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VLASOV-FOKKER-PLANCK MODELS FOR MULTILANE TRAFFIC FLOW*

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Abstract. Systems of equations of Vlasov-Fokker-Planck type are suggested for multilane traffic flow. The equations include nonlocal and time-delayed braking and acceleration terms with rates depending on the densities and relative speeds. The braking terms include lane-change probabilities. It is shown that simple natural assumptions on the structure of these probabilities lead to multivalued fundamental diagrams, consistent with traffic observations. Lane-changing behavior is the crucial ingredient in such bifurcations.

Key words. traffic flow, Vlasov-Fokker-Planck model

AMS subject classifications. 82C31, 60K30

1. Introduction

The objective of this article is the design and analysis of a system of Vlasov-Fokker-Planck equations for multilane traffic flow on a highway. Our work is partly motivated by measurements and hypotheses put forward by B. Kerner [12, 9, 10, 11]. In particular, the present model may represent the various phases and phase transitions observed by Kerner, i.e., free flow, synchronized flow/synchronized equilibria, and congested flow/moving jams. We demonstrate that realistic assumptions on lane change probabilities lead to bifurcations in fundamental diagrams which are consistent with traffic observations [11].

The explanation of multivalued fundamental diagrams is one of the challenges which a good mathematical model should meet. Ideally, the model should also allow efficient calculations on the propagation of traffic features like jams caused by lane endings, instabilities in moderate density traffic caused by lane changes, and so on.

Three types of mathematical models for traffic dynamics have found extensive attention over the last few decades. The first type are systems of ordinary differential-delay equations in which the position and speed of each individual car are dependent variables. Such models link naturally with what are known as macroscopic models; macroscopic models of the simplest kind consist of one nonlinear PDE of conservation type, linking traffic density and traffic flux by a constitutive relation (the fundamental diagram). This relation is obtained by assuming equilibrated traffic flow. Models of this type, called first order macroscopic models, are in use for practical traffic analysis and predictions on multilane freeways [3].

In second order macroscopic models there is a separate equation for the macroscopic speed (or flux, respectively), and one then has a system of two first order partial differential equations, where the flux equation is usually augmented with a relaxation term. Well-known models of this type are due to Payne-Whitham [20] and Aw-Rascle [2]. The main motivation for the introduction of the Aw-Rascle model was in fact a

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weakness of the Payne-Whitham model, namely the unrealistic prediction of negative fluxes made by this model in certain scenarios. We refer to [2] for details.

Microscopic and macroscopic models can be formulated for one-lane roads, for each lane of a multilane highway (with appropriate coupling terms to address lane changes), or for the cumulative traffic on a highway under the assumption that traffic is so homogenized that all lanes show the same traffic pattern. This latter assumption is usually made in the formulation and analysis of these models.

The third type of mathematical models used in traffic dynamics are kinetic models, where a kinetic car density $f(t, x, v)$ depending on time, space, and speed is introduced. Such models require a statistical interpretation, and it is our belief that they are unnecessary for the modeling of traffic on single-lane roads, where there can be no passing (even the necessity for second-order macroscopic models seems then questionable). However, we believe that kinetic models can contribute valuable insight for multilane traffic, and the purpose of this paper is to add evidence to this claim.

The first kinetic model for traffic flow was suggested by Prigogine and Herman [21]. As observed by Nelson and Sopasakis [18], this model does lead to multivalued fundamental diagrams, albeit the computed diagrams do not look realistic. Other kinetic models for traffic flow were suggested, for example, by Paveri-Fontana ([19] and [5]). More recently, Klar and Wegener [14, 15, 17, 1, 4] have introduced Enskog-type kinetic equations for multilane-traffic flow, and a number of kinetic and analytical studies [6, 7] on the existence and nature of equilibria and on moment closures and macroscopic equations [4, 1] have since been done. Remarkably, the Aw-Rascle model arises in this process from a suitable closure process for the momentum equations [4].

In spite of these gratifying links, the modeling of traffic dynamics with Enskog equations as done in references [14, 15] raises several fundamental questions. The formulation of such models involves assumptions which are common and accepted in particle dynamics, but call for discussion when the particles are cars. First, is it realistic to assume "vehicular chaos," even with corrective factors? Second, is it acceptable. to assume that the velocity adjustment of drivers is instantaneous, and that the new velocity is chosen from a set which is not necessarily well correlated with the speed of the leading car? In [7] it was demonstrated that equilibrium solutions arising from these models vary with the driver behavior, and so results derived from such models are sensitive with respect to assumptions on braking speed and braking behavior.

