Viscosity solutions to delay differential equations in demo-economy

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Abstract

Economic and demographic models governed by linear delay differential equations are expressed as optimal control problems in infinite dimensions. A general objective function is considered and the concavity of the Hamiltonian is not required. The value function is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation and a verification theorem is proved.

Key words: viscosity solutions, delay differential equation, vintage models.

1 Introduction

Fabbri et al. (to appear) study a family of optimal control problems driven by delay differential equations using strong solutions. Here I treat a larger class of economic and demographic problems, written as optimal control problems with delay state equation, using viscosity solutions. I use an equivalent formulation of the delay problem introducing a suitable Hilbert space and re-writing the state equation as a suitable ordinary differential equation[1] (ODE) in the Hilbert space.

Models in epidemiology and in dynamic population governed by linear delay differential equations for which a formulation in Hilbert spaces is possible are presented in Section 2. I will use a demographic model with an explicit age structure by Boucekkine et al. (2002), a vintage capital model with linear production function (AK) by Boucekkine et al. (2005)[2], a model for obsolescence and depreciation with linear production function by Boucekkine et al.

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[1]The method I use is due to Vinter and Kwong (1981) and Delfour (1986, 1980, 1984). I refer to the book by Bensoussan et al. (1992) for a systematic presentation.

[2]The model by Boucekkine et al. (2005) was also studied by Fabbri and Gozzi (submitted) using dynamic programming.

(2004) and an advertising model with delay effects by Gozzi and Marinelli (2004), Gozzi et al. (preprint), Faggian and Gozzi (2004).

I recall[3] that dynamic programming consists of four steps: (i) write the dynamic programming principle for the value function and its infinitesimal version, the HJB equation, (ii) solve the HJB equation and prove that the solution is the value function, (iii) prove a verification theorem which can involve the value function and which gives the optimal control as a function of the state finding the closed loop, (iv) solve the closed loop equation if possible, obtained after inserting the closed loop in the state equation.

The difference between Fabbri et al. (to appear) and the present work is the different study of the HJB equation. Fabbri et al. (to appear) solved the HJB equation by approximation, introducing a sequence of more regular problems that converges to the original one (Faggian, 2005*a*, *b*; Faggian and Gozzi, 2004). Here I study the existence of viscosity solutions for the HJB equation. Viscosity solutions in HJB equation allows one to avoid the concavity assumption of the Hamiltonian and of the target. Problems with multiple optimal solutions, [4] where the value function is not everywhere differentiable, are also tractable. Moreover, I do not require that the control and the state are de-coupled in the objective function (see Subsection 3.2). A verification result represents a key step in dynamic programming because it verifies whether a given admissible control is optimal or not and gives a way to construct optimal feedback controls.

On viscosity solution I have recalled that a crucial step in dynamic programming is to solve the associated HJB equation. Such a solution is used to find optimal controls in a closed-loop form. There are many definitions of solutions of a partial differential equations and in particular of the HJB equation related to optimal control problems. Which one shall we choose? In the classical works Fleming and Rishel (1975) use a regular solution: the solution of the HJB equation is a regular (C^1) function which satisfies the equation pointwise. However the solution of the HJB equation is often neither C^1 nor differentiable. Crandall and Lions (1983) defined viscosity solutions of the HJB equation in finite dimension. The idea is that the solution can be less regular, for example continuous, and the solution uses sub and super differential or test functions. Every regular solution of the HJB equation is also a viscosity solution. Many HJB equations admit viscosity solutions but no classical solutions. Under general hypotheses, in the finite dimensional case, the HJB equation related to an optimal control problem admits a unique viscosity solution which is exactly the value function of the problem. Viscosity solutions can be used to check results and to solve optimal control problems. The infinite dimensional case is more complex and the literature is scarce.

[3] A more detailed description of the method is in Fabbri et al. (to appear).

[4] I refer to Deissenberg et al. (2004) for a bibliography of such problems in economics.

The viscosity method, introduced in the study of the finite dimensional HJ equation by Crandall and Lions (1983) was extended to the infinite dimensional case (Crandall and Lions, 1985, 1986*a*, *b*, 1990, 1991, 1994*a*, *b*). Other variants of the concept of viscosity solutions of HJB equations in Hilbert spaces are given by Ishii (1993) and Tataru (1992*a*, *b*, 1994).

In partial differential equation (PDE) *with boundary control* there is no complete theory but some works on specific PDE adapting the ideas and techniques of viscosity solutions for first order HJB equations (Cannarsa et al., 1991, 1993; Cannarsa and Tessitore, 1994, 1996*a*, *b*; Gozzi et al., 2002; Fabbri, submitted). Most of these works treat the case in which the generator of the semigroup appearing in the state equation is self-adjoint.

Infinite dimensional HJB equations arising from delay differential equations (DDEs) *with delay in the control* present an unbounded term similar to the one arising in boundary control problems (Fabbri and Gozzi, submitted; Fabbri et al., to appear, use classical and strong[5] solutions). These papers do not cover the case presented here.

2 Demo-economic models

Linear delay differential equations (LDDEs) model many phenomena in epidemics (Hethcote and van den Driessche, 1995, 2000; Smith, 1983; Waltman, 1974) and in biomedicine (Bachar and Dorfmayr, 2004; Culshaw and Ruan, 2000; Luzyanina et al., 2004). A review on delay differential equations in biosciences is in Bocharova and Rihanb (2000) and Baker et al. (1999).

2.1 Three examples

Three economic models will help us to understand which assumptions can be the right one.

2.1.1 A vintage capital model with linear production function (AK)

The growth model with vintage capital and linear production function presented by Boucekkine et al. (2005) is based on the following accumulation for capital goods

$$k(s) = \int_{s-R}^{s} i(\tau)\mathrm{d}\tau$$

where $i(\tau)$ is the investment at time τ. Capital goods are accumulated for length R of time (scrapping time) and then dismissed. Investments are differentiated with respect to their ages. The production function is linear:

$$y(s) = ak(s)$$

[5] A strong solution is a suitable limit of classical solutions of approximating problems.

for some constant $a > 0$ where $y(s)$ is the output at time s. At every time s the planner splits the production into consumption $c(s)$ and investment in new capital $i(s)$:

$$y(s) = c(s) + i(s),$$

then the state equation is

$$\dot{k}(s) = i(s) - i(s - R), \quad s \in [0, +\infty)$$

which is a linear delay differential equation. The social planner maximizes the function

$$\int_0^{+\infty} e^{-\rho s} \frac{c(s)^{1-\sigma}}{1-\sigma} \mathrm{d}s = \int_0^{+\infty} e^{-\rho s} \frac{(ak(s) - i(s))^{1-\sigma}}{1-\sigma} \mathrm{d}s \tag{1}$$

Investment and consumption at time s must not be negative:

$$i(s) \geq 0, \qquad c(s) \geq 0, \quad \forall s \in [t, T] \tag{2}$$

The admissible set has the form:

$$\mathcal{A} \stackrel{\text{def}}{=} \{i(\cdot) \in L^2_{\text{loc}}([0, +\infty), \mathbb{R}) : \; 0 \leq i(s) \leq ak(s) \;\text{ a.e. in } [0, +\infty)\}.$$

where $L^2_{\text{loc}}([0, +\infty), \mathbb{R})$ is the space of all functions from $[0, +\infty)$ to $\mathbb{R}$ that are Lebesgue measurable and square integrable on all bounded intervals.

2.1.2 An advertising model with delay effects

Gozzi et al. (preprint) and Gozzi and Marinelli (2004) in the stochastic case and Faggian and Gozzi (2004) in the deterministic case (Feichtinger et al., 1994, and references therein) studied the following advertising model.

