Visualizing Newton's Method on Fractional Exponents

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Abstract -- Newton's method for finding complex solutions of the equation $z^{\alpha} - 1 = 0$ is investigated for real values of α. The bifurcations that occur as the power α varies are illuminated using computer graphics. In particular, the appearance of an attracting 2-cycle just below even powers is investigated.

Keywords
Visualization - Chaos - Newton's method

Introduction

Computer graphics allows the visualization of the dynamics of iteration in powerful ways. Pictures of bifurcations of functions of a real variable and of basins of attraction for complex dynamics play an important role in understanding the dynamics of both real and complex iteration — from period doubling to chaos; for examples, see [1-5, 9-12, 14]. This short note describes bifurcations in the complex dynamics of Newton's method that we can visualize using a sequence of images.

Visualizing the convergence of Newton's method in the complex plane has been the subject of much recent study. Often those images show the basins of attraction of Newton's method applied to a polynomial of low degree. Typically, a family of third degree polynomials is considered or $z^n - 1$ for *n* equal to 3, 4 or 5 is considered. Also, in [8] Newton's method on systems with 2 complex variables is considered. In [15] nonintegral exponents are considered and in [13] a generalization of Newton's method is discussed. The dynamics of direct iteration of $z^{\alpha} + c$ for fractional exponents, which generalize the classical Mandelbrot and Julia sets, are described in [6] and are analyzed in [7]. Instead of direct iteration, we consider the bifurcations that occur using Newton's method on $z^{\alpha} - 1$ as we vary α.

Consider the basins of attraction for the roots of $z^3 - 1$ compared to those for $z^4 - 1$. The first has three main basins of attraction around the three roots of unity with symmetric turbulence on the boundaries. The second case is quite similar except it contains four basins of attraction. In general, we expect that slight changes to α should produce only slight changes in the behavior of Newton's method on $z^{\alpha} - 1$. However, it is clear that some fundamental changes in behavior must occur as we change from three to four basins of attraction. Thus, the question of how the change from three to four basins occurs is of interest. How many basins of attraction are there for any given positive real α? Does every starting guess converge to a root? Where do the bifurcations occur? This paper takes a look at those questions. Figs. 1-8 show the convergence of Newton's method on $z^{\alpha} - 1$ for $3 \le \alpha \le 5$. Fig. 1 shows three basins of attraction and Fig. 5 shows four basins of attraction as described above.

The Computation

Newton's method for solving $f(z) = 0$ begins with an initial guess z_0 and proceeds via: $z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$. Here the function is $f(z) = z^{\alpha} - 1$ so this simplifies into

$$z_{n+1} = z_n - \frac{z_n^\alpha - 1}{\alpha\, z_n^{\alpha-1}} = (1-\frac{1}{\alpha})z_n + \frac{1}{\alpha} z_n^{\alpha-1} \ .$$

Notice that the second version of this formula can be evaluated more quickly since it involves only one power of a complex number. Also notice we can think of steps of Newton's method as iteration of the function $g(z)$: $z_{n+1} = g(z_n)$ where $g(z) = (1-\frac{1}{\alpha})z + \frac{1}{\alpha} z^{\alpha-1}$.

In all the figures the initial guess z_0 used for Newton's method corresponds to position in the picture and varies with $-2 \le \mathrm{Re}(z_0) \le 2$ and $-1.5 \le \mathrm{Im}(z_0) \le 1.5$. The color used indicates the basin of attraction and the shade indicates the iteration count modulo 3. For example, all the shades of green indicate convergence to the root 1. The shades are chosen so that when we are within the green basin we are moving in the direction of increasing iteration count if we move from the darkest, through the medium, and to the lightest shade.

All the images were computed with extended precision floating point arithmetic using 19-20 significant digits and using high precision exponents so that the range of magnitudes is $10^{\pm 4931}$. Complex exponentiation was implemented using the standard branch cut along the negative real line. A sequence, z_n, was considered convergent if successive values differed by less than 10^{-10} in magnitude. The roots were dynamically determined with a distance of discrimination of 10^{-10}. The roots and 2-cycles discovered can be independently verified as described in the next section.

Results

In Fig. 1 we see three symmetric basins of attraction that are expected when $\alpha = 3$ since the roots of $z^3 - 1$ are the three roots of unity. In Fig. 2, $\alpha = 3.3$ and the turbulent region between the left-most basins increases in size. Notice also that unlike the case when $\alpha = 3$ there are boundaries between the red and blue basins where there is no turbulence (look at some of the small red and blue regions near the negative real axis).

