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VISUALISATION OF ISOMETRIC MAPS

EBERHARD MALKOWSKY AND VESNA VELIČKOVIĆ

Abstract. **We give use our own software for differential geometry and geometry and its extensions** [4, 3, 1, 5] **and apply it to the visualisation and animation of isometries between certain surfaces.**

1. Notations

Let $D \subset \mathbb{R}^2$ be a domain and S be a surface given by a parametric representation $\vec{x}(u^i) = (x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2))$ for $(u^i) = (u^1,u^2) \in D$ with coordinate functions $x^k \in C^r(D)$ $(r \geq 1)$ and $\vec{x}_1 \times \vec{x}_2 \neq \vec{0}$ on D where $\vec{x}_k = \partial\vec{x}/\partial u^k$ for $k = 1,2$. By

$$\vec{N}(u^i) = \frac{\vec{x}_1(u^i) \times \vec{x}_2(u^i)}{\|\vec{x}_1(u^i) \times \vec{x}_2(u^i)\|}$$

we denote the surface normal vector of S at $(u^i) \in D$, and the functions $g_{ik}, L_{ik} : D \to \mathbb{R}$ with $g_{ik} = \vec{x}_i \bullet \vec{x}_k$ and $L_{ik} = \vec{x}_{ik} \bullet \vec{N}$ where $\vec{x}_{ik} = \partial^2\vec{x}/\partial u^i \partial u^k$ for $i,k = 1,2$ are called the *first* and *second fundamental coefficients of S*. We write $g = \det((g_{ik})_{i,k=1,2})$ and $L = \det((L_{ik})_{i,k=1,2})$. The functions $K, H : D \to \mathbb{R}$ with

$$K = L/g \text{ and } H = \frac{1}{2}(g_{11}L_{22} - 2g_{12}L_{12} + g_{22}L_{11})$$

are called the *Gaussian and mean curvature of S*.

Let S and S^* be surfaces with parametric representations $\vec{x}(u^i)$ and $\vec{x}^*(\tilde{u}^i)$ and $F : S \to S^*$ a map onto S^* given by functions $h^i \in C^r(D)$ with

$$\tilde{u}^i = h^i(u^1,u^2) \ (i = 1,2)$$

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and non vanishing Jacobian. We may introduce new parameters u^{*i} for S^* by putting

$$\tilde{u}^i = h^i(u^{*1}, u^{*2}) \text{ (i=1,2).}$$

Then the map F is given by $u^{*i} = u^i$ $(i = 1, 2)$, and S and S^* are said to have the same parameters.

A map $F : S \to S^*$ is called *isometric* if the length of every arc on S is the same as that of its corresponding image.

2. Isometric Maps

There are well–known, simple necessary and sufficient conditions for a map $F : S \to S^*$ to be isometric; the conditions involve relations between the first fundamental coefficients g_{ik} and g^*_{ik} of S and S^*.

Theorem 2.1. ([2, Sätze 57.1, 57.2, pp. 213, 214])
A map $F : S \to S^*$ is isometric if and only if their first fundamental coefficients g_{ik} and g^*_{ik} with respect to the same parameters (u^j) and (u^{*j}) satisfy $g_{ik}(u^j) = g^*_{ik}(u^{*j})$ for $i, k = 1, 2$. In particular, the Gaussian and geodesic curvature of a surface are invariant under isometric maps.

Since a sphere of radius r and a plane have Gaussian curvature $K = 1/r$ and $K = 0$, respectively, it is obvious from Theorem 2.1 that no part of a sphere can be mapped isometrically into a plane.

Ruled surfaces play an important role in the theory of isometric maps; a *ruled surface* is a surface that contains a family of straight line segments. It is generated by moving vectors along a curve. Let γ be a curve with a parametric representation $\vec{y}(s)$ for s in some interval I where s is the arc length along γ, and, for every $s \in I$, let $\vec{z}(s)$ be a unit vector. Then a ruled surface generated by moving the vectors $\vec{z}(s)$ along the curve γ. Writing $u^1 = s$ and u^2 for the parameter along the vectors $\vec{z}$, we obtain a parametric representation

$$\vec{x}(u^i) = \vec{y}(u^1) + u^2\vec{z}(u^1). \tag{2.1}$$

Examples for ruled surfaces are planes, cylinders, cones, hyperboloids of one sheet and hyperbolic paraboloids. The first three surfaces are so–called *torses*. A *torse* is a ruled surface which has the same tangent plane at every point of each of its generating straight lines. It is known that a surface is a torse if and only if it is a plane, cylinder, cone or tangent surface [2, Satz 58.3, p. 223].