Let $t \geq 0$ be an initial time, $T > t$ a terminal time ($T < +\infty$ here), $\gamma(s)$, with $0 \leq t \leq s \leq T$, the stock of advertising goodwill[6] of the product to be launched. The dynamics is given by the following controlled delay differential equation (DDE) with delay $R > 0$ where z is the spending in advertising:

$$\begin{cases} \dot{\gamma}(s) = a_0 \gamma(s) + \int_{-R}^0 \gamma(s + \xi) da_1(\xi) + b_0 z(s) + \int_{-R}^0 z(s + \xi) db_1(\xi), \\ \\ \gamma(t) = x; \quad \gamma(\xi) = \theta(\xi), \; z(\xi) = \delta(\xi) \;\; \forall \xi \in [t - R, t], \end{cases} \tag{3}$$

for $s \in [t, T]$, with the assumptions:

- a_0 is a constant factor of image deterioration in absence of advertising, $a_0 \leq 0$;

[6] The advertising goodwill measurement reflects a "stock of information" from current and past advertising that currently influences demand. It was first introduced by Nerlone and Arrow (1962).

- $a_1(\cdot)$ is the distribution of oblivion time, $a_1(\cdot) \in L^2([-R, 0]; \mathbb{R})$;
- b_0 is a constant advertising efficiency factor, $b_0 \geq 0$;
- $b_1(\cdot)$ is the density function of the time lag between the advertising expenditure z and the corresponding effect on the goodwill level, $b_1(\cdot) \in L^2([-R, 0]; \mathbb{R}_+)$;
- x is the level of goodwill at the beginning of the advertising campaign, $x \geq 0$;
- $\theta(\cdot)$ and $\delta(\cdot)$ are respectively the goodwill and the spending rate at the beginning, $\theta(\cdot) \geq 0$, with $\theta(0) = x$, and $\delta(\cdot) \geq 0$.

The objective function is

$$J(t, x; z(\cdot)) = \varphi_0(\gamma(T)) + \int_t^T h_0(z(s))\, ds. \tag{4}$$

where $\varphi_0(\cdot)$ and $h_0(\cdot)$ are continuous functions.

2.1.3 A model for obsolescence and depreciation

Boucekkine et al. (in preparation) presented a model of obsolescence and depreciation with linear production function. The production net of maintenance and repair costs $y(t)$ satisfies the delay differential equation:

$$y(t) = \int_{t-R}^{t} (\Omega e^{-\delta(t-s)} - \eta) i(s) \mathrm{d}s \tag{5}$$

where Ω, η and δ are real positive constants and $\eta = e^{-\delta T}\Omega$. The control variable is given by the investment $i(s)$, $0 \leq i(s) \leq y(s)$. The planner maximizes the function

$$\int_0^{+\infty} e^{-\rho s} \frac{(y(s) - i(s))^{1-\sigma}}{1-\sigma} \mathrm{d}s \tag{6}$$

for a positive constant σ and a discount factor ρ.

Boucekkine et al. (1997, 2001) treat these problems numerically.

2.2 Demographic applications

Boucekkine et al. (2004) consider a demographic model with an explicit age structure. At any time t, $h(v)$ is the human capital of the cohort born at v, $v \leq t$. $T(t)$ is the time spent at school so $t - T(t)$ is the last cohort which entered the job market at t. $A(t)$ is the maximal age attainable, $t - A(t)$ is the last cohort still at work. The aggregate stock of human capital available at time t is:

$$H(t) = \int_{t-A(t)}^{t-T(t)} h(v)e^{nv}m(t-v)\mathrm{d}v$$

where n is the population growth rate, e^{nv} the cohort size born at v, and $m(t-v)$ is the probability for an individual born at v to be alive at t. Boucekkine et al. (2002) study the case in which $A(t)$ and $T(t)$ are constant.

3 The Problem

3.1 The delay state equation

From now on I consider a fixed delay $R > 0$. With notation from Bensoussan et al. (1992), given $T > t \geq 0$ and $z \in L^2([t-R,T],\mathbb{R})$ for every $s \in [t,T]$ $z_s \in L^2([-R,0];\mathbb{R})$ is the function

$$\begin{cases} z_s \colon [-R,0] \to \mathbb{R} \\ z_s(r) \overset{\text{def}}{=} z(s+r). \end{cases} \tag{7}$$

Given an admissible control $u(\cdot) \in L^2(t,T)$, consider the delay differential equation:

$$\begin{cases} \dot{y}(s) = N(y_s) + B(u_s) + f(s) \qquad \text{for } s \in [t,T] \\ (y(t), y_t, u_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2([-R,0];\mathbb{R}) \times L^2([-R,0];\mathbb{R}) \end{cases} \tag{8}$$

where y_t and u_t are interpreted by means of Eq. (7).

$$N, B \colon C([-R,0],\mathbb{R}) \to \mathbb{R}. \tag{9}$$

In particular:

Hypothesis 3.1. $N, B \colon C([-R,0],\mathbb{R}) \to \mathbb{R}$ *are continuous linear functions.*

In the delay setting the initial data are a triple (ϕ^0, ϕ^1, ω) where ϕ^0 is the state at the initial time t, ϕ^1 is the history of the state and ω the history of the control up to time t on the interval $[t-R,t]$. In the following $f \equiv 0$.

Eq. (8) includes our three examples, namely:

- In Boucekkine et al. (2005), Fabbri and Gozzi (submitted), $N = 0$ and $B = \delta_0 - \delta_R$ so the state equation is

$$k(s) = \int_{s-R}^{s} i(r)\mathrm{d}r \tag{10}$$

- In Gozzi et al. (preprint), Gozzi and Marinelli (2004) the definitions of N and B are respectively

$$\begin{aligned} N &\colon C([-R,0]) \to \mathbb{R} \\ N &\colon \gamma \mapsto a_0\gamma(0) + \textstyle\int_{-R}^{0} \gamma(r) da_1(r) \end{aligned} \tag{11}$$

$$\begin{aligned} B &\colon C([-R,0]) \to \mathbb{R} \\ B &\colon \gamma \mapsto b_0\gamma(0) + \textstyle\int_{-R}^{0} \gamma(r) db_1(r) \end{aligned} \tag{12}$$

- In Boucekkine et al. (in preparation) $N = 0$ and

$$\begin{aligned} B &\colon C([-R,0]) \to \mathbb{R} \\ B &\colon \gamma \mapsto (\Omega - \eta)\gamma(0) - \delta\Omega \textstyle\int_{-R}^{0} e^{\delta r}\gamma(r)\mathrm{d}r \end{aligned} \tag{13}$$

Proposition 3.2. *Given an initial condition $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0)$, a control $u \in L^2_{\mathrm{loc}}[0,+\infty)$ and a function $f \in L^2([0,T]\mathbb{R})$ there exists a unique solution $y(\cdot)$ of Eq. (8) in $H^1_{\mathrm{loc}}[0,\infty)$. Moreover for all $T > 0$ there exists a constant $c(T)$ depending only on $R, T, \|N\|$ and $\|B\|$ such that*

$$|y|_{H^1(0,T)} \leq c(T)\Big(|\phi^0| + |\phi^1|_{L^2(-R,0)} + |\omega|_{L^2(-R,0)} + |u|_{L^2(0,T)} + |f|_{L^2(0,T)}\Big). \tag{14}$$

Proof. In Bensoussan et al. (1992) Theorem 3.3 page 217 for the first part and Theorem 3.3 page 217, Theorem 4.1 page. 222 and page 255 for the second statement. □

3.2 The target functional

I consider a target functional to be maximized, of the form

$$\int_t^T L_0(s, y(s), u(s))\mathrm{d}s + h_0(y(T)) \tag{15}$$

where

$$L_0 \colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

and

$$h_0 \colon \mathbb{R} \to \mathbb{R}$$

are continuous functions.