The assumption of instantaneous velocity adjustment is acceptable for rarefied gas dynamics, where particle encounters occur on truly microscopic time scales, much shorter than the time span between encounters. In traffic flow, driver interactions happen on time scales comparable to those for phase transitions. Hence the details of the interaction, i.e., reaction times, braking/acceleration behavior etc. must be part of the modeling process.

This finally brings us to the content of this paper.

In Section 2 we present a model which incorporates driver behavior into kinetic equations of Vlasov-Fokker-Planck type. For simplicity we only consider two lanes; the equations incorporate braking and acceleration terms with appropriate nonlocalities and time delays; for braking scenarios, there are lane-change probabilities which depend on relative speed (more sophisticated dependencies are likely and could be incorporated into the model, but we refrain from doing so for the sake of simplicity). There are diffusion terms which are such that for a wide range of the parameters the model permits trivial equilibria (synchronized flow); this is significant because such equilibria exist from a fundamental point of view, and they are also observed in reality for certain density regimes (see [12, 9, 10, 11]).

In Section 3 we analyse the steady homogeneous model and derive fundamental diagrams. For certain density regimes, these diagrams are multivalued.

Our model is a new type of macroscopic model inasmuch as the behavior of each driver is guided by his/her observation of macroscopic quantities. These quantities are observed with time delays (reaction time) and spatial nonlocalities depending on the driver's speed. These are realistic features which have been incorporated into other kinetic models [13, 14, 15, 16], but their implementation into macroscopic models is problematic because of the lack of an explicit speed variable. It is certainly possible to dicuss momentum equations and possible closure relations of our Vlasov-Fokker-Planck model, but we found that one has to distinguish many cases (braking vs. acceleration scenarios), and simplifications arise only under strong and not realistic assumptions (equal thresholds for braking and acceleration; low traffic temperature). Of course, under the ultimate assumption that traffic is steady and independent of the location, we arrive at the homogeneous scenario which allows the computation of the fundamental diagrams done in Section 3. In other situations, it appears that the Vlasov-Fokker-Planck model itself is better suited for numerical and analytical purposes than for momentum equations or closures of these equations.

2. The Equations

2.1. The Basic Model.

We consider traffic flow on a highway with two lanes. We label the lanes as $i = 1, 2$, where 1 is the label of the right lane and 2 the label of the left lane. The lanes will be treated as completely equivalent (this is, of course, not quite realistic, but the model is easily generalized). By $f_i(t, x, v)$ we denote the number density of cars in line i which at time t are at location x and move with speed v. Here, $x \in \mathbb{R}$ denotes the location on the highway; we assume that $v \in [0, u_{max}]$, where u_{max} is the speed limit. In the remainder of this paper we normalize $u_{max} = 1$.

We assume further that all drivers have the same constant reaction time $\tau > 0$ and observe braking and acceleration thresholds H_B and H_A given by $H_B = x + T_B v$ and $H_A = x + T_A v$. Here, v is the driver's speed and T_B and T_A are reaction times (in general different from τ). This means that a driver at x moving with speed v will brake in reaction to a traffic condition observed at $x + T_B v$, and if no condition for braking applies, he/she will accelerate (if possible) in reaction to a traffic condition observed at $x + T_A v$. Observations suggest that $T_A \geq T_B$; a driver will accelerate only if his distance to the leading car exceeds $T_A v$ and brake only if this distance becomes smaller than $T_B v$.

The total densities are $\rho_i(t, x) = \int_0^1 f_i(t, x, v)\, dv$, and we set

$$f_i(t, x, v) = \rho_i(t, x) F_i(t, x, v).$$

F_i is the probability density in v of cars at x in lane i. The macroscopic flux in lane i is given by $j_i(t, x) = \int v f_i(t, x, v)\, dv$, and the average speed in lane i is $u_i = j_i/\rho_i$. There are other important moments with physical meaning, e.g., the "traffic temperature" $e_i(t, x) = \int (v - u_i)^2 f_i(t, x, v)\, dv$. For synchronized equilibria, defined by $f_i(t, x, v) = \rho_i(t, x)\delta_{u_i}(v)$ we see immediately that $e_i = 0$.