The following result shows that torses are the only surfaces that can be mapped isometrically into a plane.

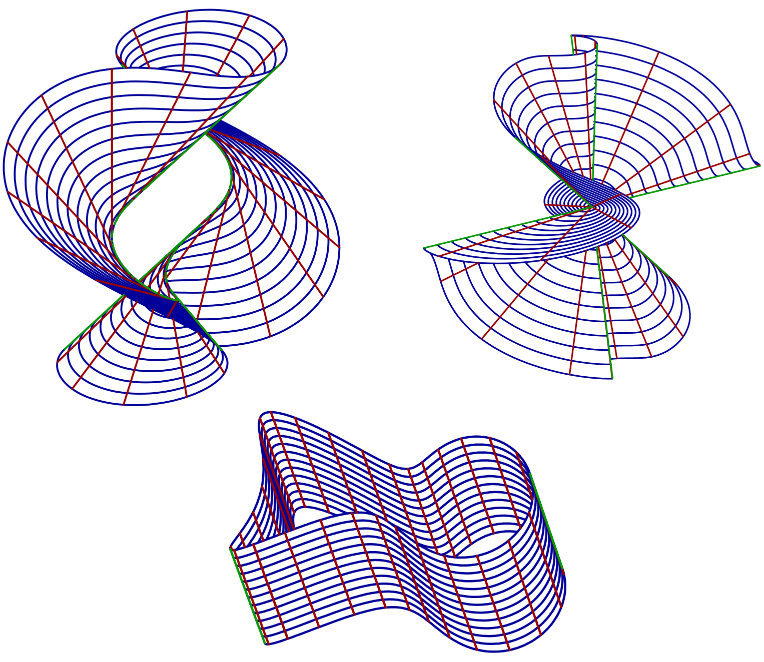


FIGURE 1. Torses: Tangent surface, Cone, Cylinder

Theorem 2.2. ([2, Satz 59.3, p. 228])
A sufficiently small part of a surface of class C^r $(r \geq 3)$ *can be mapped isometrically into a plane if it is part of a torse.*

Now we consider two more classes of surfaces that can also be mapped isometrically to one another, namely *surfaces of revolution* and *screw surfaces*.

Let γ be a curve in a plane, e. g. the x^1x^3–plane, and be given by a parametric representation $\vec{x}(t) = (r(t), 0, h(t))$ for t in some interval I where $r(t) > 0$ and $|r'(t)| + |h'(t)| > 0$ on I. Then a surface of revolution RS is generated by rotating γ about the x^3–axis. If we write $u^1 = t$ and u^2 for the angle of rotation measured anti–clockwise from the positive x^1–axis then RS has a parametric representation

$$\vec{x}(u^i) = (r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)) \quad \text{for } (u^1, u^2) \in D \subset I \times (0, 2\pi). \tag{2.2}$$

A *screw surface* is generated by the simultaneous rotation of a curve γ about a fixed axis A and the translation along A such that the speed of translation is proportional to the speed of rotation. The curve of intersection γ^* of a screw surface S with a plane through the axis A is called a *meridian of*

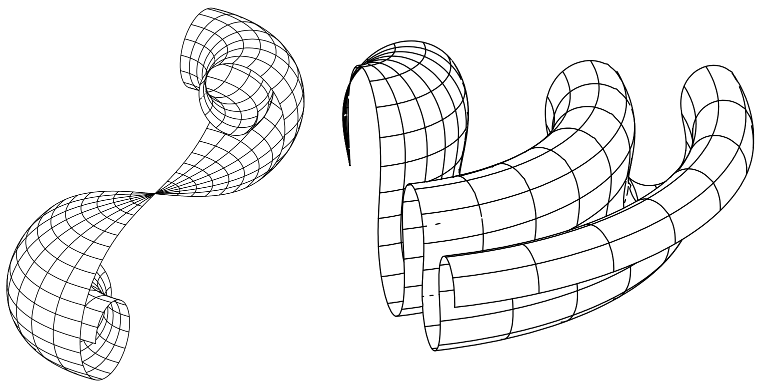


FIGURE 2. Surfaces of revolution

S. If γ^* is subjected to the same movement that generates the screw surface S then γ^* also generates S. Thus any screw surface can be generated by a planar curve which performs a screw movement around a straight line in its plane. We choose the x^3–axis as the axis A. Then γ^* can locally be represented by $x^3 = g(u^1)$ where u^1 denotes the distance between the axis A and the points of γ^*. We assume that γ^* is in the x^1x^2–plane at the beginning of the movement. Let u^2 be the angle of rotation. The translation of γ^* is parallel to the x^3–axis and proportional to u^2 by definition. Thus the screw surface S can be represented by

$$\vec{x}(u^i) = (u^1 \cos u^2, u^1 \sin u^2, cu^2 + g(u^1)). \tag{2.3}$$

where $c \neq 0$ is a constant. The u^2–lines of a screw surface are helices and its u^1–lines are its meridians.

