Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions

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Abstract

We tackle the issue of the inviscid limit of the incompressible Navier-Stokes equations when the Navier slip-with-friction conditions are prescribed on the impermeable boundaries. We justify an asymptotic expansion which involves a weak amplitude boundary layer, with the same thickness as in Prandtl's theory and a linear behavior. This analysis holds for general regular domains, in both dimensions two and three.

1 Introduction

In this paper we deal with the Navier-Stokes equations of the (homogeneous Newtonian) incompressible fluid mechanics. Most of the studies assume the validity of the Dirichlet-Stokes no-slip condition, i.e. that the velocity vanishes on the boundaries. It is striking to see that a century of agreement with experimental results had as consequence that many textbooks of fluid dynamics fails to mention that the no-slip condition remains an assumption. However this experimental fact was not always accepted in the past and an another approach was to suppose that a fluid can slide over a solid surface. In 1823 Navier proposed a slip-with-friction boundary condition and claimed that the component of the fluid velocity tangent to the surface should be proportional to the rate of strain at the surface [30]. The velocity's component normal to the surface is naturally zero as mass cannot penetrate an impermeable solid surface. Then boundary is referred as characteristic since it consists of stream lines all the time. Recent experiments, generally with typical dimensions microns or smaller, have demonstrated that the phenomenon of slip actually occurs. We refer to [26] for a review of several experiments which reveal the extremely rich possibilities for slip behavior, with dependence on various factors. For example adherence condition is no longer true -as pointed out in 1959 by Serrin [32]- when moderate pressure is involved (even when the continuum approximation still holds) such as in high attitude aerodynamics. We also stress that the Navier slip-with-friction condition was derived in the kinematic theory of gases by Maxwell. In this case when the mean free path tends to zero, so does the slip length.

The Navier slip-with-friction conditions are also used for simulations of flows in the presence of rough boundaries, such as in aerodynamics (space shuttles covered by tiles), in weather forecast (where trees, buildings, water waves have to be taken into account), in hemodynamics (cell surfaces of the endothelium). The direct simulation in the real domain is technically hard to implement and an alternative is then to reduce no-slip condition on rough boundaries to

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ad hoc boundary conditions, the so-called *wall laws*, on a smooth domain (see [4]). For some mathematical justifications we refer to a pair of recent papers by Jager and Mikelic [20], [19], the references within, especially the paper [3] by Barrenechea, Le Tallec and Valentin, the paper [1] by Achdou, Pironneau and Valentin, the papers [31] and [5] about large eddy simulations in turbulent models and the work of Bresch and Milisic [8].

It is easy to adapt the classical results about Leray-Hopf weak solutions to the non-stationary Navier-Stokes equations with Navier boundary conditions (cf. [10], [24] in 2D, [17] in 3D). In this paper we deal with the issue of the inviscid limit (as the slip length keeps fixed) which is naturally raised by the smallness of the kinematic viscosity of fluids like air and water. In 2D recent results have been obtained: Clopeau, Mikelic and Robert [10] prove the convergence to the Euler equations for a L^∞ vorticity. This result was extended to L^p vorticities, for $p > 2$, by Lopes Filho and al. [29], and to Yudovich vorticities by Kelliher [24], [23]. The methods of [10], [29], [24], [23] rely on a priori estimates on vorticity and compactness method. For this reason they seems hard to adapt to 3D. However, as observed in [17], in both dimensions two and three a direct L^2 estimate allows to show the strong L^2 convergence to the Euler solution. In this paper, we develop a descriptive method which allows to precisely describe the error, both in 2D and 3D.

As explicitly said in [29], some difficulties are linked to the existence of a boundary layer and the question of describing this boundary layer is explicitly raised. In this paper, we reply to this question by revealing the existence of a weak amplitude (velocity) boundary layer. More precisely, the boundary layer has an amplitude (in a L^∞ sense) of $O(\sqrt{\nu})$, where ν is the amplitude of the viscosity. Furthermore this boundary layer has a linear behavior and its thickness is $O(\sqrt{\nu})$, as in Prandtl's theory of no-slip boundary conditions. However Prandtl's theory of no-slip boundary conditions is still not proved. With sharp contrast with the present situation the no-slip boundary layer has a large amplitude, a non-linear behavior and may separate cf. [12] and the references therein.

We end this short introductory part with the statement of our result. We refer to the next section for the precise definitions and further comments. We introduce for $m, p \in \mathbb{N}$ the anisotropic Sobolev space $H^{m,p}$ of functions $f(x,z) \in L^2(\Omega \times \mathbb{R}_+)$ such that $\partial_x^\alpha \partial_z^q f \in L^2(\Omega \times \mathbb{R}_+)$ for all $|\alpha| \leqslant m$ and $q \in \{0, \ldots, p\}$. We denote by n a certain smooth extension inside Ω of the exterior normal to $\partial\Omega$ and φ is a smooth function equal to the distance to the boundary in a small neighborhood of the boundary. We will prove the following theorem.

Theorem 1. *Let Ω be a bounded smooth domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ and let $u_0 \in H^3$ be a divergence free vector field tangent to the boundary. Let u^ν be a weak Leray solution of the Navier-Stokes equations on Ω with Navier-slip boundary conditions and initial velocity u_0. Let $T > 0$ be such that there exists $u^0 \in C\big([0,T]; H^3(\Omega)\big)$ a smooth solution of the Euler equation on Ω with initial velocity u_0. There exists a boundary layer profile*

$$v \in L^\infty(0,T; H^{2,0}) \cap L^2(0,T; H^{2,1})$$

such that the following asymptotic expansion holds true:

$$u^\nu(t,x) \sim u^0(t,x) + \sqrt{\nu}\, v(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) \tag{1}$$

as $\nu \to 0$ with an error which is $O(\nu)$ in $L^\infty(0,T; L^2(\Omega))$ and $O(\sqrt{\nu})$ in $L^2(0,T; H^1(\Omega))$. Moreover, the function $v(t,x,z)$ vanishes for x outside a small neighborhood of the boundary, satisfies $\partial_z v \in L^\infty([0,T] \times \Omega \times \mathbb{R}_+)$ and the orthogonality condition

$$v(t,x,z) \cdot n(x) = 0 \text{ for all } (t,x,z) \in [0,T] \times \Omega \times \mathbb{R}_+. \tag{2}$$

This theorem complements the results in the noncharacteristic case (non vanishing normal velocity prescribed at the boundary) given by many authors among others by [2], [13], [36].

2 Notations and results

We consider $d = 2$ or 3 and denote by $x \in \mathbb{R}^d$ the space variable. Let Ω denote a smooth bounded domain of $\mathbb{R}^d$. We introduce a smooth function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ such that in a neighborhood $\mathcal{V}$ of $\partial\Omega$ one has that $\Omega \cap \mathcal{V} = \{\varphi > 0\} \cap \mathcal{V}$, $\Omega^c \cap \mathcal{V} = \{\varphi < 0\} \cap \mathcal{V}$, $\partial\Omega := \{\varphi = 0\} \cap \mathcal{V}$ and normalized such that $|\nabla\varphi(x)| = 1$ for all $x \in \mathcal{V}$. This implies that $\varphi(x)$ is the distance between x and $\partial\Omega$ for x in $\mathcal{V}$. We may assume, without restriction, that $\mathcal{V}$ is a tubular neighborhood of $\partial\Omega$, i.e. that $\mathcal{V}$ is of the form $\mathcal{V} := \{x \in \Omega / \ \varphi(x) < \eta\}$ for a real number $\eta > 0$. We define a smooth extension of the normal unit vector n inside Ω by taking $n := \nabla_x\varphi$. Let $\chi \in C_0^\infty(\overline{\Omega}; [0,1])$ be such that $\operatorname{supp}\chi \subset \mathcal{V}$ and $\chi = 1$ in a neighborhood of the boundary $\partial\Omega$. For a vector field $\widetilde{u}$ defined on Ω, we define the tangential part of $\widetilde{u}(x)$ to be

$$\widetilde{u}_{tan}(x) = \chi(x)[\widetilde{u} - (\widetilde{u} \cdot n)n]. \tag{3}$$

Clearly $\widetilde{u}_{tan}$ is smooth in Ω if $\widetilde{u}$ is smooth, is compactly supported in $\mathcal{V}$ and is equal to the orthogonal projection on n for x in a smaller neighborhood of the boundary (a neighborhood where $\chi = 1$).

We consider the incompressible Navier-Stokes equations:

$$\partial_t u^\nu + u^\nu \cdot \nabla u^\nu = \nabla p^\nu + \nu \triangle u^\nu \quad \text{in } \Omega, \tag{4}$$

$$\operatorname{div} u^\nu = 0 \quad \text{in } \Omega, \tag{5}$$

with the Navier slip-with-friction conditions:

$$u^\nu \cdot n = 0 \quad \text{on } \partial\Omega, \tag{6}$$

$$\left[D(u^\nu)n + \alpha u^\nu\right]_{tan} = 0 \quad \text{on } \partial\Omega, \tag{7}$$

and initial condition

$$u^\nu(0, \cdot) = u_0 \quad \text{in } \Omega, \tag{8}$$

where ν is the coefficient of kinematic viscosity, α is a scalar friction function of class C^2 (without a sign), u is the velocity, p is the pressure, $D(u)$ is the rate of strain tensor (or shear stress) defined by $D_{ij}(u) = (\partial_i u_j + \partial_j u_i)/2$. Let us denote by $L^2_\sigma(\Omega)$ the space of L^2 divergence free vector fields tangent to the boundary. The following theorem can be considered to be part of the mathematical folklore.

Theorem 2. *Assume that $u_0 \in L^2_\sigma(\Omega)$. There exists a global (in time) weak Leray solution $u^\nu \in C^0_w\big([0, +\infty); L^2_\sigma(\Omega)\big) \cap L^2_{loc}\big([0, +\infty); H^1(\Omega)\big)$ of (4)-(8) in the sense that for all divergence free tangent to the boundary vector fields $\varphi \in C_0^\infty([0, +\infty) \times \overline{\Omega})$,*

$$-\int_0^\infty \int_\Omega u^\nu \cdot \partial_t\varphi + 2\nu \int_0^\infty \int_{\partial\Omega} \alpha u^\nu \cdot \varphi + 2\nu \int_0^\infty \int_\Omega D(u^\nu) \cdot D(\varphi) + \int_0^\infty \int_\Omega u^\nu \cdot \nabla u^\nu \cdot \varphi = \int_\Omega u_0 \cdot \varphi_0$$

Moreover, this solution verifies the energy inequality

$$\|u^\nu(t)\|^2_{L^2} + 4\nu \int_0^t \int_{\partial\Omega} \alpha |u^\nu|^2 + 4\nu \int_0^t \int_\Omega |D(u^\nu)|^2 \leqslant \|u_0\|^2_{L^2}, \quad \textit{for all } t \geqslant 0. \tag{9}$$

For the sake of completeness we will give a sketch of the proof of this theorem in section 3. In the limit case $\nu = 0$, the higher derivative term $\nu \triangle u^\nu$ is dropped from the Navier-Stokes system and one formally gets the Euler system:

$$\partial_t u^0 + u^0 \cdot \nabla u^0 = \nabla p^0 \quad \text{in } \Omega, \tag{10}$$

$$\operatorname{div} u^0 = 0 \quad \text{in } \Omega, \tag{11}$$

which rules the behavior of the inviscid flows. Since the Euler system is first order, we have a reduction of the order of the equations and a corresponding reduction must be done in the number of the boundary conditions; we only impose the normal component of the velocity at the boundary:

$$u^0 \cdot n = 0 \quad \text{on } \partial\Omega, \tag{12}$$

We will work with smooth solutions to (10)-(12) whose existence is well-known to be global in time in dimension two and at least local in time in dimension three, see for example [6, 11, 21, 22, 37]. Therefore **for the remainder of the paper we will consider an initial data $\mathbf{u_0} \in \mathbf{H^3}(\boldsymbol{\Omega}) \cap \mathbf{L}^2_\sigma(\boldsymbol{\Omega})$ which is independent of the viscosity** ν. Thus there exist

$$T > 0 \text{ and } u^0 \in C([0,T]; H^3(\Omega)) \cap C^1([0,T]; H^2(\Omega)) \tag{13}$$

solution of the Euler system (10)-(11)-(12) with initial data

$$u^0|_{t=0} = u_0 \quad \text{in } \Omega, \tag{14}$$

In dimension two, the time existence T can be chosen infinite. **Let us stress that this time T is from now on assumed to be finite and fixed** (but arbitrarily large in dimension two). In this paper we tackle the issue of the convergence of the Leray solutions u^ν corresponding to the initial data u_0 toward those of the Euler system (10)-(11)-(12)-(14) when ν goes decreasing to 0. Since for the Navier-Stokes equations ($\nu > 0$), we impose an additional boundary condition on the tangential component of the velocity, one must allow a thin layer where there is a sharp transition of the fluid velocity from a solution to the Euler's equations to the Navier slip condition. The fluid develops an internal length scale so that one is faced with a singular perturbation problem. To describe this viscous boundary layer we add an extra variable z to the space-time variables t, x.