In general the equation $z^\alpha - 1 = 0$ will have roots $z = e^{2\pi i k/\alpha}$ where *k* are integers so that $2\pi k/\alpha$ is in the interval $(-\pi, \pi]$. That is, if *k* ranges over the integers satisfying $-\alpha/2 < k \le \alpha/2$ the values for *z* given above run through the roots. The three roots that are observed when $\alpha = 3.3$ correspond to $k = 0, \pm 1$. These are 1 and approximately $-0.327 \pm 0.945i$. The roots for other α's can also be computed this way.

When we increase α to $\alpha = 3.88221$ there are still only three roots but there is severe turbulence. The V-shaped turbulent region between the primary red and blue regions seen in Fig. 3 is larger than that in Fig. 2 and almost forms a "basin" of its own. In the first two figures the maximum number of iterations required for convergence was 66. In this figure the maximum number of iterations required for determining any pixel was 152,931. However, for every pixel computed the algorithm did eventually converge to one of the three roots!

A striking bifurcation occurs with a very slight increase to $\alpha = 3.88222$. Fig. 4 shows three basins of attraction in red, green and blue and a large region in yellow where Newton's method does not converge. However, in this region, Newton's method approaches an attractive 2-cycle. In the long term, values shown in yellow oscillate between values close to $r = -0.883623 + 0.154366i$ and $s = -0.883623 - 0.154366i$. No pixel required more than 1855 iterations to converge to a root or to find the 2-cycle in this figure.

Using *Mathematica*^{TM} it is easy to check numerically that *r* and *s* are 2-cycles since they are roots to the equation $h(z) = z$ where $\alpha = 3.88222$, $h(z) = g(g(z))$, and $g(z)$ was defined in the previous section. Now the fact that this is an attractive 2-cycle can be verified since $|h'(r)| = 0.984 < 1$ and likewise for s. Going back to the case when $\alpha = 3.88221$ we can find roots to $h(z) = 0$ given approximately by $u = -0.881416 + 0.154395i$ and $v = -0.883228 - 0.156458i$; notice the roots *u* and *v* are not quite conjugates. These roots also form a 2-cycle, but the 2-cycle is just barely repelling since $|h'(u)| = 1.00003 > 1$ and likewise for *v*. The fact that this derivative is just barely over one explains why such a large number of iterations were required for computing some pixels in Fig. 3.

The appearance of an attractive 2-cycle continues as α increases toward 4 but the points in the 2-cycle approach each other and the negative real axes. At $\alpha = 4$, shown in Fig. 5, these points meet at -1 and form an attractive basin shown in magenta. Any small increase in α produces yet another basin of attraction. In Fig. 6, where $\alpha = 4.1$, a cyan basin of attraction is conjugate to the magenta basin. Notice that there is no turbulence on the negative real axis which is a boundary between these basins. Also notice the fractal cyan and magenta "arches". As α increases turbulence does appear on the negative real axis. At first it is restricted to inside the unit circle. Between $\alpha = 4.7$ and $\alpha = 4.8$ the turbulence moves out along the negative real axis. Fig. 7 shows the basins of attraction when $\alpha = 4.8$. Notice that there is almost a five-fold symmetry. Finally, the expected five-fold symmetry appears in Fig. 8 where $\alpha = 5$.

In Fig. 9, the angle of the long term behaviors is shown verses the power α for $1 \le \alpha \le 11$. The angle is the angle about the origin from the positive real axis. Attractive roots are shown in blue and attractive 2-cycles are shown in green. Notice that just before α equals each even integer there is an attractive 2-cycle. The points in the 2-cycle become less distinct and approach -1; that is, the angles approach $\pm\pi$. At the even integer, the 2-cycle has become the new root at -1. As soon as α increases beyond the even integer, the root at -1 splits into two attractive roots. These roots move apart as α increases and become uniformly spaced around the unit circle when α reaches the next higher odd integer. The most striking feature in this figure is the sudden appearance of the attractive 2-cycles. The first few values of α where these bifurcations occur can be computed: 1.8163832, 3.882214033, 5.893141755, and 7.897747821.