- In Boucekkine et al. (2005); Fabbri and Gozzi (submitted) the time horizon is infinite and the objective functional was constant relative risk-aversion (CRRA):

$$\int_0^{+\infty} \frac{(Ak(s) - i(s))^{1-\sigma}}{1-\sigma}\mathrm{d}s \tag{16}$$

- In Boucekkine et al. (in preparation) the functional is constant relative risk-aversion:

$$\int_0^{+\infty} \frac{(y(s)-i(s))^{1-\sigma}}{1-\sigma} \mathrm{d}s. \tag{17}$$

- In Faggian and Gozzi (2004) the functional is concave and of the form:

$$\int_t^T l_0(s,c(s)) + n_0(s,y(s))\mathrm{d}s + m_0(y(T)). \tag{18}$$

The generality of the objective functional is one of the improvements due to viscosity solutions. Fabbri et al. (to appear) considered only objective functionals of the form

$$\int_t^T e^{-\rho s} l_0(c(s))\mathrm{d}s + m_0(y(T)) \tag{19}$$

where l_0 and m_0 are concave, and the utility function l_0 depends only on consumption (that is the control) c.

3.3 Constraints

To define the optimization problem we specify the set of admissible trajectories. In the examples a lower bound on the control variable is assumed. In Boucekkine et al. (2005), Fabbri and Gozzi (submitted), Boucekkine et al. (in preparation), the constraint $u \geq 0$ is assumed. Here the constraint is more general:

$$u \geq \Gamma_-(y) \tag{20}$$

where $\Gamma_-\colon \mathbb{R} \to (-\infty, 0]$ is continuous.

In Boucekkine et al. (2005), Fabbri and Gozzi (submitted) the investment i cannot be greater than the production $ak(t)$, in Boucekkine et al. (in preparation) $i \leq y$. Here I impose

$$u \leq \Gamma_+(y) \tag{21}$$

where $\Gamma_+\colon \mathbb{R} \to [0, +\infty)$ is a continuous function. In Boucekkine et al. (2005), Fabbri and Gozzi (submitted) $\Gamma_+(y) = Ay$, in Boucekkine et al. (in preparation) $\Gamma_+(y) = y$.

The three main components of an optimal control problem are the state equation, the target functional and the constraints.

- The state equation is a general homogeneous linear DDE, in which the derivative of the state y depends both on the history of the state y_s (where y_s means the history of y in the interval $[s-R, s]$) and on the history of the control u_s. y_s and u_s are defined as in Eq. (7):

$$\begin{cases} y_s\colon [-R, 0] \to \mathbb{R} \\ y_s(r) \overset{\text{def}}{=} y(s+r). \end{cases} \tag{22}$$

and the same for u_s. The presence of the delay in the control yields an unbounded term. In our state equation as reformulated in M^2 a non-analytic semigroup appears. Fabbri (submitted) treats viscosity solution of HJB equation with boundary term and with non-analytic semigroup but only on a very specific transport partial differential equation.

- There are state-control constraints.
- The target functional is of the form

$$\int_t^T L_0(s, y(s), u(s))\mathrm{d}s + h_0(y(T)) \tag{23}$$

where L_0 and h_0 are continuous. In Boucekkine et al. (2005), Fabbri and Gozzi (submitted) and Fabbri (to appear) the utility function is constant relative risk-aversion; in Fabbri et al. (to appear) it is concave.

4 The problem in Hilbert spaces

I recall how to rewrite the state equations of a control problem subject to a DDE as a control problem subject to an ordinary differential equation (ODE) in a suitable Hilbert space (Chapter 4 of Bensoussan et al., 1992).

I use the following notations:

- $y(\cdot)$ is the solution of the delay differential Eq. (8).
- (ϕ^0, ϕ^1, ω) is the initial datum in the delay differential Eq. (8).
- $x(\cdot)$ is the trajectory in the Hilbert space $M^2 = \mathbb{R} \times L^2[-R, 0]$ and is solution of the differential equation (28). $x^0(\cdot) = y(\cdot)$.
- $\langle a, b\rangle_{\mathbb{R}} = ab$ is the product in $\mathbb{R}$ of two real numbers $a, b \in \mathbb{R}$.
- $\langle \cdot, \cdot \rangle_{L^2}$ will indicate the scalar product in $L^2(-R, 0)$: if $\phi^1 \in L^2$ and $\psi^1 \in L^2$ the scalar product is defined as

$$\left\langle \phi^1, \psi^1 \right\rangle_{L^2} = \int_{-R}^0 \phi^1(r)\psi^1(s)\mathrm{d}s. \tag{24}$$

- The brackets $\langle \cdot, \cdot \rangle$ without index will indicate the scalar product in M^2: if $\phi = (\phi^0, \phi^1) \in M^2$ and $\psi = (\psi^0, \psi^1) \in M^2$ the scalar product is defined as

$$\langle \phi, \psi \rangle = \phi^0\psi^0 + \left\langle \phi^1, \psi^1 \right\rangle_{L^2}. \tag{25}$$

- The brackets $\langle \cdot, \cdot \rangle_{X \times X'}$ is the duality pairing between a space X and the dual X'.

- The symbol $|y|_X$ means the norm of the element y in the Banach space X.
- $\|T\|$ is the operator norm of the operator T.
- $C^1([0,T]\times M^2)$ is the set of the continuously differentiable functions $\varphi\colon [0,T]\times M^2 \to \mathbb{R}$.
- If $\varphi \in C^1([0,T]\times M^2)$ $\partial_t\varphi(t,x)$ is the partial derivative with respect to t and $\nabla\varphi(t,x)$ the differential with respect to the state variable $x \in M^2$.

Consider L the linear operator defined in Subsection 8. Under Hypothesis 3.1

Proposition 4.1. *The operator A^* defined as:*

$$\begin{cases} D(A^*) = \left\{(\phi^0,\phi^1)\in M^2 \;:\; \phi^1 \in W^{1,2}(-R,0) \text{ and } \phi^0 = \phi^1(0)\right\} \\ A^*(\phi^0,\phi^1) = (L\phi^1, D\phi^1) \end{cases} \tag{26}$$

is the generator of a C_0 semigroup on the Hilbert space $M^2 \stackrel{\text{def}}{=} \mathbb{R}\times L^2([-R,0];\mathbb{R})$

Proof. See Bensoussan et al. (1992) Chapter 4. □

From the form of $D(A^*)$ the operator B is the linear continuous functional

$$\begin{cases} B\colon D(A^*) \to \mathbb{R} \\ B\colon (\varphi^0,\varphi^1) \mapsto B(\varphi^1) \end{cases} \tag{27}$$

where $D(A^*)$ is endowed with the graph norm.[7] In the following B has this second definition. The adjoints of A^* and B are respectively A and B^*.

Eq. (8) is included into the following ordinary differential equation in the Hilbert space M^2

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}x(s) = Ax(s) + B^*z(s) \\ x(t) = x. \end{cases} \tag{28}$$

Indeed Eq. (28) admits a unique solution $x(\cdot)$ over a suitable subset of $C([0,T];M^2)$. This solution is a couple $x(s) = (x^0(s), x^1(s)) \in \mathbb{R}\times L^2(-R,0)$,[8] where $x^0(s)$ is the unique absolutely continuous solution $y(s)$

[7]For $x\in D(A^*)$ the graph norm $|x|_{D(A^*)}$ is defined as

$$|x|_{D(A^*)} = |x|_{M^2} + |A^*x|_{M^2}.$$

[8]I will write

$$x(s)_{u(\cdot),t,x} = (x^0_{u(\cdot),t,x}(s), x^1_{u(\cdot),t,x}(s))$$

to emphasize the dependence on the control and on initial data.

of Eq. (8) and x^1 a suitable transformation of the histories of the state y and of the control u (Fabbri et al., to appear, and Appendix A).

