On Visibility Graphs of Point Sets in the Plane

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Abstract

The visibility graph $\mathcal{V}(X)$ of a discrete point set $X \subset \mathbb{R}^2$ has vertex set X and an edge xy for every two points $x, y \in X$ whenever there is no other point in X on the line segment between x and y. We show that for every graph G, there is a point set $X \in \mathbb{R}^2$, such that the subgraph of $\mathcal{V}(X \cup \mathbb{Z}^2)$ induced by X is isomorphic to G. As a consequence, we show that there are visibility graphs of arbitrary high chromatic number with clique number six settling a question by Kára, Pór and Wood.

1 Introduction

The concept of a visibility graph is widely studied in discrete geometry. You start with a set of objects in some metric space, and the visibility graph of this configuration contains the objects as vertices, and two vertices are connected by an edge if the corresponding objects can "see" each other, i.e., there is a straight line not intersecting any other part of the configuration from one object to the other. Often, there are extra restrictions on the objects and on the direction of the lines of visibility.

Specific classes of visibility graphs which are well studied include bar visibility graphs (see [3]), rectangle visibility graphs (see [6]) and visibility graphs of polygons (see [1]). In this paper we consider visibility graphs of point sets.

Let $X \subset \mathbb{R}^2$ be a a discrete point set in the plane. The *visibility graph of* X is the graph $\mathcal{V}(X)$ with vertex set X and edges xy for every two points $x, y \in X$ whenever there is no other point in X on the line segment between x and y, i.e. when the point x is visible from the point y and vice versa.

Kára, Pór and Wood discuss these graphs [4], and make some observations regarding the chromatic number $\chi(\mathcal{V}(X))$ and the clique number $\omega(\mathcal{V}(X))$, the order of the largest clique. In particular, they characterize all visibility graphs with $\chi(\mathcal{V}(X)) = 2$ and $\chi(\mathcal{V}(X)) = 3$, and in both cases, $\omega(\mathcal{V}(X)) = \chi(\mathcal{V}(X))$. Similarly, they show the following proposition.

Proposition 1. *Let $\mathbb{Z}^2$ be the integer lattice in the plane, then $\omega(\mathcal{V}(\mathbb{Z}^2)) = \chi(\mathcal{V}(\mathbb{Z}^2)) = 4$.*

Note that $\mathcal{V}(\mathbb{Z}^2))$ is not perfect as it contains induced 5-cycles. Further, it is not true in general that $\omega(\mathcal{V}(X)) = \chi(\mathcal{V}(X))$—there are point sets with as few as nine points with $\omega(\mathcal{V}(X)) = 4$ and $\chi(\mathcal{V}(X)) = 5$.

For general graphs, there are examples with $\chi(G) = k$ and $\omega(G) = 2$ for any k, one famous example is the sequence of graphs M_{k-2} by Mycielski [5]. No similar construction is known for visibility graphs with bounded clique number. As their main result, Kára et al. construct a family of point sets with $\chi(\mathcal{V}(X)) \geq (c_1 \log\ \omega(\mathcal{V}(X_i)))^{c_2 \log\ \omega(\mathcal{V}(X_i))}$ for some constants c_1 and c_2 and with $\omega(\mathcal{V}(X_i))$ getting arbitrarily large. Our main result is the following theorem.

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Theorem 2. *For every graph G, there is a set of points $X \subset \mathbb{R}^2$ such that the subgraph of $\mathcal{V}(X \cup \mathbb{Z}^2)$ induced by X is isomorphic to G.*

Let G_k be a graph with $\chi(G_k) = k$ and $\omega(G_k) = 2$, and let X_k be the corresponding set given by Theorem 2. Let $Y_k \subset X_k \cup \mathbb{Z}^2$ be the subset of points contained in the convex hull of X_k. Then $\chi(\mathcal{V}(Y_k)) \geq \chi(G_k) = k$ and $\omega(\mathcal{V}(Y_k)) \leq \omega(G_k) + \omega(\mathcal{V}(\mathbb{Z}^2)) = 6$, so we get the following corollary settling the question from above raised by Kára et al.