The sequence of images in this paper shows chaotic behaviors not usually seen in images of the convergence of Newton's method. The intertwining basins with smooth boundaries observed just above $\alpha = 4$ are a result of the branch cut used in computing the complex exponential function. Those smooth boundaries do not occur for Newton's method on $z^n - 1$ with integer exponents. There are also fascinating bifurcations that occur just below even integers. Attractive 2-cycles appear for those exponents. The changeover of a 2-cycle from repelling to attractive near $\alpha = 3.88221$ can be analyzed as well as observed visually.

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Authors' biographies

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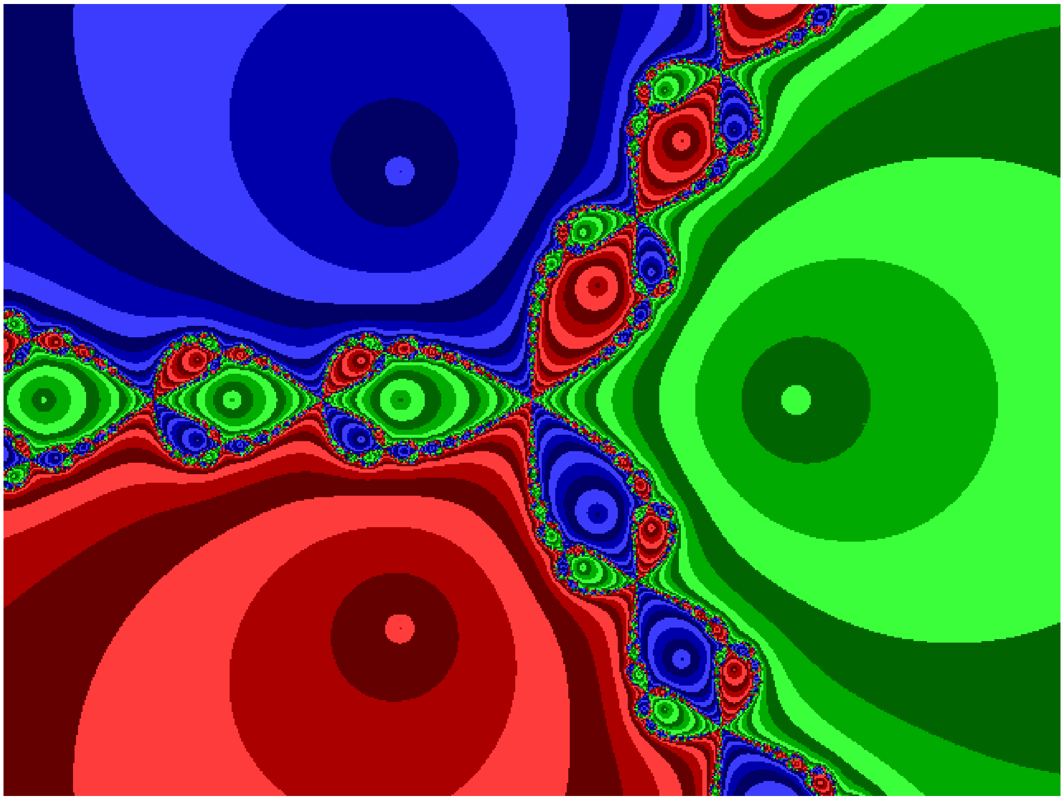

Figure 1: Newton's method on $z^3 - 1$.

