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Viscous fingering in Hele-Shaw cells

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The phenomenon of interfacial motion between two immiscible viscous fluids in the narrow gap between two parallel plates (Hele-Shaw cell) is considered. This flow is currently of interest because of its relation to pattern selection mechanisms and the formation of fractal structures in a number of physical applications. Attention is concentrated on the fingers that result from the instability when a less-viscous fluid drives a more-viscous one. The status of the problem is reviewed and progress with the thirty-year-old problem of explaining the shape and stability of the fingers is described. The paradoxes and controversies are both mathematical and physical. Theoretical results on the structure and stability of steady shapes are presented for a particular formulation of the boundary conditions at the interface and compared with the experimental phenomenon. Alternative boundary conditions and future approaches are discussed.

1. History and status

Flow in a porous medium is a challenging scientific problem of great technological importance. Even though it usually concerns viscous fluids under circumstances where the nonlinear convective terms in the equations of motion are negligible, the mathematical difficulties caused by the randomness and chaotic structure of the medium make fundamental theory as difficult as for turbulence. Moreover, experiments are hard as it is not easy to see into a porous medium or know exactly the position of a measuring probe. The difficulties are compounded when the fluid is not homogeneous. For the case of two immiscible fluids, the added complication of moving microscopic and macroscopic interfaces makes the problem even more intractable.

In about 1956, Sir Geoffrey Taylor paid a visit to the Humble Oil Company and became interested in problems of two-phase flow in porous media. He worked out the macroscopic instability which can arise when a less-viscous fluid drives a more-viscous one and which is at least partly responsible for the coneing in processes of secondary recovery in oil fields. He also realized that two-dimensional flow in a porous medium is modelled by flow in a Hele-Shaw (1898) apparatus consisting of two flat parallel plates separated by a small gap b. Then the average two-dimensional velocity $\boldsymbol{u}$ of a viscous fluid in the space between the plates is related to the pressure p by the formula

$$\boldsymbol{u} = -\frac{b^2}{12\mu}\,\mathrm{grad}\,p, \quad \mathrm{div}\,\boldsymbol{u} = 0 \tag{1.1}$$

where μ is the viscosity of the fluid. This is identical with Darcy's law for motion in a porous medium of permeability $b^2/12$. But it is, of course, an approximation valid when the gap or transverse dimension b is small compared with variations of scale a, say, in the lateral dimension parallel to the plates; see also Lamb (1932, §330).

(It is a curiosity that the original purpose of the Hele-Shaw apparatus was to model two-dimensional irrotational flow of a perfect fluid, the equations being the same with the velocity potential $-b^2p/12\mu$.)

The predicted instability was verified qualitatively and then the problem of flow in the Hele-Shaw apparatus took on a life of its own, as the finite-amplitude stages of the instability displayed a fascinating and still not completely explained phenomenon. It was observed that a characteristic feature of the intermediate stages of the growth of the instability was the formation of fingers. One finger would then grow at the expense of its neighbours, and eventually the apparatus, which was in the shape of a long narrow cell henceforth referred to as a Hele-Shaw cell, would contain just one, long, steadily propagating, stable finger. A theory of these fingers was worked out under certain assumptions about the appropriate boundary conditions on the interface between the fluids and compared with experimental observations. This work is described in a now classical paper by Saffman & Taylor (1958, henceforth called ST). In particular see figures 2, 3 and 4 of ST. Many similar photographs have been taken in recent years.

The theory was both successful and unsuccessful. On the assumption that surface tension did not affect the kinematic and dynamic boundary conditions at the interface between the two fluids, it predicted the shapes of steady fingers but found a continuum of solutions. The width of the finger was not predicted by the calculation. However, if the experimentally observed width was chosen, then the observed and calculated shapes agreed exactly in the limit of relatively large capillary number $Ca = \mu U_F/T$, where U_F is the velocity of the finger and T is the interfacial tension. This width was half the cell width. In a further development of the theory Taylor & Saffman (1958) examined the stability of the fingers to infinitesimal disturbances with the same boundary conditions as used for the calculation of steady finger shapes. It was found that the fingers were unstable. Observations, for example ST figure 12, show very stable regular fingers.

It was clear that the discrepancy between theory and experiment was due to the assumption that the dynamic boundary conditions are independent of surface tension. This was also shown by the experimental observation that the width of the finger was a function of the capillary number, approaching half the channel width or $\lambda = \frac{1}{2}$, where λ is the ratio of finger width to channel width, asymptotically as the capillary number increased, see ST figure 14. (The abscissa in the original publication are in error, the actual values are twice those originally shown.) The big puzzle was the selection mechanism which caused surface tension to choose out of the continuum of solutions the one with $\lambda = \frac{1}{2}$. This particular solution was found to possess a number of unique properties (Saffman 1959; Taylor & Saffman 1959; Jacquard & Seguier 1962), but none had physical significance. The other mystery is the reason for the observed stability when the theory which predicts instability also calculates the steady shape so accurately. Note that for small values of the capillary number for which the observed finger widths are appreciably greater than $\frac{1}{2}$, the shapes are significantly different from the calculated shapes with the same width for zero surface tension.

Attempts were made to incorporate surface tension into the theory but in 1958 the analytical resources were not sufficient to handle the nasty nonlinear boundary-value problem that resulted for the shape and stability of fingers, and computers were still in their infancy. ST did, however, calculate the effect of surface tension on the stability of a plane interface. They used the dynamic boundary condition that the pressure jump across the interface was the surface tension multiplied by the curvature of the projection of the interface on the bounding plates. It was found that a minimum critical velocity was then predicted and also a most unstable wavelength. (It should be noted that independently of the ST work, Chuoke, van Meurs & van der Poel (1959) studied both experimentally and theoretically the instability of the plane interface. They carried out experiments in both packed beds and a Hele-Shaw cell. They attribute the first published notice of the instability to Hill (1952). My recollection of work that took place nearly 30 years ago is incomplete, but it is likely that the inclusion of surface tension into the ST instability calculation followed a private communication between Chuoke and Taylor, although I think it was already under consideration by us. Bensimon *et al.* (1985) state that Taylor & Saffman saw instability of fingers at high speed but ignored it. Again, I have no recollection of this observation and cannot find any mention in our papers or correspondence, but the experiments were carried out entirely by Taylor, my contribution being the calculations, and it may be that he saw instabilities and communicated this to others. As discussed below, it is likely that ST would have seen unstable fingers if they had doubled their greatest capillary number. See *Note added in proof.*)

Interest in the fingering problem then apparently lapsed apart from some isolated numerical studies (e.g. Meng & Thomson 1978) and was not attacked again theoretically until 1980, when McLean & Saffman (1981, henceforth called MS) solved the boundary-valve problem numerically using the ST surface-tension dynamic boundary condition. They also computed the effect of surface tension on the growth rates of infinitesimal disturbances. At the same time, further experiments were done by Pitts (1980) which confirmed the original ST experiments and provided further data on profiles affected by surface tension.

The MS work was also both successful and unsuccessful. It demonstrated numerically that surface tension gave isolated solutions and eliminated the continuous infinity of solutions found by ST. Moreover the dimensionless width λ decreased monotonically to $\frac{1}{2}$ as the capillary number increased, in qualitative agreement with the observations of ST and Pitts. However, the quantitative agreement was poor, the calculated values of $\lambda-\frac{1}{2}$ at each capillary number being about one half the observed values. Moreover, the growth rates of small disturbances remained positive, so that surface tension incorporated via the ST boundary condition did not explain the observed stability. On the other hand, the agreement between the MS calculated shapes and the observed shapes was excellent when the calculated shape with the same asymptotic width was chosen.

One further point of considerable mathematical interest was raised by MS. It is natural to try to calculate the effects of small surface tension by carring out an expansion in inverse powers of the capillary number. Such an expansion turns out to be singular at the sides of the finger because of its infinite length, but the singularity can be handled by the now well-known techniques of singular perturbation theory developed 25 years ago at the California Institute of Technology by P. A. Lagerstrom, S. Kaplan and colleagues. It turns out that there is no difficulty in principle in carrying out the formal perturbation for any value of λ. That is, perturbation theory does not select any special value of λ for non-zero surface tension. Yet numerical treatment of the problem does! This conflict between perturbation theory and numerics is potentially serious, especially when one bears in mind the multitudinous applications of formal perturbation theory. Despite great effort by myself and others, it remained unresolved. However, current studies of the problem by Tanveer (1986), Combescot *et al.* (1986), Dombre, Hakim & Pomeau (1986), Hong & Langer (1986) and Shraiman (1986) indicate where the trouble lies. It seems (as suggested by MS) that there are exponentially small terms, neglected in a formal perturbation expansion, which will not satisfy the boundary conditions unless λ has a particular value.