In the next hypothesis I formalize this state-control constraint $u \in [\Gamma_-(y), \Gamma_+(y)]$:

Hypothesis 4.2. *With a control $u(\cdot)$ and the related state trajectory $x(\cdot) = (x^0(\cdot), x^1(\cdot))$ the state-control constraint is:*

$$\Gamma_-(x^0(s)) \leq u(s) \leq \Gamma_+(x^0(s)) \quad \forall s \in [t, T] \tag{29}$$

where Γ_- and Γ_+ are locally Lipschitz continuous functions

$$\begin{aligned} \Gamma_+ &: \mathbb{R} \to [0, +\infty) \\ \Gamma_- &: \mathbb{R} \to (-\infty, 0] \end{aligned} \tag{30}$$

and such that $|\Gamma_-(t)| \leq a + b|t|$ and $|\Gamma_+(t)| \leq a + b|t|$ for two positive constants a and b.

The set of admissible controls is

$$\mathcal{U}_{t,x} \stackrel{\text{def}}{=} \{u(\cdot) \in L^2(t, T) \ : \ \Gamma_-(x^0_{u(\cdot),t,x}(s)) \leq u(s) \leq \Gamma_+(x^0_{u(\cdot),t,x}(s))\} \tag{31}$$

The target functional in Eq. (15) written in the new variables is

$$\int_t^T L_0(s, x^0(s), u(s))\mathrm{d}s + h_0(x^0(T)).$$

Hence

$$J(t, x, u(\cdot)) = \int_t^T L(s, x(s), u(s))\mathrm{d}s + h(x(T)) \tag{32}$$

where

$$\begin{cases} L \colon [0, T] \times M^2 \times \mathbb{R} \to \mathbb{R} \\ L \colon (s, x, u) \mapsto L_0(s, x^0, u) \end{cases} \tag{33}$$

$$\begin{cases} h \colon M^2 \to \mathbb{R} \\ h \colon x \mapsto h_0(x^0) \end{cases} \tag{34}$$

and L and h are continuous functions. Moreover I ask that

Hypothesis 4.3. *L and h are uniformly continuous and*

$$|L(s, x, u) - L(s, y, u)| \leq \sigma(|x - y|) \quad \textit{for all } (s, u) \in [0, T] \times \mathbb{R} \tag{35}$$

where σ is a modulus of continuity.[9]

The original optimization problem is equivalent to the optimal control problem in M^2 with state equation (28) and target functional given by Eq. (32).

[9]A continuous positive function such that $\sigma(r) \to 0$ for $r \to 0^+$.

Lemma 4.4. *Under Hypothesis (4.2) and given an initial datum* $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R, 0) \times L^2(-R, 0)$ *then Eq. (8) has a unique solution* $y(\cdot)$ *in* $H^1(t, T)$*. It is bounded in the interval* $[t, T]$ *uniformly in the control* $u(\cdot) \in \mathcal{U}_{t,x}$ *and in the initial time* $t \in [0, T)$*. Let* K *be a constant such that* $|y(s)| \leq K$ *for any* $t \in [0, T)$*, any control* $u(\cdot) \in \mathcal{U}_{t,x}$ *and any* $s \in [t, T]$*.*

Proof. In Appendix A. □

Remark 4.5. *Hypothesis (4.2) implies that* $u(s) \leq a + bK$ *for all controls in* $\mathcal{U}_{t,x}$*.*

Lemma 4.6. *Under Hypothesis (4.2) the solution* $x(s)$ *of Eq. (28) satisfies*

$$|x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0 \tag{36}$$

uniformly in (t, x) *and in the control* $u(\cdot) \in \mathcal{U}_{t,x}$*.*

Proof. In Appendix A. □

The value function of the problem is defined as

$$V(t, x) = \sup_{u(\cdot) \in \mathcal{U}_{t,x}} J(t, x, u(\cdot)) \tag{37}$$

Proposition 4.7. *The value function* $V : [0, T] \times M^2 \to \mathbb{R}$ *is continuous.*

Proof. In Appendix A. □

5 Viscosity solutions for HJB equation

The HJB equation of the system is defined as

$$\begin{cases} \partial_t w(t, x) + \langle \nabla w(t, x), Ax \rangle + H(t, x, \nabla w(t, x)) = 0 \\ w(T, x) = h(x) \end{cases} \tag{38}$$

where H is defined as:

$$\begin{cases} H : [0, T] \times D(A^*) \to \mathbb{R} \\ H(t, x, p) \stackrel{\text{def}}{=} \sup_{u \in [\Gamma_-(x^0), \Gamma_+(x^0)]} \{uB(p) + L(t, x, u)\} \end{cases} \tag{39}$$

H is the *Hamiltonian* of the system.

5.1 Definition and preliminary lemma

Definition 5.1. *A function $\varphi \in C^1([0,T] \times M^2)$ is a test function and I write $\varphi \in \textsc{Test}$ if $\nabla\varphi(s,x) \in D(A^*)$ for all $(s,x) \in [0,T] \times M^2$ and $A^*\nabla\varphi\colon [0,T] \times M^2 \to \mathbb{R}$ is continuous. This means that $\nabla\varphi \in C([0,T] \times M^2; D(A^*))$ where $D(A^*)$ is endowed with the graph norm.*

Definition 5.2. *$w \in C([0,T] \times M^2)$ is a viscosity subsolution of the HJB equation (or simply a "subsolution") if $w(T,x) \le h(x)$ for all $x \in M^2$ and for every $\varphi \in \textsc{Test}$ and every local minimum point (t,x) of $w - \varphi$,*

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \le 0. \tag{40}$$

Definition 5.3. *$w \in C([0,T] \times M^2)$ is a viscosity supersolution of the HJB equation (or simply a "supersolution") if $w(T,x) \ge h(x)$ for all $x \in M^2$ and for every $\varphi \in \textsc{Test}$ and every local maximum point (t,x) of $w - \varphi$,*

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \ge 0. \tag{41}$$

Definition 5.4. *$w \in C([0,T] \times M^2)$ is a viscosity solution of the HJB equation if it is both a supersolution and a subsolution.*

Proposition 5.5. *Given $(t,x) \in [0,T] \times M^2$ and $\varphi \in \textsc{Test}$ there exists a real continuous function $O(s)$ such that $O(s) \xrightarrow{s\to t^+} 0$ and such that for every admissible control $u(\cdot) \in \mathcal{U}_{t,x}$*

$$\left| \frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} - \partial_t\varphi(t,x) - \langle A^*\nabla\varphi(t,x), x\rangle - \right.$$
$$\left. - \frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}} \,\mathrm{d}r}{s-t} \right| \le O(s) \tag{42}$$

(where $x(s)$ is the trajectory starting from x at time t and subject to the control $u(\cdot)$).

Moreover if $u(\cdot) \in \mathcal{U}_{t,x}$ is continuous in t

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \xrightarrow{s\to t^+}$$
$$\xrightarrow{s\to t^+} \partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \tag{43}$$

Proof. In Appendix A. □

$O(s)$ is independent of the control. This fact will be crucial when I prove that the value function is a viscosity supersolution of the HJB equation.

Corollary 5.6. *Given* $(t,x) \in [0,T] \times M^2$ *and* $\varphi \in \textsc{Test}$ *and an admissible control* $u(\cdot) \in \mathcal{U}_{t,x}$

$$\varphi(s,x(s)) - \varphi(t,x) = \\ = \int_t^s \partial_t\varphi(r,x(r)) + \langle A^*\nabla\varphi(r,x(r)), x(r)\rangle + \langle B(\nabla\varphi(r,x(r))), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r \tag{44}$$

where $x(s)$ *is the trajectory starting from* x *at time* t *and subject to the control* $u(\cdot)$.

5.2 The value function as viscosity solution of HJB equation

Proposition 5.7. (Bellman's optimality principle) *The value function* V, *defined in Eq. (37) satisfies:*

$$V(t,x) = \sup_{u(\cdot)\in\mathcal{U}_{t,x}} \left(V(s,x(s)) + \int_t^s L(r,x(r),u(r))\mathrm{d}r \right) \tag{45}$$

for all $s > t$ *where* $x(s)$ *is the trajectory at time* s *starting from* x *subject to control* $u(\cdot) \in \mathcal{U}_{t,x}$.