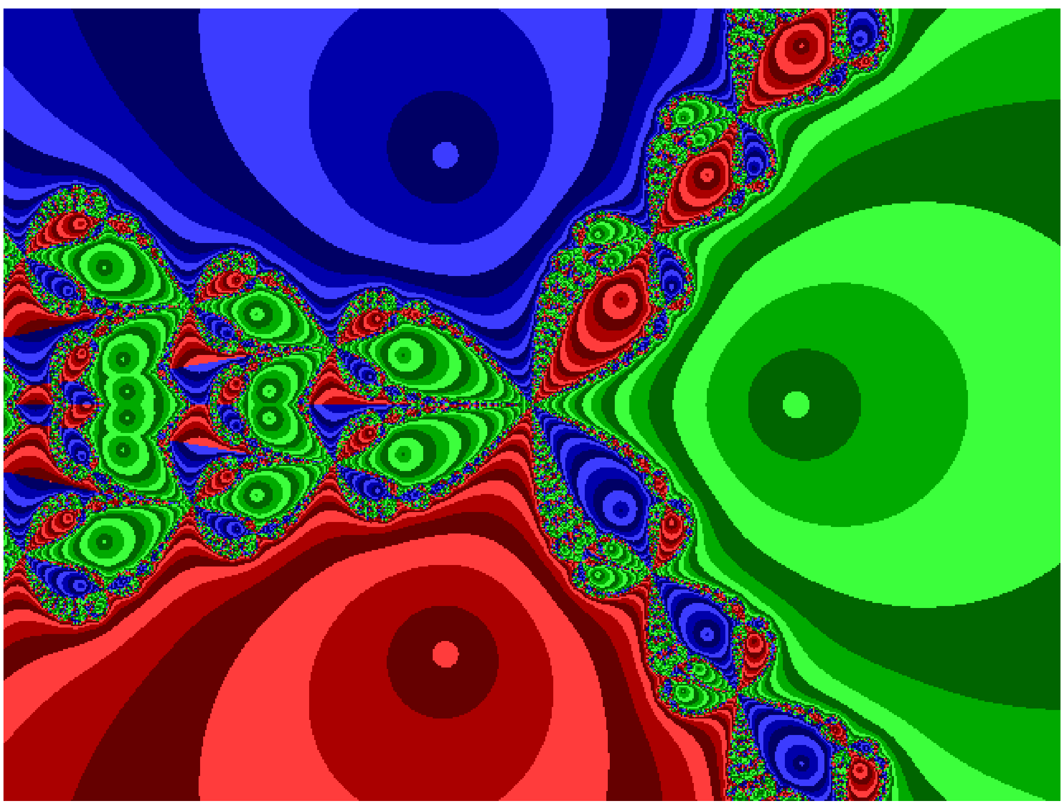

Figure 2: Newton's method on $z^{3.3} - 1$.

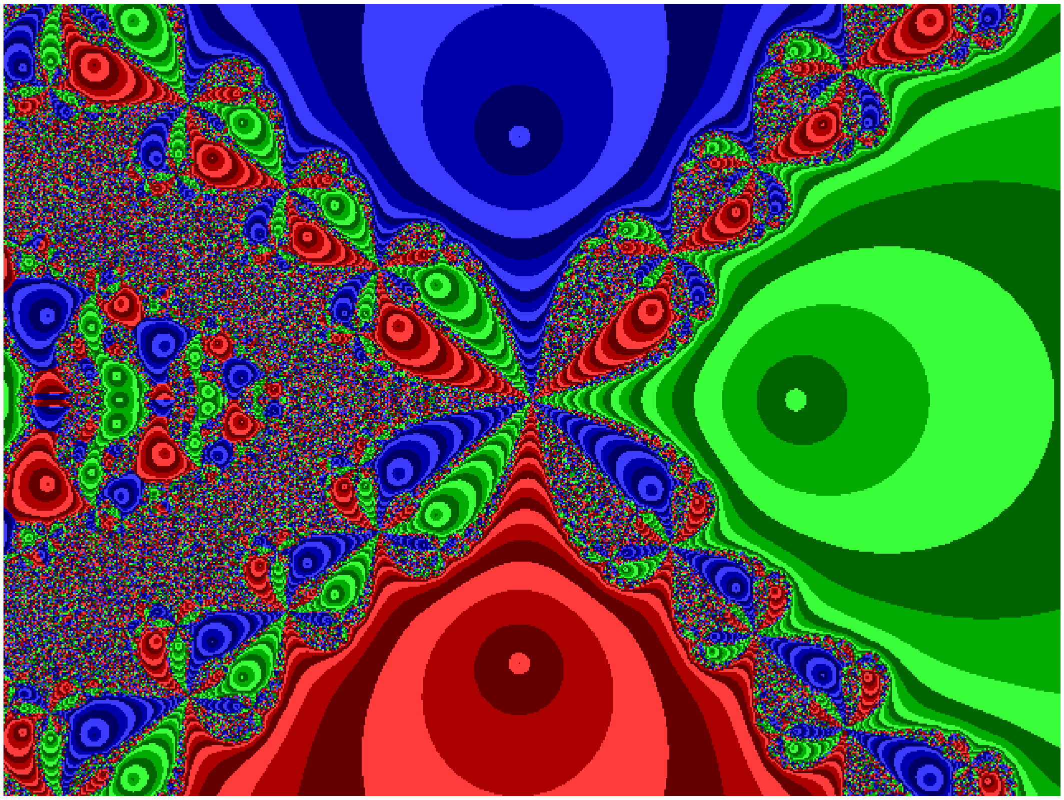

Figure 3: Newton's method on $z^{3.88221} - 1$.