Of course, the first thought in such a situation was that the numerics was wrong, and that still cannot be discounted. But it can be said that the calculation of steady shapes has been repeated to my knowledge at least four times by independent workers using independent methods, and all agree with the MS results. Romero (1981), Vanden-Broeck (1983), S. Tanveer (private communication, 1985), DeGregoria & Schwarz (1986) all find the isolated steady MS shape. However, Romero did find that the MS solutions were not unique and at least two other steady shapes existed. Vanden-Broeck in a more systematic approach produced plausible evidence for an enumerable infinity of steady shapes, all wider than the MS solution but tending to $\lambda = \frac{1}{2}$ as capillary number increases. Tanveer's numerical work supports this result. Thus the inclusion of surface tension in the ST manner changes a continuous infinity to an enumerable infinity, but still does not make the solution unique. The other solutions do not remove the disagreement between the dependence on capillary number of measured and observed values of λ. The calculated widths of the Romero solutions are at least twice the experimental values: see figure 1, where the widths as measured by ST and Pitts are shown with the theoretical values of ST and Romero. Vanden-Broeck's conclusion that there are an enumerable infinity of solutions for which $\lambda \rightarrow \frac{1}{2}$ as $\mu U_{\mathrm{F}}/T \rightarrow \infty$ is also supported by the new theoretical work referred to above.

MS investigated the limit of zero capillary number, i.e. infinite surface tension, but were unable to make any progress. Pomeau (1985) has published an asymptotic analysis for this case, but it is not clear that his asymptotic equations have any solutions. (See the remarks in §2 about Kadanoff's argument for the non-existence of fingers for too large values of the capillary number.)

The position with regard to stability of the fingers is at the moment in a state of flux. There are conflicts here between theory, computation and experiment. (The terms 'theory' and 'computation' are becoming mixed and it is perhaps advisable to give precise definitions. By theory, I mean reduction of the problem to an operator equation with steady solutions and the determination analytically or numerically of existence and spectral stability of solutions. By computation, I mean solving an initial-value problem describing the evolution in time of some initial condition; i.e. carrying out a numerical experiment or simulation.)

The first theoretical stability calculation was carried out by Taylor & Saffman (1958) for the case of zero surface tension and driving fluid of zero viscosity. They found that all fingers were spectrally unstable. The interesting feature of this calculation was that it could be done in closed form and simple analytical expressions obtained for the eigenvalues and eigenvectors. The growth rates of the high-order eigenvectors of short 'wavelength' approximated those of the short-wavelength disturbances to a plane surface. MS investigated the question of the effect of surface tension on this instability, using the ST boundary conditions. They hoped to find that surface tension stabilized the disturbances. This was not found. (Stabilization of the short-wavelength modes was not checked owing to problems with resolution.) Surface tension reduced the growth rates of the low-order modes, but did not stabilize them. The growth rate of the longest-wavelength disturbance decreased by about 10% as the parameter

$$\kappa = \frac{Tb^2\pi^2}{12\,\mu U_{\mathrm{F}}\, a^2\,(1-\lambda)^2} \tag{1.2}$$

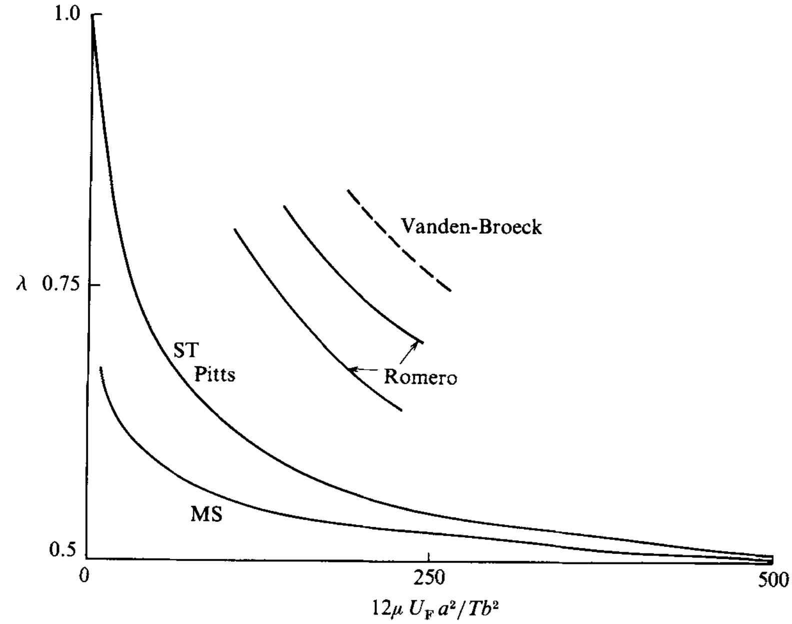

FIGURE 1. Sketch of theoretical and experimental values of dimensionless finger width λ *versus* reduced capillary number $\epsilon^{-1} = 12\mu U_F a^2/b^2T$. The value of ϵ^{-1} for the ST and Pitts experiments went up to 500.

increased from 0 to 1. The ST experiments were for $\kappa > 7.7 \times 10^{-2}$. This is unsatisfactory but not a scientific paradox (like the conflict between numerics and perturbation theory) since it may very well be because the ST boundary conditions are not realistic for unsteady flow. A more serious matter is that computations using the ST boundary conditions by DeGregoria & Schwartz (1985, 1986) showed stabilization for $\kappa \geqslant 2 \times 10^{-2}$. Since the same equations solved in different ways should not give such different results, there is a serious problem. Fortunately, it appears that this paradox may be due to an error by MS. The MS analysis has been checked by S. K. Sarkar (private communication, 1985) who found an analytical oversight in the formulation of the spectral stability calculation. Preliminary calculations by J. W. McLean (private communication, 1985) with a corrected version of the MS code indicate that the error had a major effect on the values of the eigenvalues. The longest-wavelength mode now stabilizes at $\kappa = 9 \times 10^{-2}$, see figure 2. The tentative nature of these results must be emphasized, but intensive work is now in progress to check the spectral stability using a variety of approaches. It is therefore too early to worry about the factor-4 disagreement between theory and computation, which if confirmed would suggest that the numerical schemes have appreciable numerical dissipation. The corresponding value of the abscissa in figure 1 is 440, which suggests, bearing in mind the roughly factor-2 discrepancy between theoretical and experimental values of the capillary numbers for the same width, that ST would have seen instability if they had doubled their largest speeds.

An inconvenience with the problem is the difficulty in choosing a dimensionless parameter. The theory suggests that the most suitable variable would be

$$\epsilon' = \frac{b^2T}{12\mu U_\infty a^2}, \tag{1.3}$$

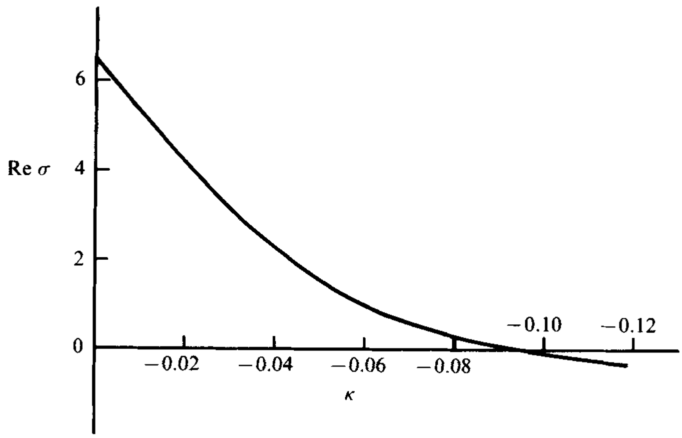

FIGURE 2. Preliminary results (J. W. McLean, private communication 1985) on theoretical dependence of growth rate of lowest mode on κ. Mode is stable for $\epsilon^{-1} < 440$.