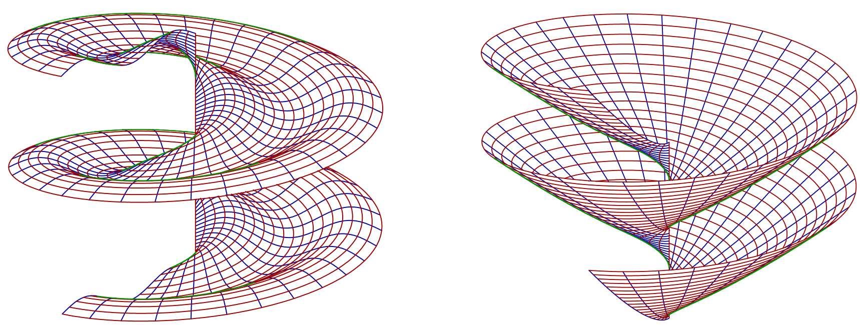


FIGURE 3. Screw surfaces

The concept of screw surfaces can be generalised to obtain surfaces with a parametric representation $\vec{x}(u^i) = (u^1 \cos u^2, u^1 \sin u^2, f(u^1, u^2))$ where $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is of class $C^r(D)$ for $r \geq 1$.

Bour's well–known theorem states that every screw surface can be mapped onto a part of a surface of revolution.

Theorem 2.3 (Bour). ([2, Satz 57.4, p. 217])
Every screw surface S can be mapped isometrically onto a surface of revolution.

We give a complete proof of this result, since it is constructive and gives a method to find a surface of revolution a given screw surface can be mapped isometrically onto; it can also be used to make animations for isometric maps between screw surfaces and surfaces of revolution.

We consider surfaces of revolution and screw surfaces given by parametric representations

$$\vec{x}(u^i) = (u^1 \cos u^2, u^1 \sin u^2, h(u^1)) \tag{2.4}$$

and

$$\bar{\vec{x}}(\bar{u}^i) = (\bar{u}^1 \cos \bar{u}^2, \bar{u}^1 \sin \bar{u}^2, c\bar{u}^2 + g(\bar{u}^1)) \tag{2.5}$$

where c is a constant.

First we introduce orthogonal parameters u^{*1} and u^{*2} on a screw surface S with a parametric representation (2.5) such that the u^{*1} lines are helices and the u^{*2} lines are their orthogonal trajectories. We put $u^{*1} = \bar{u}^1$ and $u^{*2} = h(\bar{u}^1, \bar{u}^2)$. Then the u^{*1} lines are helices and we have to determine the function h such that the u^{*2} lines are their orthogonal trajectories. Since

$$\bar{\vec{x}}_1 = (\cos \bar{u}^2, \sin \bar{u}^2, g'(\bar{u}^2)) \text{ and } \bar{\vec{x}}_2 = (-\bar{u}^1 \sin \bar{u}^2, \bar{u}^1 \cos \bar{u}^2, c),$$

the first fundamental coefficients of S with respect to the parameters $\bar{u}^1$ and $\bar{u}^2$ are given by $\bar{g}_{11} = 1 + (g'(\bar{u}^1))^2$, $\bar{g}_{12} = cg'(\bar{u}^1)$ and $\bar{g}_{22} = (\bar{u}^1)^2 + c^2$, and consequently the first fundamental form of S is

$$\begin{aligned} ds^2 &= \left(1 + \left(g'(\bar{u}^1)\right)^2\right)\left(d\bar{u}^1\right)^2 + 2cg'(\bar{u}^1)d\bar{u}^1 d\bar{u}^2 + \left(\left(\bar{u}^1\right)^2 + c^2\right)\left(d\bar{u}^2\right)^2 \\ &= \left(1 + \frac{\left(\bar{u}^1\right)^2 \left(g'(\bar{u}^1)\right)^2}{\left(\bar{u}^1\right)^2 + c^2}\right)\left(d\bar{u}^1\right)^2 \\ &\qquad + \left(\left(\bar{u}^1\right)^2 + c^2\right)\left(\frac{cg'(\bar{u}^1)}{\left(\bar{u}^1\right)^2 + c^2} d\bar{u}^1 + d\bar{u}^2\right)^2 \end{aligned}$$

