ on } \partial U \tag{26}$$

and satisfies (25), then $u = w$.

Proof. a) We first prove that w is a viscosity solution of (24). The proof is similar to the proof of Theorem 2.1. To prove that w is a subsolution choose $\psi \in C^2(U)$ and $y_0 \in U$ such that $\psi \geq w$ on U and $\psi(y_0) = w(y_0)$. Let $\tau \leq \tau_U$ be a bounded stopping time. Then by the strong Markov property we have (see e.g [1], Lemma 9.10).

$$\begin{aligned} w(y_0) &= E^{y_0}[\int_0^{\tau} F(X_t)dt + w(X_\tau)] \\ &\leq E^{y_0}[\int_0^{\tau} F(X_t)dt + \psi(X_\tau)] \\ &= \psi(y_0) + E^{y_0}[\int_0^{\tau} (L\psi + F)(X_t)dt] \end{aligned}$$

Hence $E^{y_0}[\int_0^\tau (L\psi + F(X_t))dt] \geq 0$. Dividing by $E^{y_0}[\tau]$ and letting $\tau \to 0$ we get

$$L\psi(y_0) + F(y_0) \geq 0.$$

This proves that w is a subsolution.

Similarly, to prove that w is a supersolution choose $\phi \in C^2(U)$ and $y_0 \in U$ such that $\phi \leq w$ in U, $\phi(y_0) = w(y_0)$. Let $\tau \leq \tau_U$ be a bounded stopping time. Then by the same argument as above

$$\begin{aligned} w(y_0) &= E^{y_0}[\int_0^{\tau} F(X_t)dt + w(X_\tau)] \\ &\geq \phi(y_0) + E^{y_0}[\int_0^{\tau} (L\phi + F)(X_t)dt], \end{aligned}$$

and we get $L\phi(y_0) + F(y_0) \leq 0$. Hence w is also a supersolution.
Next we verify that $u = w$ satisfies (25). If $\tau \leq \tau_U$ is a stopping time, then by the strong Markov property

$$\begin{aligned} w(X_\tau) &= E^{X_\tau}[\int_0^{\tau_U} F(X_t)dt + G(X_{\tau_U})] \\ &= E^x[(\int_\tau^{\tau_U} F(X_t)dt + G(X_{\tau_U}))|\mathcal{F}_\tau]. \end{aligned}$$

Hence by the Jensen inequality

$$|w(X_\tau)| \leq E^x[(\int_\tau^{\tau_U} |F(X_t)|dt + |G(X_{\tau_U})|)|\mathcal{F}_\tau] =: H_\tau(\omega)$$

Let $\eta : [0, \infty) \to [0, \infty)$ be convex, increasing. Then by the Jensen inequality

$$\begin{aligned} E^x[\eta(|w(X_\tau|)] &\leq E^x[\eta(E^x[(\int_\tau^{\tau_U} |F(X_t)|dt + |G(X_{\tau_U})|)|\mathcal{F}_\tau])] \\ &\leq E^x[E^x[\eta(\int_\tau^{\tau_U} |F(X_t)|dt + |G(X_{\tau_U})|)|\mathcal{F}_\tau]] \\ &= E^x[\eta(\int_\tau^{\tau_U} |F(X_t)|dt + |G(X_{\tau_U})|)] \end{aligned} \tag{27}$$

By (22) and (23) the family

$$\{\int_\tau^{\tau_U} |F(X_t)|dt + |G(X_{\tau_U})|)\}_{\tau \leq \tau_U}$$

is Q^x- uniformly integrable (u.i.) and therefore there exists a u.i. test function η (i.e a convex, increasing function η such that $\lim_{x\to\infty} \frac{\eta(x)}{x} = \infty$) and such that the expression in (27) is uniformly bounded for $\tau \leq \tau_U$. Hence $\{w(X_\tau)\}_{\tau \leq \tau_U}$is Q^x-uniformly integrable.