where U_∞ is the velocity of the fluid far ahead of the finger. This quantity is defined whether the interface has settled down to a steady state or not and can, at least in principle, be fixed experimentally. But unfortunately it is not usually measured and the usual velocity employed is the speed U_F of the finger, which is easily measured, provided the finger is moving steadily. So one uses

$$\epsilon = \frac{b^2 T}{12\mu U_F a^2} \tag{1.4}$$

If the displacing fluid were to completely expel the more viscous fluid from the gap between the plates there would be a relationship

$$U_\infty = \lambda U_F, \tag{1.5}$$

but in general this is not so. A fraction m of the more viscous fluid is left adhering to the plates, so that

$$U_\infty = (1-\bar{m})\lambda U_F, \tag{1.6}$$

where $\bar{m}$ is some average value of m around the boundary of the finger. ST were aware of the fact that the less-viscous fluid would not completely expel the more-viscous. They showed that if m = constant, the equations for the finger shape are identical with those assuming $m = 0$, provided the viscosity of the displaced fluid is $(1-m)\,\mu$. Therefore for comparison between theory based on the assumption $m = 0$ and experiment, the experimental values of ϵ should be increased by a factor $(1-m)^{-1}$.

Reinelt & Saffman (1985) studied two-dimensional slow viscous flow between parallel plates and calculated m numerically as a function of $\mu U/T$ for a two-dimensional tongue advancing with speed U. According to their calculations, $m \approx 0.15$ when $\mu U/T \approx 0.1$. (The asymptotic formula $m \sim 1.337\,(\mu U/T)^{\frac{2}{3}}$ (Bretherton 1961) appears to agree with the numerical solution of the equations only for $\mu U/T < 0.02$. On the other hand, for $\mu U/T < 0.01$, the experimental results of Fairbrother & Stubbs (1935) suggest $m \sim 0.5\,(\mu U/T)^{\frac{1}{2}}$. This disagreement between theory and experiment has not yet to my knowledge been satisfactorily explained.) In any case, as discussed by MS, only a small part of the discrepancy between experimental and theoretical values of λ can be explained by variable-tongue-thickness effects incorporated kinematically at the values of $\mu U/T$ at which the experiments were done (see MS, figure 5).

The dependence of the film thickness on the speed of the interface normal to itself is neglected in the ST formulation, which essentially assumes m = constant. In this case ϵ (or an equivalent variable) is the sole parameter. Saffman (1982) gave a formulation of the problem that did not take m to be constant, but did not obtain any solutions. However, it is clear from this work that the parameter ϵ is not sufficient if tongue-thickness effects are significant, and that ϵ and the finger capillary number $Ca = \mu U_F/T$ (or Ca and the aspect ratio a/b) are both 'control parameters' in the jargon of physics.

Romero (1981), see §4, appears to have carried out the only theoretical study in which variations of m, or more precisely the normal velocity of the interface, are incorporated into the dynamic boundary condition, but unfortunately the results are incomplete. It was found that improved qualitative agreement between theoretical and experimental values of λ could be obtained by introducing this effect. It is not known, however, if the shapes are seriously affected (remember that the ST boundary conditions give the shapes very well when the experimental width is used), or if the stability characteristics are altered, or if the degenerary properties are changed. Tabeling & Libchaber (1986) find from their experiments, with $a/b = 33.1$ (ST and Pitts used $a/b = 15.9$), that ϵ is not only the parameter, and they claim that the changes in the dynamic boundary condition associated with finite tongue thickness (film draining) bring the experiment and theory (with the effect incorporated in an average manner) into good agreement, at least for slow fingers with widths greater than 0.6. On the other hand, Park & Homsy (1985) in their experiments ($a/b = 125$) claim that ϵ is the scaling parameter. Further investigation is clearly required.

Park & Homsy find that destabilization occurs for $\epsilon^{-1} > 1200$ The revised MS theory (still, it must be emphasized, to be checked) gives $\epsilon^{-1} \approx 440$ for destabilization of the lowest mode. This ratio of experiment to theory is comparable with the disagreement between theoretical and experimental values of ϵ for the same width λ. Tabeling, Zocchi & Libchaber (1986) have reported destabilization for $\epsilon^{-1} \approx 1000$.

It appears therefore that the quantitative disagreement between theory and experiment may be due to the boundary conditions being quantitatively in error. One possible way to investigate this is to consider the stability of a plane interface. Experimental results have recently been obtained by Park, Gorrell & Homsey (1984), but the experimental measurements do not appear to be sufficiently definitive to determine the precise boundary conditions with confidence.

A more serious difficulty concerns the selection mechanism. At present, it appears that there are an enumerable infinity of solutions, and the question is what picks one particular solution out of this infinity. Actually, the degeneracy may be worse than thought. So far only symmetrical solutions have been considered in the presence of surface tension, but Taylor & Saffman (1959) showed that asymmetrical solutions existed in the absence of surface tension, and these may perhaps also exist when T is not zero. Another possible way to attack the questions is to consider finite bubbles instead of infinitely long fingers, since the geometrical singularity is then removed. Again, Taylor & Saffman (1959) demonstrated exact solutions for $T = 0$. The case of non-zero T has recently been investigated, and the results (Tanveer 1986) are most interesting. Maxworthy (1986) has recently carried out experimental observations of bubbles in Hele-Shaw cells.

It should be borne in mind that the whole problem may be artificial and due to the attempt to use perturbation theory on what is in fact a well-posed flow problem. The actual problem is the flow of a fluid of small viscosity into a rectangular cavity occupied by a viscous one. The limiting case of exactly two-dimensional flow and the case of axisymmetric flow into a cylindrical cavity were solved by Reinelt & Saffman

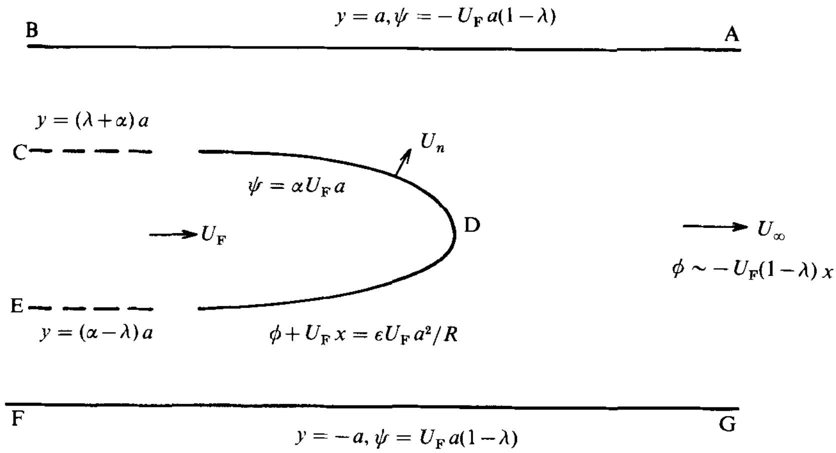

FIGURE 3. Sketch of geometry for a finger advancing into a Hele-Shaw cell, and values of the velocity potential and stream function on the boundary in a frame moving with the finger.

(1985) and there were no signs of non-uniqueness or pattern-selection problems. The evidence is that this flow, described by the creeping-flow equations and continuity of stress across the interface, is well posed. The ill posedness and the mathematical paradoxes all occur when the attempt is made to solve the rectangular problem by expanding in the small parameter b/a.

2. Analysis of steady fingers with the ST boundary conditions

For a finger propagating into a channel, the theoretical problem is sketched in figure 3. A function $\tilde{\phi}(x, y)$ is required that satisfies, in the region between the finger and the bounding walls,

$$\nabla^2 \tilde{\phi} = 0, \tag{2.1}$$

and the boundary conditions

$$\frac{\partial \tilde{\phi}}{\partial y} = 0 \quad \text{on } y = \pm a, \quad -\infty < x < \infty, \tag{2.2}$$

$$p + \frac{T}{R} = 0 \quad \text{on CDE (finger boundary)}, \tag{2.3}$$

$$\frac{\partial \tilde{\phi}}{\partial n} = U_n \quad \text{on CDE}, \tag{2.4}$$

$$\tilde{\phi} \sim U_\infty x \quad \text{as } x \to \infty. \tag{2.5}$$

The pressure p is related to the velocity potential $\tilde{\phi}$ by

$$p = \frac{-12\mu\tilde{\phi}}{b^2}. \tag{2.6}$$

This is a free-boundary-value problem for Laplaces equation; the existence of two boundary conditions (2.3) and (2.4) for a harmonic function are equivalent in principle to an equation for the moving interface. R is the radius of curvature of the interface in the plane of the flow; called the lateral curvature by MS.

