VMD, the WZW Lagrangian and ChPT:

The Third Mixing Angle

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Abstract

We show that the Hidden Local Symmetry Model, supplemented with well-known procedures for breaking flavor SU(3) and nonet symmetry, provides all the information contained in the standard Chiral Perturbation Theory (ChPT) Lagrangian $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$. This allows to rely on radiative decays of light mesons ($VP\gamma$ and $P\gamma\gamma$) in order to extract some numerical information of relevance to ChPT: a value for $\Lambda_1 = 0.20 \pm 0.04$, a quark mass ratio of $\simeq 21.2 \pm 2.4$, and a negligible departure from the Gell-Mann–Okubo mass formula. The mixing angles are $\theta_8 = -20.40° \pm 0.96°$ and $\theta_0 = -0.05° \pm 0.99°$. We also give the values of all decay constants. It is shown that the common mixing pattern with one mixing angle θ_P is actually quite appropriate and algebraically related to the η/η' mixing pattern presently preferred by the ChPT community. For instance the traditional θ_P is functionally related to the ChPT θ_8 and fulfills $\theta_P \simeq \theta_8/2$. The vanishing of θ_0, supported by all data on radiative decays, gives a novel relation between mixing angles and the violation of nonet symmetry in the pseudoscalar sector. Finally, it is shown that the interplay of nonet symmetry breaking through U(3) $\rightarrow$ SU(3)$\times$ U(1) satisfies all requirements of the physics of radiative decays without any need for additional glueballs.

I. INTRODUCTION

We have recently proposed a model for radiative decays of all light mesons [1] which gives a consistent and successful description of all reported experimental information. This covers the 14 decay modes of the kind[1] $V \to P\gamma$ and $P \to \gamma\gamma$. This vector meson dominance (VMD) based model relies on the hidden local symmetry (HLS) approach developed in Ref. [2] which introduces the vector mesons as gauge bosons of a spontaneously broken hidden local symmetry and closely resembles Seiberg's EM duality in supersymmetric QCD [3], as noted in Ref. [4] and again in [5]. Its anomalous sector [2,6] (referred to hereafter as FKTUY), describes the radiative decays of light flavor mesons. In its original form this Lagrangian is U(3) symmetric, as it possesses both nonet symmetry and SU(3) flavor symmetry.

In order to describe the full pattern of light mesons radiative decays, these unbroken schemes need to be supplemented with symmetry breaking mechanisms. Breaking the SU(3) flavor symmetry is an essential step [1]. This is performed following the mechanism proposed by Bando, Kugo and Yamawaki (BKY) [7,4] and does not depend on any additional free parameter. An additional breaking procedure [1,8,9] is needed in order to describe the observed [10] features of K^* radiative decays; it allows us to recover for this sector a structure derived by G. Morpurgo in his approach to low energy QCD [11].

An explicit form [12] of nonet symmetry breaking (NSB) for vector (V) mesons seems to play a negligible role [9] when focusing on radiative decays. Angular departures from ideal mixing are instead highly significant; they can be essentially explained by the ω_I/ϕ_I transitions[2] inherent to SU(3) VMD models like HLS. One cannot, however, completely exclude that some kind of vector NSB is hidden inside these angular effects [9].

Instead, NSB for pseudoscalar mesons (PS) is an essential ingredient [1]. It has been performed in the manner of Ref. [12], which turns out to allow couplings to singlet and octet PS components to be different. It is a purpose of the present paper to revisit the issue of how NSB can be consistently implemented within the HLS Lagrangian.

The problem of η/η' mixing [13,14] is tightly connected with the breaking of nonet symmetry. This can be performed at the level of the coupling constants [12], but this breaking can also be connected with a possible glue component inside light mesons [15–18]. Indeed, the full set of radiative decays [1] examplify the large importance of this effect. Within this context, this reference also showed that effects of such a glue component can only affect the η' meson, but cannot be disentangled from genuine nonet symmetry breaking effects without some *a priori* knowledge of one of these twin phenomena.

In connection with this particular problem, but even more closely related with the effects of symmetry breakdown in Chiral Perturbation Theory (ChPT), Kaiser and Leutwyler [19,20] advocate an η/η' mixing scheme (see Ref. [21] for a comprehensive review), more complicated than the usual one, depending on two decay constants and two mixing angles. Some

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[1] This counting does not include $\pi^0 \to \gamma\gamma$, which would serve to fix f_π. We prefered using directly the PDG recommended value [10] as for f_K.

[2] The subscript I indicates the ideal combinations.

phenomenological analyses [22,23] have investigated this new scheme. In a more axiomatic approach to QCD, Shore [24] also finds appropriate a four parameter parametrization of the η/η' mixing.

Nevertheless, the analysis of radiative decays (however, 14 independent decay modes) of Ref. [1] does not find any need for a four–parameter structure of the η/η' mixing, as if phenomenology were exhibiting several relations among the 4 decay constants [19,20,24], which might be fulfilled at the (already high) level of accuracy permitted by the data. To be more precise, Ref. [1] yields a quite satisfactory description of the data by introducing one PS mixing angle and one NSB parameter affecting the PS sector; additional departures from SU(3) flavor symmetry as per BKY [7,4] arise only through a dependence in f_K/f_π, which can hardly be considered as a (free) parameter.

More appealing, the anomalous Lagrangian of Wess, Zumino and Witten (WZW) [25,26], with SU(3) symmetry broken as explained in Refs. [1], leads to definitions of the mixing angle and decay constants as per Current Algebra and as following from the HLS–FKTUY framework. The (single) mixing angle was found in Ref. [1] to be $\simeq -10°$, and, moreover, the value for the octet decay constant is $f_8 = 0.82 f_\pi$. The relevance of these parameter values is strongly supported by an impressive agreement with experimental data within a highly constrained model (5 parameters for 14 decay modes). Interestingly, Ref. [27], relying on lattice QCD calculations reaches also a mixing angle a value $\simeq -10° \pm 2°$, with a preference for $-10.2°$.

On the other hand, there are repeated claims [28–30] that a (single) mixing angle, as coming from standard Current Algebra expressions, is quite appropriate and is found to be much less negative ($\simeq -13°$ to $\simeq -15°$) than expected from ChPT ($\simeq -20°$). A quite detailed discussion of this can be found in Ref. [31] (see also [32]) where such an angle value is derived from a bound state approach. Another Lagrangian approach [33], parent to HLS using a specific breaking scheme, recently claimed an angle value close to the previously mentioned ones ($\simeq -15.4° \pm 1.8°$).

All this seems in glaring disagreement with the expectations of ChPT [13,14,19,20,34,35]. Taking into account the special role of ChPT in low energy phenomenology, a possible contradiction between ChPT, lattice QCD calculations or the VMD conceptual framework[3] is a worrying question which must be addressed and understood. This is the purpose of the present paper, which will show that the contradiction is illusory and only due to different definitions of the same parameters in a naive understanding of the WZW approach (with encompasses the Current Algebra definitions) and in ChPT. We shall explicitly state the relationships between them.

It will be shown that the VMD approach, relying on the HLS model broken as in Ref. [1], is actually in accord with all ChPT expectations associated with the ChPT Lagrangian [19–21] $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$. This will be illustrated by deriving from a broken VMD Lagrangian model, all known leading order expressions for ChPT mixing angles and decay constants, and by deriving their expected numerical values. For the sake of conciseness, we shall frequently use NSB to refer to nonet symmetry breaking and to FSB for SU(3) flavor symmetry breaking.

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[3]Or other approaches as listed in the throughout discussion in Ref. [31].

The outline of the paper is as follows. In Section II we present a Lagrangian VMD model, based on the HLS approach, which includes both NSB and FSB. We show that there is a close connection between them. In Section III, we study the field transformation which permits us to write the kinetic energy of this broken VMD model in canonical form, in terms of renormalized fields. We show here that the field transformation of Refs. [1,9] corresponds to a first order truncation in both NSB and FSB.

Section IV gives the VMD description of the $\eta/\eta' \to \gamma\gamma$ decay which depends on one mixing angle and nonet symmetry breaking (as parametrized by x). In Section V the corresponding description is derived, starting from the BKY broken WZW Lagrangian, and it is shown that WZW and VMD coincide.

In Section VI, we derive the set of relations which allows one to define mixing angles and decay constants in accord with the standard (or extended) ChPT approach. Here we show, first that the definitions of mixing angles and decay constants from VMD/WZW and ChPT do not coincide once symmetry is broken and, second, that VMD provides expressions and values for all accessible ChPT parameters in accord with expectations. This is illustrated by several examples, including the functional relation between the VMD mixing angle θ_P and the ChPT angle θ_8.

In Section VII, we show that starting from the axial anomaly, it is possible to reconstruct the one angle mixing scheme as it arises in our broken VMD model and from the WZW Lagrangian; we comment on the previous use of ChPT predictions in phenomenological analyses of radiative decays data.

In Section VIII, we show that nonet symmetry breaking and pseudoscalar mixing angle(s) are functionally related, which is a completely new result. This allows us to perform a fit of radiative decays with only 4 free parameters. The level of nonet symmetry breaking correlated with the fit value of the pseudoscalar mixing angle is shown to remove any need for glue in the η' meson. A few other points of interest are also examined (quark mass ratio, isoscalar mass matrix, effects of NSB on PS mixing angle values). Finally, Section IX is devoted to conclusions.

II. A BROKEN HLS MODEL FOR RADIATIVE DECAYS

The model developed in Ref. [1] in order to describe all light meson radiative decays relies on breaking nonet symmetry and flavor SU(3) in the HLS Lagrangian [2], and especially in its anomalous (FKTUY) sector [6]. The breaking procedure of SU(3) flavor symmetry (referred to hereafter as FSB) in the non–anomalous HLS Lagrangian is the so–called new scheme, a variant the original BKY breaking mechanism [7] discussed in Ref. [4].

For the purpose of only studying light meson radiative decays [1], a detailed knowledge of the nonet symmetry breaking (NSB) mechanism is not needed; one only needs to know the field renormalization it would imply. The choice made in Ref. [1] was to postulate a likely form; this was determined by the O'Donnell derivation of the SU(3) – not U(3) – $VP\gamma$ couplings, which assumes only the SU(3) flavor group structure, gauge invariance and Lorentz invariance [12].

However, the way FSB and NSB in the PS sector merge together is a much stronger assumption which only relies on its impressive phenomenological success [1] when describing the full set of radiative decays of light mesons. In this section, we aim at proposing a

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Lagrangian model which provides the appropriate PS field renormalization; it allows to strongly motivate this assumption, by relating this Lagrangian to the ChPT framework.

A. Basic Ingredients

The basic ingredients of the effective Lagrangian approach to the interaction of vector and pseudoscalar mesons are the matrices V and P of the vector and pseudoscalar fields expressed in the flavor (u, d, s) basis. The vector meson field matrix V is usually written[4] in terms of ideally mixed states (ω^I, ϕ^I)

$$
V \equiv V^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} (\rho^0 + \omega^I)/\sqrt{2} & \rho^+ & K^{*+} \\ \rho^- & (-\rho^0 + \omega^I)/\sqrt{2} & K^{*0} \\ K^{*-} & \overline{K}^{*0} & -\phi^I \end{pmatrix}, \quad a = 1, ..., 8. \tag{1}
$$

Correspondingly, the pseudoscalar field matrix is usually defined as

$$
P \equiv P^{a'} T^{a'} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi_8 + \frac{1}{\sqrt{3}}\eta_0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi_8 + \frac{1}{\sqrt{3}}\eta_0 & K^0 \\ K^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\pi_8 + \frac{1}{\sqrt{3}}\eta_0 \end{pmatrix},
$$
$$
a' = 0, ..., 8 \tag{2}
$$

using the conventional octet and singlet components (π_8, η_0) for the isoscalar mesons. For definiteness, the SU(3) matrices will be denoted T^a $(a = 1, \cdots 8)$ and fulfill the normalization condition $\text{Tr}[T^a T^b] = \delta^{ab}/2$. We complete this matrix basis, by adding the unit matrix suitably normalized $T^0 \equiv \mathbf{1}/\sqrt{6}$.