Proof. In Li and Yong (1995) Chapter 6. □

Theorem 5.8. *The value function* V *is a viscosity solution of the HJB equation.*

Proof. In Appendix A. □

I cannot give a uniqueness result for the viscosity solution of the HJB equation yet. It will be an issue for future work.

6 A verification result

Lemma 6.1. *Let* $f \in C([0,T])$. *Extend* f *to* g *on* $(-\infty,+\infty)$ *with* $g(t) = g(T)$ *for* $t > T$ *and* $g(t) = g(0)$ *for* $t < 0$. *Assume there is a* $\rho \in L^1(0,T;\mathbb{R})$ *such that*

$$\left| \liminf_{h\to 0} \frac{g(t+h)-g(t)}{h} \right| \le \rho(t) \quad \text{a.e. } t \in [0,T] \tag{46}$$

Then

$$g(\beta) - g(\alpha) \ge \int_\alpha^\beta \liminf_{h\to 0} \frac{g(t+h)-g(t)}{h}\mathrm{d}t \quad \forall\, 0 \le \alpha \le \beta \le T. \tag{47}$$

Proof. In Yong and Zhou (1999) page 270. □

I first introduce a set related to a subset of the subdifferential of a function in $C([0,T] \times M^2)$. Its definition is suggested by the definition of sub- and super- solutions.

Definition 6.2. *Given $v \in C([0,T] \times M^2)$ and $(t,x) \in [0,T] \times M^2$, $Ev(t,x)$ is defined as*

$$Ev(t,x) = \{(q,p) \in \mathbb{R} \times D(A^*) \ : \ \begin{array}{l} \exists \varphi \in \textsc{Test}, \text{ such that } v - \varphi \\ \text{ attains a local minimum in } (t,x), \\ \partial_t \varphi(t,x) = q, \ \nabla\varphi(t,x) = p, \\ \text{and } v(t,x) = \varphi(t,x)\} \end{array} \tag{48}$$

Moreover $Ev(t,x)$ is a subset of the subdifferential of v.

Theorem 6.3. *Let $(t,x) \in [0,T] \times M^2$ be an initial datum ($x(t) = x$). Let $u(\cdot) \in \mathcal{U}_{t,x}$ and $x(\cdot)$ be the corresponding trajectory. Let $q \in L^1(t,T;\mathbb{R})$, $p \in L^1(t,T;D(A^*))$ be such that*

$$(q(s), p(s)) \in EV(t, x_{t,y}(s)) \quad \text{for almost all } s \in (t,T) \tag{49}$$

Moreover if $u(\cdot)$ satisfies

$$\int_t^T \langle A^* p(s), x(s)\rangle_{M^2} + \langle Bp(s), u(s)\rangle_{\mathbb{R}} + q(s)\ \mathrm{d}s \geq$$
$$\geq \int_t^T -L(s,x(s),u(s))\ \mathrm{d}s, \tag{50}$$

then $u(\cdot)$ is an optimal control at (t,x).

Proof. In Appendix A. □

A Appendix

I use the following notation of Bensoussan et al. (1992). Given N and B two continuous linear functions

$$N, B \colon C([-R,0]) \to \mathbb{R}$$

of norms respectively $\|N\|$ and $\|B\|$ (as in Hypothesis (3.1)), $\mathcal{N}$ and $\mathcal{B}$ are the applications

$$\begin{aligned} \mathcal{N}, \mathcal{B} &\colon C_c((-R,T);\mathbb{R}) \to L^2(0,T) \\ \mathcal{N}(\phi) &\colon t \mapsto N(\phi_t) \\ \mathcal{B}(\phi) &\colon t \mapsto B(\phi_t) \end{aligned} \tag{51}$$

where ϕ_t has the meaning of Eq. (7), namely

$$\begin{cases} \phi_t \colon [-R,0] \to \mathbb{R} \\ \phi_t(r) \stackrel{\text{def}}{=} z(t+r). \end{cases} \tag{52}$$

Theorem A.1. *$\mathcal{N}, \mathcal{B}\colon C_c((-R,T);\mathbb{R}) \to L^2(0,T)$ have continuous linear extensions $L^2(-R,T) \to L^2(0,T)$ of norms $\le \|N\|$ and $\le \|B\|$.*

Proof. In (Bensoussan et al., 1992) Theorem 3.3, page 217. □

Definition A.2. *Let $a < b$ two real numbers, $\mathcal{F}(a,b)$ a set of functions from $[a,b]$ to $\mathbb{R}$. For each u in $\mathcal{F}(a,b)$ and all $s \in [a,b]$, define the functions $e_-^s u$ and $e_+^s u$ as*

$$
\begin{aligned}
e_-^s u\colon [a,+\infty) \to \mathbb{R}, \quad e_-^s u(t) &= \begin{cases} u(t) & t \in [a,s] \\ 0 & t \in (s,+\infty) \end{cases} \\
e_+^s u\colon (-\infty,b) \to \mathbb{R}, \quad e_+^s u(t) &= \begin{cases} 0 & t \in (-\infty,s] \\ u(t) & t \in (s,b]. \end{cases}
\end{aligned}
$$

Using the $\mathcal{N}$ and $\mathcal{B}$ notations, Eq. (8) is rewritten as

$$
\begin{cases} \dot{y}(t) = \mathcal{N}y + \mathcal{B}u + f \\ (y(0), y_0, u_0) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0). \end{cases} \tag{53}
$$

Using e_-^s and e_+s I decompose $y(\cdot)$ and $u(\cdot)$ as $y = e_+^0 y + e_+^0 \phi^1$ and $u = e_+^0 u + e_+^0 \omega$. I separate the solution $y(t)$, $t \ge 0$ and the control $u(t)$, $t \ge 0$ from the initial functions ϕ^1 and ω:

$$
\begin{cases} \dot{y}(t) = \mathcal{N}e_+^0 y + \mathcal{B}e_+^0 u + \mathcal{N}e_-^0 \phi^1 + \mathcal{B}e_-^0 \omega + f \\ y(0) = \phi^0 \in \mathbb{R} \end{cases} \tag{54}
$$

System (54) does not directly use the initial function ϕ^1 and ω but only the sum of their images $\mathcal{N}e_-^0 \phi^1 + \mathcal{B}e_-^0 \omega$. I introduce two operators

$$
\begin{cases} \overline{N}\colon L^2(-R,0) \to L^2(-R,0) \\ (\overline{N}\phi^1)(\alpha) \stackrel{\text{def}}{=} (\mathcal{N}e_-^0 \phi^1)(-\alpha) \qquad \alpha \in (-R,0) \end{cases}
$$

and

$$
\begin{cases} \overline{B}\colon L^2(-R,0) \to L^2(-R,0) \\ (\overline{B}\omega)(\alpha) \stackrel{\text{def}}{=} (\mathcal{B}e_-^0 \omega)(-\alpha) \qquad \alpha \in (-R,0) \end{cases}
$$

The operators $\overline{N}$ and $\overline{B}$ are continuous (Bensoussan et al., 1992).

$$
\mathcal{N}e_+^0 \phi^1(t) + \mathcal{B}e_+^0 \omega(t) = (e_+^{-R}(\overline{N}\phi^1 + \overline{B}\omega))(-t) \qquad \text{for } t \ge 0.
$$

Calling

$$
\xi^1 = (\overline{N}\phi^1 + \overline{B}\omega) \tag{55}
$$

and $\xi^0 = \phi^0$, Eq. (54) and then Eq. (8) are rewritten as

$$
\begin{cases} \dot{y}(t) = (\mathcal{N}e_+^0 y)(t) + (\mathcal{B}e_+^0 u)(t) + (e_+^{-R}\xi^1)(-t) + f(t) \\ y(0) = \xi^0 \in \mathbb{R} \end{cases} \tag{56}
$$

where $\mathbb{R} \times L^2(-R,0) \ni \xi \stackrel{\text{def}}{=} (\xi^0, \xi^1)$. Eq. (56) makes sense for all $\xi \in \mathbb{R} \times L^2(-R,0)$ also when ξ^1 is not of the form (55). I have embedded the original system (8) into a family of systems of the form (56).