It is believed that the unsteady problem is well posed, at least for a finite time.

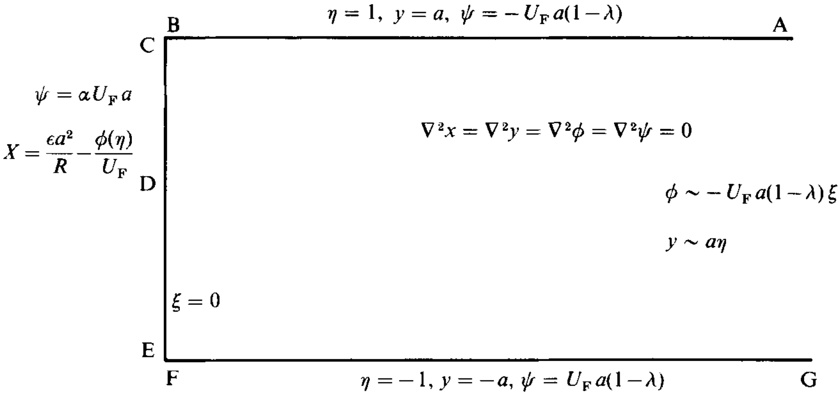

FIGURE 4. The ζ-plane and the boundary conditions for the harmonic functions x, y, ϕ, ψ.

That is, the initial-value problem with the position of the interface originally a given smooth infinite or closed curve has a unique solution for some finite time. To my knowledge this has not been proved rigorously, however. Saffman and Taylor in 1957 discovered exact unsteady solutions for $T = 0$ which developed cusps in finite time (see §3). It is suspected that these cusps do not appear if $T > 0$, but this is unknown.

As discussed in §1, the evidence is now overwhelming that for $T > 0$ the steady problem is not well posed in the sense that solutions are not unique for semi-infinite fingers.

There are many ways to analyse the shape and stability of steady fingers. We shall describe here an approach based on conformal mapping, which is a little different from that employed by ST. The finger is reduced to rest, giving a problem for the velocity potential $\phi = \tilde{\phi} - U_\infty x$ and its harmonic conjugate (the stream function) ψ as shown in the sketch, figure 3. The parameter α measures the amount of asymmetry. The dimensionless parameter ϵ (equation (1.4)) characterizes the surface tension. $2\lambda a$ is the asymptotic width of the finger.

We take as the unknown the mapping $x + iy = z = f(\zeta)$, $\zeta = \xi + i\eta$, which transforms the physical plane into the semi-infinite strip $\xi \geqslant 0$, $-1 \leqslant \eta \leqslant 1$, see figure 4.

The complex potential $w = \phi + i\psi$ can be written by inspection as a function of ζ. It is

$$w = -aU_F(1-\lambda)\left\{\zeta + \frac{2}{\pi}\log\left(1 + e^{-\pi\zeta}\right)\right\} + \frac{2}{\pi}\alpha U_F a \log\left[\frac{1 + ie^{-\frac{1}{2}\pi\zeta}}{1 - ie^{\frac{1}{2}\pi\zeta}}\right]. \tag{2.7}$$

It is easily verified that this expression satisfies the conditions satisfied by w as indicated in figures 3 and 4. An arbitrary real constant can be added to w, but without loss of generality we take it to be zero. This is equivalent to fixing the physical origin and removing the degeneracy due to Galilean invariance. Note that on the finger

$$\phi = \Phi(\eta) = -\frac{2}{\pi}U_F a(1-\lambda)\log\left[2\cos\left(\tfrac{1}{2}\pi\eta\right)\right] + \frac{\alpha}{\pi}a\log\left[\frac{1 + \sin\left(\frac{1}{2}\pi\eta\right)}{1 - \sin\left(\frac{1}{2}\pi\eta\right)}\right]. \tag{2.8}$$

When $T = 0$, i.e. $\epsilon = 0$, it follows from inspection that

$$z = -\frac{w}{U_F} + \lambda a\zeta. \tag{2.9}$$

This shape is an asymmetric finger (Taylor & Saffman 1959) with asymptotes $y = \pm\lambda + \alpha$. There is thus a doubly infinite continuum of solutions for the case $T = 0$, provided $\lambda < 1-\alpha$. (We can take $\alpha > 0$ without loss of generality.)

For $T > 0$, the mathematical problem of steady shapes is to find the analytic function $z = x+\mathrm{i}y = f(\zeta)$ which is analytic in the semi-infinite strip $\xi \geqslant 0$, $-1 \leqslant \eta \leqslant 1$, and satisfies the boundary conditions

$$y = +a \quad \text{on } \eta = \pm 1, \quad \xi > 0, \tag{2.10}$$

$$x = -\frac{\Phi(\eta)}{U_{\mathrm{F}}} + \frac{\epsilon a^2}{R} \quad \text{on } \xi = 0, \quad -1 < \eta \geqq < 1, \tag{2.11}$$

where

$$\frac{1}{R} = \frac{X'Y'' - X''Y'}{(X'^2 + Y'^2)^{\frac{3}{2}}} \tag{2.12}$$

and $' = \mathrm{d}/\mathrm{d}\eta$. Here X and Y denote the values on the end of the strip. This nonlinear boundary-value problem is highly non-trivial. To date, all published work appears to consider the case of symmetrical fingers ($\alpha = 0$) and we shall henceforth restrict ourselves to this case, but clearly the problem of asymmetrical fingers is worthy of further attention.

Before discussing a possible approach slightly different from those already used, which has the advantage that it gives plausible reasons for the counting, it is appropriate to mention an ingenious argument brought to my attention by L. Kadanoff (private communication, 1985) which shows that there is an upper bound on ϵ for solutions to exist. This argument does not use the formulation employed here but that of MS. Referring to the geometry of figure 3, we denote the distance from the nose by S and the angle between the tangent to the finger and the x-axis by θ. Then

$$\frac{\mathrm{d}X}{\mathrm{d}S} = \cos\theta, \quad \frac{\mathrm{d}\theta}{\mathrm{d}S} = \frac{1}{R}, \quad \frac{\mathrm{d}\phi}{\mathrm{d}S} = q, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{q}{R}, \tag{2.13}$$

where q is the speed of the fluid past the surface. Now differentiate the boundary condition (2.11) with respect to S and multiply by $\mathrm{d}\theta/\mathrm{d}\phi$ to give

$$\cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{\epsilon a^2}{R}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{R}\right) - \frac{1}{U_{\mathrm{F}} R}. \tag{2.14}$$

Integrate from the nose to infinity. Thus

$$\left[\frac{\epsilon a^2}{2R^2} - \sin\theta\right]_0^\infty = 1 - \frac{\epsilon a^2}{2R_0^2} = \int_0^\infty \frac{\mathrm{d}\phi}{U_{\mathrm{F}} R} > 0 \tag{2.15}$$

since ϕ must increase monotonically along the surface. (Otherwise, since the flow is unidirectional at $x = \pm\infty$, there would have to be a separation bubble which would violate the single valuedness of ϕ.) R_0 denotes the radius of curvature at the nose. Hence

$$\frac{\epsilon a^2}{2R_0^2} < 1. \tag{2.16}$$

But by geometrical considerations, $R < \lambda a$ must occur somewhere on the finger. Hence,

$$\epsilon < 2\lambda^2 < 2 \tag{2.17}$$

if the curvature is greatest at the nose. This appears to put an upper limit on the value of the surface tension T for steady fingers to exist.