The physical states $(\omega, \phi, \eta, \eta')$ are generated from the ideally mixed states by means of standard rotation angles δ_V or δ_P for vector and pseudoscalar mesons. Correspondingly, the rotation angles for the singlet and octet states to the physically observed mesons are traditionally named θ_V and θ_P. These well known relations can be found in Refs. [1,12,4,10]. The connection between ideal and physical ω and ϕ fields is treated heuristically in Ref. [4] and rigorously in Ref. [9]. We recall for further use the traditional (one angle) expression

$$
\begin{bmatrix} \eta \\ \eta' \end{bmatrix} = \begin{bmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \end{bmatrix} \tag{3}
$$

If fields undergo renormalization, the fields π_8 and η_0 in the expression above should be understood renormalized [4]. With a slightly liberalized, but obvious, notation, the expressions above for V and P can also be written

$$
V = V_8 + V_0 , \quad P = P_8 + P_0 \tag{4}
$$

which exhibit their octet and singlet component combinations, and show that nonet (U(3)) symmetry is implicitly assumed.

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[4] The sign in front of ϕ^I means that we define $\phi^I = -|s\bar{s}\rangle$.

B. Physical Motivation for Nonet Symmetry Breaking (NSB)

Referring to O'Donnell [12], NSB implies modifying Eq. (4) to

$$V = V_8 + yV_0, \quad P = P_8 + xP_0 \tag{5}$$

In this way, NSB changes the relative weight of the octet and singlet parts in *a priori* both meson sectors. Ref. [9] has recently performed a throughout study of NSB in the vector sector and clearly concluded that data were consistent with no such NSB (*i.e.* $y = 1$); it is therefore a motivated choice to neglect vector NSB and state $y = 1$ definitely.

Another way to account for nonet symmetry breaking is to assume that the singlet sector contains a component other than the standard SU(3)/U(3) singlet; we name it glue only for convenience. A possible coupling of the η/η' doublet to glue can be accounted for [1] by means of an additional angle γ, which is zero if one chooses to decouple this doublet from glue

$$
\begin{bmatrix} \eta \\ \eta' \\ \eta'' \end{bmatrix} = \begin{bmatrix} \cos\theta_P & -\sin\theta_P & 0 \\ \sin\theta_P\cos\gamma & \cos\theta_P\cos\gamma & \sin\gamma \\ -\sin\theta_P\sin\gamma & -\cos\theta_P\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \\ gg \end{bmatrix}. \tag{6}
$$

Indeed, following the analysis of Ref. [1], we do not have to introduce any coupling of the η meson to glue, which would introduce an additional angle (β in Ref. [1]). The angle γ produces a coupling of (only) the η' meson to glue. We have named η'' the possible triplet companion of the η/η' mesons, and do not attempt to identify it [5]. If $\gamma = 0$, one clearly recovers the usual mixing pattern for the η/η' system by decoupling it from glue.

When fitting the data on radiative decays of light mesons, the level of correlation between x and γ is found such that assuming glue and exact nonet symmetry ($x = 1$), or assuming no glue ($\gamma = 0$) and some NSB ($x \simeq 0.9$), provide the same description of the data [1,8]. Therefore, whether glue is required in order to describe the η' properties is still a pending question which will be addressed in the present paper (see Section VIII), when an educated guess about the value of x will be made. Let us note that the level of glue can be as large as 20% if nonet symmetry breaking is ignored [1,8] by setting $x = 1$; this conclusion has been reached also by others [39,17,18].

Clearly, in order that our conclusion on this point be of relevance, the exact meaning of x should be exhibited; a framework as reliable as ChPT is appropriate. This also motivates our goal of comparing our broken HLS–FKTUY framework to ChPT.

C. Basics of the HLS Model

We refer the reader to Ref. [2] for a comprehensive review of the HLS model. A brief account can be found in Ref. [4]. We only recall the main features here.

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[5]We shall not also attempt to include this additional singlet in the Lagrangian model to be proposed for reasons which will become clear at the end of this paper.

The HLS Lagrangian can be written $\mathcal{L}_{\mathrm{HLS}} = \mathcal{L}_A + a\mathcal{L}_V$, where

$$\mathcal{L}_A = -\frac{f_\pi^2}{4}\mathrm{Tr}\left[D_\mu\xi_L\xi_L^\dagger - D_\mu\xi_R\xi_R^\dagger\right]^2 \equiv -\frac{f_\pi^2}{4}\mathrm{Tr}[L-R]^2$$

$$\mathcal{L}_V = -\frac{f_\pi^2}{4}\mathrm{Tr}\left[D_\mu\xi_L\xi_L^\dagger + D_\mu\xi_R\xi_R^\dagger\right]^2 \equiv -\frac{f_\pi^2}{4}\mathrm{Tr}[L+R]^2 \tag{7}$$

a is a parameter which is not fixed by the theory and f_π is the usual pion decay constant (92.41 MeV). The covariant derivative is

$$D_\mu\xi_{L,R} = \partial_\mu\xi_{L,R} - igV_\mu\xi_{L,R} + ie\xi_{L,R}A_\mu Q \tag{8}$$

where A_μ is the photon field, V_μ the vector meson field matrix defined above and $Q = \mathrm{Diag}(2/3, -1/3, -1/3)$ is the quark charge matrix, e is the unit electric charge and g is the universal vector meson coupling [2]. Finally, one generally chooses the "unitary" gauge, for which

$$\xi_R = \xi_L^\dagger = \xi = \exp\left(iP/f_\pi\right). \tag{9}$$

The standard VMD model is obtained by setting $a = 2$ in the HLS Lagrangian. However, several studies of the pion form factor [40,41] favor $a \simeq 2.4$, quite inconsistent with 2. A simultaneous analysis of light meson radiative decays and vector meson leptonic decays [1,9] finds $a \simeq 2.4 - 2.5$, quite consistent with pion form factor studies.

The HLS Lagrangian is given in expanded form in Ref. [4] (see Eq. (A1), where the pseudoscalar kinetic energy term has been omitted). For the purpose of the present paper, it should be noted that the pseudoscalar singlet field η_0 undergoes no interaction and only occurs in the (omitted) kinetic energy term.

D. SU(3) Breaking Mechanism (FSB) of the HLS Model

SU(3) symmetry breaking (FSB) of the HLS Lagrangian has been introduced by Bando, Kugo and Yamawaki [7] (already referred to as BKY) and originates from Refs. [2,7]. Brief accounts and some new developments can be found in Refs. [4,42], connected more precisely with the anomalous sector [6]. We refer the reader to Refs. [1,8,7,4,42] for detailed analyses of the properties of known variants of the BKY breaking scheme. Here we will only sketch the so–called new scheme detailed in Ref. [4]. Basically, the BKY breaking of SU(3) symmetry is performed by modifying Eqs. (7) in the following way

$$\mathcal{L}_{A,V} = -\frac{f_\pi^2}{4}\mathrm{Tr}[(L \mp R)(1 + (\xi_L\epsilon_{A,V}\xi_R^\dagger + \xi_R\epsilon_{A,V}\xi_L^\dagger)/2)]^2. \tag{10}$$

which has a smooth unbroken limit. The constant matrices $\epsilon_{A,V}$ are given by $\mathrm{Diag}(0, 0, c_{A,V})$. Defining the breaking matrices $X_{A,V} = \mathrm{Diag}(1, 1, 1 + c_{A,V})$, these Lagrangian terms can be written

$$\mathcal{L}_{A,V} = -\frac{f_\pi^2}{4}\mathrm{Tr}[(L \mp R)X_{A,V}(L \mp R)X_{A,V}]. \tag{11}$$

The expanded expression of the BKY broken HLS Lagrangian can be found in Ref. [4] (see Eq. (A5) in the Appendix). One should note, among other properties of this breaking mechanism, that the pseudoscalar singlet field η_0 does not undergo interactions with any of the other fields, as in the unbroken limit. It contributes only to the kinetic energy term in $\mathcal{L}_A$

$$\mathcal{L}_A = \text{Tr}[\partial P X_A \partial P X_A] + \cdots \tag{12}$$

The basic consequence of this BKY breaking mechanism for FSB is thus to force a renormalization of the (bare) pseudoscalar field matrix P, $P \to P'$

$$P' = X_A^{1/2} P X_A^{1/2}, \tag{13}$$

in order to restore the kinetic energy term to canonical form. The field transform in Eq. (13) has a smooth limit when $X_A \to \mathbf{1}$. Additionally, we have [7,4]

$$z \equiv 1 + c_A = \left(\frac{f_K}{f_\pi}\right)^2 = 1.495 \pm 0.030 \quad , \tag{14}$$

The quantity z was named ℓ_A in Ref. [1]. We shall also use the notation $Z = 1/z \simeq 2/3$ in the following, for consistency with expressions written in Ref. [1]. It should be noted [4], that the field renormalization (Eqs. (13) and (14)) is required in order to recover the charge normalization condition $F_{K^+}(0) = 1$.

The correspondence with the usual ChPT Lagrangian is easy to establish. Indeed, concerning PS fields it turns out to consider the L_5 term of the $\mathcal{L}^{(1)}$ Lagrangian together with $\mathcal{L}^{(0)}$ and the correspondence is $z = 1 + 8L_5 m_K^2/f_\pi^2$; this gives a quite good approximation [20] of the expression for $z = [f_K/f_\pi]^2$ and $L_5 \simeq 2.14 \ 10^{-3}$.

Thus, the BKY breaking mechanism outlined above, results in a renormalization of the PS field matrix in clear correspondence with ChPT expectations. It does not result likewise in a renormalization of the vector field matrix [7,4]. However, the correspondence between the results of Morpurgo [11] and the so–called K^* model [1] could well indicate that a renormalization of the vector fields is also needed [9], but only shows up in the K^* sector. In the context of the present paper, we are actually independent of any kind of symmetry breaking in the vector sector. Let us only mention the main result of Ref. [9] which tells that angular departures from ideal mixing is an appropriate parametrization of the ω/ϕ system as long as one deals only with on–shell vector resonances.

E. Nonet Symmetry Breaking (NSB) of the HLS Model

We aim here at providing a reasonable mechanism for NSB in the PS sector within an effective Lagrangian. This was required in order to describe successfully the set of observed $VP\gamma$ and $P\gamma\gamma$ radiative decays [1].

The main problem faced in the phenomenology of radiative decays is the generalization of Eq. (13) to the case where NSB is also active. At leading order, this is solely determined by the influence of NSB on the kinetic energy part of an effective Lagrangian.

If FSB were absent, we already know that $P \rightarrow P'_8 + xP'_0$ is the required field renormalization, *i.e.* the renormalization results in a rescaling of the singlet part of the P matrix. This means that NSB should contribute specifically to the kinetic energy term which would become

$$\mathcal{L}_A = \text{Tr}[\partial P \partial P] + c\text{Tr}[\partial P_0 \partial P_0] \cdots \tag{15}$$

in the absence of FSB.

Let us now examine how we might incorporate the singlet contributions into the HLS Lagrangian. As is well known, the symmetry of the HLS Lagrangian is larger than $\text{SU}(N_f){\times}\text{SU}(N_f)$, it is actually $\text{U}(N_f){\times}\text{U}(N_f)$. However, this is unphysical. The extra vector $\text{U}(1)$ symmetry conserves baryon number and is thus desirable; moreover, as remarked above, this is supported by the data. However, the additional axial $\text{U}(1)$ symmetry is a problem as it would imply either parity doublets or a ninth light pseudoscalar (for reviews see Refs. [43–45] and recently [21]). Therefore, reducing the symmetry of the HLS Lagrangian is desirable. Introducing the chiral field $U \equiv \xi^\dagger_L \xi_R = \exp(i2P/f_\pi)$ [2], one obvious way is through determinant terms [45],

$$\mathcal{L} = \mathcal{L}_{\text{HLS}} + \frac{\mu^2 f_\pi^2}{12} \ln \det U \cdot \ln \det U^\dagger + \lambda \frac{f_\pi^2}{12} \ln \det \partial_\mu U \cdot \ln \det \partial^\mu U^\dagger \tag{16}$$

where μ is a parameter with mass dimension and we have introduced the dimensionless parameter λ to allow for nonet symmetry breaking. Considering the chiral transformation $U \rightarrow g^\dagger_L U g_R$, we see Eq. (16) is now only invariant under $\text{SU}(N_f){\times}\text{SU}(N_f)$ or when $g_L = g_R$ (i.e., U_V), as desired. Rewriting the Lagrangian we have

$$\mathcal{L} = \mathcal{L}_{\text{HLS}} + \frac{\mu^2 f_\pi^2}{12} \text{Tr} \ln U \cdot \text{Tr} \ln U^\dagger + \lambda \frac{f_\pi^2}{12} \text{Tr} \ln \partial_\mu U \cdot \text{Tr} \ln \partial^\mu U^\dagger. \tag{17}$$

Now recalling Eqs. (2) and (4), this can be rewritten

$$\mathcal{L} = \mathcal{L}_{\text{HLS}} + \mathcal{L}'_{\text{HLS}} \equiv \mathcal{L}_{\text{HLS}} + \frac{1}{2}\mu^2 \eta_0^2 + \frac{1}{2}\lambda \partial_\mu \eta_0 \partial^\mu \eta_0 \tag{18}$$

as $P_0 = \eta_0 \mathbf{1}/\sqrt{6}$ and $\text{Tr}[T^{1-8}] = 0$. Thus, through this breaking of the $U_A(1)$ symmetry, the singlet acquires a mass which is nonvanishing in the chiral limit and an additional kinetic term. As can be clearly seen, this implementation of NSB only modifies the singlet contribution to the Lagrangian kinetic energy (and mass term) without changing the usual HLS interaction Lagrangian (see Eq. (A1) in Ref. [4]).