If we put

$$\frac{cg'(\bar{u}^1)}{\left(\bar{u}^1\right)^2 + c^2} d\bar{u}^1 + d\bar{u}^2 = \eta du^{*2} \text{ where } \eta \text{ is a constant,}$$

that is, if we use the transformation

$$u^{*1} = \bar{u}^1 \text{ and } u^{*2} = h(\bar{u}^1, u^2) = \frac{1}{\eta}\left(c\int \frac{g'(\bar{u}^1)}{(\bar{u}^1)^2 + c^2}\, d\bar{u}^1 \ + \ \bar{u}^2\right)$$

for $\eta \neq 0$, then the first fundamental form of S with respect to the new parameters u^{*1} and u^{*2} is given by

(2.6)
$$ds^2 = \left(1 + \frac{(u^{*1})^2 \left(g'(u^{*1})\right)^2}{(u^{*1})^2 + c^2}\right)\left(du^{*1}\right)^2 + \eta^2\left(\left(u^{*1}\right)^2 + c^2\right)\left(du^{*2}\right)^2$$

and the parameters u^{*1} and u^{*2} are orthogonal.

To prove Bour's theorem, let RS be s surface of revolution given by a parametric representation (2.4) for $(u^1, u^2) \in D \subset (0, \infty) \times (0, 2\pi)$. Then

$$\vec{x}_1 = (\cos u^2, \sin u^2, h'(u^1)) \text{ and } \vec{x}_2 = (-u^1 \sin u^2, u^1 \cos u^2, 0)$$

and the first fundamental form of RS is given by

$$ds^2 = \left(1 + \left(h'(u^1)\right)^2\right)\left(du^1\right)^2 + \left(u^1\right)^2\left(du^2\right)^2. \tag{2.7}$$

By Theorem 2.1, a screw surface S given by a parametric representation with respect to the parameters u^{*1} and u^{*2} introduced above is isometric to the surface of revolution RS if and only if the first fundamental forms (2.6) and (2.7) are of the same form. This is the case if we put

$$u^2 = u^{*2}, \tag{2.8}$$

$$\left(u^1\right)^2 = \eta^2\left(\left(u^{*1}\right)^2 + c^2\right) \text{ for some constant } \eta \neq 0 \tag{2.9}$$

and

$$1 + \left(h'(u^1)\right)^2\left(\frac{du^1}{du^{*1}}\right)^2 = 1 + \frac{(u^{*1})^2 \left(g'(u^{*1})\right)^2}{(u^{*1})^2 + c^2}. \tag{2.10}$$

Relations (2.8), (2.9) and (2.10) enable us to find a screw surface isometric to a given surface of revolution, and conversely, a surface of revolution isometric to a given screw surface. If a surface of revolution is given, then we use (2.9) to eliminate the parameter u^1 in (2.10) and then solve (2.10) for $g'(u^{*1})$ to find the function g of the screw surface. Conversely, if a screw surface is given, we use (2.9) to eliminate the parameter u^{*1} in (2.10) and then solve (2.10) for $h'(u^1)$ to find the function h of the surface of revolution.

We apply this method to find screw surfaces that are isometric to a catenoid.

Example 2.4. We consider a catenoid given by a parametric representation

$$\vec{x}(u^i) = \left(u^1 \cos u^2, u^1 \sin u^2, a \cdot \operatorname{Arcosh}(\frac{u^1}{a})\right) \text{ where } a > 0 \text{ is a constant}$$

for $(u^1, u^2) \in D \subset (a, \infty) \times (0, 2\pi)$. Now

$$h'(u^1) = \frac{a}{\sqrt{(u^1)^2 - a^2}} \text{ and } 1 + (h'(u^1))^2 = \frac{(u^1)^2}{(u^1)^2 - a^2} \text{ for } u^1 > a.$$