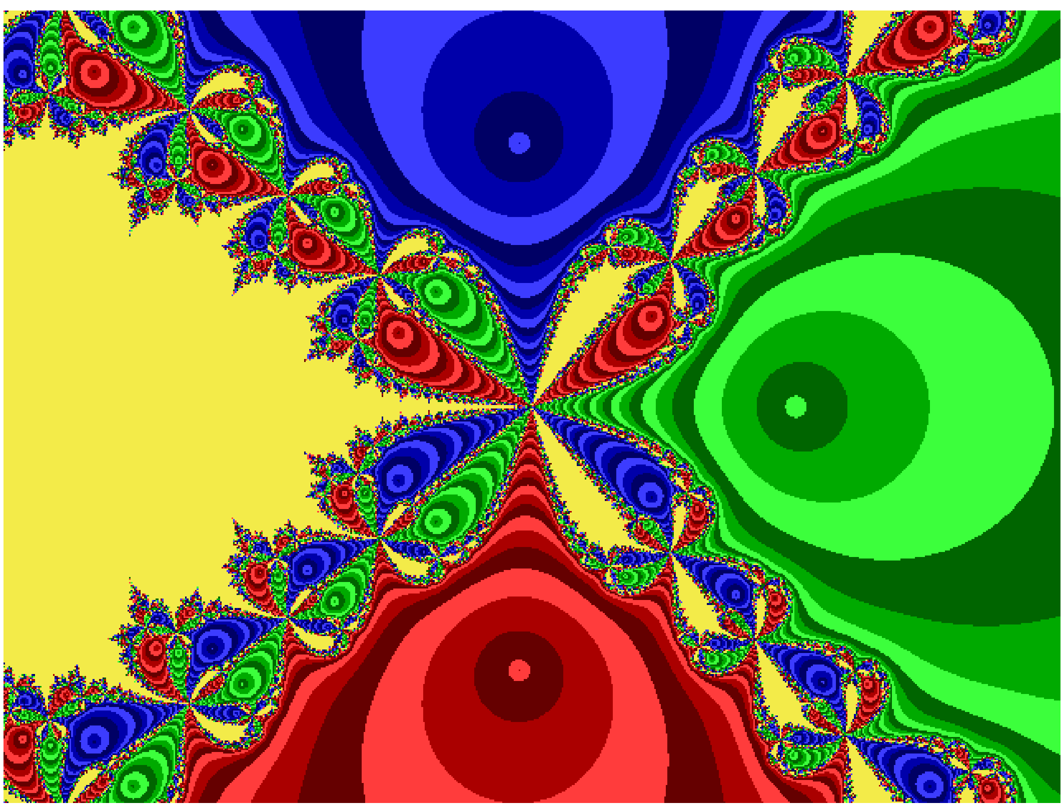

Figure 4: Newton's method on $z^{3.88222} - 1$.

Figure 5: Newton's method on $z^4 - 1$.

Figure 6: Newton's method on $z^{4.1} - 1$.