We give below a few comments on Theorem 1.

Boundary layers

Theorem 1 says that in the inviscid limit the Leray solutions u^ν of the Navier-Stokes equations (4)-(8) can be seen as the sum of the solution u^0 of the Euler system and of a boundary layer of width and amplitude $\sqrt{\nu}$. This answers to a question raised in [29]. Remark that this boundary layer has the same width as the Prandtl's one in the setting of no-slip boundary conditions. However in Prandtl's theory, the boundary layer has a large amplitude. The underlying reason is that Navier conditions (7) are of order one and involve normal derivatives of tangential components. More precisely plugging the ansatz in the right hand side of (1) instead of u^ν in the friction boundary condition (7) yields

$$0 = \big[D(u^\nu)n + \alpha u^\nu\big]_{tan} \sim \big[D(u^0)n + \alpha u^0 + \frac{1}{2}\partial_z v|_{z=0}\big]_{tan}. \tag{15}$$

So one can hope that some appropriate choice of profile v can allow us to cancel the right side of (15), at less for x on $\partial\Omega$. This is what we succeed in Proposition 5. We give some hints on our strategy just below. This supports the results of [9] about weak amplitude boundary layers in ferromagnetism and [34] where was studied the closer problem of approximating solutions of a quasilinear symmetric hyperbolic boundary value problem with maximally dissipative boundary conditions by solutions of a regularized system with arbitrary small parabolic term added, under suitable choice of boundary conditions, of mixed hyperbolic and Neumann type.

Remainder

First let us explain why the error in Theorem 1 is actually smaller than the right hand side of (1). We claim the following lemma, which will also be useful in the sequel.

Lemma 3. *There exists a constant C independent of ν such that for all $h = h(x,z)$ in $L^2_z(\mathbb{R}_+; H^1_x(\Omega))$ which vanishes for x outside the neighborhood $\mathcal{V}$,*

$$\left\| h\big(x, \frac{\varphi(x)}{\sqrt{\nu}}\big) \right\|_{L^2(\Omega)} \leqslant C\nu^{\frac{1}{4}} \|h\|_{L^2_z(\mathbb{R}_+;H^1_x(\Omega))}$$

Proof. The key idea is to use some coordinates which distinguish tangential and normal behaviors of the functions near the boundary $\partial\Omega$. It is well-known (see for example Theorem C.4.4 of [7]) that the manifold $\partial\Omega$ can be endowed by an integral compatible with the Lebesgue integral in the sense that a Fubini-type theorem holds:

$$\int_\Omega h^2\big(x, \frac{\varphi(x)}{\sqrt{\nu}}\big)dx \quad = \quad \int_0^\eta \Big(\int_{\partial\Omega} h^2\big(\sigma - sn(\sigma), \frac{s}{\sqrt{\nu}}\big)\gamma_s(\sigma)d\sigma \Big) ds$$

where $\gamma_s(\sigma)$ denotes the Jacobian of the transformation $\sigma \mapsto \sigma - sn(\sigma)$ which maps $\partial\Omega$ to $\varphi^{-1}(s)$; and η was fixed at the beginning of this section. Then we perform the change of variable $z := \frac{s}{\sqrt{\nu}}$ to get

$$\int_\Omega h^2\big(x, \frac{\varphi(x)}{\sqrt{\nu}}\big)dx \quad = \quad \sqrt{\nu}\int_0^{\frac{\eta}{\sqrt{\nu}}} I^\nu(z)dz,$$

where

$$\begin{aligned} I^\nu(z) \quad &:= \quad \int_{\partial\Omega} g^\nu(\sigma, \sqrt{\nu}z, z)\gamma_{\sqrt{\nu}z}(\sigma)d\sigma, \\ g^\nu(\sigma, s, z) \quad &:= \quad h^2\big(\sigma - sn(\sigma), z\big). \end{aligned}$$

In order to bound $I^\nu(z)$, let us first notice that, since $\partial\Omega$ is a smooth compact and η is small, we have

$$0 < \min\{\gamma_s(\sigma); 0 < s' < \eta, \sigma \in \partial\Omega\} \leqslant \max\{\gamma_s(\sigma); 0 < s < \eta, \sigma \in \partial\Omega\} < +\infty. \qquad (16)$$

We infer that

$$I^\nu(z) \leqslant C \int_{\partial\Omega} \sup_{0<s<\eta} g^\nu(\sigma, s, z)d\sigma.$$

Then, for each $(\sigma, z) \in \partial\Omega \times (0,\infty)$, we apply the Sobolev embedding $H^1(0,\eta) \hookrightarrow L^\infty(0,\eta)$ to the function $s \mapsto h\big(\sigma - sn(\sigma), z\big)$ and once more (16) to get

$$I^\nu(z) \leqslant C \int_{\partial\Omega} \Big(\int_0^\eta (h^2 + (\partial_n h)^2)(\sigma - sn(\sigma), z)\gamma_s(\sigma)ds \Big) d\sigma.$$

Using once more the Fubini principle, we get

$$I^\nu(z) \leqslant C \int_\Omega (h^2 + (\partial_n h)^2)(x,z)dx.$$

and then

$$\int_\Omega h^2\big(x, \frac{\varphi(x)}{\sqrt{\nu}}\big)dx \quad \leqslant \quad C\sqrt{\nu}\|h\|^2_{L^2_z(\mathbb{R}_+;H^1_x(\Omega))}.$$

□

As a consequence of Lemma 3 and of the regularity of the boundary layer profile v, the last term in the right hand side of (1) is $O(\nu^{\frac{3}{4}})$ in $L^\infty(0,T;L^2(\Omega))$ and $O(\nu^{\frac{1}{4}})$ in $L^2(0,T;H^1(\Omega))$; these bounds are in general optimal. These estimates improve the results of [24] in 2d and the one of [17] in both 2d and 3d. Indeed we will see (cf. Lemma 6) that $(I - \mathbb{P})(u^\nu - u^0)$ is $O(\nu)$ in $L^\infty(0,T;H^1(\Omega))$, where we denote by $\mathbb{P}$ the Leray projector, *i.e.* the L^2 orthogonal projection on the space of divergence free vector fields on Ω and tangent to the boundary $\partial\Omega$.

Higher order estimate

It could be also interesting to look for solutions more regular with respect to the time variable. For a fixed viscosity $\nu > 0$, the local existence of strong solutions of the initial boundary value problem (4)-(8) was studied by many -among others Grudd and Solonnikov by pseudo-differential methods in [14], [15], [16] (see also O. Steiger in [33]), Itoh and al. in Sobolev-Slobodetskii spaces in [35], in Hølder spaces in [18]- under the assumption that some compatibility conditions on the initial data at the boundary hold. Our conjecture is that under suitable assumptions, it could be possible to prove that (1) holds with an error which is $O(\nu)$ in $L^\infty((0,T) \times \Omega)$.

Lions' free conditions

In 2d the boundary conditions (7) can be formulated in terms of vorticity: $\omega = (2\kappa - \alpha)u \cdot \tau$ on $\partial\Omega$, where κ is the curvature of $\partial\Omega$ and τ the counterclockwise tangent vector to $\partial\Omega$. As a consequence they extend the boundary condition $\omega = 0$ on $\partial\Omega$ which was considered by J.L. Lions in [27] p. 87-98, P.L. Lions in [28] p. 129-131, Xiao and Xin in [38], Ziane in [39]. An interesting property of this Lions' free conditions is that if the initial data u_0 satisfies the Lions conditions then the solution u^0 of the Euler equation also satisfies the Lions conditions for all times. As a consequence the boundary layer v vanishes in this case. Thus the asymptotic expansion used here shed more light on the well-known fact that the Lions conditions are mathematically easier to handle, even among the Navier slip conditions. For such compatible initial data, Xiao and Xin [38] proved a H^3 convergence. As mentioned by Iftimie and Planas [17] it is impossible to extend such a result, even a H^2 convergence, for the other Navier slip conditions.

Road map of the proof

Let us now give some hints about some difficulties which occur in the proof of Theorem 1. First of all a natural way to begin the proof is to plug the ansatz (1) instead of u^ν in (4)-(8). When looking at (4), we get at the order $O(1)$ the term

$$u^0 \cdot n\partial_z v. \tag{17}$$

The other terms which appear at the order $O(1)$ cancel thanks to the Euler equation (10). A key remark is that the term (17) can indeed be seen as a term of order $O(\sqrt{\nu})$. More precisely, we rewrite

$$(u^0 \cdot n\partial_z v)|_{z=\frac{\varphi(x)}{\sqrt{\nu}}} = \sqrt{\nu}\frac{u^0 \cdot n}{\varphi(x)}(z\partial_z v)|_{z=\frac{\varphi(x)}{\sqrt{\nu}}}.$$

Thanks to (11), $u^0 \cdot n$ vanishes on the boundary $\partial\Omega$. As a consequence the function $\frac{u^0 \cdot n}{\varphi}$ is in

$$C([0,T]; H^2(\Omega)) \cap C^1([0,T]; H^1(\Omega)).$$

This would be sufficient to derive good energy estimates. Some other terms appear at the order $O(\sqrt{\nu})$ and we can be tempted to look for v such that the sum of these terms:

$$(\partial_t - \partial_z^2 + \frac{u^0 \cdot n}{\varphi(x)} z\partial_z)v + u^0 \cdot \nabla v + v \cdot \nabla u^0 + v \cdot n\partial_z v \tag{18}$$

vanishes. But there are several difficulties. First among the terms in (18) there is a non linear one:

$$v \cdot n\partial_z v. \tag{19}$$

In order to cancel this term, one hope to find v such that

$$\forall (x,z,t) \in \mathcal{V} \times \mathbb{R}_+ \times (0,T), \quad v(t,x,z) \cdot n(x) = 0. \tag{20}$$

Another motivation to look for v satisfying the condition (20) is that plugging the ansatz (1) instead of u^ν in the div free relation (5) yields at the order $O(1)$ the term

$$\partial_z v \cdot n = \partial_z (v \cdot n). \tag{21}$$

However the condition (20) holds when Ω is an half-space but not in the general case because of the terms $u^0 \cdot \nabla v + v \cdot \nabla u^0$. As a consequence, we will consider (cf. section 4.1) for v the equation

$$(\partial_t - \partial_z^2 + \frac{u^0 \cdot n}{\varphi(x)} z \partial_z) v + \big[u^0 \cdot \nabla v + v \cdot \nabla u^0)\big]_{tan} = 0.$$

Therefore we will insure condition (20) and the terms (19) and (21) will vanish. The behavior of v is thus linear. This is a sharp contrast with Prandtl's boundary layer equations, which occur when looking at the usual no-slip boundary condition. The linear behavior of the boundary layer v here can be seen as an analogy of the weakly nonlinear geometric optics. In particular when a hyperbolic system has a linearly degenerate field weak amplitude high frequency oscillations propagate linearly along it. In our setting there are no oscillations but some weak amplitude boundary layers in the neighborhood of the boundary which is characteristic for a linearly degenerate field. Of course by comparing with (18) we see that it remains to explain what happens to the normal part of $u^0 \cdot \nabla v + v \cdot \nabla u^0$. To do this we look for some pressure p^ν associated to the u^ν under the form

$$p^\nu(t,x) \sim p^0(t,x) + \nu q(t,x,\frac{\varphi(x)}{\sqrt{\nu}}).$$

Roughly speaking we look for pressure which are the sum of the Euler pressure and of a pressure boundary layer of width $\sqrt{\nu}$ and of amplitude ν. As we will seen in section 4.2 the role of q will be to compensate the normal part of $u^0 \cdot \nabla v + v \cdot \nabla u^0$.

