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VISCOUS FLOW BETWEEN PARALLEL

DISKS WITH A TIME-VARYING

GAP WIDTH AND A CENTRAL

FLUID SOURCE

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ABSTRACT

The present paper concerns the un-
steady laminar incompressible flow between
two parallel disks with a fluid source at
the center of the disk(s). Both the flow
rate of the source and the gap width between
the disks are varied arbitrarily with time
and independently of each other. Such a
flow often appears in a nozzle-flapper valve or in a statical pressure bearing such as the piston shoes of a swash type axial pis-
ton pump. Series solutions to the Navie-
r-Stokes equations are obtained, on the
basis of an asymptotic expansion in the
radial direction and a new theory of "multi-
fold series expansion" for the time variab-
le. The solutions describe precisely the
non-linear interaction between the two coe-
xisting flows. Experiments were carried out
for the case where both the gap width and
source flow were varied sinusoidally. The
solutions agree well with the experimental
results over a wide range of the flow condi-tions.

KEY WORDS

Unsteady Flow, Parallel Disks, Time-Varying
Gap Width, Time-Varying Source Flow, Multi-
fold Expansion

NOMENCLATURE

a: nondimentional amplitude of the gap
width
b: nondimentional amplitude of the flow
rate of the source
h: gap width
hm: reference gap width
p: pressure

pe: pressure outside of the disks
q: flow rate of the source
qm: reference flow rate of the source
Rqm: Reynolds number related to the source
flow [Eq.(24)]
Rw: Reynolds number related to the angular

frequency [Eq.(23)]r
: radial coordinate (see Fig. 1)
re: outer radius of the disk
rO: inner radius of the disk
u: velocity component in the r-direction
v : velocity component in the y-direction
y: coordinate in the direction vertical to
the disk (see Fig. 1)
a: first characteristic function related

to the variation of the gap widthƒÁ

: second characteristic function related
to the variation of the source flow
δ: phase lag of the source flow to the
gap width
ζ: independent variable defined by
Eq.(10)
η: independent variable defined by Eq.(3)
k:= ro/r
e.
v: kinematic viscosity of the fluid
ρ: mass density of the fluid

INTRODUCTION

The present study concerns the unsteady
laminar incompressible flow between two
parallel disks with a fluid source at the
center of the disk(s). Both the flow rate of
the source and the gap width of the disks
are varied arbitrarily with time and indepe-
ndently of each other. Such a flow can
often be seen in a statical bearing such as
the piston shoe of an axial piston pump with swash plate, in a nozzle flapper valve and
so on.

The flow is classified into the follo-
wing three types, depending on the internal impedance Zs of the flow source:
1) The time-variation of the flow rate of
the source is pre-determined (Zs is infi-
nite).
2) The time variation of the pressure of the
source is pre-determined (Zs is zero).
3) Both the flow rate and the pressure of
the source are determined as a result of
analysis (Zs is neither infinite nor zero).
Although the flow in a statical bearing
or that in a nozzle flapper valve has a

549

550

finite internal impedance of the source in
practice, the authors will begin their study
with type 1), including a constant source
flow as the first step because this type
seems easier to solve than the other cases.
Some studies have been reported on eit-
her the flow with time-varying gap width
only(1)-(5) or that with a central fluid
source only(6)-(12). The flow we deal
with, however, includes two types of flow,
one due to the gap-width variation and the
other due to source-flow variation, and
both flow interact non-linearly with each
other. Accordingly, it is easy to imagine
that, if each type of flow were solved sepa-
rately and superposed simply with each ot-
her, we could not obtain a good result be-
cause of the non-linear interaction.
In the present research, a precise
theoretical analysis is developed based on
the Navier-Stokes equations under the assum-
ptions which will be described in the follo-
wing section, by applying partly the theory
of "multifold series expansion" (13)-(17).
Experimental investigation was also made to
verify the validity and applicability of the
theory.