In the sequel we abbreviate $f = (f_1, f_2)$.

The equation for the ith lane, written in conservation form, is

$$\partial_t f_i + v\partial_x f_i + \partial_v\big(B[f_i]f_i \;-\; D[f_i]\partial_v f_i\big) \;=\; p_k[f_k]f_k \;-\; p_i[f_i]f_i. \tag{2.1}$$

Here, $k = 3 - i$ denotes the other lane and $B[f_i]$ is the braking or acceleration term, corresponding to the system of characteristic equations

$$\dot{x} = v, \quad \dot{v} = B[f_i]. \tag{2.2}$$

The f_is solving (2.1) can be interpreted as "statistical solutions" of (2.2) in the sense defined in [8]. Note that mass is omitted in the second equation; i.e., heavier cars are simply assumed to have correspondingly stronger brakes (and more power) such that their deceleration or acceleration depends in exactly the same way on $B[f_i]$ as that of lighter cars. $p_i[f]$ is the lane-changing (or passing) rate. This rate depends on traffic states in the current and adjacent lanes; we discuss these dependencies in the sequel. The lane- changing probabilities P and the lane-changing rates p are related by $p = P \cdot j$, where j is the flux. The essence of the model lies in our assumptions regarding the dependencies of $B[f_i], p_i[f_i]$, etc. on the traffic state.

2.2. The Braking/Acceleration Terms. We suggest that the braking/acceleration force $B[f_i]$ depends on the relative speed of the driver under consideration with respect to the average speed as observed by this driver a reaction time earlier and the appropriate threshold distance ahead. We emphasize that other modeling assumptions can be incorporated at this point, for example, that the driver will pay attention to the slowest driver ahead. Keeping in mind the statistical interpretation of the model, we suggest that the braking force should be given in terms of average quantities as computed from the traffic densities.

To simplify our formulas, we define

$$\begin{aligned} \rho_i^B &= \rho_i(x + T_B v, t - \tau), \qquad & u_i^B &= u_i(x + T_B v, t - \tau) \\ \rho_i^A &= \rho_i(x + T_A v, t - \tau), \qquad & u_i^A &= u_i(x + T_A v, t - \tau) \end{aligned} \tag{2.3}$$

and diffusion coefficients

$$D[f](\rho, u, v) = \sigma(\rho, u)\, |v - u|^\gamma. \tag{2.4}$$

Here, f stands for a traffic density, and ρ, u are the macroscopic density and speed associated with f. The dependencies of σ on ρ and u are addressed in Section 3. To define the braking and acceleration behavior of drivers in response to traffic situations, we now suggest the following braking/acceleration forces as functions of the traffic conditions:

$$B[f_i](t, x, v) = \begin{cases} -c_B (v - u_i^B)^2\, q_i(\rho_i^B, u_i^B, v) & v > u_i^B \\ c_A (u_i^A - v)^2 (\rho_{max} - \rho_i^A) & v \le u_i^B \text{ and } v \le u_i^A \\ 0 & \text{else} \end{cases} \tag{2.5}$$

and with $D[f](\rho, u, v)$ as defined above,

$$D[f_i](t, x, v) = \begin{cases} D[f_i](\rho_i^B, u_i^B, v) & v > u_i^B \\ D[f_i](\rho_i^A, u_i^A, v) & \text{else} \end{cases} \tag{2.6}$$

where c_B and c_A are dimensionless constants, $\gamma > 0$, and ρ_{max} is the maximum density. $q_i(...)$ is the product of the spatial density ρ and braking probability, to be introduced momentarily. Note that the explicit form of the conditions on the right-hand sides is, for example,

$$v \; > \; u_i(x + T_B v, t - \tau).$$

This means that an implicit equation for v must be solved before we know which case occurs.

We offer an interpretation of the various decisions: The first case is a braking scenario because a driver will consider braking (or a lane change) if the traffic ahead (at $x + T_B v$) is slower than he was at time $t - \tau$. We set the braking force proportional to $\rho \cdot (v - u_i^B)^2$ because the factor c_B will then be dimensionless.