It is quite interesting at this point to remark that the NSB parameter we introduce can be identified with the Λ_1 coefficient of the $\mathcal{L}^{(1)}$ contribution to the ChPT Lagrangian [19,20], as clear from Rel. (13) in [21] who carries practically the same notations as ours. Therefore, the kinetic energy of the Lagrangian, which mostly determines the PS field renormalization, meets all expectations from ChPT.

Having shown how $U_A(1)$ breaking might lead to an additional Lagrangian term, $\mathcal{L}'$ as given in Eq. (18), we now wish to explore the consequences of this. We are interested in calculating the axial currents. This can be done through an infinitesimal (axial) variation $\partial_\mu P^a \rightarrow \partial_\mu P^a + f \partial_\mu \epsilon^a$ [46]

$$J_\mu^{A,a} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \epsilon^a)} = f_\pi \frac{\partial \mathcal{L}}{\partial(\partial_\mu P^a)} = 2f_\pi \text{Tr}[T^a X_A T^b X_A]\partial P^b + \lambda f_\pi \delta^{a0}\partial\eta^0. \tag{19}$$

We see the octet components are unchanged, while the singlet component is affected by a factor of $1 + \lambda$.

III. AN EFFECTIVE LAGRANGIAN MODEL WITH FSB AND NSB AT FIRST ORDER

For the purpose of the present study, we are interested only in the PS kinetic energy part of the Lagrangian in Eq. (18), which needs to be diagonalized in order to get the explicit transform $P \to P'$, from bare to renormalized fields. Conversely, it is clear that, if NSB vanishes, the kinetic energy of the Lagrangian is rendered canonical by the transform in Eq. (13). Here, any reference to what can happen in the V sector is totally irrelevant.

A. Diagonalization of the Effective Lagrangian Kinetic Energy

The Lagrangian in Eq. (18) has a non–canonical kinetic energy, which is precisely of the form given in Eq. (15) with $c = \lambda$. Putting it into a suitable diagonal form is thus required, in order to define the physical fields in terms of the unphysical (bare) field and get the axial currents in terms of the physical fields.

It is suitable to perform diagonalization in two steps. The first step is simply to define an intermediate renormalization step by $P'' = X_A^{1/2} P X_A^{1/2}$, which puts the nonet symmetric part of the kinetic energy term into canonical form. Practically, this means that pion fields are unchanged in this renormalization, while the kaon fields absorb a f_K/f_π factor, as if NSB were absent. Concerning isoscalar mesons, these (first step) renormalized fields can be expressed in terms of the original (bare) fields through

$$\begin{bmatrix} \pi_8'' \\ \eta_0'' \end{bmatrix} = z \begin{bmatrix} B & -A \\ -A & C \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \end{bmatrix} \tag{20}$$

The parameters A, B and C depend only on the FSB parameter z already defined and they are

$$A = \frac{\sqrt{2}}{3}\frac{(z-1)}{z} \simeq 0.16 \ , \quad B = \frac{(2z+1)}{3z} \simeq 0.90 \ , \quad C = \frac{(z+2)}{3z} \simeq 0.80 \ , \tag{21}$$

where the numerical values correspond to $z \simeq 3/2$. A can be considered as the FSB characteristic size. C and B differ at first order in this breaking parameter since $\sqrt{2}(B - C) = A$. After this renormalization the kinetic energy T is still non–canonical. Using Eq. (20), T can be expressed in terms of the (intermediate, $i.e.$ double prime) fields by

$$2\text{T} = [\partial\pi_8'']^2 + [\partial\eta_0'']^2 + \lambda[A\partial\pi_8'' + B\partial\eta_0'']^2. \tag{22}$$

It is useful to define the FSB angle β:

$$\cos\beta = \frac{B}{\sqrt{A^2 + B^2}} \; , \qquad \sin\beta = \frac{A}{\sqrt{A^2 + B^2}}. \tag{23}$$

Diagonalizing Eq. (22) gives the following renormalized fields

$$\pi_8' = \cos\beta\pi_8'' - \sin\beta\eta_0''$$
$$\eta_0' = [\sin\beta\pi_8'' + \cos\beta\eta_0'']\sqrt{1 + \lambda(A^2 + B^2)}. \tag{24}$$

These field combinations, which directly follow from the eigensolutions of the quadratic form of Eq. (22), have a smooth limit when both FSB and NSB tend to zero ($\pi_8' \to \pi_8''$ and $\eta_0' \to \eta_0''$); when FSB alone tends to zero ($A \to 0$) the limit is also smooth ($\pi_8' \to \pi_8''$ and $\eta_0' \to \eta_0''\sqrt{1 + \lambda}$). However, the limit is not smooth when only NSB vanishes; indeed, Eq. (24) shows that the fields remain rotated by an angle β which is non–zero if FSB is still active. In order to cure this disease, one can choose as final renormalized fields linear combinations of the solutions in Eqs. (24), which have the desired limit properties and conserve the canonical structure of T by the diagonalization. Using $v = \sqrt{1 + \lambda(A^2 + B^2)} - 1$, these combinations are

$$\pi_8' = (1 + v\sin^2\beta)\pi_8'' + v\sin\beta\cos\beta\eta_0''$$
$$\eta_0' = v\sin\beta\cos\beta\pi_8'' + (1 + v\cos^2\beta)\eta_0'' \tag{25}$$

B. Field Transformation At First Order

We can use directly the exact transformation given by Eqs. (25); however, our breaking procedure is actually leading order in all breaking parameters. This is illustrated by having shown the connection between our breaking procedure and the $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ ChPT Lagrangian. Therefore, it is meaningful to truncate the field tranform above at leading order.

Truncating v at first order, we have $v \simeq \lambda(A^2 + B^2)/2 \simeq \lambda B^2/2$, and then

$$\begin{bmatrix} \pi_8' \\ \eta_0' \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{\lambda}{2}A^2 & \dfrac{\lambda}{2}AB \\ \dfrac{\lambda}{2}AB & (1 + \dfrac{\lambda}{2}B^2) \end{bmatrix} \begin{bmatrix} \pi_8'' \\ \eta_0'' \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & (1 + \dfrac{\lambda}{2}B^2) \end{bmatrix} \begin{bmatrix} \pi_8'' \\ \eta_0'' \end{bmatrix} \tag{26}$$

The last relation is obtained by removing breaking terms of order greater than 1. Using Eqs. (20) and (26), we can approximate the physical field (prime) combinations in terms of the bare fields by

$$\begin{bmatrix} \pi_8' \\ \eta_0' \end{bmatrix} \simeq \begin{bmatrix} B & -A \\ -A(1 + \dfrac{\lambda}{2}B^2) & C(1 + \dfrac{\lambda}{2}B^2) \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \end{bmatrix} \simeq \begin{bmatrix} B & -A \\ -A(1 + \dfrac{\lambda}{2}) & C(1 + \dfrac{\lambda}{2}) \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \end{bmatrix} \tag{27}$$

This is the physical (first order) approximation which corresponds to the field renormalization used in Ref. [1] and recalled in Eq. (29). The last matrix expression in Eq. (27) is obtained by remarking that λB^2 differs from λ by terms of order λA and then is legitimate

to neglect them at first order in further computations. The $A\lambda$ term in the lower leftmost matrix element is kept for consistency, but clearly plays a negligible role.

Therefore, the field renormalization on which the study of Refs. [1,8,9] relies is obtained from a Lagrangian model by truncating at first order in the breaking parameters. This provides an excellent fit to all light meson radiative decays, as can be seen from Ref. [1] and as will be shown below (see Subsection VIII B). We have checked that a fit to radiative decays performed as in Ref. [1] using the exact field transformation in Eq. (25) instead of its first order approximation in Eq. (27) gives indeed an improvement in fit quality, but negligible[6]. From these expressions, it is also clear that the NSB parameter x [1] is actually

$$x = 1 - \frac{\lambda}{2}B^2 \simeq \frac{1}{\sqrt{1 + \lambda B^2}} \Longrightarrow \lambda \simeq 0.20 - 0.25, \tag{28}$$

using the reference value for x (see Eq. (34)). We see the parameter λ is small. In determining the accuracy and systematic errors, the neglected orders of magnitude should be estimated from the values of A [FSB] and $(1 - x)$ [NSB]. It should be noted that, even if x carries prominently the information of NSB, it is somehow influenced by FSB as $B = 1 + A/\sqrt{2}$.

In what follows we shall approximate the change of fields by its expression at first order in the breaking parameters, which can be written

$$P = X_A^{-1/2}(P_8' + xP_0')X_A^{-1/2} \tag{29}$$

The accuracy of this expression relative to the Lagrangian defined above can be estimated at $\simeq 5\%$ by analyzing the magnitude of the neglected terms in Eq. (27).

IV. THE VMD DESCRIPTION OF $\eta/\eta' \to \gamma\gamma$ DECAYS

Following FKTUY [6], the anomalous U(3) symmetric Lagrangian describing PVV interactions is

$$\mathcal{L} = -\frac{3g^2}{4\pi^2 f_\pi}\epsilon^{\mu\nu\rho\sigma}\text{Tr}[\partial_\mu V_\nu \partial_\rho V_\sigma P]. \tag{30}$$

The $PV\gamma$ and $P\gamma\gamma$ transitions amplitudes are obtained from this Lagrangian and the non–anomalous HLS Lagrangian, needed in order to describe the direct transition of vector mesons to photons. This non–anomalous HLS Lagrangian is given in its expanded form in Ref. [4]. It should only undergo the field renormalization of Eq. (29), valid at first order in the (two) breaking parameters.

The HLS model contains the Vector Meson Dominance (VMD) assumption (for a review see [47]); it thus gives a way to relate the radiative decay modes $VP\gamma$ to each other and to the $P\gamma\gamma$ decays for light mesons, by giving a precise meaning to the equations sketched in Fig. 1.

---

[6]The improvement is larger if ones replaces the PDG value [10] for the rate $\phi \to \eta'\gamma$ by the mean value of all presently available measurements !
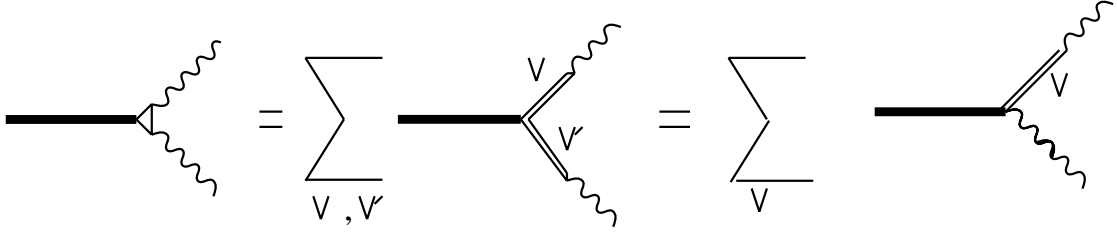
11

FIG. 1. .

Graphical representation of the relation among various kind of coupling constants. V and V' stand for the lowest lying vector mesons (ρ^0, ω, ϕ); the internal vector meson lines are propagators at $s = 0$ and are approximated by the corresponding tabulated [10] masses squared.

Propagating the field renormalization in Eq. (29) down to the FKTUY Lagrangian of Eq. (30) gives

$$\mathcal{L} = -\frac{3g^2}{4\pi^2 f_\pi}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[\partial_\mu V_\nu \partial_\rho V_\sigma X_A^{-1/2} P' X_A^{-1/2}]. \tag{31}$$

Then, the VVP Lagrangian is changed in a definite way by the renormalization procedure.