THEORY

Governing Equations
To begin with, we make the following asumptions:
1) The outer radius re of the disks is
sufficiently large compared with the inner
radius r0of the central fluid source pipe
(re>> r0).
2) The gap width h is sufficiently small
compared with the disk sizes re and ro (h<< r0).
3) The Reynolds numbers Rq and Rw of the
two coexisting flows arenottoo large,
though they are large enough so that the non-linear convective inertia effect cannot be neglected.
4) The flow rate of the source is either
kept constant or varied arbitrarily with
time.
A system of stationary cylindrical
coordinates (r, 8, y) is introduced as shown in Fig. 1. The Navier-Stokes equations and the equation of continuity for an axisymmet-
ric and radial flow of an incompressible
viscous fluid without body forces are,

(1a)

(1b)

(1c)

with the boundary conditions
u= v= 0 at y=0 (2a)

u= 0, v= dh/dt at y= h (t) (2b)

Asymptotic Expansion
First, an original form of asymptotic
series expansion is derived with respect to
r, in the region where r is sufficiently
large. From many previous researches it is
already known that the single flow 111 caused
by the gap-width variation without flow
source can be expressed as ui=rf(y,t), while
the single flow u2 due to the constant sour-
ce flow without gap-width variation as the
following form:

In the present problem, these two flows
coexist. We may, therefore, naturally have
an idea of combining the above two expres-
sions. The former expression, U1
rf(y, t), is proportional to r, and the
latter asymptotic series expansion has a
leading term of order r-1 and a common term
ratio proportional to r-2. Formally, the
former expression just corresponds, acciden-
tally and fortunately, to the next lower-
order extension of the latter asymptotic
series expansion. Thus, these two expres-
sions can be combined without any conflict
with the Navier-Stokes equations. Since the
flow is unsteady in this problem, the coef-
ficients al, a3, a5,•c naturally become
functions of y and t.
A new variable is introduced in place
of y defined by

η= y/h (t) (3)

Arranging under dimensional-analytic consi-
deration, we may formulate the following
asymptotic series expansions:

(4a)

(4b)

(4c)

where K(t) is a constant of integration with
respect to r and y to be determined later,
and

(4d)

The first term on the right-hand side

Fig. 1 Flow geometry and coordinate system

551

of Eq.(4a) corresponds to the flow caused
by the gap-width variation, while the second
and the subsequent terms are due to the
source flow. Equation (4d) is obtained by
integrating the right-hand side of Eq.(4a)
without the first term over the flow cross
sectional area. Equations (4a) and (4b)
satisfy the equation of continuity, Eq.
(1c).
Now, substituting Eqs.(4a), (4b) and
(4c) into Eqs. (1a) and (1b), and setting
each sum of coefficients with equal power of
r to zero, we obtain the following rela-
tions:

(5a)

(5b)

(5c)

(5d)

(5e)

where

(6a)

(6b)

As can be seen from Eq.(5e), P1, P2
and P3 are functions of t only. Hence, it
is possible to elinminate Pi, P2 and P3 from
Eqs.(5b),(5c) and (5d) by dif?erentiating
them with respect to ƒÅ. Finally we
obtain the following partial differential
equations for ƒÓ(ƒÅ,t), F(ƒÅ,t), G(ƒÅ,t),
… .

(7a)

(7b)

(7c)

Boundary conditions Eqs.(2a) and (2b)
are rewritten as

(8)

Multifold Expansion
First, the independent variable t is
transformed into another variable ƒÄ by the
following expression:

(9)

where ta is a constant.
By the use of this transformation, the
differentiation (h2/ v)(a/a t) in Eq.(7)
can be transformed into a standard form
ζ(∂/∂ ζ),(ζ ≧ 0), of a parabolic type.
Next, using ƒÄ, we define the following

two sets of infinite number of variable

parameters, one consisting of the deriva-
tives of the first characteristic function
α(ζ) and the other the second characteris-

tic function ƒÁ(ƒÄ):

(10a)

where

(10b)

and

(11a)

where

(11b)

Then, in Eq.(7), the differential.ope-
ration with respect to t is decomposed into
those with respect to variable parameters
Si, S2P,•c and K1, K2,•c as

(12)