I consider the case $f = 0$ from now on.

Definition A.3. *The* structural state $x(t)$ *at time* $t \geq 0$ *is defined by*

$$x(t) \stackrel{\text{def}}{=} (y(t), \overline{N}(e_+^0 y)_t + \overline{B}(e_+^0 u)_t + \Xi(t)\xi^1) \tag{57}$$

where $\Xi(t)$ *is the right translation operator defined as*

$$(\Xi(t)\xi^1)(r) = (e_+^{-R}\xi^1)(r-t) \qquad r \in [-R, 0]. \tag{58}$$

Proof of Lemma 4.4. The existence of a solution follows from Proposition 3.2. From Eq. (56), the solution of Eq. (8) is also the solution of

$$\begin{cases} \dot{y}(s) = N(e_+^t y)_s + B(e_+^t u)_s + (e_+^{-R}\xi^1)(-t) & \text{for } s \geq t \\ y(t) = \phi^0 \in \mathbb{R} \end{cases} \tag{59}$$

where $\xi^1 = (\overline{N}\phi^1 + \overline{B}\omega)$. Using Hypothesis (4.2), for every control $u(\cdot) \in \mathcal{U}_{t,x}$ and related trajectory $y(\cdot)$, the solution y_M of the ordinary differential equation

$$\begin{cases} \dot{y}_M(s) = \|N\| y_M(s) + \|B\|(a + b y_M(s)) + (e_+^{-R}\xi^1)(-t) & \text{for } s \geq 0 \\ y_M(0) = |\phi^0| \in \mathbb{R} \end{cases} \tag{60}$$

satisfies $|y(s)| \leq |y_M(s-t)|$ for all $s \in [t, T]$ and y_M is bounded on $[0, T]$. $\square$

Proof of Lemma 4.6. I prove that $|x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$, so it is enough to show that $|x^0(s) - x^0|_{\mathbb{R}} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$ and that $|x^1(s) - x^1|_{L^2} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$. The first fact is a corollary of the proof of Lemma 4.4 because $|x^0(s) - x^0| \leq y_M(s-t)$ defined in Eq. (60). Then, using the expression from Eq. (57):

$$\begin{aligned} \left|x^1(s) - x^1\right|_{L^2} &\leq \left|\Xi(s)x^1 - x^1\right|_{L^2} + \left|\overline{N}(e_+^0 y)_s\right|_{L^2} + \left|\overline{B}(e_+^0 u)_s\right|_{L^2} \leq \\ &\leq \left|\Xi(s)x^1 - x^1\right|_{L^2} + \|\overline{N}\|(s-t)^{\frac{1}{2}} K + \|\overline{B}\|(s-t)^{\frac{1}{2}}(a + Kb) \end{aligned} \tag{61}$$

where a and b are the constants of Hypothesis (4.2), K the constant of Lemma 4.4 and $\Xi(t)$ is the right translation operator defined in Eq. (58).

Moreover $\left|\Xi(s)x^1 - x^1\right|_{L^2} \xrightarrow{s \to 0} 0$ for the continuity of the translation with respect to the L^2 norm. This limit does not depend on the control. The other two terms of the right hand side of Eq. (61) are given by a constant multiplied by $(s-t)^{1/2}$ go to zero uniformly in the control. $\square$

Proof of Proposition 4.7. On $[0,T] \times M^2 \ni (t_n, x_n) \xrightarrow[\mathbb{R}\times M^2]{n\to\infty} (t,x)$, I have to estimate the terms

$$|V(t,x) - V(t,x_n)| \quad \text{ and } \quad |V(t_n,x_n) - V(t_n,x)|. \tag{62}$$

The difficulties are similar. Using arguments similar to those of Lemma 4.4[10] there exists a $M > 0$ such that, for every admissible control,

$$|x_n(s)| \le M \quad \text{ for every } s \in [t_n, T],\ n \in \mathbb{N}$$

in particular $|x_n^0(s)| \le M$. Under Hypothesis 4.2 the restrictions of Γ_+ and Γ_- in $[-M, M]$ are Lipschitz continuous for some Lipschitz constant Z. If $V(t,x) \ge V(t,x_n)$, I take an ε-optimal control $u^\varepsilon(\cdot)$ for $V(t,x)$. The problem is that $u^\varepsilon(\cdot)$ cannot be in the set $\mathcal{U}_{t,x_n}$. I approximate the control in feedback form:

$$u_n^\varepsilon(s) \stackrel{\text{def}}{=} \begin{cases} u^\varepsilon(s) & \text{if } u^\varepsilon(s) \in [\Gamma_-(x_{n\varepsilon}(s)), \Gamma_+(x_{n\varepsilon}(s))] \\ \Gamma_-(x_{n\varepsilon}(s)) & \text{if } u^\varepsilon(s) \in [\Gamma_-(x_n(s)), \Gamma_-(x_{n\varepsilon}(s))] \\ \Gamma_+(x_{n\varepsilon}(s)) & \text{if } u^\varepsilon(s) \in [\Gamma_+(x_{n\varepsilon}(s)), \Gamma_+(x_n(s))] \end{cases} \tag{63}$$

where $x_{n\varepsilon}(\cdot)$ is the solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} x_{n\varepsilon}(s) = A x_{n\varepsilon}(s) + B^* u_n^\varepsilon(s) \\ x_{n\varepsilon}(t) = x_n. \end{cases} \tag{64}$$

By definition $u^\varepsilon(s)$ is bounded, measurable, and in $L^2[0,T]$. I call $x_\varepsilon(\cdot)$ the solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} x_\varepsilon(s) = A x_\varepsilon(s) + B^* u^\varepsilon(s) \\ x_\varepsilon(t) = x. \end{cases} \tag{65}$$

and $y(\cdot) \stackrel{\text{def}}{=} x_\varepsilon(\cdot) - x_{n\varepsilon}(\cdot)$. By definition of $u_n^\varepsilon(\cdot)$

$$|u^\varepsilon(s) - u_n^\varepsilon(s)| \le Z|y^0(s)| \tag{66}$$

where $y^0(s)$ is the first component of $y(s)$. Moreover $y^0(\cdot)$ solves the following delay differential equation (using the notation of Eq. (56)):

$$\begin{cases} \dot{y}^0(s) = (\mathcal{N} e_+^0 y^0)(s) + (\mathcal{B} e_+^0 (u^\varepsilon(s) - u_n^\varepsilon))(s) + e_+^{-R}(x^1 - x_n^1)(-s) \\ y^0(t) = x^0 - x_n^0. \end{cases}$$

As in the proof of Lemma 4.4 and using Eq. (66) $|y^0(s)| \le y_M(s)|$ where y_M is the solution of the ordinary differential equation

$$\begin{cases} \dot{y}_M(s) = \|N\| y_M(s) + \|B\| y_M(s) + e_+^{-R}|x^1 - x_n^1|(-s) \\ y_M(t) = |x^0 - x_n^0| \end{cases}.$$

[10] Using the fact that $(e_+^{-R}\overline{N}\phi^1 + \overline{B}\omega)(\cdot)$ is continuous with respect to the initial data.