For the MS solutions, the largest value of ϵ/λ^2 was 1.1. In dimensional terms, and using U_∞ as the velocity scale, since in an experimental situation it is expected that either the pressure gradient ahead of the finger or the flux of fluid is specified or controlled, the inequality (2.17) gives

$$\epsilon' = \frac{b^2 T}{12\mu a^2 U_\infty} < \lambda < 1. \tag{2.18}$$

If $T/\mu U_\infty$ is too large, then presumably either the interface is straight across the gap, or steady motion is not possible, or the curvature is not of one sign and the shapes are not convex. With regard to the first two possibilities, notice that the plane interface is stable (according to the ST formulation) if $\epsilon' > 4/\pi^2$. With regard to the third possibility, S. Tanveer (private communication, 1985) has calculated the shapes of Romero fingers for ϵ up to 0.015, $\lambda = 0.94$, and finds that they cease to be convex as ϵ increases. This perhaps eliminates the upper bound on ϵ, if we allow the possibility that for large ϵ the MS solution should be replaced by a Romero one.

The accurate numerical solution of the nonlinear boundary-value problem posed by (2.10)–(2.12) is not easy. For symmetrical fingers, a possible approach is to substitute for X and Y on the finger $-1 < \eta < 1$:

$$X = \frac{2}{\pi} a(1-\lambda) \log(2\cos(\tfrac{1}{2}\pi\eta)) + a\hat{X}, \tag{2.19}$$

$$Y = a\lambda\eta + a\hat{Y}, \tag{2.20}$$

where
$$\hat{X} = \sum_{n=0}^{\infty} a_n \cos n\pi\eta, \quad \hat{Y} = -\sum_{n=1}^{\infty} a_n \sin n\pi\eta \tag{2.21}$$

are the real and imaginary parts on the boundary of an analytic function in the strip. Substitution and collocation or a spectral method would then allow numerical calculation of the coefficients. However, this approach runs into difficulties because $\hat{X}$ and $\hat{Y}$ are not analytic at the corners $\eta = \pm 1$. Substituting a form

$$\hat{Z} = \hat{X} + \mathrm{i}\hat{Y} \sim C(1+\mathrm{e}^{-\pi\zeta})^\gamma \quad \text{as } \zeta \sim \pm\mathrm{i}, \tag{2.22}$$

where C is some unknown real number and $0 < \gamma < 1$, into the boundary-value problem it is found after a little algebra that

$$\hat{X} \sim C\pi^\gamma \cos(\tfrac{1}{2}\pi\gamma)(\pm 1-\eta)^\gamma, \quad \hat{Y} \sim -C\pi^\gamma \sin(\tfrac{1}{2}\pi\gamma)(\pm 1-\eta)^\gamma \tag{2.23}$$

provided γ satisfies the transcendental equation

$$\cos(\tfrac{1}{2}\pi\gamma) = \frac{\epsilon\pi^2}{(1-\lambda)^2}\frac{\gamma^2}{4}\sin(\tfrac{1}{2}\pi\gamma). \tag{2.24}$$

(Note that $\tau = \frac{1}{2}\gamma$, $s = (\pm 1-\eta)^2$ is the change of variable to reconcile with the form of the singularity given by MS.) This means that the Fourier series (2.21) cannot be differentiated term by term to give the first and second derivatives needed for the calculation of the curvature given by (2.12), but it is necessary to subtract the singularity or stretch the coordinate system.

Since $\hat{X} = \hat{Y} = 0$ when $\epsilon = 0$, a singular perturbation expansion can be attempted. Substituting in (2.12) the expansion

$$\hat{X} = \epsilon\hat{X}_1 + \epsilon^2\hat{X}_2 + \ldots, \quad \hat{Y} = \epsilon\hat{Y}_1 + \epsilon^2\hat{Y}_2 + \ldots, \tag{2.25}$$

one finds after a little algebra that

$$\hat{X}_1 = \frac{4\pi\lambda(1-\lambda)\cos(\tfrac{1}{2}\pi\eta)}{[\{1+(2\lambda-1)\,\mathrm{e}^{-\mathrm{i}\pi\eta}\}\{1+(2\lambda-1)\,\mathrm{e}^{\mathrm{i}\pi\eta}\}]^{\frac{3}{2}}}. \tag{2.26}$$

A Fourier decomposition of (2.26) then leads to a value for $\hat{Y}_1$, or it can be expressed as an integral using the Poisson integral. The algebra and analysis quickly becomes difficult and details have not been worked out, but the indications are that the series (2.25) can be developed formally for any value of λ, as stated by MS.

It follows from the behaviour of $\hat{X}_1$ near $\eta = \pm 1$ given by (2.26) that there is a logarithmic contribution to $\hat{Y}_1$:

$$\hat{Y}_1 = \frac{2}{\pi}\hat{X}_1 \log|\pm\eta - 1|, \tag{2.27}$$

and matching with (2.23) occurs if $C = \lambda$. The perturbation expansion method therefore suggests that the degeneracy for $\epsilon = 0$ persists for $\epsilon > 0$.

Owing to the nonlinearity of the numerical problem and the difficulty in finding a satisfactory representation of the solution, it is not possible to present a rigorous argument for the correct counting. But the numerical solutions all seem to suggest strongly that, for $\epsilon > 0$, the solutions of (2.10)–(2.12) for given ϵ are isolated and that λ cannot be specified arbitrarily, although it is a multivalued function of ϵ. A plausibility argument is as follows. Suppose we guess $X(\eta)$. Then R follows from (2.11) and we have a first-order equation for $\mathrm{d}Y/\mathrm{d}\eta$. From the known behaviour of X as $\eta \rightarrow 1$, we know that

$$X' \sim \frac{2a(1-\lambda)}{\pi s} + \gamma C\pi^{\gamma}\cos\left(\tfrac{1}{2}\pi\gamma\right)s^{\gamma-1}, \tag{2.28}$$

where $s = 1-\eta$. Then (2.11) and (2.12) give, for the behaviour of Y near $\eta = 1$,

$$s^2 Y'' + sY' \approx \frac{s^{\gamma}}{\epsilon}. \tag{2.29}$$

The homogeneous solution $Y' \approx 1/s$ is not acceptable, and the solution is thus uniquely determined. Once Y is known, we have a classical boundary-value problem for y, which then gives a new value for X, and the process continues. The width λ arises as $\int_0^1 Y'\,\mathrm{d}\eta$. This argument may provide an alternative method of solution, but it has not yet been tried.

As mentioned earlier, the shapes of finite bubbles is also a problem of considerable interest. The $T = 0$ problem has the same degeneracy as the finger and one would like to know what the solution properties are for $T > 0$. One advantage of the bubble is that the geometrical singularity associated with the infinitely long sides disappears and the singular perturbation problem becomes a regular perturbation one. Tanveer (1986) finds that the degeneracy properties are the same as for the semi-infinite finger. He finds that, for the case of small bubbles, it is possible to find the general term of the expansion (2.25). It then appears that there are exponentially small terms that will not satisfy the boundary conditions except for one value of λ (the dimensionless maximum width of the bubble), which corresponds to a circular bubble. As remarked in §1, it is now believed that a similar behaviour holds for the finger and provides the resolution of the conflict between numerical and perturbation methods.

In comparing the results of steady calculations on bubbles with experiment, one caveat should be kept in mind. The ST boundary conditions may be reasonable for an advancing interface, but it is possible that they are qualitatively incorrect for a retreating interface owing to different wetting properties. Comparison with experiment may therefore not show good agreement or be of value for the purpose of validating the mathematics. It is, of course, also true for fingers that the uncertainty in the boundary conditions means that one cannot appeal to experiment to settle mathematical controversies.

3. Finger stability and unsteady solutions of the ST formulation

When $\epsilon = 0$, the methods of function theory can be used to calculate exact unsteady solutions of interfacial motion in a Hele-Shaw cell, provided one of the fluids is of negligible viscosity. (This restriction is necessary for unsteady motion because the ST transformation relating problems with viscous and non-viscous fluid in the finger then fails.) Saffman (1959) presented solutions showing how a finger forms as the asymptotic state of an initial sinusoidal disturbance to a plane interface. These have recently been generalized by Howison (1986).