The expanded form of the Lagrangian in Eq. (31) is given in the Appendix of Ref. [1]. The expressions for the two–photon decay widths of the η and η' mesons can be derived from this and the non–anomalous Lagrangian. They are [1]

$$G_{\eta\gamma\gamma} = -\frac{\alpha_{em}}{\pi\sqrt{3}f_\pi}\left[\frac{5-2Z}{3}\cos\theta_P - \sqrt{2}\frac{5+Z}{3}x\sin\theta_P\right],$$

$$G_{\eta'\gamma\gamma} = -\frac{\alpha_{em}}{\pi\sqrt{3}f_\pi}\left[\frac{5-2Z}{3}\sin\theta_P + \sqrt{2}\frac{5+Z}{3}x\cos\theta_P\right], \tag{32}$$

$$G_{\pi^0\gamma\gamma} = -\frac{\alpha_{em}}{\pi f_\pi},$$

where $Z = 1/z = [f_\pi/f_K]^2$. Actually, the last expression in Eq. (32) is a normalization condition which allows us to fix the numerical coefficient in Eq. (31). It is clear that Eq. (32) gives the two–photon decay widths in terms of f_π, f_K, x and only one mixing angle, θ_P. This will be frequently referred to as the wave–function mixing angle (see Eq. (3)). As normal these equations do not (and should not) depend on the vector meson parameters (g and δ_V).

Thus, using standard Feynman rules, the HLS model provides definite expressions for the two–photon couplings of the pseudoscalar mesons, through its anomalous (FKTUY) sector. These expressions exhibit the traditional form [13,14,49] originally obtained through Current Algebra. These couplings are related to partial widths by

$$\Gamma(X \to \gamma\gamma) = \frac{M_X^3}{64\pi}|G_{X\gamma\gamma}|^2 \quad , \quad X = \pi^0,\ \eta,\ \eta' \quad . \tag{33}$$

As a test, one can fit the parameters x and θ_P solely through radiative decays of the type $VP\gamma$ and use these values and their associated errors to predict the values for the two–photon decay widths of the η and η' mesons. The fit values used for these computations [1]

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are $x = 0.917 \pm 0.017$ and $\theta_P = -10.41° \pm 1.21°$. The results are given in Table 1 and clearly illustrate that the expressions in Eq. (32) are valid and that the $VP\gamma$ processes accurately *predict* the two–photon decay widths.

Stated otherwise, one does not need more than one angle (θ_P) in order to describe the η and η' radiative decays and this receives an especially strong support from all $VP\gamma$ modes. Additionally, despite claims [13,35,36], this angle is found to be $\simeq -10°$, in apparent (as will be seen) inconsistency with the ChPT expectation of $\simeq -20°$ [13,35,34].

Mode	VMD Fit Prediction	PDG Value	Comment	Global Fit Quality χ^2/dof Probability	(x, θ_P) Correlation
		0.514 ± 0.026	$\gamma\gamma$	$11.07/10(35\%)$	-0.34
$\eta \to \gamma\gamma$ [keV]	0.464 ± 0.026	0.46 ± 0.04	PDG mean	$9.14/10(52\%)$	-0.49
		0.324 ± 0.046	Primakoff	$14.82/10\ (13\%)$	-0.55
$\eta' \to \gamma\gamma$ [keV]	4.407 ± 0.233	4.27 ± 0.19	PDG mean		

Table 1 : Partial decay widths of the η/η' mesons, as reconstructed solely from fits to the radiative decays $VP\gamma$ (leftmost data column) and their direct measurements [10] (second data column). The third data column displays fit quality parameters when using the corresponding η measurement. The rightmost data column gives the correlation coefficient (x, θ_P) in the corresponding case.

A comment is of relevance concerning the data on $\eta \to \gamma\gamma$. One clearly sees that the $VP\gamma$ modes considered altogether clearly prefer the PDG recommended (mean) value to either of the homogeneous reported measurements. Therefore, one may guess that the (single) Primakoff effect measurement and the (fourfold) $\gamma\gamma$ measurement, both suffer from systematic errors in opposite directions. This guess is supported by the recent direct measurement of the $\eta \to \gamma\gamma$ branching fraction [48] $39.21\% \pm 0.3\%$, quite consistent with the PDG mean value. We shall revisit this issue in Section VIII.

In order to substantiate the relative quality of the three data given in Table 1, we have redone the global fit, as described in Ref. [1], changing only the $\eta \to \gamma\gamma$ data. The corresponding fit information is given in the rightmost pair of data columns in Table 1. Even if the fit probabilities are all quite acceptable, it is clear that the PDG recommanded value is indeed preferred by the full set of $VP\gamma$ decay modes. For this reason we use, from now on, the corresponding best fit results as reference values:

$$x = 0.902 \pm 0.018 \ , \quad \theta_P = -10.38° \pm 0.97° \tag{34}$$

V. THE WZW DESCRIPTION OF $\eta/\eta' \to \gamma\gamma$ DECAYS

Starting from broken HLS and FKTUY, the VMD model of Ref. [1] recovers the traditional form for the two–photon decay amplitudes, (*i.e.* the one mixing angle expressions

of Current Algebra [13,14,49]). Using these standard expressions, one indeed gets through identification with our Eqs. (32)

$$\frac{f_\pi}{\overline{f}_8} = \frac{5 - 2Z}{3} \ , \quad \frac{f_\pi}{\overline{f}_0} = \frac{5 + Z}{6} \ x \ , \tag{35}$$

where $Z = [f_\pi/f_K]^2$, and $\overline{f}_{0,8}$ denote the (Current Algebra) singlet and octet decay constants; we have already defined θ_P, the (single) mixing angle occurring in this approach. The $\overline{f}_{0,8}$ are named $\gamma\gamma$ decay constants in [31].

It is easy to check that Eqs. (32) and (35) can be derived directly from the WZW Lagrangian [25,26]. Indeed, this can be written

$$\mathcal{L}_{WZW} = -\frac{N_c e^2}{4\pi^2 f_\pi} \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma \mathrm{Tr}[Q^2 P] \tag{36}$$

(with $N_c = 3$) where $Q = \mathrm{Diag}(2/3,–1/3,–1/3)$ is the quark charge matrix, A is the electromagnetic field and P is the bare pseudoscalar field matrix. Changing to the renormalized field P' through Eq. (13) allows us to recover exactly the couplings in Eq. (32).

This illustrates clearly that, what is named f_8 in the Current Algebra [49] expressions for η/η' decays to two photons, can be expressed solely in terms of f_π and f_K, in a way which fixes its value to $\overline{f}_8 = 0.82 f_\pi$. Correspondingly, we have $\overline{f}_0 = 1.17 f_\pi$ which includes a correction of approximately 10% due to nonet symmetry breaking. The fact that the WZW Lagrangian leads to the same results as the FKTUY Lagrangian simply states their expected equivalence when deriving two–photon decay amplitudes. Stated otherwise, the structure of Eqs. (32) depends only on the BKY breaking X_A, with a small influence of PS NSB.

However, the SU(3) sector of Chiral Perturbation Theory (ChPT) [13,34,35] is well known to predict $f_8/f_\pi \simeq 1.25$ and a mixing angle of $\simeq -20°$. Then, the question is whether there is an inconsistency with respect to ChPT, or if there is a mismatch among definitions in ChPT ($f_{0,8}$) and in the VMD/WZW approach ($\overline{f}_{0,8}$), after symmetry breaking. More precisely, the question is whether $\overline{f}_8$ and $\overline{f}_0$ have actually the meaning of decay constants.

Before closing this Section, it is interesting to compare our expressions for $\overline{f}_{0,8}$ with the corresponding ones in [31] and [32]. Indeed, by identification of our Eqs. (35) with Eqs. (26), (31) and (32) of Ref. [31], we get

$$\frac{\widetilde{T}_{s\overline{s}}(0,0)}{\widetilde{T}_{\pi^0}(0,0)} = \left[\frac{f_\pi}{f_K}\right]^2 = \frac{2}{3} \tag{37}$$

for the ratio of their reduced amplitudes. Their own numerical estimate for this ratio is 0.62, in quite good agreemeent with ours. In order to reach Eq. (37), we have stated $x = 1$ in our own expressions. So, one can consider that the present Eq. (35) extends the results of Ref. [31] (and [32]) to the case when nonet symmetry is broken; the correction is however minor in this realm.

VI. A CHPT DESCRIPTION OF $\eta/\eta' \to \gamma\gamma$ DECAYS

We have seen above that, in the VMD procedure developed in Refs. [1,4,8], the expressions for f_8, f_0 and the mixing angle are the same as those obtained from the matrix

elements for $\langle \gamma\gamma | \mathcal{L}_{WZW} | \eta \rangle$ and $\langle \gamma\gamma | \mathcal{L}_{WZW} | \eta' \rangle$. We have denoted the parameters, obtained in this manner, $\overline{f}_{0,8}$ and θ_P. In ChPT, however, the corresponding quantities ($f_{0,8}$, θ_8, θ_0) are defined through other matrix elements, namely $\langle 0 | \partial^\mu J_\mu^{8,0} | \eta \rangle$ and $\langle 0 | \partial^\mu J_\mu^{8,0} | \eta' \rangle$, where the $J^{8,0}$ are the axial currents. It seems, however, traditionally admitted [13,35,51] that both sets of definitions necessarily coincide. It is this last property which is addressed now.

A. Usual ChPT Parameters from Broken VMD

The axial current defined by Eq. (19),

$$J_\mu^a = -2f_\pi \left\{ \mathrm{Tr}[T^a X_A \partial_\mu P X_A] + \delta^{a0} \lambda \partial_\mu P_0] \right\} \tag{38}$$

can be rewritten in terms of the physical fields, through the transformation in Eq. (29) (and also Eq. (27) for the isoscalar sector). We can write the matrix elements $\langle 0 | J_\mu^a | P'^a \rangle$ for $a = 1, \cdots 7$ and get the corresponding decay constants:

$$\langle 0 | J_\mu^{\pi/K} | \pi/K(q) \rangle = i f_{\pi/K} \, q_\mu \tag{39}$$

for pions and kaons, taking into account the expression for z. For the isoscalar sector, we get

$$J_\mu^8 = \frac{1+2z}{3} f_\pi \partial_\mu \pi_8^R + \frac{\sqrt{2}}{3}(1-z) f_\pi \partial_\mu \eta_0^R,$$
$$J_\mu^0 = \frac{\sqrt{2}}{3}(1-z) f_\pi \partial_\mu \pi_8^R + \frac{2+z}{3} f_\pi x(1+\lambda) \partial_\mu \eta_0^R, \tag{40}$$

with obvious notations, and where we have used $z = [f_K/f_\pi]^2 = 1/Z$. From above, we know that $x = 1/\sqrt{1 + \lambda B^2}$, is influenced by FSB ($B^2 \simeq 0.8$). Moreover, the occurrence of simply $(1 + \lambda)$ – without any dependence upon z – is certainly due to the fact that NSB in the Lagrangian of Eq. (18) does not undergo SU(3) breaking effects. Therefore, it is consistent to consider that $1 + \lambda \simeq 1/x^2$ and make the (first order) approximation $x(1 + \lambda) = 1/x$.

One should note the occurrence in Eq. (40) of singlet field contributions to the octet axial current and, conversely, of octet field contribution into the singlet axial current. Additionally, these terms vanish in the limit of unbroken SU(3) flavor symmetry ($z = 1$) as expected.

These axial currents allow to define the following matrix elements

$$\langle 0 J_\mu^8 | \pi^8(q) \rangle = i f_8 q_\mu, \quad \langle 0 | J_\mu^0 | \eta^0(q) \rangle = i f_0 q_\mu$$
$$\langle 0 | J_\mu^8 | \eta_0(q) \rangle = i b_8 q_\mu, \quad \langle 0 | J_\mu^0 | \pi^8(q) \rangle = i b_0 q_\mu \tag{41}$$

with

$$f_8 = \frac{(1+2z)}{3} f_\pi = (1.33 \pm 0.02) f_\pi, \quad b_8 = \frac{\sqrt{2}}{3}(1-z) f_\pi = (-0.24 \pm 0.01) f_\pi$$
$$f_0 = \frac{(2+z)}{3x} f_\pi = (1.29 \pm 0.03) f_\pi, \quad b_0 = \frac{\sqrt{2}}{3}(1-z) f_\pi = (-0.24 \pm 0.01) f_\pi \tag{42}$$

where the quoted errors are statistical only. So, at leading order in NSB, we have $b_0 = b_8$.