Note that the differentiation of ƒÓ with
respect to K's are not needed because this
equation does not include ƒÁ.
Owing to the transformations a= S1,
γ=K1 and Eq.(12), α (t)[or h (t)], γ (t)
[or q (t)] and t have vanished from Eq.(7).
This means that the unknown functions φ and
{F,G,…} have been transformed from fun-
ctions of (η,t) into those of (η; S1, S2,
…) and (η; S1, S2,…; K1, K2,…), respe-
ctively.
In this flow phenomenon, all flow cha-
racteristics are determined by giving the
form of two characteristic functions a(t)
and ƒÁ(t). It should be noted that to
give the form of a(t) and ƒÁ(t) is equiva-
lent to determine the values of independent
variables S1, S2,•c and Ki, K2,•c [these
variables are composed of the derivatives of
a(t) and ƒÁ(t)], as can be seen from the
Taylor's expansion theorem.
Accordingly, we can solve Eqs. (7) and
(8) in a generalized form, without any res-
triction on the functional forms of h(t) and

q(t), by expanding the functions ƒÓ, F,
G, in the following forms of "multi-
fold series" of the independent variable
parameters Si, S2,•c, K1, K2,•c:

(13a)

(13b)

(13c)

552

In these "multifold series" the Si- and
K1-terAns belong to the first order column, the S14-, S2-, S1K1-, K12- and K2-terms to
the second and so rorth, and all the series
terms belonging to a column are of the same order of magnitude.
Now, substituting Eqs. (12) and (13)
into Eqs. (7) and (8), and equating the
coefficients with similar forms of variable

parameters to zero, we obtain the following
ordinary differential equations which deter-
mines the functions01,, F0,
G0,,h1,and j1,:

(14a)

(14b)

(14c)

(14d)

(14e)

Boundary conditions are given as fol-
lows:

(15)

where Y represents any function except Oland Fo.
As can be seen from the above equa-
tions, ƒÓ1, F0, G0,
hi, and j1, are all functions
of alone and free from any other factors
related to the individual actual flow condi-

tions. Hereafter we will call them "univer-
sal coefficient-functions". Consequently,
if once these equations have been solved,
the results can be applied universally for
all kinds of, and arbitrary, variations of
the gap width and the flow rate of the
source with time.
Although analytical solutions for Eqs.
(14) and (15) exist, their derivations are
lengthy ad laborious, especially for the
higher-order functions. In the pesent re-
search, therefore, we have solved Eqs. (14)
and (15) numerically by means of Runge-
Kutta-Gill's method. Fourteen places (deci-
mal digits) in significant figure of arit-
hmetic operation were retained throughout
the computation, and the step size applied
was 0.00025.
The important results are shown in
Table 1. In Table 1, the relations

zƒÅƒÅ (0)=-ZƒÅƒÅ(1) and ZƒÅƒÅƒÅ(0)=ƒÅƒÅƒÅƒÅ(1) result
from the symmetry of the radial-flow veloci-
ty profile.

Solution for Pressure and Flow Force
When Eqs. (7) and (8) are solved using
the above mentioned method, we can obtain
important flow characteristics applicable to
practical problems(18), i.e, pressure dis-
tribution p and flow force W exerted on the
disks. Here, only the final, nondimen-
sional expressions for p and W are shown

(16)

Table 1 Universal numerical values related
to the fundamental flow characteristics (the
notation Z represents every universal
coefficient-function)

553

(17)

where

(18)

P1*, W1*: values in the region re≧r≧rO

P2*, W2*: values in the region rO≧r≧0

Applying the boundary conditions, Eq.
(8), to Eqs. (5b),(5c) and (5d), and substi-
tuting Eqs. (13) and the values in Table 1,
the coefficient-functions P1, P2, P3 are
expressed as follows:

(19)

EXPERIMENT

Fascilities and Instrumentation
Figure 2 shows the experimental appara-
tus used. Water was supplied from a head
tank through a flow control valve and two
fine wire screens into the test section
formed by two parallel disks. The test
section was set submerged within a water
tank. The water levels of both the head
tank and water tank were kept constant by
overflow devices; the head difference bet-
ween both levels was 12 m.
A mechanism converting rotation into
reciprocating motion, which was driven by a
variable-speed electric motor through a
timing belt, was used to oscillate sinusoi-
dally the upper disk of the test section.
The source flow was varied by changing
the restriction of a spool valve using a cam
attached to the axis of the same electric