Taylor & Saffman (1958) analysed the stability of steady symmetrical fingers to infinitesimal disturbances. We again work in the frame of reference in which the undisturbed finger is stationary, see figures 3 and 4. We put

$$z(\zeta, t) = z_0(\zeta)+z_1(\zeta, t),\ w(\zeta, t) = w_0(\zeta)+w_1(\zeta, t) \tag{3.1}$$

where suffix 0 refers to the solution for the undisturbed finger. The pressure boundary condition (2.11) is the same for steady and unsteady motion. But the kinematic boundary condition now becomes the condition that the normal velocity of the interface is the velocity of the fluid normal to the interface, i.e.

$$\frac{\partial \Psi}{\partial s} = \frac{\partial X}{\partial t}\sin\theta - \frac{\partial Y}{\partial t}\cos\theta, \tag{3.2}$$

which can be written

$$\frac{\partial \Psi}{\partial \eta} = \frac{\partial X}{\partial t}\frac{\partial Y}{\partial \eta} - \frac{\partial Y}{\partial t}\frac{\partial X}{\partial \eta}, \tag{3.3}$$

since $\sin\theta = \partial Y/\partial s$, $\cos\theta = \partial X/\partial s$. Capital letters again refer to values on the finger $\xi = 0$.

The linearized eigenvalue problem for infinitesimal disturbances with time dependence $e^{\sigma t}$ is

$$\frac{\partial \Psi_1}{\partial \eta} = \sigma\left[X_1\frac{\partial Y_0}{\partial \eta} - Y_1\frac{\partial X_0}{\partial \eta}\right], \tag{3.4}$$

$$X_1 = -\frac{\Phi_1}{U_F} - \epsilon a^2\frac{R_1}{R_0^2}. \tag{3.5}$$

We now discuss the TS solution for $\epsilon = 0$, for which

$$X_0 = \frac{2a}{\pi}(1-\lambda)\log\left(2\cos\left(\tfrac{1}{2}\pi\eta\right)\right), \quad Y_0 = \lambda a\eta. \tag{3.6}$$

It is clear from (3.5) that

$$\Phi_1 = -U_F X_1, \quad \Psi_1 = -U_F Y_1. \tag{3.7}$$

Then (3.4) gives

$$-\frac{\partial Y_1}{\partial \eta} = \frac{\sigma a}{U_F}\left[\lambda X_1 + (1-\lambda)\tan\left(\tfrac{1}{2}\pi\eta\right) Y_1\right]. \tag{3.8}$$

Let us first consider symmetrical disturbances of the form

$$X_1 = \sum_{n=0}^{\infty} a_n \cos n\pi\eta, \quad Y_1 = -\sum_{n=1}^{\infty} a_n \sin n\pi\eta. \tag{3.9}$$

Substitution into (3.8) and employing elementary trigonometric identities gives

$$\{\sigma a(2\lambda-1)-U_F\pi\}a_1 = -2\lambda a\sigma a_0, \tag{3.10}$$

$$\{\sigma a(2\lambda-1)-(n+1)U_F\pi\}a_{n+1} = (n\pi U_F - a\sigma)a_n, \quad n \geqslant 1. \tag{3.11}$$

The solution $\sigma = 0, a_0 \neq 0$, $a_n = 0$ for $n \geqslant 1$, is the trivial uniform displacement of the finger parallel to its axis. From (3.11), we see by inspection that the eigenvalues are

$$\sigma_N = N\pi U_F/a, \quad N \geqslant 1 \tag{3.12}$$

and the corresponding eigenvectors are finite sums of the form

$$X_1^{(N)} = \sum_{n=0}^{N} a_n^{(N)} \cos n\pi\eta, \quad Y_1^{(N)} = \sum_{n=1}^{N} a_n^{(N)} \sin n\pi\eta \tag{3.13}$$

and are therefore complete. Since, however, the high-order eigenvectors, which are those with the shortest wavelengths, grow fastest, an arbitrary disturbance can be expected to become singular in a finite time. (This phenomenon is reminiscent of the behaviour of vortex sheets.) For a general disturbance to remain describable by the Fourier-series representation, it is necessary that the initial values of the Fourier coefficients decay at least exponentially fast in N.

It is to be noted that a simple exact solution exists for $N = 1$. If we put

$$X_1 = a_0(t) + a_1(t)\cos \pi\eta, \quad Y_1 = -a_1(t)\sin \pi\eta, \tag{3.14}$$

then it is readily verified that we have an exact solution of (3.3) if

$$\dot{a}_1 = \frac{\pi U_F}{a}a_1 + \frac{\pi}{a}a_1\dot{a}_0. \tag{3.15}$$

$$\dot{a}_0 = \frac{\pi}{\lambda a}a_1\dot{a}_1 + \frac{\dot{a}_1(1-\lambda)}{\lambda}. \tag{3.16}$$

It follows from these two equations that

$$\dot{a}_1\left(1+\frac{\pi a_1}{a}\right)\left(1-\frac{\pi a_1}{\lambda a}\right) = \frac{\pi U_F}{a}. \tag{3.17}$$

Thus the amplitude of the disturbance grows, until after a finite time $\dot{a}_1 = \infty$, when either $a_1 = \lambda a/\pi$ or $a_1 = -a/\pi$, depending on the initial sign of a_1. The value of a_1 then becomes imaginary, and the solution ceases to make sense. For $a_1 = \lambda a/\pi$, $Y \approx \eta^3$ near the vertex and there is a cusp since $X \approx -\eta^2$. For $a_1 = a/\pi$, the finger bulges out, has a negative curvature at the nose, and cusps appear off the axis: see figure 5. (This solution was discovered by Taylor and myself in 1957 and written up as an appendix to the original draft of ST. For reasons not now remembered, it was decided that it lacked sufficient interest to be published at that time, perhaps because the observed fingers were stable.) Cusped solutions have been discussed recently by Howison (1986 and references cited there), Shraiman & Bensimon (1984), Sarkar (1985) and shown to be generic.

An analysis can also be carried out for asymmetrical disturbances to symmetrical fingers. Instead of (3.9), we take the disturbance to be

$$X_1 = \sum_{n=0}^{\infty} a_n \sin(n+\tfrac{1}{2})\pi\eta, \quad Y_1 = \sum_{n=0}^{\infty} a_n \cos(n+\tfrac{1}{2})\pi\eta \tag{3.18}$$

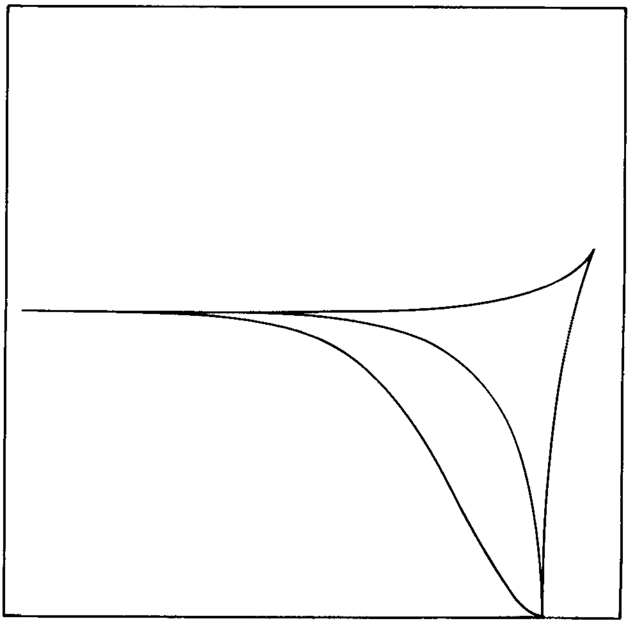

FIGURE 5. Shapes of cusped solutions for $\lambda = \frac{1}{2}$ which arise from infinitesimal perturbations to the smooth steady solution shown between the cusped shapes. The curves have been translated horizontally to have the nose at the same position.

and substitute into (3.8). We find, for $n \geqslant 0$,

$$\pi(n+\tfrac{1}{2})\,a_n + \pi(n+\tfrac{3}{2})\,a_{n+1} = \frac{a\sigma}{U_{\mathrm{F}}}\{a_n - (1-2\lambda)\,a_{n+1}\}. \tag{3.19}$$

Thus the eigenvalues are

$$\frac{a\sigma}{U_{\mathrm{F}}} = \pi(N+\tfrac{1}{2}). \tag{3.20}$$

It is expected that exact asymmetrical solutions exist, but do not appear to have been explicitly calculated to date.