Let us mention another technical difficulty in the proof. When plugging the ansatz (1) instead of u^ν in (5) yields at the order $O(\sqrt{\nu})$ the term div v and this term has no reason to vanish. As a consequence we will look for some more accurate asymptotic development of the Leray velocities under the form

$$u^\nu(t,x) \sim u^0(t,x) + \sqrt{\nu} v(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \nu w(t,x,\frac{\varphi(x)}{\sqrt{\nu}}), \tag{22}$$

where w is a velocity boundary layer. Thus plugging the ansatz (22) instead of u^ν in (5) yields at the order $O(\sqrt{\nu})$ the term

$$\partial_z w \cdot n + \operatorname{div} v \tag{23}$$

instead of div v. We will be able to define w such that the term (23) vanishes. Indeed we will look for rather special profile w which are collinear to the normal n. It would be sufficient to cancel the term (23) and matches with the difficulties caused by the boundary conditions.

The idea is then to look for velocity u^ν of the form:

$$u^\nu(t,x) = u^0(t,x) + \sqrt{\nu}\, v\big(t,x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + \nu\, w\big(t,x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + \nu R^\nu(t,x). \tag{24}$$

We also expand the pressure p^ν as follows

$$p^\nu(t,x) = p^0(t,x) + \nu\, q\big(t,x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + \nu \pi^\nu(t,x). \tag{25}$$

Then we will estimate the remainder R^ν in the spaces $L^\infty(0,T;L^2)$ and $L^2(0,T;H^1)$. We decompose $R^\nu = \mathbb{P}R^\nu + (I-\mathbb{P})R^\nu$ where $\mathbb{P}$ denotes the Leray projector, *i.e.* the L^2 orthogonal projection on the space of divergence free vector fields on Ω and tangent to the boundary. We will show first that $(I-\mathbb{P})R^\nu$ is bounded in $L^\infty(0,T,H^1)$ independently of ν thanks to a Neumann problem for the Laplacian. Then we will estimate $\|\mathbb{P}R^\nu\|_{L^\infty(0,T;L^2)}$ and $\sqrt{\nu}\|\mathbb{P}R^\nu\|_{L^2(0,T;H^1)}$ which is the main part of the proof and relies on several tricky estimates.

3 Some remarks about Leray weak solutions associated to the Navier conditions

In this section we indicate a sketch of the proof of Theorem 2. We recall first the following identity from [17, Lemma 2.2]

$$-\int_\Omega \triangle\widetilde{u}\cdot\widetilde{v} = 2\int_\Omega D(\widetilde{u})\cdot D(\widetilde{v}) - 2\int_{\partial\Omega}[D(\widetilde{u})n]_{tan}\cdot\widetilde{v}, \tag{26}$$

where $\widetilde{u}$ and $\widetilde{v}$ are smooth vector fields such that $\widetilde{v}$ is divergence free and tangent to the boundary. Therefore, multiplying the equation of motion (4) by u^ν and integrating in space and time yields the following formal energy equality

$$\|u^\nu(t)\|^2_{L^2} + 4\nu\int_0^t\int_{\partial\Omega}\alpha|u^\nu|^2 + 4\nu\int_0^t\int_\Omega |D(u^\nu)|^2 = \|u^\nu_0\|^2_{L^2}, \quad \text{for all } t\geqslant 0. \tag{27}$$

This implies an *a priori* bound for u^ν in the space $L^\infty_{loc}\big([0,+\infty),L^2(\Omega)\big)\cap L^2_{\mathrm{loc}}\big([0,+\infty),H^1(\Omega)\big)$ even if α is not positive. Indeed, the boundary term is dominated by the other two terms on the left-hand side of the above relation:

$$\begin{aligned}-4\nu\int_0^t\int_{\partial\Omega}\alpha|u^\nu|^2 &= -4\nu\int_0^t\int_\Omega \operatorname{div}(n\alpha|u^\nu|^2)\leqslant C\int_0^t\|u^\nu\|_{L^2}\|u^\nu\|_{H^1}\\ &\leqslant C\int_0^t\|u^\nu\|_{L^2}(\|u^\nu\|_{L^2}+\|D(u^\nu)\|_{L^2})\leqslant C\int_0^t\|u^\nu\|^2_{L^2}+\nu\int_0^t\|D(u^\nu)\|^2_{L^2}\end{aligned}$$

We used above the second Korn inequality: $\|u^\nu\|_{H^1}\simeq\|u^\nu\|_{L^2}+\|D(u^\nu)\|_{L^2}$. Plugging this in (27) results in

$$\|u^\nu(t)\|^2_{L^2} + 3\nu\int_0^t\int_\Omega|D(u^\nu)|^2\leqslant\|u^\nu_0\|^2_{L^2}+C\int_0^t\|u^\nu\|^2_{L^2}, \quad \text{for all } t\geqslant 0.$$

The Gronwall inequality clearly implies an a priori bound for u^ν in $L^\infty_{loc}\big([0,+\infty);L^2(\Omega)\big)$ and for $D(u^\nu)$ in $L^2_{loc}\big([0,+\infty);L^2(\Omega)\big)$. Another application of the Korn inequality gives an a priori bound for u^ν in $L^\infty_{loc}\big([0,+\infty);L^2(\Omega)\big)\cap L^2_{\mathrm{loc}}\big([0,+\infty);H^1(\Omega)\big)$. A standard Galerkin procedure implies the existence of a solution with this regularity. We next justify that this solution can be assumed to verify the energy inequality. We recall first the standard argument to justify the energy inequality. Let u^ν_n be the approximate solution obtained via the Galerkin method. We write the energy equality (27) that holds true for u^ν_n and pass to the limit $n\to\infty$. We observe that the right-hand side passes to the limit while for the left-hand side we use the standard argument of liminf (if $x_n\rightharpoonup x$ weakly then $\|x\|\leqslant\liminf\|x_n\|$) to obtain the energy inequality. This works if α is positive, but an additional argument is required if α is allowed to change sign. Indeed, we cannot us the liminf argument for the middle term on the left-hand side of (27) since this term is not positive anymore and cannot be viewed as a norm. However, the remedy is quite simple.

From the Galerkin method, we know that u^ν_n is bounded in $L^2_{\mathrm{loc}}\big([0,+\infty),H^1(\Omega)\big)$ and converges strongly to u^ν as $n\to\infty$ in $L^\infty_{loc}\big([0,+\infty);H^{-1}(\Omega)\big)$. From the interpolation inequality $\|u^\nu-u^\nu_n\|_{H^{3/4}}\leqslant C\|u^\nu-u^\nu_n\|^{1/8}_{H^{-1}}\|u^\nu-u^\nu_n\|^{7/8}_{H^1}$ we infer that $u^\nu_n\to u^\nu$ strongly in the space $L^{16/7}_{loc}\big([0,+\infty);H^{3/4}(\Omega)\big)$. By trace theorems we infer that $u^\nu_n\to u^\nu$ in $L^2_{loc}\big([0,+\infty);L^2(\partial\Omega)\big)$. Therefore

$$\int_0^t\int_{\partial\Omega}\alpha|u^\nu_n|^2\longrightarrow\int_0^t\int_{\partial\Omega}\alpha|u^\nu|^2 \qquad \text{as } n\to\infty$$

and this allows to pass to the limit as explained above and obtain the energy inequality (9).

4 Boundary layer profiles and the remainder

We will proceed in three steps to determine the boundary layer profiles.

4.1 First velocity boundary layer profile v

In this first step, we define a boundary layer profile v as a solution of a linear initial boundary value problem in the variables t, x, z. Roughly speaking, the system is parabolic (of second order) with respect to the variable z, which lies on the half-line $\mathbb{R}_+$. Moreover there is a convection term which involves the slow variable x by mean of $u^0 \cdot \nabla_x$. The slow variable x lies in the neighborhood $\mathcal{V}$ of $\partial\Omega$ meaning that $v(t,x,z)$ vanishes whenever $x \notin \mathcal{V}$. The boundary conditions on $\{z=0\}$ are inhomogeneous, of mixed type Dirichlet-Neumann. There is no source term and the initial data vanishes. We show that this linear initial boundary value problem admits a unique strong solution and so we define the boundary layer profile v as it.

We define $v(t,x,z)$ to be the solution of the equation

$$\partial_t v + (v \cdot \nabla u^0 + u^0 \cdot \nabla_x v)_{tan} + fz\partial_z v - \partial_z^2 v = 0, \qquad t>0,\ x \in \Omega,\ z>0 \tag{28}$$

supplemented with the following boundary conditions at $z=0$:

$$\partial_z v(t,x,0) - \big[\partial_z v(t,x,0)\cdot n(x)\big] n(x) = g(t,x) \quad \text{and} \quad v(t,x,0)\cdot n(x) = 0 \qquad t>0,\ x\in\Omega. \tag{29}$$

where

$$f(t,x) = \frac{u^0(t,x)\cdot n(x)}{\varphi(x)} \quad \text{and} \quad g(t,x) = -2\big[D(u^0(t,x))n + \alpha u^0(t,x)\big]_{tan}$$

The initial velocity vanishes identically:

$$v(0,x,z) \equiv 0, \qquad x \in \Omega,\ z>0. \tag{30}$$

Lemma 4. *We have that*

$$f, g \in C^0\big([0,T]; H^2(\Omega)\big) \cap C^1\big([0,T]; H^1(\Omega)\big). \tag{31}$$

Proof. The assertion about the function g follows from (13), the smoothness of the boundary and the fact that α is C^2. The assertion about f looks like a Hardy inequality. However since we could not find any reference for this precise statement let us give a few explanations for the sake of completeness. The main difficulty lies in the estimate of the normal derivatives of f near the boundary $\partial\Omega$. Let us show how to adapt the Hardy argument to our context by proving that $\partial_n^2 f_\chi$ is in $C^0\big([0,T]; L^2(\Omega)\big)$ where f_χ denotes the function $f_\chi(t,x) = \chi(x) f(t,x)$. We can assume without loss of generality that the constant η defining the neighborhood $\mathcal{V}$ of the boundary is sufficiently small so that if $\sigma \in \partial\Omega$ and $s \in [0,\eta]$, then $d(\sigma - sn, \partial\Omega) = s$ (see [7, Paragraph C.4.2]). As in the proof of Lemma 3 we are going to use curvilinear coordinates writing for each $t \in [0,T]$

$$\int_\Omega (\partial_n^2 f_\chi)^2(t,x)dx = \int_{(0,+\infty)\times\partial\Omega} (\partial_s^2 \widetilde{f}_\chi(t,\sigma,s))^2 \gamma_s(\sigma) ds d\sigma$$

where $\widetilde{f}_\chi$ denotes the function $\widetilde{f}_\chi(t,\sigma,s) := f(t, \sigma - sn(\sigma))$. From (12) we infer that

$$\widetilde{f}_\chi(t,\sigma,s) = \int_0^1 \partial_s \widehat{f}_\chi(t,\sigma,\xi s) d\xi$$

where $\widehat{f}_\chi(t,\sigma,s) := \chi(\sigma - sn(\sigma)) u^0(t, \sigma - sn(\sigma)) \cdot n(\sigma - sn(\sigma))$. By differentiating and by using the Cauchy-Schwarz inequality we deduce that

$$(\partial_s^2 \widetilde{f}_\chi(t,\sigma,s))^2 \leqslant \int_0^1 \xi^4 (\partial_s^3 \widehat{f}_\chi(t,\sigma,\xi s))^2 d\xi.$$

By using the Fubini principle and (16), we deduce that

$$\begin{aligned}\int_\Omega (\partial_n^2 f_\chi)^2(t,x)dx &\leqslant C\int_{(0,+\infty)\times\partial\Omega} (\partial_s^3 \widehat{f}_\chi(t,\sigma,s'))^2 \gamma_{s'}(\sigma)ds'd\sigma \\ &\leqslant C\|u^0(t,\cdot)\|^2_{H^3(\Omega)}\end{aligned}$$

hence $\partial_n^2 f_\chi$ is in $C^0\big([0,T];L^2(\Omega)\big)$. We proceed in the same way with the other derivatives of f to end the proof. □

For $k,m,p\in\mathbb{N}$, we introduce the following weighted anisotropic semi-norm of v:

$$\|v\|_{k,m,p} = \Big(\sum_{|\alpha|\leqslant m}\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha\partial_z^p v|^2\,dx\,dz\Big)^{\frac{1}{2}}$$

and we denote by $H^{k,m,p}$ the weighted anisotropic Sobolev space with norm given by

$$\|v\|^2_{H^{k,m,p}} = \sum_{j=0}^p \|v\|^2_{k,m,j} = \sum_{|\alpha|\leqslant m, j\leqslant p}\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha\partial_z^j v|^2\,dx\,dz$$

The following proposition summarizes the main properties of the boundary layer v:

Proposition 5. *There exists exactly one solution v of* (28)-(29)-(30) *on* $[0,T]$. *This unique solution verifies that*

$$v\in L^\infty(0,T;H^{k,2,0})\cap L^2(0,T;H^{k,2,1}) \tag{32}$$

for all $k\in\mathbb{N}$ and

$$\partial_z v\in L^\infty([0,T]\times\Omega\times\mathbb{R}_+). \tag{33}$$

Moreover the solution $v(t,x,v)$ vanishes for x outside the neighborhood $\mathcal{V}$, and satisfies the orthogonality condition (2).