First (2.9) yields $u^1 du^1 = \eta^2 u^{*1} du^{*1}$, that is $du^1/du^{*1} = \eta^2 \cdot u^{*1}/u^1$. Substituting this in (2.10), eliminating u^1 and solving for $g'(u^{*1})$, we obtain

$$\eta^4 \frac{(u^{*1})^2}{\eta^2\left((u^{*1})^2 + c^2\right) - a^2} = 1 + \frac{(u^{*1})^2 (g'(u^{*1}))^2}{(u^{*1})^2 + c^2}$$

for $u^1 > \sqrt{a^2/\eta^2 - c^2}$ and $c < a/|\eta|$, and

$$(g'(u^1))^2 = \frac{\left((u^{*1})^2 + c^2\right)\left((u^{*1})^2(\eta^2 - 1) - (c^2 - \frac{a^2}{\eta^2})\right)}{(u^{*1})^2\left((u^{*1})^2 + c^2 - \frac{a^2}{\eta^2}\right)}.$$

We choose $\eta = 1$ and put $k^2 = a^2 - c^2$ for $a \geq c$. Then we have

$$(g'(u^{*1}))^2 = \frac{\left((u^{*1})^2 + c^2\right) k^2}{(u^{*1})^2\left((u^{*1})^2 - k^2\right)} \text{ for } u^{*1} > k > 0,$$

that is

$$g'(u^{*1}) = k\sqrt{\frac{\left((u^{*1})^2 + c^2\right)}{(u^{*1})^2\left((u^{*1})^2 - k^2\right)}}.$$

This yields

$$g(u^{*1}) = k \log\left(\sqrt{\left(u^{*1}\right)^2 + c^2} + \sqrt{\left(u^{*1}\right)^2 - k^2}\right) \\ - c \arctan\left(\frac{k}{c}\sqrt{\frac{\left(u^{*1}\right)^2 + c^2}{\left(u^{*1}\right)^2 - k^2}}\right) + \tilde{d} \text{ where } \tilde{d} \text{ is a constant.}$$

We observe that we may choose $\tilde{d} = 0$, since a change in $\tilde{d}$ only results in a movement of the screw surface in the direction of the x^3–axis.

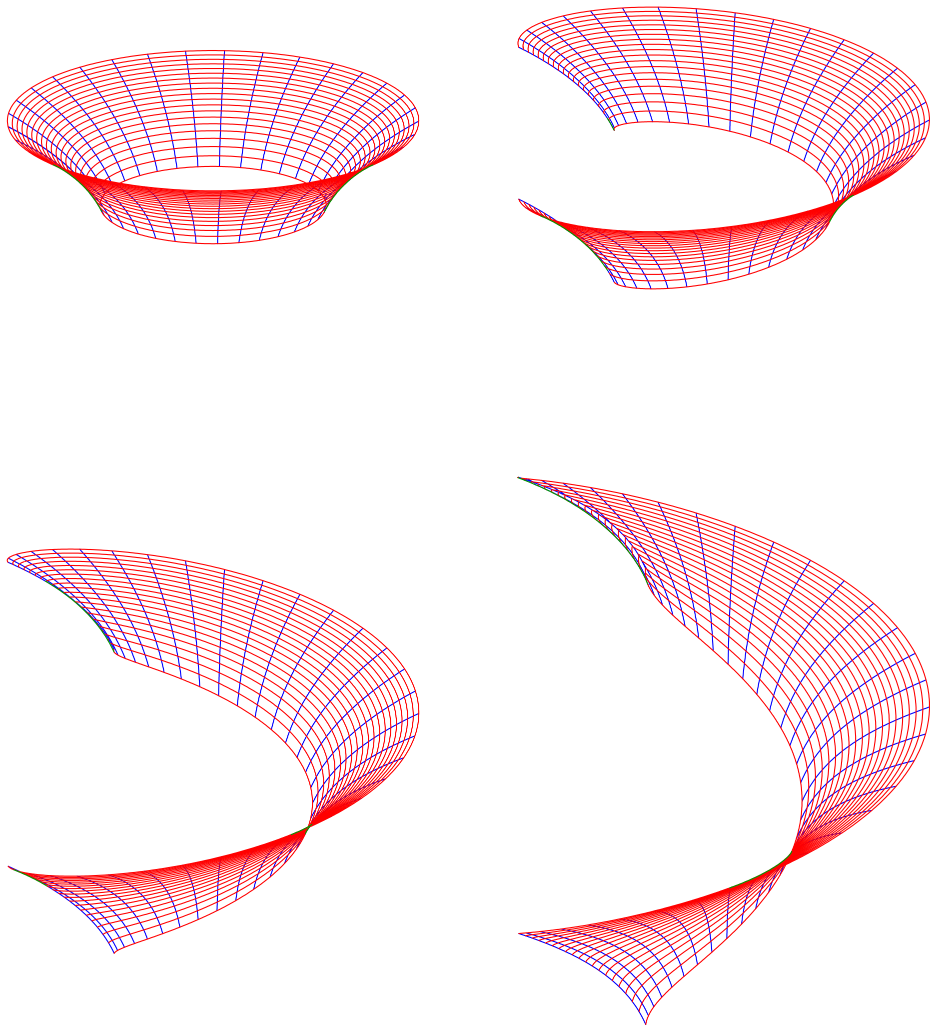
For every k with $0 < k \leq a$, that is for every c with $0 \leq c < a$, we obtain a screw surface S_k with

