Voltage Stability of Electric Power Systems

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Thierry Van Cutsem • Costas Vournas

Voltage Stability of Electric Power Systems

Springer

Thierry Van Cutsem
University of Liège
Belgium

Costas Vournas
National Technical University
Athens, Greece

Series Editors:
M.A. Pai
Department of Electrical and
Computer Engineering
University of Illinois at Urbana Champaign
Urbana, IL 61801
USA

A.M. Stankovic
Department of Electrical and
Computer Engineering
Northeastern University
Boston, MA 02115
USA

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FOREWORD

Angle stability had been the primary concern of the utilities for many decades. In the 80's due to the declining investments in new generation and transmission facilities, the system became stressed, resulting in a new phenomena hitherto largely ignored, namely voltage stability. Thus the role of reactive power in maintaining proper voltage profile in the system began receiving attention. Several instances of voltage collapse around the world only heightened the interest in this topic. Initially, treated as a static concept, the importance of dynamics of the machines, exciters, tap changers as well as dynamics of the load were found to affect voltage stability significantly.

This monograph addresses all these issues in depth from a rigorous analytical perspective as well as practical insight. Besides being a useful resource for engineers in industry, it will serve as a starting point for new researchers in this field. In the evolving scenario of a restructured power industry, the issues of voltage stability will be more complex and challenging to solve.

Profs. T. Van Cutsem and C. Vournas have worked in this research area extensively and have also offered short courses on this topic in the past.

I have great pleasure in welcoming this monograph in our power electronics and power system series.

M. A. Pai
University of Illinois
Urbana, IL

PREFACE

The idea of writing this book has probably its root in the enjoyable discussions we had in the Spring of 1994 during a sabbatical visit in Liège. Two years later, a couple of common papers and the encouragement of Prof. Pai provided the motivation for embarking on this adventure.

We undertook it courageously, in the middle of other obligations and with the help of Internet to bridge the 2500 km that separate our cities. It has been a year-long, difficult but enriching journey and for us it will remain memorable.

And so it will for our families ! Our very first thanks go to them: Marie-Paule, François, Nicolas, and Olivier on the one part, Malvina on the other. Their patience in coping with us and our absences during this period is heartily recognized.

We are grateful to Prof. Pai, the series editor, not only for his kind encouragement, but also for helping us in reviewing the text and doing his best to improve our English.

Sincere thanks are also due to our colleagues and graduate students, in particular to Dr. Patricia Rousseaux, at the University of Liège, George Manos and Basil Nomikos, at NTUA, for carefully reading our draft and suggesting corrections and improvements.

Finally, the valuable technical support of George Efthivoulidis, at NTUA, as well as his help in improving LaTeXstyles, are thankfully acknowledged.

PART I

COMPONENTS AND PHENOMENA

1

INTRODUCTION

"Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte"[1]
Blaise Pascal

1.1 WHY ANOTHER BOOK?

There was a time when power systems, and in particular transmission systems could afford to be overdesigned. However, in the last two decades power systems have been operated under much more stressed conditions than was usual in the past. There is a number of factors responsible for this: environmental pressures on transmission expansion, increased electricity consumption in heavy load areas (where it is not feasible or economical to install new generating plants), new system loading patterns due to the opening up of the electricity market, etc. It seems as though the development brought about by the increased use of electricity is raising new barriers to power system expansion.

Under these stressed conditions a power system can exhibit a new type of unstable behaviour characterized by slow (or sudden) voltage drops, sometimes escalating to the form of a collapse. A number of such voltage instability incidents have been experienced around the world. Many of them are described in [Tay94]. As a consequence, voltage stability has become a major concern in power system planning and operation.

As expected, the power engineering community has responded to the new phenomenon and significant research efforts have been devoted to developing new analysis tools

[1] (speaking of a letter) I made this one longer, only because I had not enough time to make it shorter

and controlling this type of instability. Among the early references dealing with the subject are textbooks on power system analysis devoting a section to voltage stability [ZR69, Wee79, Mil82] as well as technical papers [WC68, Nag75, Lac78, BB80, TMI83, BCR84, Cal86, KG86, Cla87, CTF87, Con91]. A series of three seminars on this specific topic [Fin88, Fin91, Fin94] has provided a forum for the presentation of research advances. Several CIGRE Task Forces [CTF93, CTF94a, CTF94b, CWG98] and IEEE Working Group reports [IWG90, IWG93, IWG96] have offered a compilation of techniques for analyzing and counteracting voltage instability. More recently, a monograph [Tay94] as well as one chapter of a textbook [Kun94] have been devoted to this topic.