motor that drove the disk.
The two circular disks, which formed
the test section, were made of 15 mm-thick
transparent plastic plates with 300 mm outer
diameter (2re). The upper disk had in its
central portion a fluid source pipe with 30
mm inner diameter (2ro). Consequently, in
the present experimment K was set to 0.1,
which fulfills the first assumption 1) in
the previous section. Sharp edge at the
central opening in the upper disk was roun-
ded off in 2 mm radius to keep off flow
separation from there. The water tank, in
which the test section was set submerged,
was a waster depth of 670 mm and 700 mm x 700 mm bottom sizes and sufficiently larger
than the test section, so that the flow
within the test section was almost complete-
ly unaffected by the existence of the water
tank walls.
The time-varying gap width between the
two disks, the pressures and the flow force
acting on the lower disk surface, were mea-
sured respectively by a displacement tran-
sducer (capacitor type), pressure transdu-
cers (water-proofed, strain-gage type; a
capacity of 0.02 MPa) and a load cell (wa-
ter-proofed, strain-gage type; a capacity of
490 N). The outputs from these transducers
were amplified and fed to a low pass filter
to cut off high frequency noises from the
electric motor and other disturbing sources;
then they were recorded by a data logger
and fed to a micro-computer in order to be
carried out data processing. The load cell
was supported and protected by a precisely-made sliding guide, which was sufficiently
low resistant for axial forces to be mea-
sured but extremely high rigid for any other
possible lateral forces. The fluid
pressures were measured on the lower disk
surface at five posotions of r= 20, 30, 50,
80 and 120 mm, the respective positions
being on different radii at angular inter-
vals of 60 deg.
In this experiment, the gap width and
the source flow were varied sinusoidally
according to the following equations:

(20)

(21)

Fig. 2 Schematic diagram of the experi-
mental setup

554

The angular frequency of h(t) was set equal
to that of q(t), the phase difference 6 be-
tween them was varied by changing the angu-
lar position of the cam to the motor shaft.
The angular frequency w was varied by chan-
ging the motor speed. The nondimêntional
amplitude b of the flow was varied by adju-
sting the opening of the flow control valve
set in series with the spool valve. Conse-
quently, it was not possible to vary b and
qmindependently of each other. However,
this little hindered the present purpose of
experiment, i.e., verification of the vali-
dity of the theory.
The frequency w /(2ƒÎ) was varied
between 0.1 and 3 Hz, 6 between -180 and
+90 deg, a between 0.25 and 0.57, b between
0.26 and 0.56, hm between 1.0 and 2.0mm,
and qm between 90 and 270 cm3/s, respective-
ly.
The value of q(t) was calculated from
the area (varied with a cam) of the spool
valve restriction and the differential pres-
sure (measured by a semi-conductor type
differential pressure transducer) immediate-
ly upstream and downstream of the spool
valve, assuming quasi-steady flow.

Experimental Results and Comparison with the
Theory

Non-linear interaction. As already
stated, simple superposition of the solution
of the flow with time-varying gap width and
that of the source flow will not result in a

good result because of non-linear interac-
tion. The interaction may grow stronger as
the following two kinds of the Reynolds
numbers corresponding to the two coexisting
flows increase together:

(22)

(23)

under the condition that the two flows are
kept in a balance of comparable effects.
From the expression for mean radial flow
velocity in the gap

(24)

it follows that the two flows are of
comparable effects when

(25)

The mean value of r* is considered to be
1/•ã2 because this radial position divides
the disk surface into two (inner and outer)
parts of equal areas. The quantity S1 is
of the order of allw and h* of unity.
Consequently, Eq. (25) becomes

(26)

The calculated results for the flow
force W* are shown in Figs. 3, where the
source flow is kept constant. In the fi-
gures, the results of the approximate supe-
rposition theory which neglects the non-
linear interaction are shown as the "simple
theory". As can be seen from these figures,
the differencebetween the present and sim-
ple theories becomes most remarkable in Fig.

3(b), i.e., when (aRto)/4=Rn. This fact justifies the conjecture des%ribed above. Figure 3(d) shows a comparison of an experi-mental result with both the simple and pre-sent theories for the case of (aRw)/4 Rn. As can be seen, while the simple solutia which does not take into account the inte-

raction produces a remarkable error, the present solution which takes it into account agrees very well with the experimental re-sult. This fact just demonstrates the vali-dity of the present theory.