One readily observes a mismatch between the VMD/WZW definition for f_8 and f_0 (see Eq. (35)) and the ChPT definitions above; this mismatch is both algebraic and numerical. Otherwise, *this* f_8 corresponds to the standard ChPT definition and it has its expected value [13,34,35,51].

In order to switch to the matrix elements for $\langle 0|J_\mu^{0,8}|\eta/\eta'\rangle$, we use Eq. (3) together with the notations of Kaiser and Leutwyler [19,20]

$$\langle 0|J_\mu^{0,8}|\eta/\eta'(q)\rangle = iF_{\eta/\eta'}^{0,8}q_\mu \tag{43}$$

and find

$$
\begin{aligned}
F_\eta^8 &= F^8\cos\theta_8 = f_8\cos\theta_P - b_8\sin\theta_P = (1.269\pm0.008)f_\pi \\
F_{\eta'}^8 &= F^8\sin\theta_8 = f_8\sin\theta_P + b_8\cos\theta_P = -(0.472\pm0.021)f_\pi \\
F_\eta^0 &= -F^0\sin\theta_0 = b_0\cos\theta_P - f_0\sin\theta_P = (0.001\pm0.023)f_\pi \\
F_{\eta'}^0 &= F^0\cos\theta_0 = b_0\sin\theta_P + f_0\cos\theta_P = (1.315\pm0.026)f_\pi
\end{aligned} \tag{44}
$$

using the reference parameter values for x and θ_P of Eq. (34). It should be stressed here that $F^{0/8}$ and $\theta_{0/8}$ differ from $f^{0/8}$ and θ_P only by terms of order b_0 and b_8. These cannot be neglected consistently if one keeps terms of order $\sin\theta_P$, which are numerically of the same order. Eqs. (44) lead to

$$
\begin{aligned}
F^8 &= (1.36\pm0.01)f_\pi \qquad F^0 = (1.32\pm0.03)f_\pi \\
\theta_8 &= -20.40°\pm0.96° \qquad \theta_0 = -0.05°\pm0.99°
\end{aligned} \tag{45}
$$

In the exact SU(3) limit ($z = 1$), the expressions in Eq. (44) imply $\theta_0 = \theta_8 = \theta_P$. Thus, the difference between them is, indeed, an effect of SU(3) flavor symmetry breaking, with only a marginal (numerical) influence of the nonet symmetry breaking parameter x.

Now, the values given in Eq. (45) can indeed be compared with ChPT expectations, as these expressions correspond to the standard ChPT definition of mixing parameters. The value for F^8 compares impressively to the parameter free prediction of Ref. [20] ($1.34f_\pi$ with no quoted error). The value for F^0 is harder to estimate theoretically because of its scale dependence; however, from the information given in Refs. [19,20], $F^0 \simeq 1.3f_\pi$ seems in an acceptable range. The value $\theta_8 = -20.40°\pm0.96°$ is impressively consistent with all reported ChPT expectations (for example, $-20°\pm4°$ from Ref. [34], $-20.5°$ from Ref. [19]).

For θ_0 the situation is unclear because the accuracy of the reported theoretical expectation [19] $\theta_0 \simeq -4°$ is lacking. Then, it is not possible to compare rigorously our result in Eq. (45) with it; we show just below that the difference with ChPT (if any) is due to non–leading terms in breaking parameters.

B. Further Comparison of VMD with ChPT

One can ask about the correspondence of the expressions for the ChPT parameters coming from our VMD Lagrangian model of currents with their usual ChPT expressions in terms of f_π and f_K. We show here that all expressions we can get in our HLS–FKTUY approach and all reported ChPT expectations to which they can be compared coincide

surely at leading order in breaking parameters. Let us illustrate with a few examples, mostly referred to in Ref. [19,21].

From Eqs. (44), one can easily derive

$$[F^8]^2 = [F^8_\eta]^2 + [F^8_{\eta'}]^2 = \left[\frac{(1+2z)}{3}\right]^2 f_\pi^2 + \frac{2}{9}(1-z)^2 f_\pi^2 \ . \tag{46}$$

Let us define

$$z = 1 + 2\varepsilon \ , \quad x = 1 + \delta \ . \tag{47}$$

Neglecting terms of order ε^2, we have $\varepsilon = f_K/f_\pi - 1 \simeq 0.22$ and $\delta = x - 1 \simeq -0.10$ as values for these perturbations to exact SU(3) flavor and nonet symmetries. Using Eq. (47), Eq. (46) gives

$$[F^8]^2 = \left[1 + \frac{8\varepsilon}{3} + \frac{8\varepsilon^2}{3} + \mathcal{O}(\varepsilon^3)\right] f_\pi^2 \ . \tag{48}$$

which can be rewritten

$$3[F^8]^2 = 4f_K^2 - f_\pi^2 + \mathcal{O}(\varepsilon^2). \tag{49}$$

This gives the same leading term as its ChPT expression (see Eq. (11) in Ref. [19]).

Concerning F^0, Eqs. (44) give at leading order

$$[F^0]^2 = \left[(1 + \frac{4\epsilon}{3})(1-2\delta) + \mathcal{O}(\epsilon^2)\right] f_\pi^2 = \frac{2f_K^2 + f_\pi^2}{3}(1-2\delta) + \mathcal{O}(\epsilon^2) \tag{50}$$

which compares quite well with the Feldmann expression [21]

$$[F^0]^2 = \frac{2f_K^2 + f_\pi^2}{3} + f_\pi^2 \Lambda_1 \tag{51}$$

from which one derives, neglecting terms of order $\mathcal{O}(\epsilon\delta)$

$$\Lambda_1 = -2\delta = \frac{1}{x^2} - 1 = 0.20 \pm 0.04 \tag{52}$$

at the mass scale defined by radiative decays. One should note that this implies that Λ_1 is nothing but what was named λ as it should since nonet symmetry breaking is done in accordance with the ChPT $\mathcal{L}_1$ Lagrangian. Eq. (52) may serve to estimate, at the same scale, the other OZI violating parameters [19,21] Λ_2 and Λ_3 from phenomenology[7]:$\Lambda_3 \simeq -0.03 \pm 0.02$, $\Lambda_2 \simeq 0.31 \pm 0.02$. These quantities are certainly scale dependent [19,20], however this dependence is expected smooth [21].

On the other hand, Eq. (44) also gives

---

[7] The symbol $\simeq$ is used because theoretical uncertainties of the relations among the Λ_i are not quoted.

$$F^0 F^8 \sin(\theta_8 - \theta_0) = f_8 b_0 + f_0 b_8. \tag{53}$$

Using the expressions in Eq. (42) and in Eq. (47), this is

$$3 F^0 F^8 \sin(\theta_0 - \theta_8) = 2\sqrt{2}(f_K^2 - f_\pi^2)\left[1 + \varepsilon - \frac{\delta}{2}\right]. \tag{54}$$

which coincides at leading order with the corresponding quantity given in Ref. [19] (see Eq. (13) there). One should note, however, that the leading correction is increased with respect to neglecting deviations from nonet symmetry, from 22% to 27%. On the other hand, one can check that

$$F^0 = 1 + \frac{2}{3}\varepsilon \quad, \quad \text{and} \quad F^8 = 1 + \frac{4}{3}\varepsilon \tag{55}$$

in units of f_π, which also corresponds to the expectation that F^0 and F^8 differ only at first non–leading order [19]. Of course, leading and non–leading in our expressions refers to the small perturbation parameters of our model, ε and δ defined above.

Therefore, at first order in breaking parameters, all expressions (and numerical values) deduced from our HLS broken Lagrangian meet all expectations of ChPT at the same order, namely, for F^0, F^8, θ_8 and θ_0. Furthermore, using our fit [1] to radiative decays, we can provide Λ_1 with a reliable value.

One should however note that higher order corrections play some role; indeed, at leading order, F^0 and F^8 differ by about 15% (see Eq. (55) just above), while the numerical value in Eq. (45) gives a difference of only $\simeq 3\%$.

The single divergence – if any – concerns θ_0. However, this is fully due to higher order corrections, the ChPT estimates of which are still missing. Indeed, this can be checked directly on Eq. (53) which gives $\theta_0 - \theta_8 \simeq -20°$, while Eq. (54) gives $\theta_0 - \theta_8 \simeq -15°$, as obtained in Refs. [19,20], *i.e.* $-14°$ to $-16°$. With this respect, it should be remarked that Eq. (54) undergoes higher order corrections of more than 20%.

The importance of having an improved theoretical estimate of θ_0 should be stressed. Indeed, it should allow to reject several modellings (see Table 1 in Ref. [21]), as the values proposed for θ_0 range between $0°$ and $-30°$. Our broken HLS–FKTUY framework points out that the condition $\theta_0 \simeq 0$ is well fulfilled by all data on radiative decays, as will be further illustrated in Subsection VIII B.

C. Relations between θ_8 , θ_0 and θ_P

Equations (44) and (42) together give

$$\tan\theta_8 = \frac{f_8 \tan\theta_P + b_8}{f_8 - b_8 \tan\theta_P} = \tan(\theta_P + \varphi_P) , \quad \tan\varphi_P = \sqrt{2}\frac{(1-z)}{(1+2z)} \tag{56}$$

which corresponds to $\varphi \simeq -10.02°$, not influenced by NSB at this order. On the other hand, the same equations provide also

$$\tan\theta_0 = \frac{f_0 \tan\theta_P - b_0}{f_0 + b_0 \tan\theta_P} = \tan(\theta_P - \psi_P) , \quad \tan\psi_P = \sqrt{2}\frac{(1-z)}{(2+z)}x \tag{57}$$

Eqs. (56) and (57) together give

$$\theta_8 + \theta_0 = 2\theta_P + \frac{\sqrt{2}}{9}(1-z)^2 + \frac{\sqrt{2}}{3}(1-z)(1-x) \tag{58}$$

up to terms of orders $\mathcal{O}[(1-z)^3]$ and $\mathcal{O}[(1-z)^2(1-x)]$, which represent a few percents only. This makes explicit the connection between VMD/WZW and ChPT angles. Th. Feldmann has recently obtained a weaker form of the above relations in [21] (see its Eq. (84)), in the sense that NSB was neglected.

Let us now consider the condition $\theta_0 = 0$ as exact, which is clearly close to real life (see Eq. (45)). One can easily check that this condition can be cast into the form

$$\tan\varphi_P = K\tan\theta_P , \tag{59}$$

where $K = (2+z)/[(1+2z)x]$ differs from unity by only 3%, which is likely well inside our (systematic) model errors. Therefore, Eq. (56) can be written

$$\tan\theta_8 = \tan\left[\theta_P + \arctan\left(K\tan\theta_P\right)\right] \simeq \tan 2\theta_P . \tag{60}$$

Thus we have $\theta_P = \theta_8/2$ already with an accuracy of order 1.5%. A slightly more accurate expression is obtained by expanding the relation above

$$\theta_8 \simeq (K+1)\,\theta_P , \tag{61}$$

taking into account the (observed) smallness of θ_P. So, at the level of accuracy permitted by the data, there is a strict equivalence in using θ_8 or θ_P. Furthermore, phenomenology indicates that higher order corrections should decrease the magnitude of θ_0.

D. Estimate of the Quark Mass Ratio

Since there is now some reason to trust the reliability of our numerical results in Eq. (44), we can use the following equation [19]

$$3\left\{[F_\eta^8 M_\eta]^2 + [F_{\eta'}^8 M_{\eta'}]^2\right\} = 4[f_K M_K]^2 \frac{2S}{(S+1)} - [f_\pi M_\pi]^2(2S-1) \tag{62}$$

in order to extract the ratio S of the strange quark mass to the mean value of the non–strange quark masses $(S = m_s/\hat{m})$. This equation is actually second degree and thus potentially admits a spurious solution. The M_i terms are the corresponding meson masses.