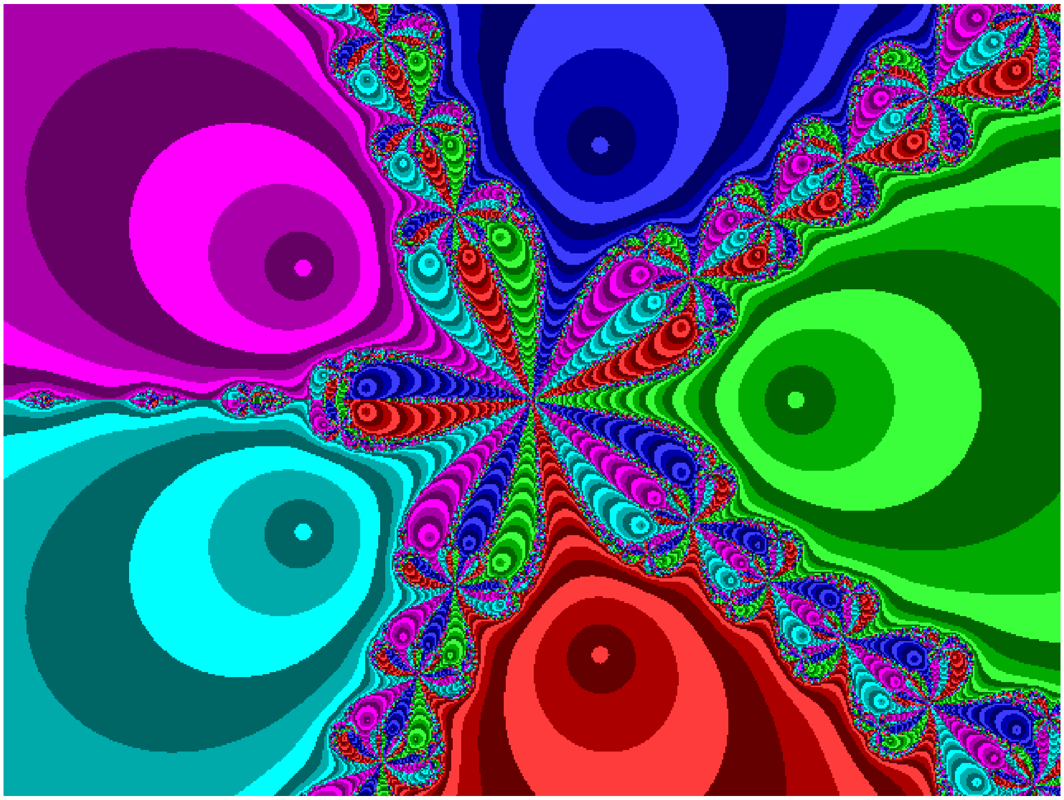

Figure 7: Newton's method on $z^{4.8} - 1$.

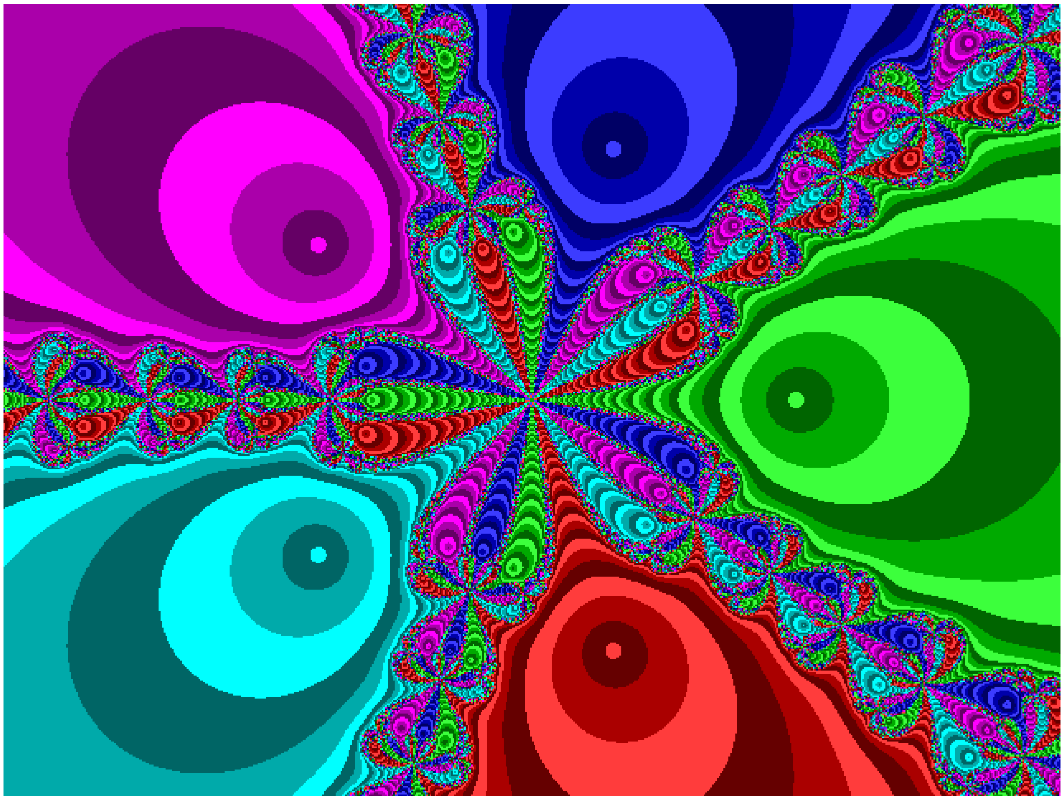

Figure 8: Newton's method on $z^{5} - 1$.