If this braking scenario does not apply, the acceleration scenario may apply: If the leading car is at $x + T_A v$ and moves at a speed greater than v, then the driver will accelerate. No lane changes are considered in this situation. We set the acceleration force proportional to $(u_i^A - v)^2(\rho_{max} - \rho_i^A)$ because this will again make c_A dimensionless, and will annihilate all acceleration at maximal density. In general we have $T_A \geq T_B$ (for numerical values see [13]) and it is therefore possible that neither scenario occurs. In this case, the driver won't brake or accelerate, and only the diffusion term (due to the driver's inability to observe speeds with accuracy) remains. In our computations, we also choose $c_A < c_B$ as braking interactions are stronger than acceleration interactions. Furthermore, $\sigma << c_A$, as diffusion occurs on smaller scales than braking or acceleration. σ must have the dimension of $[\rho u^{3-\gamma}]$ for unit reasons. We choose not to make σ dimensionless, as this gives us the freedom to discuss varying γ. However, we will choose specific dependencies of σ on ρ and u (consistent with elementary traffic facts) in our derivation of fundamental diagrams in Section 3.

Finally, we define

$$q(\rho, u, v) \; = \; \rho \left(1 - P_i(u, v)\right)$$

where the *lane-change probabilities* $P_i(\ldots)$ are

$$P_i(u, v) \; = \; \begin{cases} \left(\frac{v-u}{u_{max}-u}\right)^\delta & v \; > \; u \\ 0 & v \; \leq \; u \end{cases}$$

with $\delta > 0$. Note that lane-changes will only be considered in braking scenarios (we simply set them equal to 0 in the other cases), and we assume that the lane-change probabilities are proportional to a suitable power of the normalized relative speed. The powers γ and δ will be specified below.

We emphasize that this dependence of the lane-change probabilities is just a first guess; in reality, lane-change probabilities will not only depend on relative velocities on the lane under consideration, but also on the relative velocity with respect to the adjacent lane as well as on the densities on both lanes. Such further dependencies can be easily incorporated into the model, but they make any analytical treatment of the model harder. We will demonstrate in Section 3 that the present simple dependence of P_i suffices to explain multivalued fundamental diagrams; it has been speculated that the observed multivalued diagrams in real traffic mesurements [11] are a consequence of lane-changing behavior, and our analysis in Section 3 supports this possibility.

3. Steady and Spatially Homogeneous Equations

In the case where the traffic flow is identical on both lanes, steady and independent of x, the complicated system (2.1) collapses to one much simpler equation (this is because $t-$ and $x-$ derivatives vanish, the lane-changing terms on the right cancel, and the equations for both lanes are the same). We are left with

$$\partial_v\big(B[f_i]f_i \;-\; D[f_i]\partial_v f_i\big) \;=\; 0 \qquad \text{i.e.,} \qquad B[f_i]f_i \;-\; D[f_i]\partial_v f_i \;=\; C \;\in\; \mathbb{R}.$$

We want $f(v) = \delta_u(v)$ to be a solution of the homogeneous equation (these are the synchronized equilibria which any realistic model should admit). If $\gamma > 1$, it then follows that $C = 0$. Furthermore, the third case stated in (2.5) cannot occur. We obtain the system

$$\begin{aligned} c_B(v-u)^2 q(\rho,u,v) f \;+\; \sigma(v-u)^\gamma\,\partial_v f \;&=\; 0 \qquad v > u \\ c_A(u-v)^2(\rho_{max}-\rho) f \;-\; \sigma(u-v)^\gamma\,\partial_v f \;&=\; 0 \qquad v \le u. \end{aligned}$$

We will consider this system for general $\gamma \in (0,2)$, although $\delta_u(v)$ is a solution in the sense of distributions only if $\gamma > 1$. We are not particularly concerned that these synchronous equilibria are not included for smaller γ; after all, the factors in the diffusion and lane-changing terms are only guesses; the true functions which should occur there are probably more complicated and depend on additional variables. It is easy to adjust the dependence of σ such that the synchronous equilibria will exist for all cases, but we would lose the relative simplicity of the present model. Our subsequent numerical studies suggest that a diffusion factor σ worth studying would grow like $(v-u)^\gamma$ with $\gamma > 1$ for small $v-u$, and saturate at some level for moderate to large $v-u$. However, the saturation level would be yet another parameter.

Real data on the kind of uncertainty embodied in the diffusion function σ would require extensive psychological studies.