We postpone the proof of this result until the next section. We continue now with the construction of the boundary layer profile.

4.2 The pressure boundary layer profile q

We define a pressure boundary layer profile $q=q(t,x,z)$ as the unique function which vanishes for $z\to+\infty$ and such that

$$(v\cdot\nabla u^0 + u^0\cdot\nabla_x v)\cdot n = \partial_z q \quad \text{for } (x,z,t) \text{ in } \Omega\times\mathbb{R}_+\times[0,T]. \tag{34}$$

Since v vanishes for $x\in\Omega\setminus\mathcal{V}$, we observe that q also vanishes for $x\in\Omega\setminus\mathcal{V}$.

4.3 The second velocity boundary layer profile w

We finally construct a vector field $w=w(t,x,z)$ proportional to n, which vanishes for $z\to+\infty$ and such that

$$\operatorname{div}_x v + n\cdot\partial_z w = 0. \tag{35}$$

Equivalently,

$$w=\overline{w}\,n, \qquad \overline{w}(t,x,z) = -\int_z^\infty \operatorname{div}_x v(t,x,y)\,dy. \tag{36}$$

Clearly w vanishes for $x\in\Omega\setminus\mathcal{V}$.

4.4 The remainder R^ν

In this paragraph we use the previously introduced boundary layer profiles to write down the equation of the remainder. The remainder is defined by the expansion (24) of the velocity u^ν since the pressure p^ν is expanded according to (25). We observe now that the equation of v given in (28) can be written under the form

$$\partial_t v + v\cdot\nabla u^0 + u^0\cdot\nabla_x v + \frac{u^0\cdot n}{\varphi} z\partial_z v - |n|^2\partial_z^2 v = n\partial_z q. \tag{37}$$

This follows from (34) and using that $v\cdot n = 0$ everywhere and that $|n| = 1$ on the support of v.

We observe next that for a function $h(x,z)$ the following formulas hold true

$$\nabla_x\Big[h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big)\Big] = \nabla_x h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + \frac{n(x)}{\sqrt{\nu}}\partial_z h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big)$$

and

$$\triangle_x\Big[h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big)\Big] = \triangle_x h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + 2\frac{n(x)}{\sqrt{\nu}}\cdot\nabla_x\partial_z h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + \frac{\triangle\varphi}{\sqrt{\nu}}\partial_z h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + \frac{|n(x)|^2}{\nu}\partial_z^2 h\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big)$$

For the remaining of this subsection, all functions depending on z are evaluated at $z = \varphi(x)/\sqrt{\nu}$ unless otherwise stated. Plugging the expansions (24) and (25) into the equation of motion (4), using the equation for v given in (37), remembering that $n = \nabla\varphi$ and $v\cdot n = 0$ and repeatedly using the two above formulas we obtain that the remainder R^ν must verify the following equation:

$$\begin{aligned}
&\partial_t R^\nu - \nu\triangle R^\nu + u^\nu\cdot\nabla R^\nu + R^\nu\cdot\nabla u^0 + \sqrt{\nu}R^\nu\cdot n\,\partial_z w + R^\nu\cdot n\,\partial_z v + \sqrt{\nu}R^\nu\cdot\nabla_x v\\
&= -\partial_t w + \triangle u^0 + \sqrt{\nu}\triangle_x v + 2n\cdot\nabla_x\partial_z v + \nu\triangle_x\big[w(x,\varphi/\sqrt{\nu})\big] - u^\nu\cdot\nabla_x w - w\cdot\nabla u^0 - \frac{1}{\sqrt{\nu}}u^0\cdot n\partial_z w\\
&\quad - \sqrt{\nu}w\cdot n\,\partial_z w - w\cdot n\,\partial_z v - v\cdot\nabla_x v + \triangle\varphi\,\partial_z v - \sqrt{\nu}w\cdot\nabla_x v + \nabla_x q + \nabla_x\pi^\nu.
\end{aligned}\tag{38}$$

Similarly, using that u^ν is divergence free, that $\partial_z v\cdot n = 0$, the expansion (24) and the definition of w given in (35) we obtain that

$$\operatorname{div} R^\nu = -\operatorname{div}_x w. \tag{39}$$

We determine next the boundary conditions for the remainder. Using (6), the expansion (24) and the orthogonality condition (2) we get that

$$R^\nu(t,x)\cdot n(x) = -w(t,x,0)\cdot n(x), \qquad \forall x\in\partial\Omega. \tag{40}$$

We now use the second boundary condition (7). Observe first that for a vector field $\widetilde{u}(x,z)$ one has the identity

$$\Big\{D_x\Big[\widetilde{u}\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big)\Big]n(x)\Big\}_{tan} = \Big[D_x\widetilde{u}\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big)n(x) + \frac{1}{2\sqrt{\nu}}\partial_z\widetilde{u}\big(x,\frac{\varphi(x)}{\sqrt{\nu}}\big)\Big]_{tan}, \qquad \forall x\in\partial\Omega.$$

Using several times this identity, the boundary condition (7), the expansion (24), the boundary condition (7), the fact that v is orthogonal to n and the first boundary condition for v in (29), we obtain a second boundary condition for the remainder R^ν:

$$\big[D(R^\nu)n + D_x(w)n + \alpha R^\nu + \alpha w + \frac{\alpha}{\sqrt{\nu}}v + \frac{1}{\sqrt{\nu}}D_x(v)n\big]_{tan} = 0, \qquad \forall x\in\partial\Omega.$$

From (36) we observe next that

$$D_x(w)n = \frac{1}{2}\nabla_x \overline{w} + \overline{w}D(n)n + \frac{1}{2}n\,\partial_n \overline{w}, \qquad \forall x \in \partial\Omega.$$

We finally deduce that

$$\big[D(R^\nu)n + \frac{1}{2}\nabla_x \overline{w} + \alpha R^\nu\big]_{tan} = (b_\nu)_{tan}, \qquad \forall x \in \partial\Omega \tag{41}$$

where we defined

$$b_\nu(x) = -\overline{w}(x,0)D(n)n - \alpha w(x,0) - \frac{\alpha}{\sqrt{\nu}}v(x,0) - \frac{1}{\sqrt{\nu}}D_x v(x,0)n, \qquad x \in \overline{\Omega}. \tag{42}$$

We end this section by observing that the remainder vanishes at time $t = 0$:

$$R^\nu(0,x) = 0, \quad \forall x \in \Omega.$$

5 Estimates for the boundary layer v

We prove now Proposition 5. We assume throughout this section that the time variable belongs to the interval $[0,T]$. We denote by C a generic constant that may depend on u^0, Ω, k and T. Unless otherwise specified, the L^p norms of v are assumed to be taken with respect to x and z and the L^p norms of the t,x dependent functions f, g, u^0 are taken with respect to x only. Since the equation of v is linear, the existence and uniqueness of solutions will follow from the estimates below. We show first the orthogonality condition (2) i.e. that v is orthogonal to n everywhere and not only for $z = 0$. Taking the scalar product of (28) with n implies that

$$\partial_t(v\cdot n) + fz\partial_z(v\cdot n) - \partial_z^2(v\cdot n) = 0.$$

An immediate L^2 estimate in the z variable using that $v\cdot n$ vanishes for $z = 0$ and f is uniformly bounded implies that $v\cdot n$ vanishes for all $z \geqslant 0$. Indeed, taking the product of the above relation with $v\cdot n$ and integrating in (x,z) yields after a couple of integrations by parts in the z variable:

$$\partial_t\|v\cdot n\|^2_{L^2(\Omega\times\mathbb{R}_+)} + 2\|\partial_z(v\cdot n)\|^2_{L^2(\Omega\times\mathbb{R}_+)} = \iint_{\Omega\times\mathbb{R}_+} f|v\cdot n|^2\,dx\,dz \leqslant \|f\|_{L^\infty([0,T]\times\Omega)}\|v\cdot n\|^2_{L^2(\Omega\times\mathbb{R}_+)}$$

so, by the Gronwall lemma,

$$\|v(t)\cdot n\|^2_{L^2(\Omega\times\mathbb{R}_+)} \leqslant \|(v\cdot n)\big|_{t=0}\|^2_{L^2(\Omega\times\mathbb{R}_+)}\exp(t\|f\|_{L^\infty([0,T]\times\Omega)}) = 0.$$

This shows the orthogonality property (2). This also shows that $\partial_z v$ is orthogonal to n, so that the boundary condition for v given in (29) can be expressed under the form

$$\partial_z v(t,x,0) = (\partial_z v)_{tan} = g(t,x), \quad t \in [0,T],\ x \in \Omega. \tag{43}$$

We observe next that $v(t,x,z)$ vanishes for $x \notin \operatorname{supp}\chi$, in particular for $x \in \Omega\setminus\mathcal{V}$. Indeed, let us fix $x_0 \notin \operatorname{supp}\chi$. We view Equation (28) as a PDE in the (t,z) variables:

$$\partial_t v(t,x_0,z) + f(t,x_0)z\partial_z v(t,x_0,z) - \partial_z^2 v(t,x_0,z) = 0, \qquad t \in [0,T],\ z > 0. \tag{44}$$

The $(\dots)_{tan}$ part from (28) vanishes since, by relation (3), every tangential part vanishes for x outside the support of the cutoff function χ. The boundary condition (43) implies that

$$\partial_z v(t,x_0,0) = 0 \qquad \forall t \in [0,T].$$

Multiplying relation (44) by $v(t,x_0,z)$ and integrating in z yields

$$\partial_t\|v(t,x_0,\cdot)\|^2_{L^2(\mathbb{R}_+)}+2\|\partial_z v(t,x_0,\cdot)\|^2_{L^2(\mathbb{R}_+)}=\int_{\mathbb{R}_+} f(t,x_0)|v(t,x_0,z)|^2\,dz \leqslant \|f\|_{L^\infty([0,T]\times\Omega)}\|v(t,x_0,\cdot)\|^2_{L^2(\mathbb{R}_+)},$$

so, by the Gronwall lemma and since v vanishes at time $t=0$, we get that $v(t,x_0,z)=0$ for all $t\in[0,T]$ and $z\geqslant 0$.

We continue now with the energy estimates for v. We start with the L^2 estimates.

L^2 estimates for v.