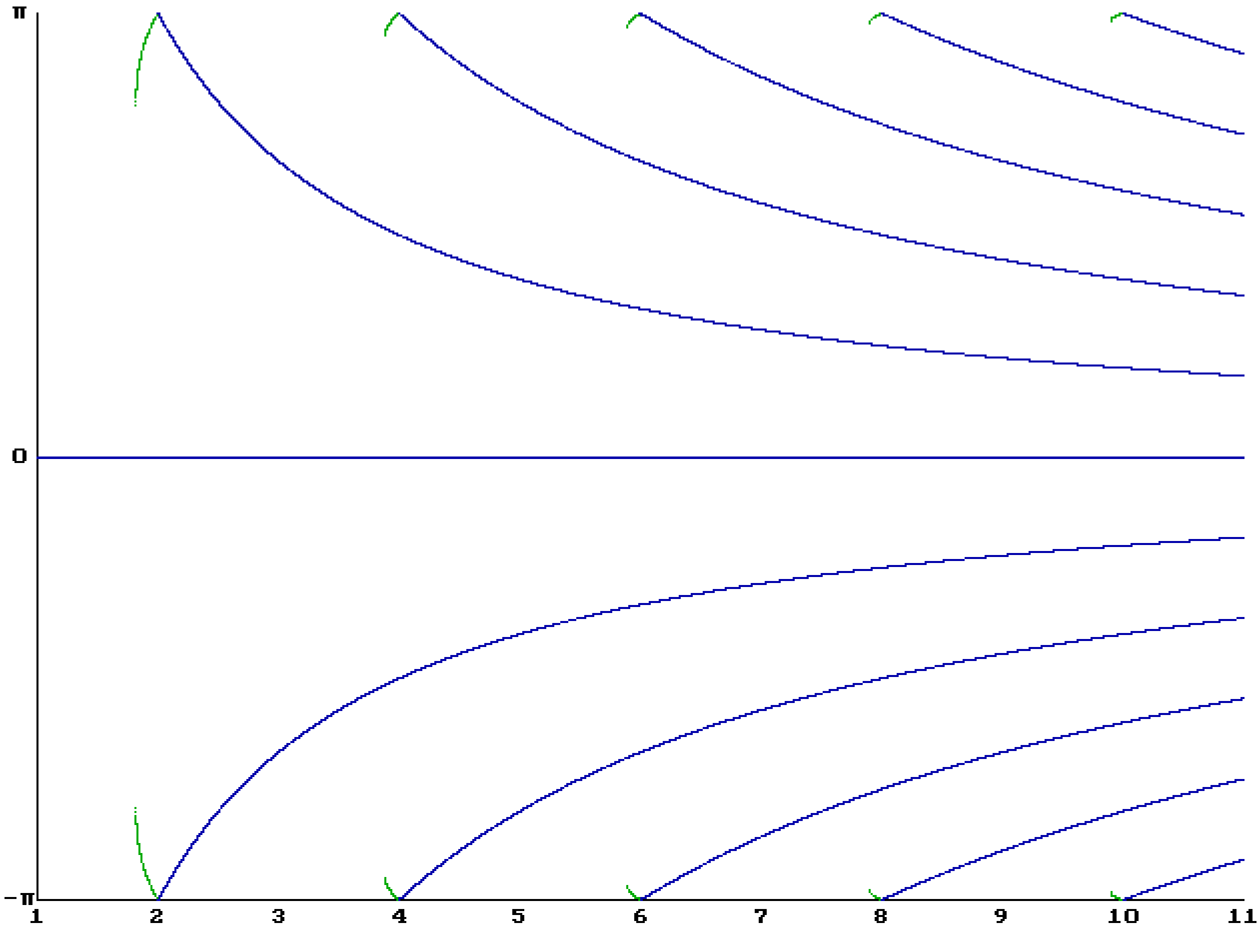

Figure 9: Angle of long term behavior of Newton's method on $z^{\alpha}-1$ for $1 \le \alpha \le 11$.