Corollary 3. *For every k, there is a finite point set $Y \subset \mathbb{R}^2$, such that $\chi(\mathcal{V}(Y)) \geq k$ and $\omega(\mathcal{V}(Y)) = 6$.*

2 Proof of the Theorem

Let G be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. We will show the following lemma in the Section 3.

Lemma 4. *For M large enough, there is a set of prime numbers $\{p_{ij} : 1 \leq i < j \leq n\}$ with the following properties:*

1. $2^M < p_{ij} < 2^{M+1}$.
2. *For $1 \leq k \leq n$, let $P_k = 2^{n_k} \prod_{i=1}^{k-1} p_{ik} \prod_{j=k+1}^{n} p_{kj}$, and choose $n_k \in \mathbb{Z}$ such that $\lfloor \log_2 P_k \rfloor = nM + 2k$. Then $p_{k\ell}$ is the only number in $\{p_{ij} : 1 \leq i < j \leq n\}$ which divides $P_\ell - P_k$ for $1 \leq k < \ell \leq n$.*

Note that $\prod_{i=1}^{k-1} p_{ik} \prod_{j=k+1}^{n} p_{kj} < 2^{(n-1)(M+1)} < 2^{nM}$, and thus $n_k > 0$ and $P_k \in \mathbb{Z}$ for all k. From this, we can construct the set of points X in Theorem 2:

$$X = \{x_i : 1 \leq i \leq n\} \subset \mathbb{R}^2, \text{ with } x_i = \left(2^{-nM} P_i \,,\; i \, \frac{\prod\limits_{k<j} (P_j - P_k)}{\prod\limits_{kj \in E(G)} p_{kj}} \right).$$

Before we prove the lemma, we will show that this point set has the properties stated in the theorem. For $1 \leq i < \ell \leq n$, let $m_{i\ell}$ be the slope of the line through x_i and x_ℓ. Then

$$m_{i\ell} = \frac{\ell - i}{P_\ell - P_i} \cdot \frac{2^{nM} \prod\limits_{k<j} (P_j - P_k)}{\prod\limits_{kj \in E(G)} p_{kj}}.$$

There are no three colinear points in X, as

$$2^{nM+2i+1} \leq P_{i+1} - P_i < 2^{nM+2i+3},$$

thus $m_{i(i+1)} > m_{(i+1)(i+2)}$, and therefore $m_{i\ell} > m_{ik}$ for $i < \ell < k$. Thus, $\mathcal{V}(X)$ is complete, and it remains to show that there is an integer point on the line segment between x_i and x_ℓ if and only if $i\ell \notin E(G)$. To establish this goal, we will look at the intersections of the line segment from x_i to x_ℓ ($i < \ell$) with the integer gridlines parallel to the y-axis.

Let $s \in \mathbb{Z}$, with $2^{-nM}P_i < s < 2^{-nM}P_\ell < 2^{2n+1}$. As $2^{2j} \leq 2^{-nM}P_j < 2^{2j+1}$ for every j, such an s exists. Let $z_{i\ell}^s = (s, y_{i\ell}^s)$ be a point on the line segment from x_i to x_ℓ. Then

$$y_{i\ell}^s = i\,\frac{\prod\limits_{k<j}(P_j - P_k)}{\prod\limits_{kj\in E(G)} p_{kj}} + (s - 2^{-nM}P_i)m_{i\ell}$$

$$= \underbrace{i\,\frac{\prod\limits_{k<j}(P_j - P_k)}{\prod\limits_{kj\in E(G)} p_{kj}}}_{(1)} + \underbrace{s\,\frac{\ell - i}{P_\ell - P_i}\cdot\frac{2^{nM}\prod\limits_{k<j}(P_j - P_k)}{\prod\limits_{kj\in E(G)} p_{kj}}}_{(2)} + \underbrace{P_i\,\frac{\ell - i}{P_\ell - P_i}\cdot\frac{\prod\limits_{k<j}(P_j - P_k)}{\prod\limits_{kj\in E(G)} p_{kj}}}_{(3)}.$$

The expression (1) is an integer since p_{kj} divides $P_j - P_k$. By the same argument , (3) is an integer—just note further that $p_{i\ell}$ divides P_i. It remains the analysis of (2).