In addition to the synchronized equilibria solutions, which exist for $\gamma > 1$, there is a continuous differentiable solution to the system if $\gamma \in (0,2)$. This solution is

$$f(v) = \begin{cases} c(\rho,u)\,\exp\left(\beta(v-u)^{3-\gamma}\left[\left(\frac{v-u}{u_{max}-u}\right)^\delta \frac{1}{3-\gamma+\delta} - \frac{1}{3-\gamma}\right]\right) & v > u \\ c(\rho,u)\,\exp\left(-\alpha\frac{(u-v)^{3-\gamma}}{3-\gamma}\right) & v \le u \end{cases},$$

with the constants

$$\alpha \;=\; \frac{c_A(\rho_{max}-\rho)}{\sigma} \qquad \text{and} \qquad \beta \;=\; \frac{c_B\rho}{\sigma}.$$

and a factor $c(\rho,u)$ determined from the normalization $\int f(v)dv = \rho$. Explicit knowledge of $c(\rho,u)$ is not necessary for the subsequent calculations.

While $\rho \in [0,\rho_{max}]$ is a real parameter, u and ρ are coupled by the fundamental diagram. This diagram is obtained from the first two momentum equations

$$\rho u \;=\; \int_0^{u_{max}} u f(v)\,dv \;=\; \int_0^{u_{max}} v f(v)\,dv$$

which gives us the following relationship between ρ and u:

$$0 = \int_0^u (u-v) \exp\left(-\alpha \frac{(u-v)^{3-\gamma}}{3-\gamma}\right) dv \tag{3.1}$$
$$- \int_u^{u_{max}} (v-u) \exp\left(\beta (v-u)^{3-\gamma} \left[\left(\frac{v-u}{u_{max}-u}\right)^\delta \frac{1}{3-\gamma+\delta} - \frac{1}{3-\gamma}\right]\right) dv.$$

Evaluating both integrals seperately, we have to find the roots of

$$R(u) := \int_0^{u_{max}} J(u,s)\, ds$$

with

$$J(u,s) = u^2(u_{max}-s) \exp\left(-\frac{\alpha}{3-\gamma}\left(\frac{u}{u_{max}}\right)^{3-\gamma} (u_{max}-s)^{3-\gamma}\right)$$
$$- (u_{max}-u)^2 s \exp\left(\beta \left[\frac{u_{max}-u}{u_{max}}\right]^{3-\gamma} s^{3-\gamma} \cdot \left[\left(\frac{s}{u_{max}}\right)^\delta \frac{1}{3-\gamma+\delta} - \frac{1}{3-\gamma}\right]\right).$$

As we can deduce from (3.1), we have $R(0) < 0$ and $R(u_{max}) > 0$ for any $\rho \in (0, \rho_{max})$. Therefore, there will always exist at least one $u^* \in (0, u_{max})$ with $R(u^*) = 0$.

3.1. First Example. For our first numerical example we choose $\gamma = 1$ and $\delta = 1$, although $\delta_u(v)$ is then no longer a solution in the sense of distributions. The

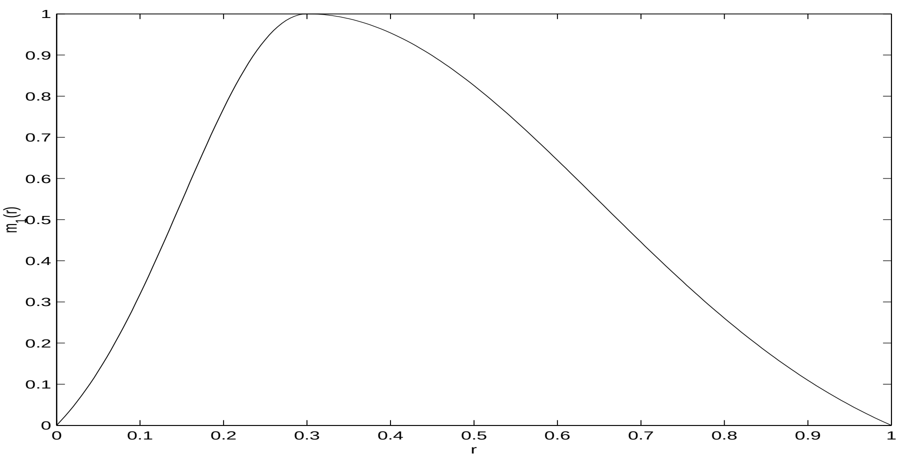

FIG. 3.1. *The function $m_1(r)$.*

function $R(u)$ simplifies to

$$R(u) = \frac{1}{\alpha}\left(1 - \exp\left(-\frac{\alpha}{2}u^2\right)\right) - \int_0^{u_{max}-u} s \exp\left(\beta s^2 \left[\frac{s}{3(u_{max}-u)} - \frac{1}{2}\right]\right) ds.$$