The open and controversial question at present is the effect of surface tension (i.e. ϵ) on these unsteady solutions of the ST formulation. As mentioned in §1, the stability analysis was carried out by MS, but unfortunately there was an analytical oversight in the formulation (discovered by Sarkar.) The results of a recalculation by McLean for the lowest-order symmetrical mode using the correct equations were shown in figure 2. It must again be strongly emphasized that these results are tentative. The high-order modes are presumably stabilized by surface tension, but low-order modes of order greater than one may be more unstable, so stabilization may not occur until κ is larger than the value shown in figure 2. This would make worse the agreement with Degregoria & Schwartz, who find numerically stabilization for $\kappa \approx 2 \times 10^{-2}$.

An important mathematical question is the spectrum of the eigenvalue problem for infinitesimal disturbances. This is discrete, and the eigenvectors are complete, if $\epsilon = 0$. One wishes to know if this is so for $\epsilon > 0$, or if a continuous spectrum, with improper eigenvectors, appears and replaces the discrete spectrum. Consider, for example, the straight sides of the finger. Introduce a disturbance of wavelength $2\pi/k$, so that the side of the finger has the shape

$$y = \lambda a + \delta\, \mathrm{e}^{\mathrm{i}kx+\sigma t}, \quad \delta \ll 1. \tag{3.21}$$

The velocity potential in finger-fixed coordinates is

$$\phi = -U_F x + \delta' \mathrm{e}^{\mathrm{i}kx+\delta t} \cosh k(a-y). \tag{3.22}$$

The kinematic boundary condition gives

$$\delta' = \frac{\delta(-\sigma + U_F \mathrm{i}k)}{\sinh ka(1-\lambda)}. \tag{3.23}$$

The linearized dynamic boundary condition is

$$\phi + U_F x = -\epsilon a^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{3.24}$$

from which it follows that

$$\sigma = U_F \mathrm{i}k - \epsilon U_F a^2 k^2 \tanh ka(1-\lambda). \tag{3.25}$$

There are thus modes on the straight sides which are convected with the flow relative to the finger, or are fixed in space, which decay exponentially in time. These improper eigenfunctions may correspond to a continuous spectrum, but since it is damped it does not imply stability if the discrete spectrum still exists.

Kessler & Levine (1985) claim that the spectrum for $\epsilon > 0$ is fundamentally different from that for $\epsilon = 0$, and that the spectrum is continuous and damped, except for the single discrete eigenvector with neutral eigenvalue corresponding to uniform translation parallel to the walls. Spectral theory of non-self-adjoint singular operators is not well understood, and at present it is not possible to say if this claim is correct or is nonsense. If correct, it is then argued that instability results from the threshold for finite-amplitude instability tending to zero as $\epsilon \to 0$.

It is hoped that future work, both numerical and analytical, may be able to distinguish the different possibilities. Appeal to experiment is irrelevant, as there is no guarantee that the ST boundary conditions are correct, and the 'spectral theory' is likely to be sensitive to the boundary conditions.

The exact solutions shown in figure 5 also make the Kessler & Levine claim implausible, as it implies that the exact unsteady solutions described by (3.14)–(3.17) are not approximations to an $\epsilon > 0$ flow, even in the initial stages when the finger is smooth and before the cusps develop. This would be a remarkable singular perturbation phenomenon if true.

4. What are the proper boundary conditions?

The study of interfacial motion using the ST formulation leads to a controversial analytical and numerical free-boundary-value problem. The results do appear to have some bearing on the experimental phenomenon, but there is sufficient disagreement and uncertainty for one to wonder if at least part of the problem may be due to a fundamental inadequacy or incompleteness of the ST formulation. (Bad physics produces bad mathematics.) ST were aware of the drawbacks of their formulation, but their inability in 1957 to solve any but the simplest problems made the question moot.

Modern computers and improved techniques of numerical analysis, which enabled MS, Romero, and subsequent workers to solve ST formulations, make it clear that the question of more general boundary conditions should be addressed. A qualitative discussion was given by Saffman (1982), and recently Reinelt (1986) appears to have successfully derived the correct quantitative boundary conditions for advancing fingers when the displaced fluid wets the plates.

Again for simplicity, we consider the case of a finger of negligible viscosity and zero gravitational effects. This is not because we believe that the more general case is unimportant – on the contrary – but the general case is harder, or appears so, and it makes sense to start with the easiest problem.) Then, the flow around the finger in the plane of the plates can be divided into four regions.

In region I outside the finger, we have the standard Hele-Shaw equations for the components $u(x, y)$, $v(x, y)$ of mean velocity.

$$u = -\frac{b^2}{12\mu}\frac{\partial p}{\partial x}, \quad v = -\frac{b^2}{12\,\mu}\frac{\partial p}{\partial y}, \tag{4.1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \tag{4.2}$$

In region III, which is the region occupied by the finger not too far from the nose, and region IV, which is far from the nose, we have to consider the motion of the viscous fluid in a film of total thickness mb, where $m = m(x, y)$, adhering to the plates, in which the average velocity (u', v') satisfies

$$u' = -\frac{m^2b^2}{12\mu}\frac{\partial p'}{\partial x}, \quad v' = -\frac{m^2b^2}{12\mu}\frac{\partial p'}{\partial y}, \tag{4.3}$$

with equation of continuity

$$\frac{\partial m}{\partial t} = \frac{\partial (mu')}{\partial x}+\frac{\partial (mv')}{\partial y}. \tag{4.4}$$

Continuity of force across the surfaces of the fingers parallel to the plate leads to the final equation

$$\tfrac{1}{2}bT\nabla^2 m + p' = 0. \tag{4.5}$$

These equations are approximations valid for $b/a \ll 1$. It has also been assumed that there are no tangential gradients of surface tension. The difference between regions III and IV lies in the different relative magnitudes of the terms in equations (4.3)–(4.5). The existence of these two separate regions was first noted by D. A. Reinelt (private communication, 1984).

Finally, region II is a 'boundary layer' of width b in which u, v, p match with u', v', p'. In the boundary layer, velocity components normal to the plates are important and a solution of the creeping-flow equations is required to provide the proper matching.

Figure 6 shows shapes of the interface in the transverse plane. Case (*a*) is the idealized picture corresponding to the ST formulation. It is assumed here that region II is of zero thickness, and is a sharp plane interface perpendicular to the plates. In this case, $m = 0$,

$$P = P' - \frac{T}{R}, \tag{4.6}$$

$$\boldsymbol{U}\cdot\boldsymbol{n} = \boldsymbol{U}_{\mathrm{F}}\cdot\boldsymbol{n} \equiv U_n, \tag{4.7}$$

where P, P' are the limits of the pressure in regions I and III respectively on the finger boundary, $\boldsymbol{U}$ is the limit of the velocity $\boldsymbol{u}$ in region I on the finger boundary and U_n is the normal velocity of the interface. R is the lateral curvarture.

Case (*b*) is perhaps a more realistic transverse shape, which takes account of a wetting angle. D. A. Reinelt (private communication, 1984) has pointed out that when the flow is static the effect of transverse curvature on the shape of the interface can be calculated as a straightforward perturbation expansion in b/R. The

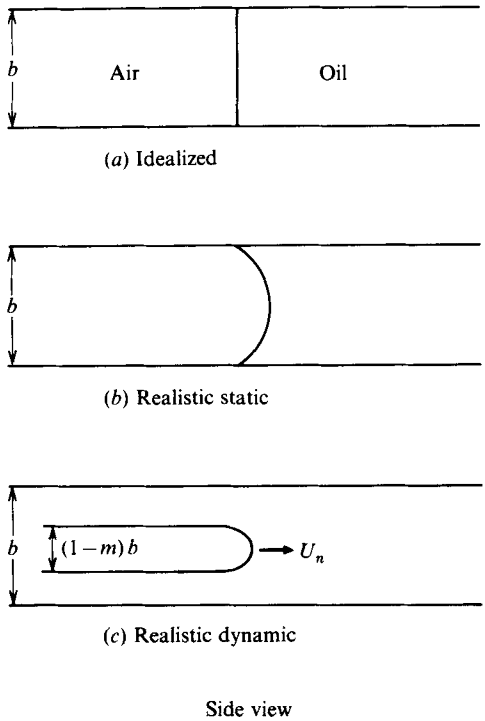

FIGURE 6. Possible interface shapes in the narrow gap.

mathematical problem is to calculate the axisymmetrical shape between parallel plates such that the sum of the principal curvatures is constant subject to a given contact angle. The result is

$$P = P' - \frac{sT}{b}\cos\varphi - \frac{T}{R\cos\varphi}\left\{\frac{\pi}{4} - \frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\right\} + O\left(\frac{bT}{R^2}\right) \tag{4.8}$$

where φ is the contact angle. For zero contact angle, one recovers the result of Park & Homsy (1984),

$$P = P' - \frac{2T}{b} - \frac{\frac{1}{4}\pi T}{R} \tag{4.9}$$

according to which complete wetting of the sidewalls produces an effective surface tension smaller by a factor $\frac{1}{4}\pi$. Note that this makes worse the agreement between MS theory and experiment for the widths of steady fingers. According to figure 1, a doubling of the pressure drop due to lateral curvature is needed to bring MS theory and experiment into rough agreement.