Let $k\in\mathbb{N}^*$, multiply the equation of v given in (28) by $(1+z^{2k})v$, integrate in x and z and remember that $v\cdot n=0$ to obtain

$$\frac{1}{2}\partial_t\|v\|^2_{k,0,0}+\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})v\cdot\nabla u^0\cdot v\,dx\,dz+\iint_{\Omega\times\mathbb{R}_+}(z+z^{2k+1})f\partial_z v\cdot v\,dx\,dz - \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_z^2 v\cdot v\,dx\,dz=0.$$

We used above that u^0 is divergence free and tangent to the boundary to deduce that $\int_\Omega u^0\cdot\nabla v\cdot v\,dx=0$. Since $v\cdot n=0$, integrating by parts in z, using the boundary condition (43) and the regularity hypothesis (31) we get

$$\begin{aligned}\partial_t\|v\|^2_{k,0,0}+2\|v\|^2_{k,0,1}&=-2\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})v\cdot\nabla u^0\cdot v\,dx\,dz\\&\quad+\iint_{\Omega\times\mathbb{R}_+}(1+(2k+1)z^{2k})f|v|^2\,dx\,dz-2\int_\Omega v(x,0)\cdot g(x)\,dx\\&=-2\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})v\cdot\nabla u^0\cdot v\,dx\,dz\\&\quad+\iint_{\Omega\times\mathbb{R}_+}(1+(2k+1)z^{2k})f|v|^2\,dx\,dz+2\int_{\Omega\times\mathbb{R}_+}\partial_z v(x,z)\cdot g(x)\,dx\,dz\\&\leqslant C\|v\|^2_{k,0,0}+C\|v\|_{k,0,1}\|g(x)(1+z^{2k})^{-\frac{1}{2}}\|_{L^2(\Omega\times\mathbb{R}_+)}\\&\leqslant C\|v\|^2_{k,0,0}+\|v\|^2_{k,0,1}+C.\end{aligned}\tag{45}$$

We used above that f and ∇u^0 are uniformly bounded. The Gronwall inequality implies that

$$\|v(t)\|^2_{k,0,0}+\int_0^t\|v(\tau)\|^2_{k,0,1}\,d\tau\leqslant e^{Ct}.\tag{46}$$

H^m_x, $m=1,2$, estimates for v.

For a vector field $\widetilde{u}$, we denote by $\mathcal{D}^m_x(\widetilde{u})$ a linear combination of components of $\widetilde{u}$ and derivatives with respect to x of order $\leqslant m$ of such components with coefficients components of n and derivatives of n. From the definition of the tangential part given in (3), we observe that if α is a multi-index then

$$\partial^\alpha(\widetilde{u}_{tan})=(\partial^\alpha\widetilde{u})_{tan}+\mathcal{D}^{|\alpha|-1}_x(\widetilde{u}).\tag{47}$$

Let now $k \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^d$, $|\alpha| = m$, $m = 1, 2$. Apply ∂_x^α to the equation of v, multiply by $(1+z^{2k})\partial_x^\alpha v$ and integrate in x and z to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})|\partial_x^\alpha v|^2\,dx\,dz \\
&\quad = \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\partial_z^2\partial_x^\alpha v\cdot\partial_x^\alpha v\,dx\,dz - \iint_{\Omega\times\mathbb{R}_+} (z+z^{2k+1})\partial_x^\alpha(f\partial_z v)\cdot\partial_x^\alpha v\,dx\,dz \\
&\qquad - \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\partial_x^\alpha\big[(v\cdot\nabla u^0 + u^0\cdot\nabla_x v)_{tan}\big]\cdot\partial_x^\alpha v\,dx\,dz \equiv I_1 - I_2 - I_3. \quad (48)
\end{aligned}$$

We now estimate each of these terms. Integrating I_1 by parts with respect to z we get

$$\begin{aligned}
I_1 = &-\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})|\partial_z\partial_x^\alpha v|^2\,dx\,dz - 2k\iint_{\Omega\times\mathbb{R}_+} z^{2k-1}\partial_z\partial_x^\alpha v\cdot\partial_x^\alpha v\,dx\,dz \\
&- \int_\Omega \partial_z\partial_x^\alpha v(x,0)\cdot\partial_x^\alpha v(x,0)\,dx \equiv -\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})|\partial_z\partial_x^\alpha v|^2\,dx\,dz - I_{11} - I_{12}.
\end{aligned}$$

Clearly,

$$|I_{11}| \leqslant \frac{1}{4}\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})|\partial_z\partial_x^\alpha v|^2\,dx\,dz + C\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})|\partial_x^\alpha v|^2\,dx\,dz.$$

Next, thanks to (43),

$$\begin{aligned}
|I_{12}| &= \Big|\int_\Omega \partial_x^\alpha g(x)\cdot\partial_x^\alpha v(x,0)\,dx\Big| \\
&= \Big|\iint_{\Omega\times\mathbb{R}_+} \partial_x^\alpha g(x)\cdot\partial_x^\alpha\partial_z v(x,z)\,dx\,dz\Big| \\
&\leqslant \|(1+z^{2k})^{\frac12}\partial_x^\alpha\partial_z v\|_{L^2(\Omega\times\mathbb{R}_+)}\|\partial_x^\alpha g\|_{L^2(\Omega)}\|(1+z^{2k})^{-\frac12}\|_{L^2(\mathbb{R}_+)} \\
&\leqslant \frac{1}{4}\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})|\partial_z\partial_x^\alpha v|^2\,dx\,dz + C.
\end{aligned}$$

We conclude the estimate for I_1:

$$I_1 \leqslant -\frac{1}{2}\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})|\partial_z\partial_x^\alpha v|^2\,dx\,dz + C\|v\|_{k,m,0}^2 + C. \quad (49)$$

We now bound I_2 and consider separately the cases $m = 1$ and $m = 2$. Assume first that $m = |\alpha| = 1$. One has that

$$I_2 = \iint_{\Omega\times\mathbb{R}_+} (z+z^{2k+1})f\,\partial_x^\alpha\partial_z v\cdot\partial_x^\alpha v\,dx\,dz + \iint_{\Omega\times\mathbb{R}_+} (z+z^{2k+1})\partial_x^\alpha f\,\partial_z v\cdot\partial_x^\alpha v\,dx\,dz \equiv I_{21} + I_{22}.$$

As in the L^2 estimates, one has the bound

$$|I_{21}| \leqslant C\|v\|_{k,m,0}^2$$

Next, since $|\alpha| = 1$,

$$\begin{aligned}
|I_{22}| &\leqslant \int_{\mathbb{R}_+} (z+z^{2k+1})\|\partial_x^\alpha f\|_{L^6(\Omega)}\|\partial_z v\|_{L^3(\Omega)}\|\partial_x^\alpha v\|_{L^2(\Omega)}\,dz \\
&\leqslant C\|f\|_{H^2}\int_{\mathbb{R}_+} \|(1+z^{2k+4})^{\frac12}\partial_z v\|_{L^2(\Omega)}^{\frac12}\|(1+z^{2k})^{\frac12}\partial_z v\|_{H^1(\Omega)}^{\frac12}\|(1+z^{2k})^{\frac12}\partial_x^\alpha v\|_{L^2(\Omega)}\,dz \\
&\leqslant C\|v\|_{k+2,0,1}^{\frac12}\|v\|_{k,1,1}^{\frac12}\|v\|_{k,1,0}
\end{aligned}$$

We used above the interpolation inequality $\|h\|_{L^3(\Omega)} \leqslant C\|h\|_{L^2(\Omega)}^{\frac{1}{2}}\|h\|_{H^1(\Omega)}^{\frac{1}{2}}$.

Consider now the case $m = 2$. When we decompose I_2, we have an additional term

$$I_2 = \iint_{\Omega\times\mathbb{R}_+} (z + z^{2k+1}) f\,\partial_x^\alpha \partial_z v \cdot \partial_x^\alpha v\,dx\,dz + \iint_{\Omega\times\mathbb{R}_+} (z + z^{2k+1}) \partial_x^\alpha f\,\partial_z v \cdot \partial_x^\alpha v\,dx\,dz$$
$$+ \iint_{\Omega\times\mathbb{R}_+} (z + z^{2k+1}) \mathcal{D}_x(f) \partial_z \mathcal{D}_x(v) \cdot \partial_x^\alpha v\,dx\,dz \equiv \widetilde{I}_{21} + \widetilde{I}_{22} + \widetilde{I}_{23}.$$

As in the case $m = 1$, we can bound

$$|\widetilde{I}_{21}| \leqslant C\|v\|_{k,m,0}^2$$

and

$$|\widetilde{I}_{23}| \leqslant C\|v\|_{k+2,1,1}^{\frac{1}{2}}\|v\|_{k,2,1}^{\frac{1}{2}}\|v\|_{k,2,0}$$

Now,

$$\begin{aligned}
|\widetilde{I}_{22}| &\leqslant \int_{\mathbb{R}_+} (z + z^{2k+1}) \|\partial_x^\alpha f\|_{L^2(\Omega)}\|\partial_z v\|_{L^\infty(\Omega)}\|\partial_x^\alpha v\|_{L^2(\Omega)}\,dz \\
&\leqslant C\|f\|_{H^2} \int_{\mathbb{R}_+} \|(1 + z^{2k+4})^{\frac{1}{2}}\partial_z v\|_{H^1(\Omega)}^{\frac{1}{2}}\|(1 + z^{2k})^{\frac{1}{2}}\partial_z v\|_{H^2(\Omega)}^{\frac{1}{2}}\|(1 + z^{2k})^{\frac{1}{2}}\partial_x^\alpha v\|_{L^2(\Omega)}\,dz \\
&\leqslant C\|v\|_{k+2,1,1}^{\frac{1}{2}}\|v\|_{k,2,1}^{\frac{1}{2}}\|v\|_{k,2,0},
\end{aligned}$$

where we used the interpolation inequality $\|h\|_{L^\infty(\Omega)} \leqslant C\|h\|_{H^1(\Omega)}^{\frac{1}{2}}\|h\|_{H^2(\Omega)}^{\frac{1}{2}}$.

We conclude that for $m = 1, 2$, we can bound

$$\begin{aligned}
|I_2| &\leqslant C\|v\|_{k,m,0}^2 + C\|v\|_{k+2,m-1,1}^{\frac{1}{2}}\|v\|_{k,m,1}^{\frac{1}{2}}\|v\|_{k,m,0} \\
&\leqslant C\|v\|_{k,m,0}^2 + C\|v\|_{k+2,m-1,1}^2 + \eta\|v\|_{k,m,1}^2,
\end{aligned} \tag{50}$$

where η is a sufficiently small constant to be chosen later. We are left with the estimate of I_3. In view of (47)

$$\begin{aligned}
I_3 &= \iint_{\Omega\times\mathbb{R}_+} (1 + z^{2k}) \partial_x^\alpha \big[(v \cdot \nabla u^0 + u^0 \cdot \nabla_x v)_{tan}\big] \cdot \partial_x^\alpha v\,dx\,dz \\
&= \iint_{\Omega\times\mathbb{R}_+} (1 + z^{2k}) \Big\{ \big[\partial_x^\alpha (v \cdot \nabla u^0 + u^0 \cdot \nabla_x v)\big]_{tan} + \mathcal{D}_x^{m-1}(v \cdot \nabla u^0 + u^0 \cdot \nabla_x v) \Big\} \cdot \partial_x^\alpha v\,dx\,dz \\
&= \iint_{\Omega\times\mathbb{R}_+} (1 + z^{2k}) \Big\{ \partial_x^\alpha (v \cdot \nabla u^0 + u^0 \cdot \nabla_x v) + \mathcal{D}_x^{m-1}(v \cdot \nabla u^0 + u^0 \cdot \nabla_x v) \\
&\qquad\qquad - \big[\partial_x^\alpha (v \cdot \nabla u^0 + u^0 \cdot \nabla_x v) \cdot n\big] n \Big\} \cdot \partial_x^\alpha v\,dx\,dz.
\end{aligned}$$

Applying the differentiation operator ∂_x^α to the equality $n \cdot v = 0$, we infer that $n \cdot \partial_x^\alpha v = \mathcal{D}_x^{m-1}(v)$. Consider now the case $m = 1$. Expanding the integrand in I_3, we find that

$$I_3 = \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) \big[\mathcal{D}_x(v)\mathcal{D}_x(u^0) + v\mathcal{D}_x^2(u^0)\big] \mathcal{D}_x(v)\,dx\,dz + \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) u^0 \cdot \nabla_x \partial_x^\alpha v \cdot \partial_x^\alpha v\,dx\,dz$$
$$+ \iint_{\Omega\times\mathbb{R}_+} (1 + z^{2k}) u^0 \cdot \nabla_x \partial_x^\alpha v \cdot n\ \mathcal{D}_x^0(v)\,dx\,dz.$$

Since u^0 is divergence free and tangent to the boundary, the second integral on the right-hand side vanishes. The third integral can be integrated by part by taking ∇_x out from the term $\nabla_x \partial_x^\alpha v$.