b) Next we prove uniqueness. By Theorem 3.3 in [2] it follows that the viscosity solution is unique if U is bounded. To prove the result in the general case consider

$$U(N) = \{x \in U; |x| < N\} \text{ for } N = 1, 2, ...$$

Suppose v is a viscosity solution of (24) and (26). Define

$$v_N = v|_{\overline{U(N)}}$$

Then trivially $u = v_N$ is a viscosity solution of

$$\begin{aligned} Lu + F &= 0 \text{ in } U(N) \\ u &= v_N \text{ on } \partial U(N) \end{aligned}$$

Since $U(N)$ is bounded it follows by uniqueness and by a) that

$$v_N(x) = E^x[\int_0^{\tau_{U(N)}} F(X_t)dt + v_N(X_{\tau_{U(N)}})]$$

Since $v_N = v$ on $\overline{U(N)}$, $v = G$ on ∂U and $\tau_{U(N)} \to \tau_U < \infty$ a.s. as $N \to \infty$, we get

$$\begin{aligned} v(x) &= \lim_{N\to\infty} v_N(x) \\ &= \lim_{N\to\infty} E^x[\int_0^{\tau_{U(N)}} F(X_t)dt + v_N(X_{\tau_{U(N)}})] \\ &= E^x[\int_0^{\tau_U} F(X_t)dt + G(X_{\tau_U})] = w(x), \end{aligned}$$

by (26). This completes the proof of Proposition 3.1. □

We now proceed with the proof of the uniqueness of the viscosity solution of (16)-(17).

Lemma 3.1. *Let v be a viscosity supersolution of* (16). *Then*

$$v(y) \geq g(y) \text{ for all } y \in S$$

Proof. Fix $y \in S$ and let $V \subset S$ be a bounded neighbourhood of y. Then v has a minimum point $\underline{x}$ in $\overline{V}$. Hence we can find a function $\phi \in C^2(S)$ such that $\phi \leq v$ on S and $\phi(\underline{x}) = v(\underline{x})$. So by Definition 2.1, part b) we conclude that

$$v(\underline{x}) = \phi(\underline{x}) \geq g(\underline{x})$$

Since the neighbourhood V can be made arbitary small we conclude by continuity that

$$v(y) \geq g(y).$$

Since y is arbitary, the lemma is proved. □

We are now ready for the main result in this section. If v is a viscosity solution of (16) we define

$$A = A_v = \{x \in S; v(x) > g(x)\} \tag{28}$$

THEOREM 3.1 (Uniqueness). *Let v be a viscosity solution of the variational inequality*

$$\min\{-Lv(x) - f(x), v(x) - g(x)\} = 0 \quad \text{for all} \quad x \in S \tag{29}$$

$$v(x) = g(x) \text{ for all } x \in \partial S \tag{30}$$

with the property that

$$\{v(X_\tau)\}_{\tau \leq \tau_S} \text{ is } Q^x - \text{ uniformly integrable for all } x \in \overline{S}. \tag{31}$$

Assume that

$$\tau_A < \infty \ a.s\ Q^x \ for\ all\ x \in S \tag{32}$$

Then

$$v(x) = \Phi(x) \ for\ all\ x \in S.$$

Proof. We first observe that v is a viscosity solution of

$$Lv + f = 0 \text{ in } A \tag{33}$$

with the boundary values

$$v = g \text{ on } \partial A. \tag{34}$$

This is a direct consequence of Definition 2.1 combined with the definition (28). Hence by Proposition 3.2 b) we conclude that

$$v(x) = E^x[\int_0^{\tau_A} f(X_t)dt + g(X_{\tau_A})] \text{ for all } x \in A$$