The real interface is more likely to have the shape shown in figure 6(*c*), especially if the capillary number is not small. We then expect that the jump conditions connecting regions I and III across the boundary layer II will have the form (Saffman 1982)

$$\Delta P = P - P' = \frac{2T}{p} c_p\left(\frac{\mu U_n}{T}, \frac{b}{R}\right), \tag{4.10}$$

$$U_n - \boldsymbol{U}\cdot\boldsymbol{n} = (U_n - \boldsymbol{U}'\cdot\boldsymbol{n})\, M\left(\frac{\mu U_n}{T}, \frac{b}{R}\right) \tag{4.11}$$

The functions c_p and M are to be found from analysis of the flow in region II. Note that, as found experimentally by Tabeling & Libchaber (1986) and Tabeling *et al.* (1986), the expected separate dependence on capillary number and curvature implies that for the general case there is not a single 'control parameter'.

For the case of a plane interface, i.e. $R = \infty$, advancing with speed U, the problem was investigated using perturbation methods for small capillary number $Ca = \mu U/T$ by Bretherton (1961), and numerically for finite Ca by Reinelt & Saffman (1985). The details of the calculation are quite interesting, and use a powerful overlapping grid technique suggested by H. O. Kreiss. The geometry is like that shown in figure 3, but the stream function satisfies the biharmonic equation instead of Laplace's equation, and the boundary conditions are zero slip at solid surfaces and continuity of normal and tangential stress at free boundaries. One can raise the same questions about well posedness. The numerics, which is rather more difficult, does give only one solution and there are no indications of others. But no attempt was made to find non-symmetric solutions, and it is an open question whether these exist. (The calculation was also done for an axisymmetric finger propagating in a capillary tube of circular cross-section. Here the numerical results agree very closely with the experimental results of Taylor (1961). Since the equations for this problem are exact, within the low-Reynolds-number approximation, this agreement confirms the accuracy of the numerical method.) The results are, however, insufficient for the Hele-Shaw problem since they do not contain any of the lateral curvature effects.

Reinelt's (1986) calculation takes these into account. He writes

$$c_p = c_{p_0} + c_{p_1}\frac{b}{R} + \dots, \tag{4.12}$$

$$M = M_0 + M_1\frac{b}{R} + \dots, \tag{4.13}$$

where c_{p_0}, c_{p_1}, M_0 and M_1 are functions only of Ca. He calculates these dependencies by perturbation theory for small Ca and numerically for larger Ca. His results can be represented roughly for $Ca < 1$ by approximations

$$c_{p_0} = -1 - 3.80C^{\frac{2}{3}}\,\mathrm{e}^{-4C^{\frac{1}{3}}} - 2C, \tag{4.14}$$

$$c_{p_1} = -\frac{\pi}{4} + 4.07C^{\frac{2}{3}}\,\mathrm{e}^{-4C^{\frac{1}{3}}} - C, \tag{4.15}$$

$$M_0 = 1.3375C^{\frac{2}{3}}\,\mathrm{e}^{-3C^{\frac{1}{3}}} + 0.3C, \tag{4.16}$$

$$M_1 = -1.3375\frac{\pi}{4}C^{\frac{2}{3}}\,\mathrm{e}^{-3C^{\frac{1}{3}}} - 0.2C. \tag{4.17}$$

For small Ca and $b/R = 0$, these reduce to those of Bretherton (1961). Note that these expressions are for advancing interfaces. It is expected that similar, but different, results will hold for retreating interfaces such as those at the back of a finite bubble, in which case m will be determined by conditions at the advancing interface and there will be results like (4.14) and (4.15) for the pressure drop.

The expressions (4.14)–(4.17) provide boundary conditions for (4.1)–(4.5), and the next task is to solve these. This remains to be done.

Romero (1981) studied solutions for the case $M = 0$, with the dynamic boundary condition

$$\Delta p = -\frac{T}{R} - \beta U_n. \tag{4.18}$$

He found that for fixed values of $Tb^2/\mu U_F a^2$, increasing β brought the MS curve of λ versus $\mu U_F/T$ into qualitative agreement with the experimental values. Unfortunately, quantitative details were not given, as it was thought that the procedure was too *ad hoc*, and it was believed that variations in M should be included for completeness, even though MS had concluded that effects of non-zero M would be insignificant. Tabeling & Libchaber (1986) and Tabeling *et al.* (1986) combine the Bretherton formula with the Park & Homsy expression to give an effective surface tension T^* defined by

$$\Delta p = -\frac{T^*}{R} = -\frac{\pi T}{4R} - 2\left(\frac{T}{b}\right) 3.8 C^{\frac{2}{3}}. \tag{4.19}$$

They claim that replacing T in the MS results by an averaged (lumped) T^* gives much better agreement, provided $\mu U_F/T$ is not too large. The evidence is that film-thickness effects on the boundary conditions are responsible for the disagreement between the MS calculations and the experiments. But it remains to be seen if the mathematical difficulties are also removed by incorporating U_n into Δp and M.

As pointed out by D. A. Reinelt (private communication, 1984), it is not necessary to solve the above equations for region III. The precise scaling is not tidy since it depends upon the relative magnitudes of b/a and Ca, but irrespective of which is the larger, it follows that $p' \ll \mu U_F/b$. Consequently, $u' \ll U_F$ and $v' \ll U_F$ and, from (4.4), $\partial m/\partial t \sim 0$. That is, $m = m(y)$ and we can take

$$\boldsymbol{U}\cdot\boldsymbol{n} = (1-M)\, U_n \tag{4.20}$$

In region IV there is a slow readjustment with $p' \approx bTm/a^2$ from (4.5), $v' \approx m^3b^3T/\mu a^3$ from (4.3), and then $mU_F/l \approx m^3b^3T/\mu a^4$ from (4.4), i.e.

$$\frac{l}{a} \sim \frac{\mu U_F}{T}\frac{a^3}{b^3}, \tag{4.21}$$

where l is the longitudinal lengthscale for the relaxation of film thickness to occur.

Many specific details remain to be worked out, but it appears that the basic physics of the ST fingers when the displacing fluid is relatively inviscid will cease to be a mystery. It is expected that use of proper boundary conditions will explain the recent observations of $\lambda < \frac{1}{2}$ reported by Tabeling *et al.* (1986). The crucial role of the boundary conditions is emphasized by the fascinating experimental results of Ben-Jacob *et al.* (1985). They introduced anisotropy by etching a grid on one of the plates. This led to a very different structure. Instead of fingers whose dynamics was dominated by tip splitting, dendritic patterns formed marked by extensive side-branching. Similar phenomena have been reported by Y. Couder & C. Basdevant (private communication, 1986) in the absence of anisotropy for fingers which have a small bubble attached to their tips. There is evidence (e.g. Maher 1985; Nittman, Daccord & Stanley 1985) that finite viscosity of the displacing fluid or non-Newtonian effects may also lead to interesting phenomena.

A referee has pointed out that similar difficulties apply to the appropriate boundary conditions for the Reynolds equations of lubrication theory when cavitation occurs (e.g. Coyne & Elrod 1970).

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Note added in proof. Professor Tony Maxworthy has kindly pointed out that ST did in fact see unstable fingers, since they say on p. 323 '... till at high speeds of flow the tongue or finger of the advancing fluid itself breaks down and divides into smaller fingers.'