I have

$$y_M(s) = |x^0 - x_n^0| e^{(\|N\|+\|B\|)(s-t)} + \int_s^t e^{(\|N\|+\|B\|)(s-\tau)} e_+^{-R} |x^1 - x_n^1|(-\tau) \mathrm{d}\tau \leq$$
$$\leq C\|x - x_n\|_{M^2} \quad (67)$$

for all $s \in [t,T]$,

$$|x_\varepsilon^0(s) - x_{n\varepsilon}^0(s)| \leq C\|x - x_n\|_{M^2} \quad \text{for all } s \in [t,T]$$

and

$$|u^\varepsilon(s) - u_n^\varepsilon(s)| \leq ZC\|x - x_n\|_{M^2} \quad \text{for all } s \in [t,T]$$

Hence, by the uniform continuity of L

$$|L(s, x_\varepsilon^0(s), u^\varepsilon(s)) - L(s, x_{n\varepsilon}^0(s), u_n^\varepsilon(s)) \leq \sigma(\|x - x_n\|_{M^2}) \quad \text{for all } s \in [t,T]$$

For the continuity of h (using $\sigma(\cdot)$ for a generic modulus),

$$J(t, x, u^\varepsilon(\cdot)) - J(t, x_n, u_n^\varepsilon(\cdot)) \leq \sigma(\|x - x_n\|_{M^2})$$

and then

$$|V(t,x) - V(t,x_n)| = V(t,x) - V(t,x_n) \leq \varepsilon + \sigma(\|x - x_n\|_{M^2})$$

I conclude for the arbitrariness of ε. □

Proof of Proposition 5.5. I write

$$\frac{\varphi(s, x(s)) - \varphi(t,x)}{s-t} = I_t + I_0 + I_1 \stackrel{\text{def}}{=} \partial_t \varphi(\xi^t(s), \xi^x(s)) +$$
$$+ \left\langle \nabla\varphi(t,x), \frac{x(s) - x}{s-t} \right\rangle + \left\langle \nabla\varphi(\xi^t(s), \xi^x(s)) - \nabla\varphi(t,x), \frac{x(s)-x}{s-t} \right\rangle \quad (68)$$

where $[t,T] \times M^2 \ni \xi(s) = (\xi^t(s), \xi^x(s))$ is a point of the line segment connecting (t,x) and $(s, x(s))$. Thanks to Lemma 4.6, $|x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$, so $|\xi(s) - (t,x)|_{\mathbb{R}\times M^2} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$ and in particular

$$|\xi^x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0 \text{ uniformly in } u(\cdot) \in \mathcal{U}_{t,x} \quad (69)$$

and then

$$|\xi(s) - (t,x)|_{[t,T]\times M^2} \leq |s - t| + |\xi^x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0$$
$$\text{uniformly in } u(\cdot) \in \mathcal{U}_{t,x}. \quad (70)$$

By definition of the test function

$$\nabla\varphi\colon [0,T] \times M^2 \to D(A^*) \quad \text{and it is continuous.} \tag{71}$$

Then

$$|\nabla\varphi(\xi^t(s), \xi^x(s)) - \nabla\varphi(t,x)|_{D(A^*)} \xrightarrow{s\to t^+} 0 \tag{72}$$

uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$.

The state equation (28) can be extended (Faggian, 2001/2002) to an equation in $D(A^*)'$ of the form

$$\begin{cases} \dot{x}(s) = A^{(E)}x(s) + B^*u(s) \\ x(t) = x \end{cases} \tag{73}$$

where $A^{(E)}$ is an extension of A and, from Lemma 4.4 and Remark 4.5, $|B^*u(s)|_{D(A^*)'} \leq |B|_{D(A^*)'}|a + bK|$. The solution of Eq. (73) in $D(A^*)'$ is also (Pazy, 1983):

$$x(s) = e^{(s-t)A^{(E)}}x + \int_t^s e^{(s-r)A^{(E)}}B^*u(r)\mathrm{d}r. \tag{74}$$

Because $x \in X \subseteq D(A^{(E)})$ a constant C depending on x is chosen so as, for all admissible controls and all $s \in [t,T]$,

$$\frac{|x(s) - x|_{D(A^*)'}}{s-t} \leq C. \tag{75}$$

By Eqs. (72) and (75), $|I_1| \xrightarrow{s\to t^+} 0$ uniform in $u(\cdot) \in \mathcal{U}_{t,x}$. Thanks to the convergence $\xi(s) \to (t,x)$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$, $I_t = \partial_t\varphi(\xi^t(s), \xi^x(s)) \xrightarrow{s\to t^+} \partial_t\varphi(t,x)$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$. It remains to show that

$$\begin{aligned} &\Bigg| \frac{\langle \nabla\varphi(t,x), x(s) - x\rangle}{s-t} - \langle A^*\nabla\varphi(t,x), x\rangle - \\ &\qquad\qquad - \frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r}{s-t}\Bigg| = \\ &= \left|\left\langle \nabla\varphi(t,x), \left(\frac{x(s)-x}{s-t} - A^{(E)}x - \frac{\int_t^s B^*u(r)\mathrm{d}r}{s-t}\right)\right\rangle_{D(A^*)\times D(A^*)'}\right| \leq O(s) \end{aligned} \tag{76}$$

uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$.

From Eq. (74) $\frac{x(s)-x}{s-t}$ in $D(A^*)'$ is expressed explicitly as:

$$\frac{x(s)-x}{s-t} = \frac{(e^{(s-t)A^{(E)}} - \mathbb{1})x}{s-t} + \frac{\int_t^s e^{(s-r)A^{(E)}}B^*u(r)\mathrm{d}r}{s-t} \tag{77}$$

I need to estimate:

$$\left|\frac{x(s)-x}{s-t} - A^{(E)}(x) - \frac{\int_t^s B^*u(r)\mathrm{d}r}{s-t}\right|_{D(A^*)'} =$$
$$= \left|\frac{(e^{sA^{(E)}}-\mathbb{1})x}{s-t} - A^{(E)}(x) + \frac{\int_t^s \left(e^{(s-r)A^{(E)}}-\mathbb{1}\right) B^*u(r)\mathrm{d}r}{s-t}\right|_{D(A^*)'} \tag{78}$$

where the term $\frac{(e^{sA}-\mathbb{1})x}{s-t} - A^{(E)}(x) \xrightarrow[D(A^*)']{s\to t^+} 0$, because $x \in M^2 \in D(A^{(E)})$ (the convergence is uniform in $u(\cdot) \in \mathcal{U}_{t,x}$ because it does not depend on $u(\cdot)$) and the second term is estimated, using Lemma 4.4, with

$$\frac{\int_t^s |u(r)| \left|\left(e^{(s-r)A^{(E)}}-\mathbb{1}\right) B\right|_{D(A^*)'} \mathrm{d}r}{s-t} \leq (aK+b) \sup_{r\in[t,s]} \left|\left(e^{(s-r)A^{(E)}}-\mathbb{1}\right) B\right|_{D(A^*)'} \tag{79}$$

which goes to zero (the estimate is uniform in the control). As $\nabla\varphi(t,x) \in D(A^*)$, the proof is complete.

Eq. (43), with $u(\cdot)$ continuous, is a simple corollary of the proof of the first part. Indeed if $u(\cdot)$ is continuous

$$\frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r}{s-t} \to \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \tag{80}$$

and the claim is proved. □

Proof of Theorem 5.8.
Subsolution:
Let (t,x) be a local minimum of $V-\varphi$ for $\varphi \in$ TEST. Assume that $(V-\varphi)(t,x)=0$ and $u \in [\Gamma_-(x^0), \Gamma_+(x^0)]$. Consider a continuous control $u(\cdot) \in \mathcal{U}_{t,x}$ such that $u(t)=u$.[11] $x(s)$ is the trajectory starting from (t,x) and subject to $u(\cdot) \in \mathcal{U}_{t,x}$. For $s>t$ with $s-t$ small enough:

$$V(s,x(s)) - \varphi(s,x(s)) \geq V(t,x) - \varphi(t,x) \tag{81}$$

and thanks to the Bellman principle of optimality

$$V(t,x) \geq V(s,x(s)) + \int_t^s L(r,x(r),u(r))\mathrm{d}r. \tag{82}$$

Then

$$\varphi(s,x(s)) - \varphi(t,x) \leq V(s,x(s)) - V(t,x) \leq -\int_t^s L(r,x(r),u(r))\mathrm{d}r, \tag{83}$$

[11] It exists: for example if $u>0$ the control $u(s) = \frac{u}{\Gamma_+(x^0)}\Gamma_+(x^0(s))$ until $\Gamma_+(x^0(s) > 0$ and then equal to 0: because Γ_+ is locally Lipschitz and sublinear, everything works.