Using the mass values for neutral mesons and the information given in Ref. [10], we get

$$\frac{m_s}{\hat{m}} = 21.23 \pm 2.42 \quad \text{or} \quad \frac{m_s}{\hat{m}} = 2.5^{+1.3}_{-0.7} \tag{63}$$

The first solution compares well to the expectation of Current Algebra (25.9) and to the estimate (26.6) of Ref. [19] ; it is also in impressive agreement with the A. Pich estimate [36,37] 22.6 ± 3.3. The magnitude of the uncertainties (about 10%) is dominated by the errors on decay constants; errors due to choosing the neutral masses for the relevant mesons have not been accounted for.

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VII. VMD, THE WZW LAGRANGIAN AND CHPT

We have shown in Section III that the HLS model can be consistently extended in order to include nonet symmetry breaking along with SU(3) breaking effects. The connection with the ChPT Lagrangian $\mathcal{L}^0 + \mathcal{L}^1$ has been exhibited and proved successful. It has been shown reasonable to restrict to first order effects. The analysis of our HLS–FKTUY model with respect to ChPT, tells us that the BKY SU(3) breaking mechanism [7,4] is justified. The most delicate assumption of the model for radiative decays of Ref. [1] is the PS field renormalization in Eq. (29); the diagonalization procedure above has shown that it is indeed perfectly justified at leading order in breaking parameters.

Instead of radiative decays which exhibit a huge sensitivity [1] to NSB (the parameter x), the basic ChPT parameters just considered exhibit only a marginal influence of NSB.

Comparing the two sets of parameters, both derived from within a common VMD framework, the single clear conclusion is that there is a mismatch between the VMD/WZW customary definitions of decay constants and mixing angles and that currently stated within ChPT. Before commenting on phenomenological issues, we first show that this mismatch is a pure effect of SU(3) breaking which could have been foreseen since the work of Kaiser and Leutwyler [19,20].

A. From ChPT back to VMD/WZW

The question of whether one can move back from the standard angles and decay constants of (extended) ChPT to the VMD/WZW framework is, of course, of special relevance. Indeed, we have already shown that, starting from our VMD/WZW model, the observed mixing angle of $\simeq -10°$ (for instance) was quite consistent will all expectations of ChPT, especially $\theta_8 \simeq -20°$. The proof of the converse, *i.e.* deriving the WZW expressions from the axial anomaly, can be done on general grounds and is outlined in the Appendix. We detail here the algebra in order to illustrate the connection between ChPT concepts and the standard VMD parameters, and also test the consistency of having used first order approximation for the field transform.

The basic idea is to remark that the divergence of axial currents is given by the axial anomaly at $q \to 0$. In the case of two–photon decays, it takes the form

$$\langle 0|\partial_\mu J^{\mu,a}|\gamma\gamma\rangle = N\langle 0|\text{Tr}[T^a F_{\alpha\beta}\tilde{F}^{\alpha\beta}]|\gamma\gamma\rangle \quad , \qquad a = 0, 3, 8 \tag{64}$$

where the axial currents are given in Eq. (19), $F_{\alpha\beta}$ is the photon field strength and $\tilde{F}_{\alpha\beta}$ its dual, T^a are the SU(3) flavor matrices and N is a normalization factor. Saturating the left–hand side of this expression with the nonet P of the (lightest) pseudoscalar mesons, we have

$$\langle 0|\partial_\mu J^{\mu,a}|\gamma\gamma\rangle = \sum_P \langle 0|\partial_\mu J^{\mu,a}|P\rangle \frac{1}{M_P^2} G_{P\gamma\gamma} = N\, \text{Tr}[T^a Q^2] \quad , \qquad a = 0, 3, 8 \tag{65}$$

where we have denoted by $G_{P\gamma\gamma}$ the decay amplitude of pseudoscalar mesons to two photons and Q is the quark charge matrix.

As we limit the sum to the lowest pseudoscalar mesons, the single intermediate state for ∂J^3 is $P = \pi^0$ and then

$$\langle 0|\partial J^3|\pi^0\rangle \frac{1}{M_{\pi^0}^2} G_{\pi^0\gamma\gamma} = N \ \mathrm{Tr}[T^a Q^2] \tag{66}$$

With $\langle 0|\partial J^3|\pi^0\rangle = f_\pi M_{\pi^0}^2$, and because the last term in Eq. (32) gives $G_{\pi^0\gamma\gamma}$, Eq. (66) provides the normalization $N = 6\alpha/\pi$.

For ∂J^0 and ∂J^8, there are two possible intermediate states (the η and η' mesons) and Eq. (65) gives:

$$
\begin{aligned}
F_\eta^8 G_{\eta\gamma\gamma} + F_{\eta'}^8 G_{\eta'\gamma\gamma} &= \frac{\alpha}{\pi\sqrt{3}}, \\[2mm]
F_\eta^0 G_{\eta\gamma\gamma} + F_{\eta'}^0 G_{\eta'\gamma\gamma} &= \frac{2\sqrt{2}\alpha}{\pi\sqrt{3}},
\end{aligned}
\tag{67}
$$

Inverting these relations gives

$$
\begin{aligned}
G_{\eta\gamma\gamma} &= \frac{\alpha_{em}}{\pi\sqrt{3}} \frac{1}{F_\eta^8 F_{\eta'}^0 - F_\eta^0 F_{\eta'}^8} \left[F_{\eta'}^0 - 2\sqrt{2} F_{\eta'}^8 \right] \\[2mm]
G_{\eta'\gamma\gamma} &= \frac{\alpha_{em}}{\pi\sqrt{3}} \frac{1}{F_\eta^8 F_{\eta'}^0 - F_\eta^0 F_{\eta'}^8} \left[-F_\eta^0 + 2\sqrt{2} F_\eta^8 \right]
\end{aligned}
\tag{68}
$$

These expressions are the $\eta/\eta' \to \gamma\gamma$ amplitudes in terms of the Kaiser–Leutwyler parameters $F^{0/8}$ and $\theta_{0/8}$. They can be reexpressed in terms of θ_P and f_0, f_8, b_0, b_8 by means of Eq. (44) and become

$$
\begin{aligned}
G_{\eta\gamma\gamma} &= \frac{\alpha_{em}}{\pi\sqrt{3}} \frac{1}{f_0 f_8 - b_0 b_8} \left[(b_0 - 2\sqrt{2} f_8)\sin\theta_P + (f_0 - 2\sqrt{2} b_8)\cos\theta_P \right], \\[2mm]
G_{\eta'\gamma\gamma} &= \frac{\alpha_{em}}{\pi\sqrt{3}} \frac{1}{f_0 f_8 - b_0 b_8} \left[-(b_0 - 2\sqrt{2} f_8)\cos\theta_P + (f_0 - 2\sqrt{2} b_8)\sin\theta_P \right].
\end{aligned}
\tag{69}
$$

These expressions exhibit the standard Current Algebra structure. Having denoted the coefficients there by $\overline{f}_{0,8}$, we clearly have

$$
\frac{1}{\overline{f}_8} = \frac{f_0 - 2\sqrt{2} b_8}{f_0 f_8 - b_0 b_8} \quad , \qquad \frac{1}{\overline{f}_0} = -\frac{1}{2\sqrt{2}} \frac{b_0 - 2\sqrt{2} f_8}{f_0 f_8 - b_0 b_8} \tag{70}
$$

This proves that $\overline{f}_8$ and $\overline{f}_0$ would coincide with the η/η' decay constants f_8 and f_0 , only if b_0 and b_8 were zero, *i.e.* if SU(3) were not broken ; in this case we would have $f_8 = x f_0 = f_\pi$. Using the expressions in Eq. (42) and truncating at leading order in $(z - 1)$ and $(x - 1)$, it is easy to check that we get the $\overline{f}_i$ as given in Eq. (35).

The expressions in Eq. (70) are interesting in this regard: they clearly show that the mismatch originates from the fact that b_0 and b_8 are non–zero when SU(3) symmetry is broken ($z \neq 1$), which is basically the point of Refs. [19,20]. Therefore, the usual expressions of Current Algebra do not directly give the isoscalar meson decay constants when SU(3) is broken.

Nevertheless, this does not prevent Current Algebra from being the most economic formulation for the study of radiative decays, as it involves only two parameters (θ_P and x) instead of four highly correlated parameters (see Eqs. (68)). Moreover, we shall see in

the next Section how x and θ_P are actually related by the observed smallness of θ_0. This means that the Current Algebra formulation, suitably used, depends (practically) on a single parameter; this can be chosen equivalently as either of x (or Λ_1) or θ_P. In this case, the two–photon decay amplitudes for η and η' become a constrained system (2 equations, 1 parameter)!

If one takes into account the smallness of θ_0 expected from ChPT and from VMD estimates (see Eq. (44)), Eq. (68) provides quite an interesting result

$$
\begin{aligned}
G_{\eta\gamma\gamma} + G_{\eta'\gamma\gamma}\tan\theta_8 &= \frac{\alpha}{\pi\sqrt{3}}\frac{1}{F^8\cos\theta_8} + \mathcal{O}(\theta_0), \\
G_{\eta'\gamma\gamma} &= \frac{\alpha}{\pi\sqrt{3}}\frac{2\sqrt{2}}{F^0} + \mathcal{O}(\theta_0),
\end{aligned}
\tag{71}
$$

together with $\theta_P = \theta_8/2 + \mathcal{O}(\theta_0)$.

B. Phenomenological Consequences

The consequences of the mismatch mentioned above are rather practical. Usually, the experimental determination of the pseudoscalar parameters (f_0, f_8, θ) is derived using the (quite standard) VMD/WZW Eqs. (32), which are nothing but the former equations of Current Algebra [13,14,49].

However, generally, the model reconstruction quality [38,51] is defined by comparing fit results for $\overline{f_8}$, $\overline{f_0}$ or θ_P with ChPT numerical expectations (F^8, F^0, θ_8). This leads to highly confusing situations as shown by the discussions in Refs. [31,50] when justifying the value found for the mixing angle ($\theta_P \simeq -13°$) which mainly differs from our because of their neglecting NSB in the PS sector. The problem is strictly the same with decay constants [31].

Sometimes, numerical ChPT expectations for F^8 are attributed to what has been named $\overline{f_8}$ in order to to constrain the WZW two–photon equations [13,14,51]; this mechanically (and artificially) pushes the mixing angle to $\simeq -20°$, as can be seen from Fig. 1 in [15].

What has been illustrated above is that such a phenomenological approach and such a theoretical treatment, are intrinsically inconsistent, as also noted by Ref. [21]. Stated otherwise, it is meaningless to compare (or constrain) the Current Algebra equations (given also by Eqs. (32)) using the PS decay constants of ChPT or the value of the ChPT angle θ_8, as traditionally done.

To be more specific, we have proved that $\overline{f_8} = (0.82 \pm 0.01)f_\pi$, $\overline{f_0} = (1.17 \pm 0.02)f_\pi$ and a (single) $\theta_P \simeq -10°$ which describes the η/η' mixing at the wave function level are consistently derived from VMD and/or the WZW Lagrangian after applying FSB (and NSB). Moreover, all this is perfectly consistent with all ChPT expectations at first order in breaking parameters: mainly $F^8 \simeq (1.25 - 1.35)f_\pi$ and $\theta_8 \simeq -20°$. VMD has been able to provide new information ($\theta_0 \simeq 0$ and $F^0 \simeq 1.3 f_\pi$, $\Lambda_1 \simeq 0.20$) of relevance for ChPT. A more refined comparison should wait until higher order ChPT estimates become available.

As a side remark, one should also recall, from Ref. [4], that SU(3) breaking does not affect the box anomalies for $\gamma\pi^+\pi^-\eta/\eta'$. It is clear, however, from Eq. (39) in this reference, that nonet symmetry breaking can play some (numerically) minor role. Thus, all existing

analyses [38,51] of the anomaly equations [49] have to be redone from scratch, at least for consistency, knowing that the expected parameter values are not (directly) the ChPT ones. Related to this point, it is clear now that the Chanowitz equations [49], correctly understood, do not point any longer to a failure of QCD, as incorrectly deduced in Ref. [38] because of the confusing angle problem mentioned above.

VIII. FEEDBACK FROM THE CHPT PARAMETRIZATION

The ChPT parameter values we have obtained (see Eq. (45)) allow for several remarks of importance which are to be discussed in this Section.