We infer that the third integral is of the same type as the first one. As $u^0 \in L^\infty(0,T;Lip(\Omega))$ we can therefore bound

$$
\begin{aligned}
|I_3| &\leqslant C\int_{\mathbb{R}_+}(1+z^{2k})\|\mathcal{D}_x(v)\|^2_{L^2(\Omega)}\,dz + C\int_{\mathbb{R}_+}(1+z^{2k})\|v\|_{L^4(\Omega)}\|\mathcal{D}_x^2(u^0)\|_{L^4(\Omega)}\|\mathcal{D}_x(v)\|_{L^2(\Omega)}\,dz\\
&\leqslant C\|v\|^2_{k,1,0} + C\|u^0\|_{H^3}\int_{\mathbb{R}_+}(1+z^{2k})\|v\|^2_{H^1(\Omega)}\,dz\\
&\leqslant C\|v\|^2_{k,1,0}
\end{aligned}
$$

On the other hand, in the case $m=2$ a similar analysis can be performed and we obtain that

$$
\begin{aligned}
|I_3| &= \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\big[\mathcal{D}_x^2(v)\mathcal{D}_x(u^0)+\mathcal{D}_x(v)\mathcal{D}_x^2(u^0)+v\mathcal{D}_x^3(u^0)\big]\mathcal{D}_x^2(v)\,dx\,dz\\
&\leqslant \int_{\mathbb{R}_+}(1+z^{2k})\big(\|\mathcal{D}_x^2(v)\|_{L^2(\Omega)}\|\mathcal{D}_x(u^0)\|_{L^\infty(\Omega)}+\|\mathcal{D}_x(v)\|_{L^4(\Omega)}\|\mathcal{D}_x^2(u^0)\|_{L^4(\Omega)}\\
&\qquad\qquad + \|v\|_{L^\infty(\Omega)}\|\mathcal{D}_x^3(u^0)\|_{L^2(\Omega)}\big)\|\mathcal{D}_x^2(v)\|_{L^2(\Omega)}\,dz\\
&\leqslant C\int_{\mathbb{R}_+}(1+z^{2k})\|v\|^2_{H^2(\Omega)}\,dz\\
&\leqslant C\|v\|^2_{k,2,0}.
\end{aligned}
$$

We conclude that in both cases $m=1$ and $m=2$, the following bound holds:

$$
|I_3| \leqslant C\|v\|^2_{k,m,0}. \tag{51}
$$

Putting together estimates (48), (49), (50) and (51), we get that

$$
\partial_t\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha v|^2\,dx\,dz + \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z\partial_x^\alpha v|^2\,dx\,dz \leqslant C + C\|v\|^2_{k,m,0} + C\|v\|^2_{k+2,m-1,1} + \eta\|v\|^2_{k,m,1}.
$$

Summing over $1\leqslant|\alpha|=m$ and adding the result to (45), we conclude that

$$
\partial_t\|v\|^2_{k,m,0} + \|v\|^2_{k,m,1} \leqslant C + C\|v\|^2_{k,m,0} + C\|v\|^2_{k+2,m-1,1} + C\eta\|v\|^2_{k,m,1}.
$$

Choosing $\eta = \frac{1}{2C}$ and applying the Gronwall lemma we finally get

$$
\|v(t)\|^2_{k,m,0} + \frac{1}{2}\int_0^t\|v(\tau)\|^2_{k,m,1}\,d\tau \leqslant Ce^{Ct}\Big(t + \int_0^t\|v(\tau)\|^2_{k+2,m-1,1}\,d\tau\Big). \tag{52}
$$

From the L^2 estimates stated in (46), the left-hand side of (52) when $m=0$ and $k\in\mathbb{N}^*$ is bounded up to time T. Applying (52) for $m=1$ and the result for $m=0$ and $k+2$, we next deduce that the left-hand side is bounded for $m=1$ and $k\in\mathbb{N}^*$. We deduce similarly that it is bounded for $m=2$ too, and this proves that the solution v belongs to the space $L^\infty(0,T;H^{k,2,0})\cap L^2(0,T;H^{k,2,1})$.

Uniform estimates for $\partial_z v$.

We adapt an a priori estimate discovered by Kiselev and Ladyshenskaya [25] in the setting of pressureless Navier-Stokes equations and based on a maximum principle. We set

$$
\alpha := 6\|\nabla u_0\|_{L^\infty([0,T]\times\Omega)} + 2\|f\|_{L^\infty([0,T]\times\Omega)} \text{ and } w(t,x,z) := \partial_z v(t,x,z)e^{-\alpha t}.
$$

From the equation of v given in (28) and the orthogonality condition (2) we deduce that w satisfies the equation

$$\partial_t|w|^2 + 2(w\cdot\nabla u^0 + u^0\cdot\nabla_x w)\cdot w + fz\partial_z(|w|^2) + 2f|w|^2 - \partial_z^2(|w|^2) + 2|\partial_z w|^2 + 2\alpha|w|^2 = 0.$$

If the function $|w|^2$ attains its maximum on $[0,T]\times\overline{\Omega}\times\mathbb{R}_+$ in a point P whose coordinates (t_P, x_P, z_P) satisfy $t_P>0$, $z_P>0$ then in this point P one has that

$$\partial_t(|w|^2)\geq 0,\ \partial_z(|w|^2)=0,\ u^0\cdot\nabla_x(|w|^2)=0 \text{ (since } u^0 \text{ is tangent to } \partial\Omega) \text{ and } \partial_z^2(|w|^2)\leqslant 0$$

so that

$$2\alpha|w|^2 \leqslant -2(w\cdot\nabla u^0)\cdot w - 2f|w|^2 \leqslant \alpha|w|^2,$$

which is impossible unless $w=0$. Using the initial and boundary conditions (29) and (30) for the cases $t_P=0$ or $z_P=0$ we then deduce the uniform estimate (33) and thus complete the proof of Proposition 5.

6 Estimates of the remainder

We start by using the regularity of v expressed in (32) as well as in Lemma 3 to deduce some bounds on various norms of the boundary layer profiles that will be used in the sequel. We observe first that if $H(x,z)$ is a sufficiently regular function defined on $\Omega\times\mathbb{R}_+$, $p\in[2,\infty]$, $m\geqslant \frac{3}{2}-\frac{3}{p}$ if $p<\infty$ and $m>\frac{3}{2}$ if $p=\infty$, then

$$\|H(x,\varphi/\sqrt{\nu})\|_{L^p(\Omega)} \leqslant \big\|\|H\|_{L^\infty_z(\mathbb{R}_+)}\big\|_{L^p_x(\Omega)} \leqslant \big\|\|\partial_z H\|_{L^1_z(\mathbb{R}_+)}\big\|_{L^p_x(\Omega)} \leqslant \big\|\|\partial_z H\|_{L^p_x(\Omega)}\big\|_{L^1_z(\mathbb{R}_+)}$$
$$\leqslant \|(1+z)\partial_z H\|_{L^2_z(\mathbb{R}_+;L^p_x(\Omega))}\|(1+z)^{-1}\|_{L^2(\mathbb{R}_+)} \leqslant C\|H\|_{1,m,1}, \quad (53)$$

where we used the Sobolev embedding $H^m(\Omega)\hookrightarrow L^p(\Omega)$.

We collect now several estimates on the first boundary layer profile by using the known regularity of v, relation (53) and Lemma 3:

$$\|(1+z)\nabla_x v\big|_{z=\varphi/\sqrt{\nu}}\|_{L^2(\Omega)} \leqslant C\nu^{\frac14}\|(1+z)\nabla_x v\|_{L^2_z(\mathbb{R}_+;H^1_x(\Omega))}$$
$$\leqslant C\nu^{\frac14}\|v\|_{1,1,0} \text{ bounded in } L^\infty(0,T), \quad (54)$$
$$\|v\|_{L^1_z(\mathbb{R}_+;H^2_x(\Omega))} \leqslant C\|(1+z)v\|_{L^2_z(\mathbb{R}_+;H^2_x(\Omega))} \leqslant C\|v\|_{1,2,0} \text{ bounded in } L^\infty(0,T), \quad (55)$$
$$\|z\partial_z v\|_{L^1_z(\mathbb{R}_+;H^2_x(\Omega))} \leqslant C\|v\|_{2,2,1} \text{ bounded in } L^2(0,T), \quad (56)$$
$$\|\nabla_x\partial_z v\|_{L^2_z(\mathbb{R}_+;H^1_x(\Omega))} \leqslant C\|v\|_{0,2,1} \text{ bounded in } L^2(0,T), \quad (57)$$
$$\|\nabla_x v(x,\varphi/\sqrt{\nu})\|_{L^6(\Omega)} \leqslant C\|v\|_{1,2,1} \text{ bounded in } L^2(0,T), \quad (58)$$
$$\|v(x,\varphi/\sqrt{\nu})\|_{L^\infty(\Omega)} \leqslant C\|v\|_{1,2,1} \text{ bounded in } L^2(0,T), \quad (59)$$
$$\|\triangle_x v(x,\varphi/\sqrt{\nu})\|_{L^2(\Omega)} \leqslant C\|v\|_{1,2,1} \text{ bounded in } L^2(0,T), \quad (60)$$

Next, from relation (36) one has that

$$\|\nabla_x w(x,\varphi/\sqrt{\nu})\|_{L^2(\Omega)} + \|\overline{w}(x,0)\|_{H^1(\Omega)}$$
$$\leqslant C\|v\|_{L^1_z(\mathbb{R}_+;H^2_x(\Omega))} \leqslant C\|v\|_{1,2,0} \text{ bounded in } L^\infty(0,T), \quad (61)$$
$$\|w(x,\varphi/\sqrt{\nu})\|_{L^6(\Omega)} \leqslant C\|\nabla_x v\|_{L^1_z(\mathbb{R}_+;L^6_x(\Omega))} \leqslant C\|v\|_{1,2,0} \text{ bounded in } L^\infty(0,T), \quad (62)$$
$$\|\partial_z w(x,\varphi/\sqrt{\nu})\|_{L^6(\Omega)} = \|n\operatorname{div}_x v(x,\varphi/\sqrt{\nu})\|_{L^6(\Omega)} \text{ bounded in } L^2(0,T), \quad (63)$$

We now estimate the H^1 norm of b_ν defined in (42). With similar arguments as above one can prove that

$$\sqrt{\nu}\|b_\nu\|_{H^1(\Omega)} \leqslant C\sqrt{\nu}\|\overline{w}\|_{L^\infty_z(\mathbb{R}_+;H^1(\Omega))} + C\|v\|_{L^\infty_z(\mathbb{R}_+;H^2(\Omega))} \leqslant C\|v\|_{1,2,1} \text{ bounded in } L^2(0,T). \quad (64)$$

Finally, from the definition of q given in (34) we infer that

$$\|\nabla_x q(x,\varphi/\sqrt{\nu})\|_{L^2(\Omega)} \leqslant C\|v\|_{L^1_z(\mathbb{R}_+;H^2(\Omega))} \leqslant C\|v\|_{1,2,0} \text{ bounded in } L^\infty(0,T). \quad (65)$$

From now on, all functions depending on z are assumed to be evaluated in $z=\varphi/\sqrt{\nu}$ unless otherwise specified. We will now estimate the remainder R^ν in the spaces $L^\infty(0,T;L^2)$ and $L^2(0,T;H^1)$. We denote by $\mathbb{P}$ the Leray projector, *i.e.* the L^2 orthogonal projection on the space of divergence free vector fields tangent to the boundary. We decompose $R^\nu=\mathbb{P}R^\nu+(I-\mathbb{P})R^\nu$ and show first that $(I-\mathbb{P})R^\nu$ is bounded in H^1 independently of ν. More precisely, we start by proving the following lemma.