One important aspect of the voltage stability problem, making its understanding and solution more difficult, is that the phenomena involved are truly nonlinear. As the stress on the system increases, this nonlinearity becomes more and more pronounced. This makes it necessary to look for a new theoretical approach using notions of nonlinear system theory [Hil95].

In this general framework the objective of our book is twofold:

- formulate a unified and coherent approach to the voltage stability problem, consistent with other areas of power system dynamics, and based on analytical concepts from nonlinear systems theory;
- use this approach in describing methods that can be, or have been, applied to solve practical voltage stability problems.

To achieve these two goals, we rely on a variety of power system examples. We start from simple two-bus systems, on which we illustrate the essence of the theory. We proceed with a slightly more complex system that is detailed enough to capture the main voltage phenomena, while still allowing analytical derivations. We end up with simulation examples from a real-life system.

1.2 VOLTAGE STABILITY

Let us now address a fundamental question: *what is voltage stability ?*

Convenient definitions have been given by IEEE and CIGRE Working Groups, for which the reader is referred to the previously mentioned reports. However, at this early point we would like to define voltage *instability* within the perspective adopted throughout this book:

> *Voltage instability stems from the attempt of load dynamics to restore power consumption beyond the capability of the combined transmission and generation system.*

Let us follow this descriptive definition word by word:

- *Voltage*: as already stated, the phenomenon is manifested in the form of large, uncontrollable voltage drops at a number of network buses. Thus the term "voltage" has been universally accepted for its description.
- *Instability*: having crossed the maximum deliverable power limit, the mechanism of load power restoration becomes unstable, reducing instead of increasing the power consumed. This mechanism is the heart of voltage instability.
- *Dynamics*: any stability problem involves dynamics. These can be modelled with either differential equations (continuous dynamics), or with difference equations (discrete dynamics). We will refer later to the misconception of labeling voltage stability a "static" problem.
- *Loads* are the driving force of voltage instability, and for this reason this phenomenon has also been called *load instability*. Note, however, that loads are not the only players in this game.
- *Transmission* systems have a limited capability for power transfer, as is well known from circuit theory. This limit (as affected also by the generation system) marks the onset of voltage instability.
- *Generation*: generators are not ideal voltage sources. Their accurate modelling (including controllers) is important for correctly assessing voltage stability.

One term also used in conjunction with voltage stability problems is *voltage collapse*. In this book we use the term "collapse" to signify a sudden catastrophic transition that is usually due to an instability occurring in a faster time-scale than the one considered. As we will see, voltage collapse may, or may not be the final outcome of voltage instability.

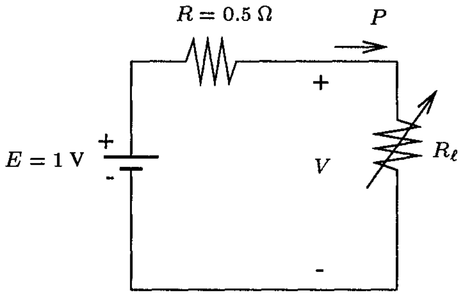

Figure 1.1 DC system

On the rôle of reactive power

The reader may have noticed that we did not include in the above definition of voltage instability the important concept of *reactive power.* It is a well-known fact that in AC systems dominated by reactances (as power systems typically are) there is a close link between voltage control and reactive power. However, by not referring to reactive power in our definition, we intend not to overemphasize its rôle in voltage stability, where *both* active and reactive power share the leading rôle.

The decoupling between active power and phase angles on the one hand, and reactive power and voltage magnitudes on the other hand, applies to normal operating conditions and cannot be extended to the extreme loading conditions typical of voltage instability scenarios.

The following example illustrates that there is no "cause and effect" relationship between reactive power and voltage instability. Consider the system of Fig. 1.1 made up of a DC voltage source E feeding through a line resistance R a variable load resistance R_ℓ.