A. A Hidden Relation between x and θ_P/θ_8

At the level of accuracy permitted by the whole set of radiative decays of light mesons, the results gathered in Eq. (45) indicate that $\theta_0 = 0$ is well fulfilled experimentally. At its level of accuracy ($\theta_0 = -0.05° \pm 1°$), one can even ask oneself whether this relation is only approximate; this means that θ_0 does not undergo significant effects of SU(3) breaking, as opposed to θ_8 and θ_P. As remarked in Ref. [19], this also means that, in the sense that $|\eta\rangle$ is orthogonal to $J^0|0\rangle$, the η meson is practically pure octet. But as shown above, the same $|\eta\rangle$ happens also to be a mixture of $|\pi^8\rangle$ and $|\eta^0\rangle$ with an angle $\theta_P \simeq -10°$. This illustrates the duality of definitions from another point of view.

The numerical result $\theta_0 = 0$ indicates that the state mixing angle θ_P fulfills

$$\tan \theta_P = \sqrt{2} \frac{(1-z)}{(2+z)} \, x \tag{72}$$

to good accuracy; this calls for several important remarks.

- By providing a definite value for x, Eq. (72) allows to address the issue of a possible glue content inside the η' (see, for instance, Table II in Ref. [1] or Fig. 2 in Ref. [8]). Indeed, this equation is purely a consequence of η physics and Refs. [1,8] have shown that no glue was required inside the η meson. As a matter of consequence, Eq. (72) is not influenced by a possible glue content in the η' meson.

- Eq. (72) reveals an unexpected *algebraic* relation[8] between the mixing angles θ_P/θ_8 and the nonet symmetry breaking parameter x. It will be checked explicitly in Subsection VIII B using the whole set of radiative decays.

---

[8] As a matter of consequences, there is a numerical correlation (x, θ_P) in fit procedures, which has been completely missed in our study in Ref. [1]. The correlation coefficient is given in Table 1. This is smaller than what could have been expected, but the sign is consistent, if one remarks that θ_P and x carry opposite signs.

- Eq. (72) allows, for the first time, for a constrained fit to solely the $\eta \to \gamma\gamma$ and $\eta' \to \gamma\gamma$ partial widths. Indeed, symmetry breaking effects in terms of singlet and octet components are completely determined by the BKY procedure, as shown in Section V. Only x is still free. Now, Eq. (72) tells that θ_P is not an independent additional parameter and is fixed by the value x. So, the single undetermined parameter is either θ_P or x.

  We have performed the exercise and got $\theta_P = -10.34° \pm 0.22°$ (corresponding to $x = 0.903 \pm 0.017$); using the PDG value of the η rate – confirmed by the recent measurement of Ref. [48]. We get $\chi^2/dof = 0.8 \; 10^{-4}/1$ (a 99% probability), while the Primakov measurement gives $\chi^2/dof = 6.99/1$ (0.8% probability), and the $\gamma\gamma$ measurement gives $\chi^2/dof = 3.89/1$ (4.8% probability). So, the correlation is indeed observed; the reconstruction quality indicates however that the Primakoff measurement is affected by large systematics; this effect is present to a lesser extend in the measurement of the $\gamma\gamma$ Experiments. Therefore, a new accurate measurement of $\eta \to \gamma\gamma$ would be welcome.

To our knowledge, it is the first time such a relation as Eq. (72) is reported. If $z = 1$ (then all θ's vanish), x does not vanish but becomes unconstrained. Traditionally, the wave–function mixing angle is expressed in terms of PS meson masses ; our expression (72) tells that the mixing angle can also be expressed as a function of f_K/f_π with some influence of NSB.

From the point of view of ChPT, it should be stressed that θ_P is the variable which emphasizes the most the existence of nonet symmetry breaking; using the terminology of ChPT, it is directly proportional to $x = 1/\sqrt{1 + \Lambda_1}$, which exhibits the scale dependence of θ_P.

B. A New Global Fit to Radiative Decays

From Eq. (72), we have been led to conclude that x and θ_P are algebraically related, at least to a very good approximation. In order to check this in its full realm, we have redone the fits given in Ref. [1], requiring additionally this functional relation. This turns out to describe all radiative decays in terms of only 4 independent parameters (g, θ_P, θ_V, ℓ_T) or (g, x, θ_V, ℓ_T), which is, by far, the most constrained fit of the 14 radiative decay modes ever attempted[9]. We use the constant approximation for δ_V following the conclusion of our study [9].

The fit quality obtained when setting up the constraint is $\chi^2/dof = 9.14/11$, and does not exhibit any degradation compared to the fit quality reached when releasing this constraint

---

[9]As, now, we know that θ_P and θ_8 are functionally related, and that $\theta_8 \simeq -20°$ is equivalent to the favored [1] $\theta_P \simeq -10°$, we could, in principle, fix θ_8 (and then θ_P) to its ChPT expectation. We have not perform this exercise, as the accuracy on the ChPT estimate of θ_8 is still poor [34] and its sensitivity to NSB somewhat unclear. Nevertheless, it indicates that, from first principles, one can perform a fit to the 14 radiative decays, with remaining free fitting parameters referring only to vector mesons properties.

$\chi^2/dof = 9.13/10$. The difference, in this last case, with Ref. [1] is simply the use inside the fit of the PDG mean value for $\eta \to \gamma\gamma$ instead of its mean value from all experiments (including the Primakoff effect measurement). In all cases, we have used the so–called [1] K^* model, also commented on in Ref. [9]. Therefore, Eq. (72) is indeed intrinsically present in the full data set examined sofar.

Practically, the fit returns all parameters at the values obtained when leaving x and θ_P unrelated, even θ_P which changes from $\theta_P = -10.38° \pm 0.97°$ to $\theta_P = -10.32° \pm 0.20°$. The sharp reduction of the statistical error is an effect of removing the correlations by accounting explicitly for the functional relation in Eq. (72). The corresponding value for x is 0.901 (to be compared with the fit value $x = 0.902$ mentioned above). This numerical value has an important consequence, as will be seen in the next subsection. The fit branching fractions with x and θ_P left free or related are given in Table 2 altogether with the data recommended by the PDG [10], all used in the fit procedure.

When setting the $x - \theta_P$ relation, we do not observe any degradation in the quality of the description of the various branching fractions. Comparing the two sets of predictions in Table 2, one should note the sharp reduction of the statistical errors produced by having switched on the $x - \theta_P$ relation in all decay modes involving the η meson. This also affects the modes involving η', though to a lesser extent.

The increased accuracy we observe is not an artifact, but a trivial consequence of using independent (fit) variables instead of correlated ones. This explains why, even if it were theoretically better motivated, the use of correlated quantities like $F^0_{\eta/\eta'}$ and $F^8_{\eta/\eta'}$ is not recommended in numerical analyses, as this results in unaccurate uncertainties.

One should also note that recent measurements [52] of $\phi \to \eta'\gamma$ by the CMD–2 Collaboration at VEPP–2M, using new η' decay modes, confirm the central value of Ref. [53] rather than that of Ref. [54] reported by the PDG [10]; the agreement with the fit values we always get for this mode [1] is thus improved. Indeed, the new measurement ($[5.8 \pm 1.8] \times 10^{-4}$) reported by Ref. [52] (or by Ref. [53]) would provide a minimum χ^2 smaller than reported above by one unit; it has not been used in the fit in order to keep consistency with all 1998 recommended values [10].

C. The x–θ_P Relation Kills Glue in the η' Meson

In the previous study of Refs. [1,8] it was shown that the correlation between glue and nonet symmetry breaking was huge. As stated several times above, accounting generally for glue coupling to the η/η' system implies that two angles have to be introduced in addition to θ_P. One (β) is such that $\beta = 0$ implies that the η meson does not couple to glue, the other (γ) is such that $\gamma = 0$ implies a decoupling of the η' from glue.

However Table IV in Ref. [1] or Fig. 2 in Ref. [8], clearly show that: **i/** whatever the value of the nonet symmetry breaking parameter x, the angle β is not observed to deviate sensitively from zero; **ii/** additionally, for $x \simeq 0.9$ the angle γ is also consistent with zero.

We have seen above that it was indeed appropriate to parametrize PS NSB by x and given its most probable (fit) value 0.901. So, we can conclude that within the picture presented in this paper, there is no signal for a glue component inside the η and the η' mesons, but instead there is a significant signal for deviation from exact nonet symmetry: $x = 0.901$ relative to 1 reveals a 5σ significance level.

This is confirmed by performing the fit with γ and x, now (numerically) decorrelated because of the functional relation in Eq. (72). In this case, the fit quality is strictly unchanged $\chi^2/\mathrm{dof} = 9.14/10$ and the minimum is reached for $\gamma = -0.02° \pm 18°$; this shows that no glue component inside the η' is required by the data.

The conclusion would be quite different [1,8,17,18,39] if NSB could have been neglected. Indeed the level of correlation between standard nonet symmetry breaking (the parameter x) and glue is such that one can easily misidentify the former as being the later. So, even if our data set cannot exclude the existence of glue coupled to the η/η' system, it exclude presently its need.

D. θ_P and the Isoscalar Mass Matrix

In light of the above, the ChPT picture happens to be consistent with the standard one angle state mixing scheme of fields (or wave–functions):

$$
\begin{bmatrix} \eta \\ \eta' \end{bmatrix} = \begin{bmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \end{bmatrix} \simeq \begin{bmatrix} \cos\dfrac{\theta_8}{2} & -\sin\dfrac{\theta_8}{2} \\ \sin\dfrac{\theta_8}{2} & \cos\dfrac{\theta_8}{2} \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \end{bmatrix} \tag{73}
$$

The most accurate value for θ_P comes out from fit to all radiative decays with the $x - \theta_P$ correlation set up. Indeed, from the subsection just above, we know that it is legitimate to neglect coupling to glue. Actually, one cannot assert that glue (or any other singlet state) is not present inside the η/η' system, but what is shown in Refs. [1,8] for η, and just above for η', is that no glue contribution is required. A kind of minimum complexity argument, then leads to state $\beta = \gamma = 0$ and to decouple glue from the η/η' system. The angle value is $\theta_P = -10.32° \pm 0.20°$, a hardly constrained value.

With this, it is possible to revisit the determination of the isoscalar mass matrix [13,35,51]. Indeed we know that the mass matrix $\mathcal{M}$:

$$
\mathcal{M} = \begin{bmatrix} m_{88}^2 & m_{08}^2 \\ m_{08}^2 & m_{00}^2 \end{bmatrix} \tag{74}
$$

admits the following eigenvectors:

$$
\begin{cases} v_\eta = (\cos\theta_P , -\sin\theta_P) \\ v_{\eta'} = (\sin\theta_P , \cos\theta_P) \end{cases} \tag{75}
$$

with eigenvalues M_η^2 and $M_{\eta'}^2$. This gives $m_{88}^2 = 0.320 \pm 0.001$, $m_{00}^2 = 0.898 \pm 0.001$ and $m_{08}^2 = -0.109 \pm 0.004$ in units of GeV^2. This corresponds to $m_{00} = 0.948 \pm 0.001$ GeV, and $m_{88} = 0.566 \pm 0.001$ GeV, the former close to the η' mass and the latter close to the η mass as one might expect from the value of θ_P. The off–diagonal term can be written $m_{08}^2 = -0.45 M_K^2$. The solution favored by VMD phenomenology is a very small deviation from the classical Gell–Mann–Okubo formula [13,35,51]. More precisely, from:

$$M_{\pi^8}^2 = \frac{(4M_K^2 - M_\pi^2)}{3}\left[1 + \Delta\right] \tag{76}$$

(with $M_{\pi^8} \equiv m_{88}$) one extracts $\Delta \simeq 0.01$, where most of the error is due to choosing the mass values for K and π.

E. Broken VMD, ChPT and the Third Mixing Angle

We have shown that the HLS model, after applying FSB and NSB, yields the structure of the ChPT Lagrangian $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$. To be more specific, this concerns the PS kinetic energy part and the Λ_2 and L_8 terms [20,21] are not considered; they would influence the PS mass term which is outside the realm of this paper. Therefore, one can state that $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ is equivalent to the HLS Lagrangian broken as shown in Section III and the relationship exhibited in Section VI C between θ_8 and θ_P is not accidental.

Expressing the $\eta/\eta' \to \gamma\gamma$ coupling constants in terms of θ_P, we have seen that this is the mixing angle which traditionally parametrizes the Current Algebra expressions. It is clearly a natural parametrization of VMD instead of the two mixing angles θ_0 and θ_8. For this purpose, it is quite interesting to compare our Eqs. (32) with the corresponding Eqs.(39) in Ref. [21], expressed in terms of θ_0 and θ_8.