The case where the source flow is kept constant. In this case b=0 in Eq. (21), and the flow characteristics are governed by the three parameters, i.e., R. , Rqm and a. Measurements were carried out for various

sets of values of these parameters (R0 a). Some representative results are shoin in Fig. 4 for pressures and in Fig. 5 for flow forces. In these figures, the theore-tical results of the new series solutions

given in the previous section are also dis-played.
It can be seen from examination that

the present new sseries solutions agree very well with the experimental results over the range in which the series solutions are rapidly convergent. Namely, in the case of Figs. 4(a), 4(c), 5(a) and 5(b), the solu-tions are rapidly convergent and really good agreements are attained between the theory and experiments. On the other hand, in the
cases of Figs. 4(b), 4(d), and 5(c), the
solutions are found to be no longer rapidly convergent and correspondingly some disag-reement begins to appear.
Thus, there is naturally a finite range of applicability for the present finitely-
truncated series solutions. Therefore, for
the convenience of application, we have provided a practical guide which indicates roughly the range of applicability, by com-paring the theory with the many experimental results obtained through this experimental

Fig.3 Comparison between the present and
the approximate superposition theories: the results of the nondimensional force W. for the case of sinusoidaliy oscillating gap-
width variation [in (d) an experimental
result is shown together].

555

investigation. Some examples are shown in
Fig. 6.

The case where the source flow is va-
ned sinusoidally. The flow characteris-
tics in this case are subjected to a, b,ƒÂ
RƒÖ and Ra. Some representative results are
shown in Figs. 7 and 8. Theoretical results

(a) r*= 0.13

(c) r*= 0.80

(b) r*= 0.13

(d) r*= 0.80

Fig. 4 Theoretical and experimental results
of the pressure for the case of sinusoidally
oscillating gap-width variation

(b)

(a)

(c)

Fig. 5 Theoretical and experimental results
of the flow forces for the case of
sinusoidally oscillating gap-width variation

(a) (b)

Fig.6 Range of aplicability of the solu -
tions for the case of sinusoidally oscilla -
ting gap-width variation: (a) pressure (a=
0.38); (b) force (a = 0.43)

(a)

(c)

(e)

(b)

(d)

(f)

Fig. 7 Experimental and theoretical results of the pressure for the case of sinusoidally oscillating gap-width and source-flow varia-tions

(a)

(c)

(e)

(b)

(d)

(f)

Fig. 8 Theoretical and experimental results of the flow force for the case of sinusoida-lly oscillating gap-width and source-flow variations

556

of the present series solutions are also
shown in the same figures. Although it was
difficult to select the values of the para-
meters in round figures because of more
parameters than the previous case and the
limitation of experimental instrumentation,
we tried to pick up the parameters so that
the readers may be able to estimate the
effect of only one of the parameters, com-
paring any two graphs of (a) through (f) in Figs. 3 and 4.
As can be seen from these figures, the
present theoretical solutions calculated
from finitely-truncated series agree very
well with the experiments like as the pre-
ceding case, and the consistency of the
theory is confirmed. The applicable range
may be estimated from Fig.6, with IR (t)Imam
taken in place of R,, and in fact this'has
been confirmed by many experimental results.

CONCLUSIONS

A theoretical analysis and experimental
results have been presented for the unsteady
laminar incompressible flow between parallel
disks with an arbitrarily varying gap width
and a central source of arbitrarily varying
(or constant) flow rate.
New series solutions to the Navier-
Stokes equations are obtained, making use of
a technique of asymptotic series expansion
in the radial direction and on the basis of
a new theory of "multifold series expan-
sion"(17) with respect to the time variable.
The solutions can describe precisely the
important interaction phenomenon between the
two coexisting flows, one caused by the gap-
width variation and the other due to the
source. This interaction phenomenon is
produced through the non-linear convective

inertial forces, and its complicated charac-
teristics have not been made clear up to
date: the present theory is the first to
throw light upon it.
Next, as a typical example, the parti-
cular case of sinusoidally oscillating gap-
width and source-flow variations was taken
up, and the experiments for measurements of
the pressures and the flow force acting on
the disk surface were carried out.
From comparison of the theory with the expe-
riments, the validity of the new theory has
been verified. The solutions given by the
new theory agree well with the experimental
results even in severe cases when the appro-
ximate "superposition theory" which neglects
the interaction effect produces a remarkable
error.

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