We assume that R_ℓ is automatically varied by a control device, so as to achieve a power consumption setpoint P_o. For instance it could be governed by the following ordinary differential equation:

$$\dot{R}_\ell = I^2 R_\ell - P_o \tag{1.1}$$

It is well known that the maximum power that can be transferred to the load corresponds to the condition $R_\ell = R$ and is given by:

$$P_{max} = \frac{E^2}{4R} \tag{1.2}$$

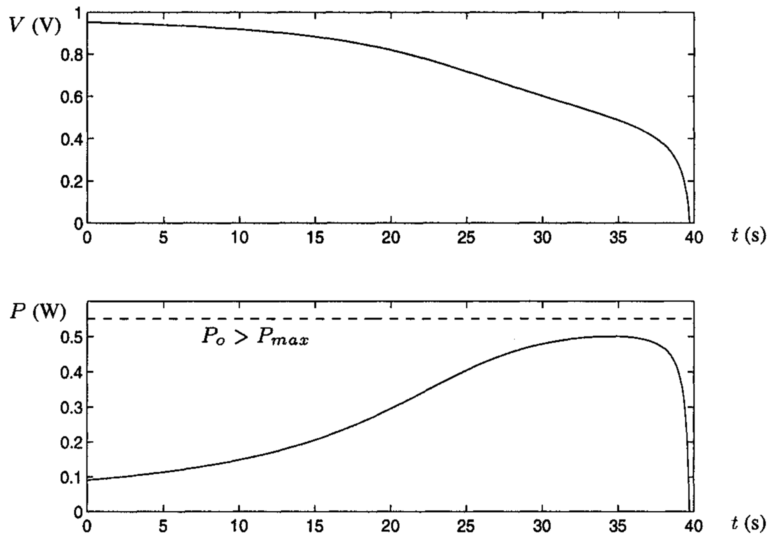

Figure 1.2 Voltage instability in a DC system

If the demand P_o is made larger than P_{max} the load resistance will decrease below R and voltage instability will result after crossing the maximum power point. A typical simulation for this case is shown in Fig. 1.2.

This simple paradigm has the major characteristics of voltage instability, although it does not involve reactive power. In actual AC power systems, reactive power makes the picture much more complicated but it is certainly not the only source of the problem.

1.3 POWER SYSTEM STABILITY CLASSIFICATION

We now place voltage stability within the context of power system stability in general. Table 1.1 shows a classification scheme based upon two criteria: time scale and driving force of instability.

The first power system stability problems encountered were related to generator rotor angle stability, either in the form of undamped electromechanical oscillations, or in the form of monotonic rotor acceleration leading to the loss of synchronism. The former type of instability is due to a lack of *damping* torque, and the latter to a lack of *synchronizing* torque.

Table 1.1 Power System Stability Classification

Time scale	Generator-driven	Load-driven
short-term	rotor angle stability transient / steady-state	short-term voltage stability
long-term	frequency stability	long-term voltage stability

The first type of instability is present even for small disturbances and is thus called *steady-state* or *small-signal* stability. The second one is initiated by large disturbances and is called *transient* or *large-disturbance* stability. For the analysis of steady-state stability it is sufficient to consider the linearized version of the system around an operating point, typically using eigenvalue and eigenvector techniques. For transient stability one has to assess the performance of the system for a set of specified disturbances.

The time frame of rotor angle stability is that of electromechanical dynamics, lasting typically for a few seconds. Automatic voltage regulators, excitation systems, turbine and governor dynamics all act within this time frame. The relevant dynamics have been called *transient dynamics* in accordance with transient stability, generator transient reactances, etc. However, this may create misinterpretations, since "transient" is also used in "transient stability" to distinguish it from "steady-state stability", which also belongs to the same time frame. For this reason we prefer to refer to the above time frame of a few seconds as the *short-term time scale.*

When the above mentioned short-term dynamics are stable they eventually die out some time after a disturbance, and the system enters a slower time frame. Various dynamic components are present in this time frame, such as transformer tap changers, generator limiters, boilers, etc. The relevant transients last typically for several minutes. We will call this the *long-term time scale.*

In the long-term time scale we can distinguish between two types of stability problems:

1. *frequency* problems due to generation–load imbalance irrespective of network aspects within each connected area;
2. *voltage* problems, which are due to the electrical distance between generation and loads and thus depend on the network structure.

In modern power systems, frequency stability problems can be encountered after a major disturbance has resulted in islanding. Since we have assumed that the electromechanical oscillations have died out, frequency is common throughout each island and the problem can be analyzed using a single-bus equivalent, on which all generators and loads are connected. The frequency instability is related to the active power imbalance between generators and loads in each island[2].

Voltage stability, on the other hand, requires a full network representation for its analysis. This is a main aspect separating the two classes of long-term stability problems. Moreover, as suggested by the definition we gave in Section 1.2, voltage instability is load driven.