If $i\ell \notin E(G)$, then (2) is an integer. Therefore, $z_{i\ell}^s \in \mathbb{Z}^2$, and $x_i x_\ell \notin E(\mathcal{V}(X \cup \mathbb{Z}^2))$. If $i\ell \in E(G)$, observe that $p_{i\ell} > 2^M > \max\{\ell - i, s\}$, so $p_{i\ell}$ does not divide s or $\ell - i$. Clearly, $p_{i\ell}$ does not divide 2^{nM}, and by Lemma 4, it does not divide any of the $P_j - P_k$ other than $P_\ell - P_i$. Thus, (2) is not an integer, $z_{i\ell}^s \notin \mathbb{Z}^2$ for all s considered, and $x_i x_\ell \in E(\mathcal{V}(X \cup \mathbb{Z}^2))$, proving Theorem 2. □

3 Proof of Lemma 4

By an inequality of Finsler [2], there are more than $2^M/(3(M+1)\ln 2) > 2n^3$ prime numbers in the interval from 2^M to 2^{M+1}.

We will pick the p_{ij} sequentially in the order $p_{12}, p_{13}, \dots, p_{1n}, p_{23}, \dots, p_{(n-1)n}$, with the following conditions given by the lemma:

(a) p_{ij} is a prime number with $2^M < p_{ij} < 2^{M+1}$.

(b) p_{ij} is different from all primes picked before.

(c) p_{ij} does not divide $P_k - P_\ell$ for all $1 \leq \ell < k < i$.

(d) If $j = n$, no $p_{k\ell}$ divides $P_i - P_r$ for $\{k, \ell\} \neq \{i, r\}$.

Assume that we have picked numbers up to but not including p_{ij} according to (a)-(d), and we want to pick p_{ij}. Consider first the case that $j < n$. There were less than $\binom{n}{2}$ primes selected before, and each $P_k - P_\ell$ has at most n prime divisors greater than 2^M, thus at most $\binom{n}{2} + n\binom{n}{2} < n^3$ of the choices are blocked, and we can find p_{ij} according to (a)-(c).

If $j = n$, pick p_{ij} according to (a)-(c), and assume that $p_{k\ell}$ divides $P_i - P_r$ for some $\{k, \ell\} \neq \{i, r\}$ (i.e., condition (d) is violated). We have $k \neq i$ as all $p_{i\ell}$ divide P_i, otherwise $p_{i\ell}$ also divides P_r and thus $r = \ell$, a contradiction. Similarly, $\ell \neq i$.

Pick another number p'_{ij} according to (a)-(c). If $p_{k\ell}$ divides $P'_i - P_r$, then $p_{k\ell}$ divides $P'_i - P_i = (p'_{ij} - p_{ij})P_i/p_{ij}$, and thus $p_{k\ell}$ divides $p'_{ij} - p_{ij}$. But this is impossible since $|p'_{ij} - p_{ij}| < 2^M < p_{k\ell}$. Therefore, each $p_{k\ell}$ can block at most one choice for p_{ij} this way, so in total at most $\binom{n}{2}$ further choices are blocked by condition (d), and we can always find a number p_{ij} with (a)-(d). This concludes the proof of the lemma. □

4 Further Questions

We have shown that there are visibility graphs with $\chi(\mathcal{V}(X)) \geq k$ and $\omega(\mathcal{V}(X)) = 6$ for every k. For all visibility graphs with $\omega(\mathcal{V}(X)) \leq 3$, we know that $\chi(\mathcal{V}(X)) = \omega(\mathcal{V}(X))$. The only cases left to consider are $\omega(\mathcal{V}(X)) = 4$ and $\omega(\mathcal{V}(X)) = 5$. A similar technique of combining a visibility graph with $\omega(\mathcal{V}(X)) = 3$ with a graph G with $\omega(G) = 2$ and large chromatic number will not work, since the visibility graphs with $\omega(\mathcal{V}(X)) = 3$ are too simple (all but at most two of their vertices are collinear unless $\mathcal{V}(X)$ is a special graph on six vertices). It would be no surprise to us if the chromatic number of visibility graphs with $\omega(\mathcal{V}(X)) = 5$ is bounded.

Finally, one could look for smaller point sets with $\chi(\mathcal{V}(X)) \geq k$ and $\omega(\mathcal{V}(X)) = 6$, as our sets tend to be very large.

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