which implies, dividing by $(s-t)$,

$$\frac{\varphi(s,x(s))-\varphi(t,x)}{s-t} \leq -\frac{\int_t^s L(r,x(r),u(r))\mathrm{d}r}{s-t}. \tag{84}$$

Using Proposition 5.5,

$$\partial_t\varphi(t,x)+\langle A^*\nabla\varphi(t,x),x\rangle+\langle B(\nabla\varphi(t,x)),u(t)\rangle_{\mathbb{R}} \leq -L(t,x,u) \tag{85}$$

hence

$$\partial_t\varphi(t,x)+\langle A^*\nabla\varphi(t,x),x\rangle+(\langle B(\nabla\varphi(t,x)),u\rangle_{\mathbb{R}}+L(t,x,u)) \leq 0 \tag{86}$$

Taking the $\sup_{u\in[\Gamma_-(x^0),\Gamma_+(x^0)]}$ I obtain the subsolution inequality:

$$\partial_t\varphi(t,x)+\langle A^*\nabla\varphi(t,x),x\rangle+H(t,x,\nabla\varphi(t,x)) \leq 0 \tag{87}$$

Supersolution:
Let (t,x) be a maximum for $V-\varphi$ and such that $(V-\varphi)(t,x)=0$. For $\varepsilon>0$ take $u(\cdot)\in\mathcal{U}_{t,x}$ an ε^2-optimal strategy.[12] $x(s)$ is the trajectory starting from (t,x) and subject to $u(\cdot)\in\mathcal{U}_{t,x}$. For $(s-t)$ small enough

$$V(t,x)-V(s,x(s)) \geq \varphi(t,x)-\varphi(s,x(s)) \tag{88}$$

and from ε^2 optimality

$$V(t,x)-V(s,x(s)) \leq \varepsilon^2+\int_t^s L(r,x(r),u(r))\mathrm{d}r \tag{89}$$

so

$$\frac{\varphi(s,x(s))-\varphi(t,x)}{s-t} \geq \frac{-\varepsilon^2-\int_t^s L(r,x(r),u(r))\mathrm{d}r}{s-t} \tag{90}$$

For $(s-t)=\varepsilon$

$$\frac{\varphi(t+\varepsilon,x(t+\varepsilon))-\varphi(t,x)}{\varepsilon} \geq -\varepsilon-\frac{\int_t^{t+\varepsilon} -L(r,x(r),u(r))\mathrm{d}r}{\varepsilon} \tag{91}$$

and from Proposition 5.5 a $O(\varepsilon)$ with $O(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$ is taken independently on the control $u(\cdot)\in\mathcal{U}_{t,x}$, such that:

$$\begin{aligned}&\partial_t\varphi(t,x)+\langle A^*\nabla\varphi(t,x),x\rangle+\\&\quad+\frac{\int_t^{t+\varepsilon}\langle B(\nabla\varphi(t,x)),u(r)\rangle_{\mathbb{R}}+L(r,x(r),u(r))\mathrm{d}r}{\varepsilon} \geq -\varepsilon+O(\varepsilon).\end{aligned} \tag{92}$$

The supremum over u in the integral, when $\varepsilon\to 0$, gives

$$\partial_t\varphi(t,x)+\langle A^*\nabla\varphi(t,x),x\rangle+H(t,x,\nabla\varphi(t,x)) \geq 0 \tag{93}$$

[12] ε^2-optimal means that $J(t,x,u(\cdot)) \geq V(t,x)-\varepsilon^2$.

Then V is a supersolution of the HJB equation. V is both a viscosity supersolution and a viscosity subsolution of the HJB equation and, by definition, it is a viscosity solution of the HJB equation. □

Proof of Theorem 6.3. The function

$$\begin{cases} \Psi\colon [t,T] \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \\ \Psi\colon s \mapsto \left(\langle A^*p(s), x(s)\rangle_{M^2}, \langle Bp(s), u(s)\rangle_{\mathbb{R}}, q(s), L(s,x(s),u(s))\right) \end{cases} \tag{94}$$

is in $L^1(t,T;\mathbb{R}^4)$ from Lemma 4.4. The set of the right-Lebesgue point is of full measure. I choose a point $\bar{s}$ in this set. I keep choosing $\bar{s}$ in a full measure set if I assume that Eq. (49) is satisfied at $\bar{s}$. I set $\bar{x} := x(\bar{s})$ and I consider a function $\varphi \equiv \varphi^{\bar{s},\bar{x}} \in \textsc{Test}$ such that $V \geq \varphi$ in a neighborhood of $(\bar{s},\bar{x})$, $V(\bar{s},\bar{x}) - \varphi(\bar{s},\bar{x}) = 0$ and $(\partial_t)(\varphi)(\bar{s},\bar{x})) = q(\bar{s})$, $\nabla\varphi(\bar{s},\bar{x}) = p(\bar{s})$. Then for $\tau \in (\bar{s},T]$ and $(\tau - \bar{s})$ small enough,

$$\frac{V(\tau,x(\tau)) - V(\bar{s},\bar{x})}{\tau-\bar{s}} \geq \frac{\varphi(\tau,x(\tau)) - \varphi(\bar{s},\bar{x})}{\tau-\bar{s}} \geq \tag{95}$$

for Proposition 5.5

$$\geq \partial_t\varphi(\bar{s},\bar{x}) + \frac{\int_{\bar{s}}^{\tau} \langle B\nabla\varphi(\bar{s},\bar{x}), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r}{\tau-\bar{s}} + \langle A^*\nabla\varphi(\bar{s},\bar{x}), x\rangle + O(\tau-\bar{s}). \tag{96}$$

Because of the choice of $\bar{s}$ I know that

$$\frac{\int_{\bar{s}}^{\tau} \langle B\nabla\varphi(\bar{s},\bar{x}), u(r)\rangle_{\mathbb{R}}\,\mathrm{d}r}{\tau-\bar{s}} \xrightarrow{\tau\to\bar{s}^+} \langle B\nabla\varphi(\bar{s},\bar{x}), u(\bar{s})\rangle_{\mathbb{R}}. \tag{97}$$

For almost every $\bar{s}$ in $[t,T]$

$$\begin{aligned} \liminf_{\tau\downarrow\bar{s}} \frac{V(\tau,x(\tau)) - V(\bar{s},x(\bar{s}))}{\tau-\bar{s}} \geq& \\ \geq \langle B\nabla\varphi(\bar{s},x(\bar{s})), u(\bar{s})\rangle_{\mathbb{R}} +& \\ + \partial_t\varphi(\bar{s},x(\bar{s})) + \langle A^*\nabla\varphi(\bar{s},x(\bar{s})), x(\bar{s})\rangle =& \\ = \langle Bp(\bar{s}), u(\bar{s})\rangle_{\mathbb{R}} + q(\bar{s}) + \langle A^*\nabla p(\bar{s}), x(\bar{s})\rangle& \end{aligned} \tag{98}$$

then Lemma 6.1 holds true and

$$\begin{aligned} V(T,x(T)) - V(t,x) \geq& \\ \geq \int_t^T \langle Bp(\bar{s}), u(\bar{s}))\rangle_{\mathbb{R}} + q(\bar{s}) + \langle A^*\nabla p(\bar{s}), x(\bar{s})\rangle\,\mathrm{d}\bar{s} \geq& \end{aligned} \tag{99}$$

using Eq. (50)

$$\geq \int_t^T -L(r,x(r),u(r))\mathrm{d}r. \tag{100}$$

Hence

$$V(t,x) \leq V(T,x(T)) + \int_t^T L(r,x(r),u(r))\mathrm{d}r =$$
$$= h(x(T)) + \int_t^T L(r,x(r),u(r))\mathrm{d}r \quad (101)$$

then $(x(\cdot),u(\cdot))$ is an optimal pair. □

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