We now make a special choice for the diffusion factor; the motivation is that there should be no diffusion in certain limit scenarios such as standing traffic, zero car density, etc. If no reasonable dependencies of σ to these ends are enforced, one gets unrealistic fundamental diagrams at the endpoints. We choose

$$\sigma(\rho, u) = \sigma_c \, \rho_{max} \, u_{max}^2 \, m_1\left(\frac{\rho}{\rho_{max}}\right) \, m_2\left(\frac{u}{u_{max}}\right)$$

with a positive constant σ_c, a non-negative function $m_1(r)$ depicted in Figure 3.1 and

$$m_2(s) = s(1-s).$$

m_1 consists of two linked Gaussian distributions and is chosen such that diffusion vanishes in the limit scenarios where $\rho = 0$, $\rho = \rho_{max}$. Furthermore we define $m_2(s)$ similarly, so that the diffusion vanishes also for $u = 0$, $u = u_{max}$. We repeat that such choices are essential to obtain fundamental diagrams which are consistent with the basic facts that there is no flux when there are no cars, and there is also no flux in standing traffic.

The fundamental diagram with $c_A = 5$, $c_B = 25$ and $\sigma_c = 0.25$ is shown in Figure 3.2. We note that the numerical determination of the solution for large densities becomes increasingly difficult due to the small values of the involved integrals. Furthermore, in Figure 3.3 we plot some equilibrium distributions for $\rho = 0.25$. We have normalized such that $\rho_{max} = 1$ and $u_{max} = 1$.

3.2. Second Example. In this second example $\delta_u(v)$ is in fact a solution. To satisfy the condition $\gamma \in (1, 2)$ we choose $\gamma = 1.5$ and $\delta = 1$. Also, we make an adjustment for dimensional consistency and set

$$\sigma(\rho, u) = \sigma_c \, \rho_{max} \, u_{max}^{3/2} \, m_1\left(\frac{\rho}{\rho_{max}}\right) \, m_2\left(\frac{u}{u_{max}}\right)$$

where m_1, m_2 are defined above. Again we assume $\rho_{max} = 1$, $u_{max} = 1$, and, slightly different from above, $c_A = 5$, $c_B = 20$, $\sigma_c = 0.5$. The calculated fundamental diagram is shown in Figure 3.4. Again we calculate some distribution functions for $\rho = 0.25$. These are plotted in Figure 3.5.

4. Conclusions

- In the above examples, we obtained 3-valued fundamental diagrams. We expect that by considering more complicated (but realistic) dependencies on additional variables of the lane-changing probabilities, multivalued fundamental diagrams with more than three values will arise.
- Similarly, by adjusting parameters the "multivalued" flow regime between the critical densities ρ_1 and ρ_2 can be fitted to experimental data.
- The time-dependent and spatially inhomogeneous case will be investigated in future work.