$$g(u^{*1}) = k \log\left(\sqrt{\left(u^{*1}\right)^2 + c^2} + \sqrt{\left(u^{*1}\right)^2 - k^2}\right) \\ - c \arctan\left(\frac{k}{c}\sqrt{\frac{\left(u^{*1}\right)^2 + c^2}{\left(u^{*1}\right)^2 - k^2}}\right).$$

which is isometric to the catenoid. If $k = a$, that is $c = 0$, then we obtain the original catenoid

$$g(u^{*1}) = a \log\left(u^{*1} + \sqrt{(u^{*1})^2 - a^2}\right) = a\mathrm{Arsinh}\left(\frac{u^{*1}}{a}\right) \text{ for } u^{*1} > a.$$

If $k = 0$, that is $c = a$, then $g'(u^{*1}) = 0$ and we obtain a helikoid.



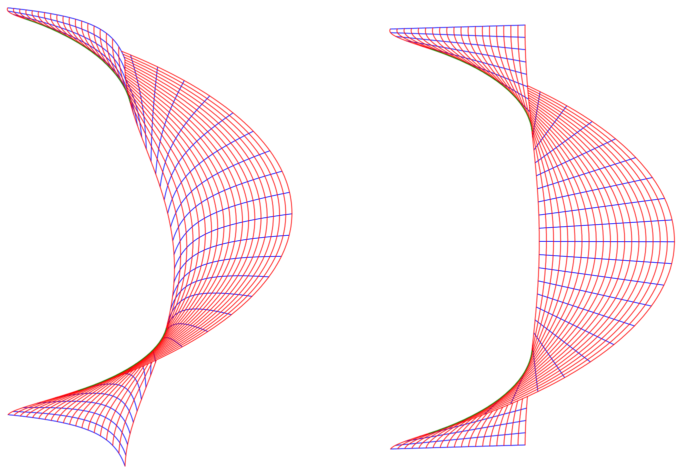


FIGURE 4. Isometric map from a catenoid to a screw surface

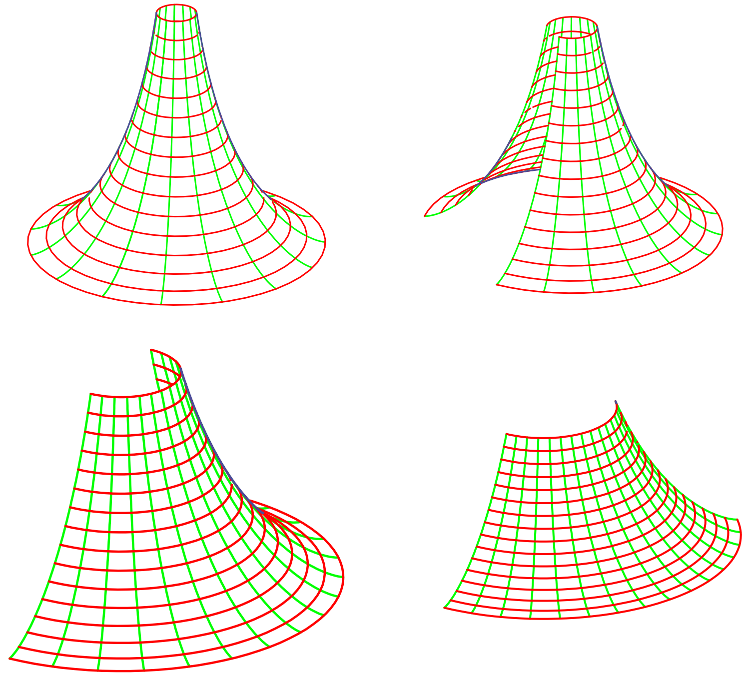


FIGURE 5. Isometric map of the pseudo-sphere

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