Therefore, as it is possible to provide θ_P a clear physical meaning through a mathematically well defined procedure, it is a quite legitimate choice. Its main virtue is to allow for a straigthforward handling of symmetry breaking effects in the VMD Lagrangians (HLS and FKTUY) as shown in Refs. [4,1]. The present paper has shown that the corresponding parametrization of VMD is easy to connect with the ChPT conceptual framework.

F. Nonet Symmetry Breaking and the Third Mixing Angle

Let us make a final (practical) remark on NSB. Ref. [1], and Ref. [38] before, clearly proved that nonet symmetry breaking plays numerically a major role in accounting for radiative decays of light mesons within a relatively simple and constrained framework. Even if small in absolute magnitude ($\delta \simeq -0.10$), the effect is statistically significant ($\simeq 5\sigma$). At the present level of experimental accuracy, the data description quality is sensitive to this improvement.

Quite interestingly, neglecting NSB in coupling expressions does not result in a dramatic change of the fitted mixing angle value, but mostly of the fit probabilities. Ref. [1] has thus obtained $\theta_P = -14° \pm 1°$ with, however, a degraded fit quality ($\chi^2/dof = 32/9$). Table 1 in Ref. [1] clearly show that all other physics parameters have unchanged fit values.

This kind of angle value has been obtained in several other approaches ; their common feature is their neglecting NSB in the pseudoscalar sector. Ref. [29] thus obtains $\theta_P = -18.2° \pm 1.4°$ in their fit to $VP\gamma$ processes and $\theta_P = -12.3° \pm 2.0°$ in their fit to the $P\gamma\gamma$ modes[10]. Refs. [30,31] within the bound state approach and dealing with 2–photon

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[10]One should note that, even if both fit qualities are *separately* good, the two angle values are

processes only also get a mixing angle of $\theta_P \simeq -12°$. The model of Ref. [33] finds a fit solution $\theta_P \simeq -15°$ (with no quoted fit quality) ; its freedom is however much larger than ours and the relation between their breaking scheme and ours – which now is motivated by ChPT concepts – is unclear.

Instead, in our approach, the best solutions to separately $VP\gamma$ and $P\gamma\gamma$ processes co-incide and both correspond to $\theta_P = -10.3° \pm 0.2°$. Additionally, we should remind that, in our approach, all $P\gamma\gamma$ couplings depend at leading order only on f_K/f_π and θ_P or the NSB parameter x. From this point of view, we consider that the result reported in [27] (a prefered $\theta_P \simeq -10.2°$) gives our approach a strong support from lattice QCD estimates.

It should thus be stressed that NSB in the pseudoscalar sector is exhibited, not so much by sharp changes in numerical values of physics parameters, but rather by their improved fit probabilities and, therefore, their actual accuracy. Indeed, as clear from this Section, and the previous ones, it was shown that the main ChPT parameters have values which are not sharply sensitive to having $x \neq 1$.

This is also the reason why the numerical results of Ref. [31] are so close of ours, even if these authors neglect NSB within the limited data set they consider. To be more specific, effects of the small value of δ (or of Λ_1) are competing with SU(3) breaking, always by modifying non–leading corrections; for instance, the magnitude of the correction terms to $\theta_0 - \theta_8$ (see Eq. (54), for instance).

IX. CONCLUSION

In a previous work, we were faced with a paradoxical problem. Using the HLS model and its anomalous FKTUY sector, together with a definite breaking scheme, it is possible to achieve quite a satisfactory description of all radiative decays, including $\eta/\eta' \to \gamma\gamma$. Within this framework, it was moreover possible to predict accurately these last rates, using only numerical information obtained by fitting the $VP\gamma$ processes in isolation. This quite satisfactory pattern was obscured by some strange results: the (single) pseudoscalar mixing angle was found at $\theta_P \simeq -10°$ and the octet decay constant was $f_8 = 0.82 f_\pi$, both in obvious disagreement with ChPT expectations. This meets independent analyses of other authors like Kekez, Klabučar, Scadron, Bramon ; it received recently a clear support of lattice QCD computations of the UKQCD group.

The origin of this disagreement has been investigated. Starting from a broken VMD based Lagrangian, we have shown how to deduce the (WZW) η/η' two–photon amplitudes on the one hand, and the expectation values $\langle 0|J^{0,8}|\eta/\eta'\rangle$ on the other hand (J's are the axial currents), which gives the customary (ChPT) definition of decay constants.

The disagreement reported above has been traced back to inconsistent definitions for the same parameters provided by ChPT and the WZW Lagrangian (through the former definitions of Current Algebra). This inconsistency is a pure consequence of breaking SU(3) symmetry. For instance, it was shown that none of the two angles of extended ChPT can

different enough that one can guess that the fit quality of a *global* description would be certainly degraded and providing $\theta_P \simeq -15°$.

appear *as such* in the two–photon decay amplitudes. In some way, besides the angles θ_8 and θ_0 recently introduced by Kaiser and Leutwyler, the standard angle θ_P, which still describes the η/η' wave–function mixing, goes on playing an important role, the main one in radiative decays. It has been shown that $\theta_P \simeq \theta_8/2$, within a ChPT motivated Lagrangian framework, perfectly accounting for all data on radiative decays and all accessible ChPT expectations. Moreover, analysis of numerical correlations has illustrated why the use of θ_P is more appropriate in order to get *accurate* measurements of physics parameters, including the standard ChPT ones.

We thus have clearly illustrated that, the HLS–FKTUY model, supplemented with SU(3) flavor breaking à la BKY and nonet symmetry breaking, as was introduced in Ref. [1] was equivalent to the ChPT Lagrangian $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$. Besides the successful description of all radiative decays, broken VMD thus meets all the requirements of extended ChPT (the two mixing angles and the two decay constants) with good accuracy. It was then shown that the relevant (and self–consistent) angle pattern is $\theta_0 \simeq 0°$, $\theta_8 \simeq -20°$ and $\theta_P \simeq -10°$. Correspondingly, we have simultaneously $\overline{f_8} = 0.82f_\pi$ when using the WZW (or Current Algebra) definition and $f_8 = 1.33f_\pi$, when using standard ChPT definition. Subsequently, we have shown that the $\overline{f}_i$ cannot be interpreted as isoscalar meson decay constants because of flavor SU(3) breakdown. Therefore, VMD phenomenology is indeed able to provide ChPT with quite reliable input. With this respect, an important feedback from ChPT would be a more precise estimate of θ_0 and/or a (reliable) theoretical error.

This study has led us to several additional conclusions:

i/ Because $\theta_0 = 0$ is observed with a quite impressive precision, it has been possible to relate nonet symmetry breaking (the parameter x) and the mixing angle θ_P. To our knowledge such a relation has never been reported.

ii/ The nonet symmetry breaking parameter λ which weights the additional singlet contribution to the Lagrangian is small ($\lambda \simeq 0.20 \pm 0.04$). It coincides with the usual OZI breaking parameter Λ_1 of ChPT.

iii/ As consequence of $x = 0.901$, it has been shown, that no glue component is needed inside the η' meson.

iv/ By relating x and θ_P, the condition $\theta_0 = 0$ allows to account for observed correlations in fitting radiative decays. Additionally, this leads us to propose a 4–parameter model to account for all data on radiative decays (including $K^{*\pm} \to K^\pm \gamma$); this is by far the most constrained model ever proposed and we proved that it is quite successful.

v/ The quark mass ratio deduced from VMD information is $m_s/\hat{m} = 21.2 \pm 2.4$.

vi/ Departure from the classical Gell–Mann–Okubo quadratic mass relation is observed at only the percent level.

Process	x and θ_P Related	x and θ_P Unrelated	PDG
$\rho \to \pi^0\gamma$ ($\times 10^4$)	5.16 ± 0.03	5.16 ± 0.03	6.8 ± 1.7
$\rho \to \pi^\pm\gamma$ ($\times 10^4$)	5.12 ± 0.03	5.12 ± 0.03	4.5 ± 0.5
$\rho \to \eta\gamma$ ($\times 10^4$)	3.16 ± 0.05	3.19 ± 0.10	$2.4^{+0.8}_{-0.9}$
$\eta' \to \rho\gamma$ ($\times 10^2$)	33.3 ± 1.26	34.5 ± 2.1	30.2 ± 1.3
$K^{*\pm} \to K^\pm\gamma$ ($\times 10^4$)	9.80 ± 0.94	9.80 ± 0.93	9.9 ± 0.9
$K^{*0} \to K^0\gamma$ ($\times 10^3$)	2.32 ± 0.02	2.32 ± 0.02	2.3 ± 0.2
$\omega \to \pi^0\gamma$ ($\times 10^2$)	8.49 ± 0.05	8.49 ± 0.05	8.5 ± 0.5
$\omega \to \eta\gamma$ ($\times 10^4$)	7.81 ± 0.11	7.88 ± 0.23	6.5 ± 1.0
$\eta' \to \omega\gamma$ ($\times 10^2$)	2.83 ± 0.11	2.94 ± 0.19	3.01 ± 0.30
$\phi \to \pi^0\gamma$ ($\times 10^3$)	1.28 ± 0.12	1.27 ± 0.12	1.31 ± 0.13
$\phi \to \eta\gamma$ ($\times 10^2$)	1.28 ± 0.02	1.27 ± 0.04	1.26 ± 0.06
$\phi \to \eta'\gamma$ ($\times 10^4$)	0.59 ± 0.02	0.60 ± 0.03	$1.2^{+0.7}_{-0.5}$
$\eta \to \gamma\gamma$ ($\times 10^2$)	38.87 ± 0.75	39.3 ± 1.8	39.21 ± 0.34
$\eta' \to \gamma\gamma$ ($\times 10^2$)	2.09 ± 0.08	2.17 ± 0.10	2.11 ± 0.13

Table 2 : Radiative decay branching fractions. The first two data columns display the fit results using the K^* model of Refs. [1,8]; in the first data column (present work) the x–θ_P of Eq. (72) is switched on while, in the second one, it is not. The data for $\eta \to \gamma\gamma$ is the recommended value [10]. The last data column displays the accepted values from the 1998 Review of Particle Properties [10].

APPENDIX A: THE ANOMALOUS DECAY TERM

We shall now briefly discuss the equivalence of the anomalous decay amplitude $T(P \to \gamma\gamma)$ as calculated either from the divergence of the axial current, or the WZW Lagrangian [46]. Let us recall the form of the axial current given in Eq.(19). We then take the divergence of this to obtain[11]

$$\partial^\mu J_\mu^a = 2f\text{Tr}[T^a X_A T^b X_A]\partial^2 P^b + \lambda f \delta^{a0}\partial^2 P^0. \tag{A1}$$

Now let us turn to the equations of motion for the pseudoscalar fields, namely,

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu P)} - \frac{\partial\mathcal{L}}{\partial P} = 0, \tag{A2}$$

leading to (allowing for a pseudoscalar mass term)

$$2\text{Tr}[T^a X_A T^b X_A]\partial^2 P^b + \lambda\delta^{a0}\partial^2 P^0 = m_P^2 P^a - \frac{C}{f}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\text{Tr}[Q^2 T^a], \tag{A3}$$

where C is a well known dimensionless constant [26].

We are now in a position to show that the amplitude obtained form the axial current is equivalent to that obtained from the anomalous Lagrangian. First let us consider $\langle AA|\partial^\mu J_\mu^a|0\rangle$. As a total divergence this vanishes, in accordance with the fact that there are no truly massless particles in the spectrum (we are not considering the chiral limit). So using this along with Eqs. (A1) and (A3) we have

$$fm_P^2\langle AA|P^a|0\rangle = C\langle AA|F_{\mu\nu}\tilde{F}^{\mu\nu}\text{Tr}[Q^2 T^a]|0\rangle. \tag{A4}$$

As $q \to 0$ $\langle X|P|0\rangle = \langle X|P\rangle$, hence, as m_P^2 is absorbed in the definition of the amplitude, we have

$$T(P^a \to AA) = \frac{C}{f}\langle AA|F_{\mu\nu}\tilde{F}^{\mu\nu}\text{Tr}[Q^2 T^a]|0\rangle. \tag{A5}$$

We see that the result we have obtained, starting with the axial current, is the same as one would obtain from the anomalous Lagrangian term. This equivalence extends to VVP interactions.

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[11] In this Section we use for conciseness the notation $P = \sum_{a=0,\cdots 8} P^a T^a$; thus for instance $P^0 = \eta_0$.

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