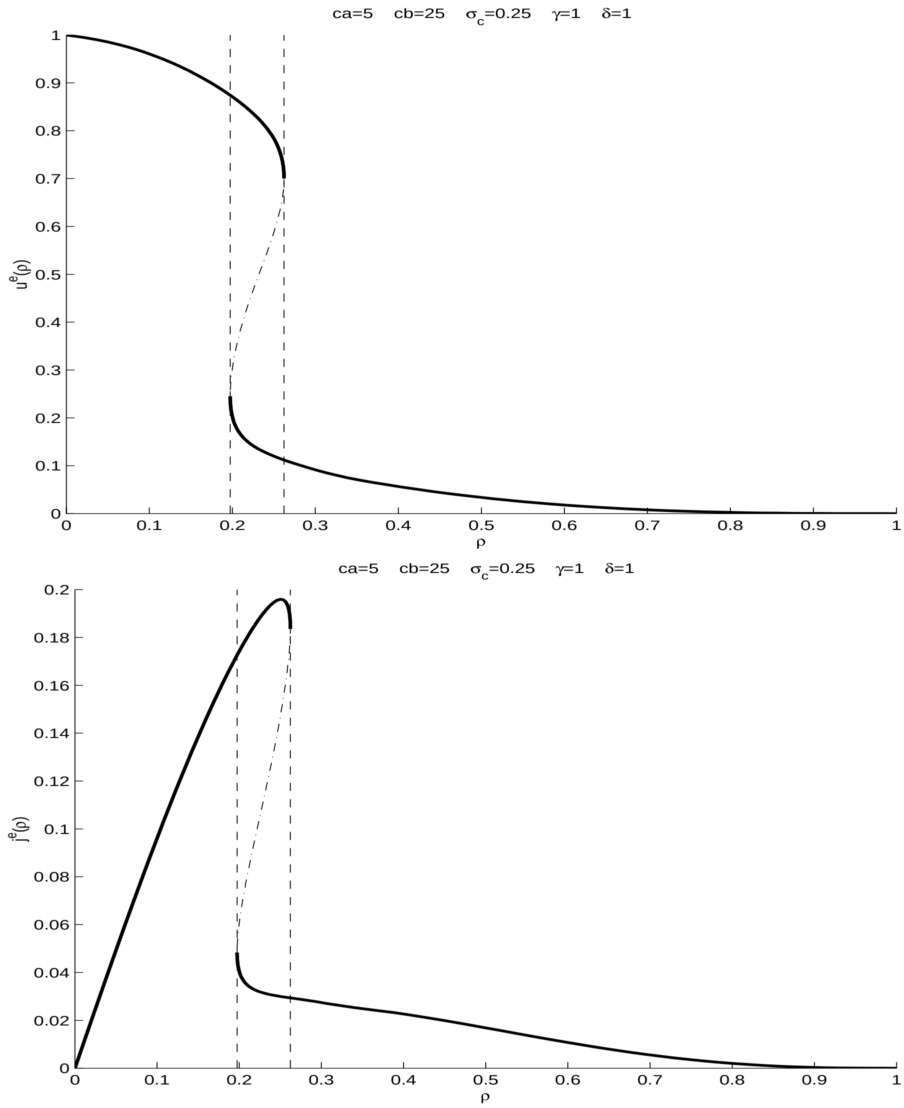

FIG. 3.2. *Equilibrium velocity and flux for* $\gamma = \delta = 1$.

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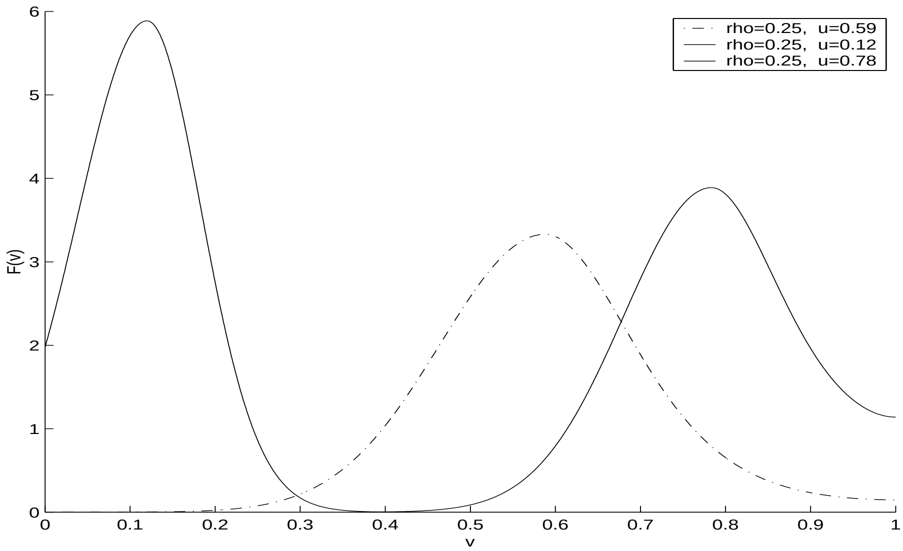

FIG. 3.3. *The distribution function $F(\rho, v)$ for the special choice $\rho = 0.25$ and its associated equilibrium velocities. In this case, we achieve three different distributions.*