Lemma 6. *The family $(I-\mathbb{P})R^\nu$ is bounded in $L^\infty(0,T;H^1(\Omega))$.*

Proof. Since $u^\nu \in C^0_w\big([0,+\infty);L^2_\sigma(\Omega)\big)\cap L^2_{\text{loc}}\big([0,+\infty);H^1(\Omega)\big)$, we immediately deduce that $R^\nu\in L^2(0,T;H^1(\Omega))$ so $R^\nu(t)\in H^1(\Omega)$ for almost every $t\in[0,T]$. It is therefore sufficient to show that there exists a constant C independent of ν such that if $R^\nu(t)\in H^1(\Omega)$ then $\|(I-\mathbb{P})R^\nu(t)\|_{H^1}\leqslant C$. Let t be such a time. For notational convenience, we drop the dependence on t in the rest of this proof but we keep in mind that we assume that $R^\nu\in H^1(\Omega)$

By the well-known properties of the Leray projector, we know that there exists a scalar function $\rho\in H^2(\Omega)$ such that $(I-\mathbb{P})R^\nu=\nabla\rho$, that is

$$R^\nu=\mathbb{P}R^\nu+\nabla\rho.$$

Taking the divergence of the above relation and also the scalar product with n restricted to the boundary, we infer from (36), (39) and (40) that ρ verifies the following Neumann problem

$$\begin{aligned}\triangle\rho &= -\operatorname{div}_x w(x,\varphi/\sqrt{\nu}) \quad \text{in} \quad \Omega\\ \frac{\partial\rho}{\partial n} &= -\overline{w}(x,0) \quad \text{on} \quad \partial\Omega.\end{aligned}$$

The standard regularity theory for the Neumann problem for the Laplacian implies that

$$\|\nabla\rho\|_{H^1}\leqslant C\|\triangle\rho\|_{L^2(\Omega)}+C\|\frac{\partial\rho}{\partial n}\|_{H^{\frac12}(\partial\Omega)}$$

for some constant C that depends only on Ω. We therefore deduce from (61) that

$$\|(I-\mathbb{P})R^\nu\|_{H^1(\Omega)}=\|\nabla\rho\|_{H^1(\Omega)}\leqslant C\|\operatorname{div}_x w(x,\varphi/\sqrt{\nu})\|_{L^2(\Omega)}+C\|\overline{w}(x,0)\|_{H^1(\Omega)}\leqslant C.$$

This concludes the proof of the lemma. □

We will now estimate $\|\mathbb{P}R^\nu\|_{L^2}$ which is the main part of the proof. In order to avoid estimating the unknown pressure term $\nabla\pi^\nu$, we need to multiply the equation of R^ν given in (38) by $\mathbb{P}R^\nu$ and integrate in space and from 0 to t. The regularity at hand is not sufficient to be able to do that. Indeed, the integral $\int_0^t\int_\Omega u^\nu\cdot\nabla R^\nu\cdot\mathbb{P}R^\nu$ is not convergent. We would like to be able to write

$$\begin{aligned}\int_0^t\int_\Omega u^\nu\cdot\nabla R^\nu\cdot\mathbb{P}R^\nu &= \int_0^t\int_\Omega u^\nu\cdot\nabla(I-\mathbb{P})R^\nu\cdot\mathbb{P}R^\nu+\int_0^t\int_\Omega u^\nu\cdot\nabla\mathbb{P}R^\nu\cdot\mathbb{P}R^\nu\\ &= \int_0^t\int_\Omega u^\nu\cdot\nabla(I-\mathbb{P})R^\nu\cdot\mathbb{P}R^\nu.\end{aligned}$$

We observe by Lemma 6 that the last integral is convergent. "We used" the usual cancellation property to say that the last integral on the first line of the relation above vanishes. This cancellation property cannot be used directly since that integral is not convergent. However, there is a classical trick that allows to do just that at the price of assuming an energy inequality on the solution and of obtaining at the end an inequality instead of an equality. The idea is the following. Let $\widetilde{u}^\nu(t,x) = u^0(t,x) + \sqrt{\nu}\, v\big(t,x,\frac{\varphi(x)}{\sqrt{\nu}}\big) + \nu\, w\big(t,x,\frac{\varphi(x)}{\sqrt{\nu}}\big)$ so that $u^\nu = \widetilde{u}^\nu + \nu R^\nu$. The equation of R^ν given in (38) was deduced by writing a PDE for $\widetilde{u}^\nu$ and subtracting it from the PDE for u^ν. We would like to multiply the equation of R^ν by $\mathbb{P}R^\nu$, *i.e.* to write $\mathrm{Eqn}(R^\nu)\cdot \mathbb{P}R^\nu = 0$, where $\mathrm{Eqn}(R^\nu)$ is the PDE verified by R^ν. This is not legitimate for the reason explained above. However, we can write

$$\begin{aligned}\nu^2\,\mathrm{Eqn}(R^\nu)\cdot\mathbb{P}R^\nu &= [\mathrm{Eqn}(u^\nu) - \mathrm{Eqn}(\widetilde{u}^\nu)]\cdot(u^\nu - \mathbb{P}\widetilde{u}^\nu)\\ &= \mathrm{Eqn}(u^\nu)\cdot u^\nu - \mathrm{Eqn}(u^\nu)\cdot\mathbb{P}\widetilde{u}^\nu - \mathrm{Eqn}(\widetilde{u}^\nu)\cdot u^\nu + \mathrm{Eqn}(\widetilde{u}^\nu)\cdot\mathbb{P}\widetilde{u}^\nu.\end{aligned}$$

Since $\widetilde{u}^\nu$ is sufficiently smooth, all terms on the right-hand side make sense except for $\mathrm{Eqn}(u^\nu)\cdot u^\nu$. But the energy inequality (9) verified by u^ν is equivalent to saying that $\mathrm{Eqn}(u^\nu)\cdot u^\nu \leqslant 0$. Therefore, even though we cannot make sense of $\mathrm{Eqn}(R^\nu)\cdot\mathbb{P}R^\nu = 0$, the energy inequality (9) allows us to say that $\mathrm{Eqn}(R^\nu)\cdot\mathbb{P}R^\nu \leqslant 0$ if we replace the divergent integral $\int_0^t\int_\Omega u^\nu\cdot\nabla R^\nu\cdot\mathbb{P}R^\nu$ by $\int_0^t\int_\Omega u^\nu\cdot\nabla(I-\mathbb{P})R^\nu\cdot\mathbb{P}R^\nu$. In conclusion, we can multiply (38) by $\mathbb{P}R^\nu$ in the manner described above to obtain

$$\frac{1}{2}\|\mathbb{P}R^\nu(t)\|_{L^2}^2 - \nu\int_0^t\int_\Omega \triangle R^\nu\cdot\mathbb{P}R^\nu \leqslant \sum_{k=1}^{9}\int_0^t A_k \tag{66}$$

where

$$\begin{aligned} A_1 &= -\int_\Omega u^\nu\cdot\nabla(I-\mathbb{P})R^\nu\cdot\mathbb{P}R^\nu, & A_2 &= -\frac{1}{\sqrt{\nu}}\int_\Omega u^0\cdot n\,\partial_z w\cdot\mathbb{P}R^\nu,\\ A_3 &= -\nu\int_\Omega w\cdot\nabla_x w\cdot\mathbb{P}R^\nu, & A_4 &= -\nu\int_\Omega R^\nu\cdot\nabla_x w\cdot\mathbb{P}R^\nu,\\ A_5 &= \nu\int_\Omega \triangle\big[w(x,\varphi/\sqrt{\nu})\big]\cdot\mathbb{P}R^\nu, & A_6 &= -\int_\Omega \partial_t w\cdot\mathbb{P}R^\nu,\\ A_k &= -\int_\Omega F_k\cdot\mathbb{P}R^\nu,\quad k\in\{7,8,9\}, \end{aligned}$$

and

$$\begin{aligned} F_7 = (u^0+\sqrt{\nu}v)\cdot\nabla_x w + w\cdot\nabla u^0 + v\cdot\nabla_x v + \sqrt{\nu}w\cdot\nabla_x v\\ + \sqrt{\nu}w\cdot n\,\partial_z w - \triangle u^0 - 2\partial_n\partial_z v - \triangle\varphi\partial_z v - \nabla_x q - \sqrt{\nu}\triangle_x v, \end{aligned}$$

$$F_8 = w\cdot n\,\partial_z v + R^\nu\cdot n\,\partial_z v + R^\nu\cdot\nabla u^0,$$

and

$$F_9 = \sqrt{\nu}R^\nu\cdot(n\,\partial_z w + \nabla_x v)$$

We now estimate each of the terms in (66). We handle first the Laplacian term. By (26) and (41)

$$\begin{aligned}-\nu\int_\Omega \triangle R^\nu\cdot\mathbb{P}R^\nu =& 2\nu\int_\Omega D(R^\nu)\cdot D(\mathbb{P}R^\nu) - 2\nu\int_{\partial\Omega}\big[D(R^\nu)n\big]_{tan}\cdot\mathbb{P}R^\nu\\ =& 2\nu\|D(R^\nu)\|_{L^2}^2 + 2\nu\int_\Omega D(R^\nu)\cdot D[(\mathbb{P}-I)R^\nu] + 2\nu\int_{\partial\Omega}(\alpha R^\nu - b_\nu + \frac{1}{2}\nabla_x\overline{w})\cdot\mathbb{P}R^\nu\end{aligned}$$

We observe now by the Stokes formula that

$$\int_{\partial\Omega}(\alpha R^\nu - b_\nu)\cdot \mathbb{P}R^\nu = \int_\Omega \operatorname{div}\big[n(\alpha R^\nu - b_\nu)\cdot \mathbb{P}R^\nu\big]$$
$$\leqslant C\|\alpha R^\nu - b_\nu\|_{H^1}\|\mathbb{P}R^\nu\|_{L^2} + C\|\alpha R^\nu - b_\nu\|_{L^2}\|\mathbb{P}R^\nu\|_{H^1}$$
$$\leqslant C\|R^\nu\|_{L^2}\|R^\nu\|_{H^1} + C\|R^\nu\|_{H^1}\|b_\nu\|_{H^1}$$
$$\leqslant C\|R^\nu\|_{L^2}\|R^\nu\|_{H^1} + \eta\|R^\nu\|^2_{H^1} + C\|b_\nu\|^2_{H^1},$$

where η is a sufficiently small constant independent of ν to be chosen later. Next,

$$\begin{aligned}\int_{\partial\Omega}\nabla\overline{w}\cdot\mathbb{P}R^\nu &= \int_\Omega \operatorname{div}[n\nabla\overline{w}(x,0)\cdot\mathbb{P}R^\nu]\\
&= \int_\Omega \operatorname{div}(n)\nabla\overline{w}(x,0)\cdot\mathbb{P}R^\nu + \sum_i\Big(\int_\Omega n_i\nabla\overline{w}(x,0)\cdot\partial_i\mathbb{P}R^\nu + \int_\Omega n_i\nabla\partial_i\overline{w}(x,0)\cdot\mathbb{P}R^\nu\Big)\\
&= \int_\Omega \operatorname{div}(n)\nabla\overline{w}(x,0)\cdot\mathbb{P}R^\nu + \sum_i\Big(\int_\Omega n_i\nabla\overline{w}(x,0)\cdot\partial_i\mathbb{P}R^\nu - \int_\Omega \partial_i\overline{w}(x,0)\nabla n_i\cdot\mathbb{P}R^\nu\Big)\\
&\leqslant C\|\overline{w}(x,0)\|_{H^1}\|R^\nu\|_{H^1}\\
&\leqslant C\|R^\nu\|_{H^1}.\end{aligned}$$

We used above (61) and the fact that $\mathbb{P}R^\nu$ is divergence free and tangent to the boundary. The above relations together with Lemma 6 now imply that

$$-\nu\int_\Omega \triangle R^\nu\cdot\mathbb{P}R^\nu \geqslant 2\nu\|D(R^\nu)\|^2_{L^2} - C\nu\|R^\nu\|_{L^2}\|R^\nu\|_{H^1} - C\nu\|R^\nu\|_{H^1} - \eta\nu\|R^\nu\|^2_{H^1} - C\nu\|b^\nu\|^2_{H^1}.$$

To complete the estimate of the Laplacian term, it remains to recall the second Korn inequality which states that $\|R^\nu\|_{H^1}\simeq\|R^\nu\|_{L^2}+\|D(R^\nu)\|_{L^2}$. Consequently, there exists a constant δ_0 independent of ν such that

$$\begin{aligned}-\nu\int_\Omega \triangle R^\nu\cdot\mathbb{P}R^\nu &\geqslant 2\delta_0\nu\|R^\nu\|^2_{H^1} - C\|R^\nu\|^2_{L^2} - C\nu\|R^\nu\|_{L^2}\|R^\nu\|_{H^1} - C\nu\|R^\nu\|_{H^1}\\
&\qquad - \eta\nu\|R^\nu\|^2_{H^1} - C\nu\|b^\nu\|^2_{H^1}\\
&\geqslant (\delta_0-\eta)\nu\|R^\nu\|^2_{H^1} - C\|R^\nu\|^2_{L^2} - f_0^\nu,\end{aligned}\tag{67}$$

where, according to (64), f_0^ν is a function of time which is bounded in $L^1(0,T)$ independently of ν.