Now, when referring to voltage stability we can identify dynamic load components with the tendency to restore their consumed power in the time-frame of a second, i.e. in the short-term time scale. Such components are mainly induction motors and electronically controlled loads, including HVDC interconnections. We have thus to introduce a *short-term voltage stability* class alongside generator rotor angle stability. Since these two classes of stability problems belong to the same time scale, they require basically the same complexity of component models and sometimes distinction between the two in meshed systems becomes difficult [VSP96]. In other words, in the short-term time scale, there is not a clear-cut separation between load-driven and generator-driven stability problems, as there is as between frequency and long-term voltage stability.

It should be noted that the identification of the driving force for an instability mechanism in Table 1.1 does not exclude the other components from affecting this mechanism. For instance, load modelling does affect rotor angle stability, and, as we will show in this book, generator modelling is important for a correct voltage stability assessment.

Each of the four major stability classes of Table 1.1 may have its own further subdivisions, like the ones we have already seen in the case of generator rotor angle stability. We can thus identify small-signal and large-disturbance forms of voltage stability. Note, however, that this distinction is not as important as in the case of rotor angle stability, where transient and steady-state stability relate to different problems. Thus, although the small-signal versus large-disturbance terminology exists and is in accordance with the above stability classification we will not use it extensively in this book. We see voltage stability as a single problem on which one can apply a combination of both linearized and nonlinear tools.

---

[2] Note that the counterpart of frequency stability in the short-term time scale is rotor angle stability, since in this time scale there is no common frequency

Another point to be made here deals with the distinction between dynamic and "static" aspects. In fact, long-term voltage stability has been many times misunderstood as a "static" problem. The misconception stems from the fact that static tools (such as modified power flow programs) are acceptable for simpler and faster analysis. Voltage stability, however, is dynamic by nature, and in some cases one has to resort to dynamic analysis tools (such as time-domain methods). One should thus avoid to confuse means with ends in stability classification.

1.4 STRUCTURE OF THIS BOOK

The book consists of two parts.

Part I deals with phenomena and components. It includes Chapters 2, 3, and 4, each dealing with one of the three major aspects of the voltage stability problem according to our definition of Section 1.2.

We start with transmission aspects in *Chapter 2*, because it is the limits on power transfer that set up the voltage stability problem. In this chapter we review the problem of maximum deliverable power in AC systems and concentrate on a number of transmission components that are linked to voltage stability, such as compensation, off-nominal tap transformers, etc.

Chapter 3 reviews the basics of generator modelling, including significant details, such as the effect of saturation on capability limits. Frequency and voltage controls are also reviewed, as well as the various limiting devices that protect generators from overloading. We finally consider how generator limits affect the maximum deliverable power of the system.

In *Chapter 4* we focus on the driving force of voltage instability, i.e. load dynamics. We first give a general framework of load restoration and then we proceed with the analysis of three major components of load restoration, namely induction motors, load tap-changers and thermostatic load. Finally we discuss aggregate generic load models.

Part II of the book deals with the description of voltage instability mechanisms and analysis methods.

We first provide in *Chapter 5* a summary of the mathematical background from nonlinear system theory necessary for the analysis of later chapters. This includes the notions of bifurcation, singularity, and time-scale decomposition.

In *Chapter 6* we discuss general modelling requirements for voltage stability analysis, and illustrate them using a simple but fully detailed example.

Chapter 7 gives the basic voltage stability theory in terms of three closely linked concepts: loadability limits, bifurcations, and sensitivities. For the most part, this chapter deals with smooth parameter changes. The effect of discontinuities, especially those caused by the overexcitation limiters of synchronous generators is explicitly taken into account.

In *Chapter 8* we concentrate on large, abrupt disturbances and describe one by one the possible mechanisms of losing stability, whether in the long-term, or the short-term time scale. We also concentrate on countermeasures applicable to each type of instability. The detailed example introduced in Chapter 6 is used to illustrate some of the key instability mechanisms.

Finally, in *Chapter 9* we give a representative sample of criteria and computer methods for voltage stability analysis. After a brief review of security concepts, we consider methods for contingency evaluation, loadability limit computation and determination of secure operation limits. We end up with examples from a real-life system.

At the end of some chapters we provide problems. Some of them are straightforward applications of the presented methods. Other problems refer to the examples and test cases given in the text. Finally, some are at the level of research topics. The authors would be pleased to receive suggestions and exchange views on all these.

1.5 NOTATION

We give below a short list of notation conventions used in this book.