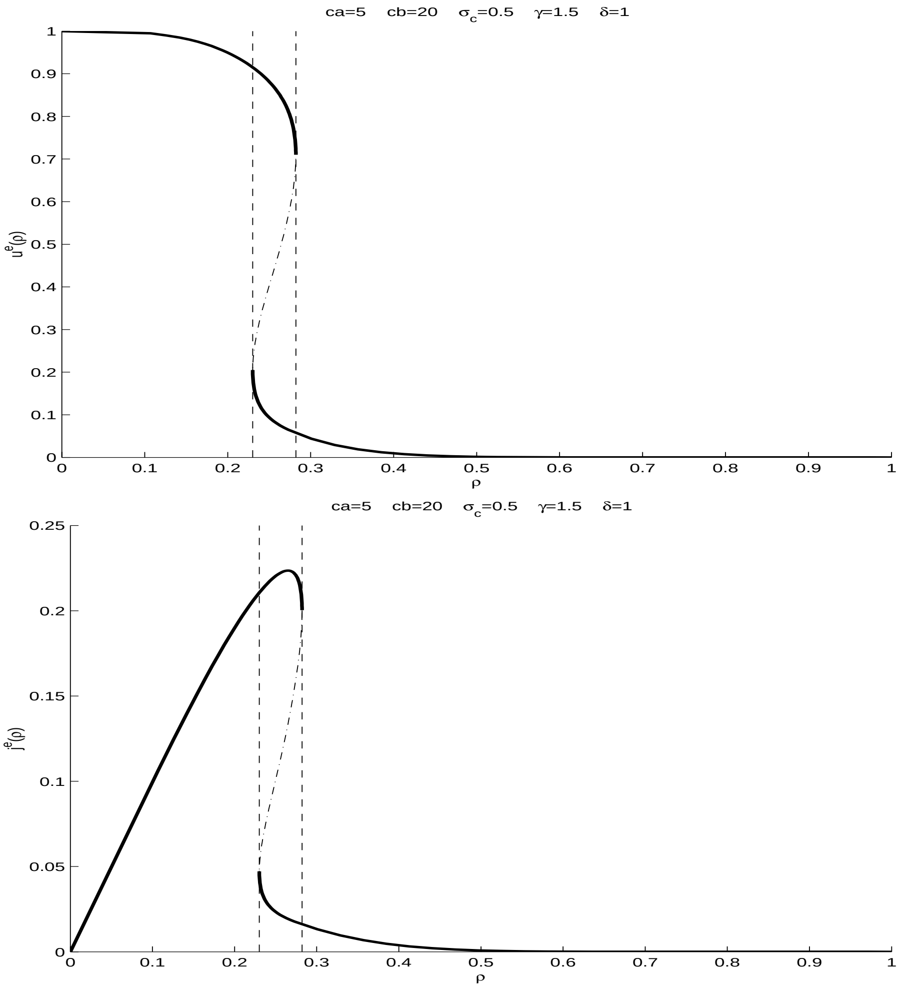

FIG. 3.4. *Equilibrium velocity and flux for $\gamma = 1.5$ and $\delta = 1.5$.*

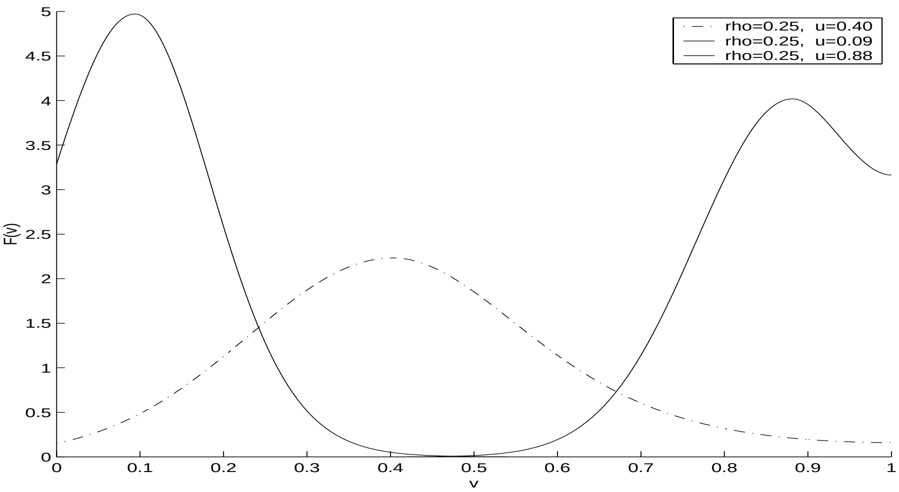

FIG. 3.5. *The distribution function $F(\rho, v)$ for the special choice $\rho = 0.25$ and its associated equilibrium velocities. In this case, we achieve three different distributions.*

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