Therefore,

$$v(x) \leq \Phi(x). \tag{35}$$

To get the opposite inequality define

$$S(N) = \{x \in S; |x| < N\}$$

and put

$$v_N = v|_{\overline{S(N)}}, \quad N = 1, 2, ...$$

Then v_N is a viscosity solution of

$$\min\{-Lv_N - f, v_N - g\} = 0 \text{ in } S(N) \tag{36}$$

and

$$v_N = v|_{\partial S(N)} \text{ on } \partial S(N). \tag{37}$$

Then by the comparison theorem on bounded sets (Theorem 3.3 in [2]) and Lemma 3.3 we conclude that

$$v_N \geq u_N \text{ in } S(N), \tag{38}$$

where u_N is the (unique) solution of (36) with the boundary condition

$$u_N = g \text{ on } \partial S(N) \tag{39}$$

Hence by Theorem 2.1 we have

$$v_N(x) \geq \sup_{\tau \leq \tau_{S(N)}} E^x[\int_0^{\tau} f(X_t)dt + g(X_\tau)]$$

Letting $N \to \infty$ we get

$$v(x) \geq \Phi(x). \tag{40}$$

Combining (35) and (40) we get the result. □

4 An Example

Theorem 3.4 is useful as a verification theorem for optimal stopping problems. We illustrate this with the following example:
Let a, c, ρ be positive numbers such that

$$\frac{4\rho a}{1+2\rho a^2} < c < \frac{1}{a} \tag{41}$$

Define

$$h(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 1 - cx & \text{for } 0 < x < a \\ 1 - ca & \text{for } a \leq x \end{cases}$$

Consider the optimal stopping problem

$$\Phi(s,w) = \sup_{\tau \geq 0} E^{s,w}[e^{-\rho(s+\tau)} h(W_\tau)] \tag{42}$$

where $\rho > 0$ is constant. This is a problem of the form (1) with $S = \boldsymbol{R}^2$, $T = \infty$, $f = 0$,

$$dX_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dW_t; \quad X_0 = x = (s,w)$$

and

$$g(x) = g(s,w) = e^{-\rho s} h(w).$$

In this case the generator L of X_t is

$$L = \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial w^2}$$

and therefore the variational inequality for (42) is

$$\min\{-L\Phi(x), \Phi(x) - g(x)\} = 0 \text{ for all } x \in \boldsymbol{R}^2 \tag{43}$$

We now use some intuition to *guess* that the continuation region has the form

$$D = \{(t,w), 0 < w < w_1\}$$

for some $w_1 > 0$. In the view of (43) the value function Φ should solve the equation

$$L\Phi = 0 \text{ for } 0 < w < w_1 \tag{44}$$

We guess that Φ is of the form

$$J(s,w) = e^{-\rho s} K(w)$$

which substituted in (44) gives

$$\frac{1}{2}K''(w) - \rho K(w) = 0 \text{ for } 0 < w < w_1$$

The general solution of this equation is

$$K(w) = C_1 e^{\sqrt{2\rho}w} + C_2 e^{-\sqrt{2\rho}w}; 0 < w < w_1$$

We now determine C_1, C_2 and w_1 by using the following 3 equations:

(*Continuity at* $w = 0$) $K(0) = 1$, i.e.

$$C_1 + C_2 = 1 \tag{45}$$

(*Continuity at* $w = w_1$) $K(w_1) = 1 - ca$, i.e.

$$C_1 e^{\sqrt{2\rho}w_1} + C_2 e^{-\sqrt{2\rho}w_1} = 1 - ca \tag{46}$$

(C^1 *at* $w = w_1$) $K'(w_1) = 0$, i.e.

$$\sqrt{2\rho}C_1 e^{\sqrt{2\rho}w_1} - \sqrt{2\rho}C_2 e^{-\sqrt{2\rho}w_1} = 0 \tag{47}$$