- Phasors are shown as capital letters with an overline, e.g. $\bar{I}, \bar{V}$.
- Phasor magnitudes are shown by the same capital letter without the overline, e.g. I, V.
- Lowercase bold letters, e.g. $\mathbf{x}, \mathbf{y}$, correspond to column vectors. Superscript ^{T} denotes transpose. Therefore row vectors are written as $\mathbf{x}^T, \mathbf{y}^T$.
- A collection of phasors in a column vector is represented as a capital bold letter with an overline, e.g. $\bar{\mathbf{I}}$.
- Matrices are normally shown as bold capital letters, e.g. $\mathbf{A}, \mathbf{J}$.
- Jacobian matrices are shown as a bold letter (indicating the vector function) with a bold subscript (indicating the vector with respect to which we differentiate). Thus:

$$\mathbf{f_x} = \left[\frac{\partial f_i}{\partial x_j}\right]$$

- Time derivatives appear with a dot, e.g. $\dot{x}$.

2

TRANSMISSION SYSTEM ASPECTS

"Maybe I can't define stability, but I know it when I see it !"[1]
Carson W. Taylor

In this chapter we analyze the rôle played by the transmission system in voltage stability.

We first deal with two basic notions: the maximum power that can be delivered to loads and the relationship between load power and network voltage. Then we briefly and qualitatively explain how these two basic properties may result in voltage instability. Next, we discuss the effect of components that affect the transmission capability, series and shunt compensation on one hand, transformers with adjustable tap ratio on the other hand. We also introduce the notion of VQ curves that express the relationship between voltage and reactive power at a given bus.

Most of the material of this chapter is based on the analysis of a simple single-load infinite-bus system, which allows easy analytical derivations and provides insight into the problem. Basic concepts introduced in this chapter will be generalized in later chapters to large system of arbitrary complexity.

2.1 SINGLE-LOAD INFINITE-BUS SYSTEM

We consider the simple system of Fig. 2.1, which consists of one load fed by an infinite bus through a transmission line. By definition, the voltage magnitude and frequency

[1] Panel Session presentation at the 1997 IEEE/PES Winter Power Meeting

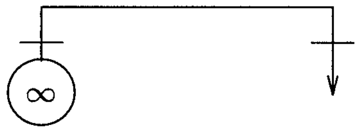

Figure 2.1 Single-load infinite-bus system

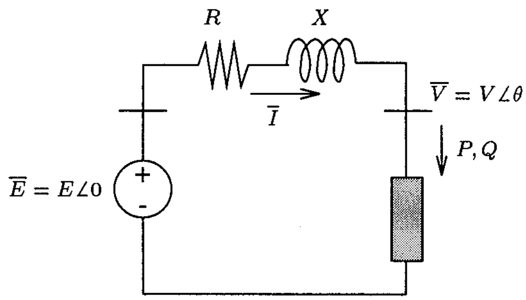

Figure 2.2 Circuit representation

are constant at the infinite bus. We assume balanced 3-phase operating conditions, so that the per phase representation is sufficient. We also consider steady-state sinusoidal operating conditions, characterized by phasors and complex numbers. The phase reference is arbitrary and need not be specified at this stage.

This leads to the circuit representation of Fig. 2.2. The infinite bus is represented by an ideal voltage source E. The transmission line is represented by its series resistance R and reactance X, as given by the classical pi-equivalent. The line shunt capacitance is neglected for simplicity (the effects of shunt capacitors are considered later in Section 2.6.2). The transmission impedance is:

$$Z = R + jX$$

Alternatively, we may think of E and Z as the Thévenin equivalent of a power system as seen from one bus. Note that, because power generators are not pure voltage sources, the Thévenin emf somewhat varies as more and more power is drawn from the system; we will however neglect this variation in a first approximation and consider a constant emf E as mentioned previously.

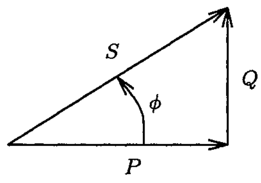

Figure 2.3 Definition of angle ϕ

Finally, let us recall that the *load power factor* is given by:

$$\mathrm{PF} = \frac{P}{S} = \frac{P}{\sqrt{P^2+Q^2}} = \cos\phi$$

where P, Q and S are the active, reactive and apparent powers and ϕ is the angle defined in Fig. 2.3.

2.2 MAXIMUM DELIVERABLE POWER

As pointed out in the Introduction, voltage instability results from the attempt of loads to draw more power than can be delivered by the transmission and generation system. In this section we focus on determining the maximum power that can be obtained at the receiving end of the simple system of Fig. 2.2, under various constraints.