Estimate of A_1. One has that

$$\begin{aligned}A_1 &= -\int_\Omega u^\nu\cdot\nabla(I-\mathbb{P})R^\nu\cdot R^\nu\\
&\leqslant \|u^\nu-u^0\|_{L^6}\|(I-\mathbb{P})R^\nu\|_{H^1}\|R^\nu\|_{L^3} + \|u^0\|_{L^\infty}\|(I-\mathbb{P})R^\nu\|_{H^1}\|R^\nu\|_{L^2}\\
&\leqslant C\|\sqrt{\nu}v+\nu w\|_{L^6}\|R^\nu\|^{\frac12}_{L^2}\|R^\nu\|^{\frac12}_{H^1} + C\nu\|R^\nu\|^{\frac12}_{L^2}\|R^\nu\|^{\frac32}_{H^1} + C\|R^\nu\|_{L^2}\\
&\leqslant \eta\nu\|R^\nu\|^2_{H^1} + (1+\|R^\nu\|^2_{L^2})f_1^\nu,\end{aligned}\tag{68}$$

where we used the interpolation inequality $\|R^\nu\|_{L^3}\leqslant C\|R^\nu\|^{\frac12}_{L^2}\|R^\nu\|^{\frac12}_{H^1}$, relations (59), (62) and f_1^ν is a function of time which is bounded in $L^1(0,T)$ independently of ν.

Estimate of A_2. Thanks to (36) and (54),

$$A_2 = -\frac{1}{\sqrt{\nu}}\int_\Omega u^0\cdot n\,\partial_z w\cdot\mathbb{P}R^\nu\,dx = -\int_\Omega\frac{u^0\cdot n}{\varphi}\,(z\operatorname{div}_x v)\big|_{z=\varphi/\sqrt{\nu}}\,n\cdot\mathbb{P}R^\nu\,dx \leqslant C\|\mathbb{P}R^\nu\|_{L^2}.\tag{69}$$

Estimate of A_3 and A_4. In view of (61) and (62)

$$A_3 = -\nu \int_\Omega w \cdot \nabla_x w \cdot \mathbb{P}R^\nu \leqslant \nu \|w\|_{L^6} \|\nabla_x w\|_{L^2} \|\mathbb{P}R^\nu\|_{L^3} \leqslant C\nu \|\mathbb{P}R^\nu\|_{L^2}^{\frac{1}{2}} \|\mathbb{P}R^\nu\|_{H^1}^{\frac{1}{2}}$$
$$\leqslant \eta\nu \|R^\nu\|_{H^1}^2 + C\|\mathbb{P}R^\nu\|_{L^2}^{\frac{2}{3}} \quad (70)$$

and

$$A_4 = -\nu \int_\Omega R^\nu \cdot \nabla_x w \cdot \mathbb{P}R^\nu \leqslant \nu \|\nabla_x w\|_{L^2} \|R^\nu\|_{L^4}^2 \leqslant C\nu \|R^\nu\|_{L^2}^{\frac{1}{2}} \|R^\nu\|_{H^1}^{\frac{3}{2}}$$
$$\leqslant \eta\nu \|R^\nu\|_{H^1}^2 + C\|R^\nu\|_{L^2}^2. \quad (71)$$

We used above the interpolation inequality $\|R^\nu\|_{L^4} \leqslant C\|R^\nu\|_{L^2}^{\frac{1}{4}} \|R^\nu\|_{H^1}^{\frac{3}{4}}$.

Estimate of A_5. We integrate by parts to obtain

$$A_5 = \nu \int_\Omega \triangle\big[w(x, \varphi/\sqrt{\nu})\big] \cdot \mathbb{P}R^\nu = -\nu \int_\Omega \nabla\big[w(x, \varphi/\sqrt{\nu})\big] \cdot \nabla \mathbb{P}R^\nu + \nu \int_{\partial\Omega} \partial_n\big[w(x, \varphi/\sqrt{\nu})\big] \cdot \mathbb{P}R^\nu$$
$$\equiv -I_1 + I_2.$$

From (61) and (63) we can deduce that

$$|I_1| \leqslant \nu \|\nabla\big[w(x, \varphi/\sqrt{\nu})\big]\|_{L^2} \|\mathbb{P}R^\nu\|_{H^1} \leqslant \eta\nu \|R^\nu\|_{H^1}^2 + f_2^\nu$$

where f_2^ν is a time dependent function bounded independently of ν in $L^1(0, T)$. On the other hand, one has from (36) that $\partial_n\big[w(x, \varphi/\sqrt{\nu})\big] = \partial_n\big[\overline{w}(x, \varphi/\sqrt{\nu})\big] n + \overline{w} \partial_n n$ and since $\mathbb{P}R^\nu$ is tangent to the boundary we get that

$$I_2 = \nu \int_{\partial\Omega} \overline{w}(x, 0)\, \partial_n n \cdot \mathbb{P}R^\nu \leqslant \nu \|\overline{w}(x, 0)\|_{L^2(\partial\Omega)} \|\mathbb{P}R^\nu\|_{L^2(\partial\Omega)} \leqslant C\nu \|\overline{w}(x, 0)\|_{H^1} \|R^\nu\|_{H^1}$$
$$\leqslant \eta\nu \|R^\nu\|_{H^1}^2 + C,$$

where we used (61). We infer that

$$A_5 \leqslant 2\eta\nu \|R^\nu\|_{H^1}^2 + f_2^\nu + C. \quad (72)$$

Estimate of A_6. We use (28) and (36) to write

$$\begin{aligned} -A_6 = \int_\Omega \partial_t w \cdot \mathbb{P}R^\nu &= \int_\Omega \partial_t \overline{w}\, n \cdot \mathbb{P}R^\nu\, dx \\ &= -\int_\Omega \int_{\varphi/\sqrt{\nu}}^\infty \partial_t \operatorname{div}_x v(x, z)\, dz\ n \cdot \mathbb{P}R^\nu\, dx \\ &= \int_\Omega \int_{\varphi/\sqrt{\nu}}^\infty \operatorname{div}_x \big\{\big[v(x, z) \cdot \nabla u^0 + u^0 \cdot \nabla_x v(x, z)\big]_{tan}\big\}\, dz\ n \cdot \mathbb{P}R^\nu\, dx \\ &\quad + \int_\Omega \int_{\varphi/\sqrt{\nu}}^\infty \operatorname{div}_x\big[f\, z \partial_z v(x, z)\big]\, dz\ n \cdot \mathbb{P}R^\nu\, dx \\ &\quad - \int_\Omega \int_{\varphi/\sqrt{\nu}}^\infty \operatorname{div}_x \partial_z^2 v(x, z)\, dz\ n \cdot \mathbb{P}R^\nu\, dx \\ &\equiv J_1 + J_2 - J_3 \end{aligned}$$

We bound each of these terms. First, from (31), (55) and (56) we infer that

$$|J_1 + J_2| \leqslant C\|n \cdot \mathbb{P}R^\nu\|_{L^2} \big(\|v\|_{L^1_z(\mathbb{R}_+; H^2(\Omega))} + \|z \partial_z v\|_{L^1_z(\mathbb{R}_+; H^2(\Omega))}\big) \leqslant \|\mathbb{P}R^\nu\|_{L^2}^2 + f_3^\nu,$$

where f_3^ν is bounded in $L^1(0,T)$. Finally, we use Lemma 3 to bound

$$|J_3| = \Big|\int_\Omega \operatorname{div}_x \partial_z v\big|_{z=\varphi/\sqrt{\nu}}\, n\cdot \mathbb{P}R^\nu\,dx\Big| \leqslant C\|\operatorname{div}_x \partial_z v\big|_{z=\varphi/\sqrt{\nu}}\|_{L^2}\|\mathbb{P}R^\nu\|_{L^2}$$
$$\leqslant C\nu^{\frac14}\|\operatorname{div}_x \partial_z v\|_{L^2_z(\mathbb{R}_+;H^1_x(\Omega))}\|\mathbb{P}R^\nu\|_{L^2} \leqslant \|\mathbb{P}R^\nu\|_{L^2}^2 + f_4^\nu,$$

where, by (57), f_4^ν is bounded in $L^1(0,T)$. Therefore,

$$A_6 \leqslant 2\|\mathbb{P}R^\nu\|_{L^2}^2 + f_3^\nu + f_4^\nu. \tag{73}$$

Estimate of A_7. We observe that according to Lemma 3 and to relations (54), (57), (58), (59), (60), (61), (62), (63), (65) one has that $\|F_7\|_{L^2}$ is bounded in $L^1(0,T)$ independently of ν. Therefore, we can estimate

$$|A_7| = \Big|\int_\Omega F_7\cdot \mathbb{P}R^\nu\Big| \leqslant \|F_7\|_{L^2}\|\mathbb{P}R^\nu\|_{L^2} \leqslant \|F_7\|_{L^2} + \|F_7\|_{L^2}\|\mathbb{P}R^\nu\|_{L^2}^2. \tag{74}$$

Estimate of A_8. One can write

$$\begin{aligned}|A_8| =\Big|\int_\Omega F_8\cdot \mathbb{P}R^\nu\Big| &\leqslant C\|R^\nu\|_{L^2}\|\partial_z v\|_{L^\infty}(\|w\|_{L^2} + \|R^\nu\|_{L^2}) + C\|R^\nu\|_{L^2}^2\\ &\leqslant C\|R^\nu\|_{L^2}^2(1+\|\partial_z v\|_{L^\infty}) + C\|\partial_z v\|_{L^\infty}^2\|w\|_{L^2}^2\\ &\leqslant C\|R^\nu\|_{L^2}^2 + C\end{aligned} \tag{75}$$

where we used (33) and (62).

Estimate of A_9. In view of (58) and (63) we can write

$$\begin{aligned}|A_9| =\Big|\int_\Omega F_9\cdot \mathbb{P}R^\nu\Big| &\leqslant C\sqrt{\nu}\|R^\nu\|_{L^3}^2\|n\partial_z w + \nabla_x v\|_{L^3}\\ &\leqslant C\sqrt{\nu}\|R^\nu\|_{L^2}\|R^\nu\|_{H^1}\|n\partial_z w + \nabla_x v\|_{L^3} \leqslant \eta\nu\|R^\nu\|_{H^1}^2 + f_5^\nu\|R^\nu\|_{L^2}^2,\end{aligned} \tag{76}$$

where the time dependent function f_5^ν is bounded in $L^1(0,T)$ independently of ν.

Collecting now relations (66), (67), (68), (69), (70), (71), (72), (73), (74), (75), (76) and using Lemma 6 to write $\|R^\nu\|_{L^2} \leqslant \|\mathbb{P}R^\nu\|_{L^2} + \|(I-\mathbb{P})R^\nu\|_{L^2} \leqslant \|\mathbb{P}R^\nu\|_{L^2} + C$, we finally obtain that

$$\|\mathbb{P}R^\nu(t)\|_{L^2}^2 + 2\nu(\delta_0 - 7\eta)\int_0^t \|R^\nu(\tau)\|_{H^1}^2\,d\tau \leqslant \int_0^t g_1^\nu(\tau)\,d\tau + \int_0^t g_2^\nu(\tau)\|\mathbb{P}R^\nu(t)\|_{L^2}^2\,d\tau, \qquad \forall t\in[0,T],$$

where g_1^ν and g_2^ν are bounded independently of ν in $L^1(0,T)$. Choosing now $\eta = \delta_0/14$ and applying the Gronwall lemma yields the desired conclusion:

$$\|\mathbb{P}R^\nu(t)\|_{L^2}^2 + \nu\delta_0\int_0^t \|R^\nu(\tau)\|_{H^1}^2\,d\tau \leqslant \int_0^t g_1^\nu(\tau)\,d\tau\,\exp\Big(\int_0^t g_2^\nu(\tau)\,d\tau\Big), \qquad \forall t\in[0,T].$$

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