By combining (45)-(47) we get

$$C_1 = \frac{1}{2}(1 - \sqrt{ca(2 - ca)}) > 0, \quad C_2 = 1 - C_1 > 0$$

and

$$w_1 = \frac{1}{\sqrt{2\rho}} \ln(\frac{1 - ca}{2C_1}) > 0.$$

We conclude that with these values of C_1, C_2 and w_1, our candidate for the value function Φ is

$$J(w) = e^{-\rho s} K(w), \tag{48}$$

where

$$K(w) = \begin{cases} 1 & \text{for } w \leq 0 \\ C_1 e^{\sqrt{2\rho}w} + C_2 e^{-\sqrt{2\rho}w} & \text{for } 0 < w < w_1 \\ 1 - ca & \text{for } w_1 \leq w \end{cases} \tag{49}$$

The optimal stopping time τ^* corresponding to this value function is

$$\tau^* = \tau_D = \inf\{t > 0; W_t \notin (0, w_1)\} \tag{50}$$

To check if this candidate actually is the value function, it suffices to verify that it is a viscosity solution of (43). First we note that since

$$\begin{aligned}\lim_{w\to 0+} K'(w) &= \sqrt{2\rho}(C_1 - C_2) \\ &= -\sqrt{2\rho ac(2-ac)} > -c \\ &= \lim_{w\to 0+} h'(w)\end{aligned}$$

(by (41)), we have

$$J(x) > g(x) \iff x \in D.$$

a) We first verify that J is a viscosity *subsolution* of (43). By our construction of J it is enough to verify (18) at the points $y_0 = (s,0)$ and $y_0 = (s,w_1)$. Since $J(y_0) = g(y_0)$ at these points, it is trivial that (18) holds when $\psi(y_0) = g(y_0)$.

b) Next we verify that J is a viscosity *supersolution* of (43) and again it is enough to verify this at $y_0 = (s,0)$ and $y_0 = (s,w_1)$.

(i) If $y_0 = (s,0)$ then no $\phi \in C^2(\boldsymbol{R}^2)$ exists such that $\phi(s,0) = J(s,0) = 1$ and $\phi \leq J$ on $\boldsymbol{R}^2$, because

$$\lim_{w\to 0+} \frac{\partial J}{\partial w}(s,w) = e^{-\rho s}\sqrt{2\rho ac(2-ca)} < 0 = \lim_{w\to 0-} \frac{\partial J}{\partial w}(s,w).$$

Hence (19) holds trivially.

(ii) Let $y_0 = (s,w_1)$ and choose $\phi \in C^2(\boldsymbol{R}^2)$ such that $\phi(y_0) = J(y_0) = 1 - ca$ and $\phi \leq J$ on $\boldsymbol{R}^2$. Then the map

$$(s,w) \to H(s,w) := \phi(s,w) - J(s,w);\, s \in \boldsymbol{R}, w \leq w_1$$

has a local maximum at all points (s,w_1) and therefore by the first order conditions

$$\frac{\partial \phi}{\partial s}(s,w_1) = \frac{\partial J}{\partial s}(s,w_1) \quad \text{and} \quad \frac{\partial \phi}{\partial w}(s,w_1) \geq \frac{\partial J}{\partial w}(s,w_1) \tag{51}$$

and by the second order conditions

$$\frac{\partial^2 \phi}{\partial w^2}(s,w_1) \leq \lim_{w\to w_1^-} \frac{\partial^2 J}{\partial w^2}(s,w). \tag{52}$$

Since $LJ(s,w) = 0$ for $0 < w < w_1$ it follows from (51) and (52) that

$$L\phi(s,w_1) \leq 0$$

and hence (19) holds at $y_0 = (s,w_1)$.

We conclude that our candidate (48) -(49) is indeed a viscosity solution of (43) and hence it is the value function of the optimal stopping problem (42). Moreover, the optimal stopping time τ^* is given by (50).

Note that in this example the value function is not C^1 at $w = 0$, so the "high contact" or "smooth fit" principle does not hold, moreover the solution could not have been found by using the verification theorem in [1].

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