2.2.1 Unconstrained maximum power

For the sake of simplicity we start by assuming that the load behaves as an impedance. In fact we will show later on that this choice does not affect the results. We denote the load impedance by:

$$Z_\ell = R_\ell + jX_\ell$$

where R_ℓ and X_ℓ are the load resistance and reactance, respectively.

We first revisit a classical derivation of circuit theory known as the load adaptation problem [CDK87] or maximum power transfer theorem: assuming that both R_ℓ and X_ℓ are free to vary, find the values which maximize the *active* power consumed by the load.

The current $\bar{I}$ in Fig. 2.2 is given by:

$$\bar{I} = \frac{\bar{E}}{(R + R_\ell) + j(X + X_\ell)}$$

and the active power consumed by the load:

$$P = R_\ell I^2 = \frac{R_\ell E^2}{(R + R_\ell)^2 + (X + X_\ell)^2} \tag{2.1}$$

Maximizing P over the two variables R_ℓ and X_ℓ, the necessary extremum conditions are:

$$\begin{aligned} \frac{\partial P}{\partial R_\ell} &= 0 \\ \frac{\partial P}{\partial X_\ell} &= 0 \end{aligned}$$

which after some calculations yields:

$$\begin{aligned} (R + R_\ell)^2 + (X + X_\ell)^2 - 2R_\ell(R + R_\ell) &= 0 \\ -R_\ell(X + X_\ell) &= 0 \end{aligned}$$

The solution to these equations, under the constraint $R_\ell > 0$, is unique:

$$R_\ell = R \tag{2.2a}$$

$$X_\ell = -X \tag{2.2b}$$

or in complex form:

$$Z_\ell = Z^*$$

One easily checks that this solution corresponds to a maximum of P. In other words:

> load power is maximized when the load impedance is the complex conjugate of the transmission impedance.

Under the maximum power conditions, the impedance seen by the voltage source is $R + R_\ell + jX + jX_\ell = 2R$, i.e. it is purely resistive and the source does not produce any reactive power. The corresponding load power is:

$$P_{max} = \frac{E^2}{4R} \tag{2.3}$$

and the receiving-end voltage:

$$V_{maxP} = \frac{E}{2}$$

where the subscript $maxP$ denotes a value under maximum active power condition.

The unconstrained case is not well suited for power system applications. The first problem is that in a transmission system the resistance R can be negligible compared to the reactance X. Now, making R tend to zero, the optimal load resistance (2.2a) also goes to zero, while the maximum power (2.3) goes to infinity. The two results might seem in contradiction: however, as R and R_ℓ go to zero, the current I goes to infinity (since $X + X_\ell = 0$) and so does the power $R_\ell I^2$! This is obviously unrealistic.

Even when taking into account the nonzero transmission resistance R, the above result is not directly applicable to power systems. Indeed, a highly capacitive load would be required to match the dominantly inductive nature of the system impedance. A modified derivation, closer to power system applications is made by assuming that the power factor of the load is specified. This case is dealt with in the next subsection.

2.2.2 Maximum power under a given load power factor

Specifying the load power factor $\cos\phi$ is equivalent to having a load impedance of the form:

$$Z_\ell = R_\ell + jX_\ell = R_\ell + jR_\ell \tan\phi$$

which now leaves R_ℓ as the single degree of freedom for maximizing the load power.

The current $\bar{I}$ is now given by:

$$\bar{I} = \frac{\bar{E}}{(R + R_\ell) + j(X + R_\ell \tan\phi)}$$

and the load active power by:

$$P = R_\ell I^2 = \frac{R_\ell E^2}{(R + R_\ell)^2 + (X + R_\ell \tan\phi)^2} \tag{2.4}$$

The extremum condition is:

$$\frac{\partial P}{\partial R_\ell} = 0$$

or, after some calculations:

$$(R^2 + X^2) - R_\ell^2(1 + \tan^2\phi) = 0 \tag{2.5}$$

which is equivalent to:

$$|Z_\ell| = |Z|$$

The second derivative is given by:

$$\frac{\partial^2 P}{\partial R_\ell^2} = -2R_\ell(1+\tan^2\phi)$$

which is always negative, thereby indicating that the solution is a maximum. In other words:

> under constant power factor, load power is maximized when the load impedance becomes equal in magnitude to the transmission impedance.

The optimal load resistance and reactance are thus given by:

$$\begin{aligned} R_{\ell maxP} &= |Z|\cos\phi \\ X_{\ell maxP} &= |Z|\sin\phi = R_{\ell maxP}\tan\phi \end{aligned}$$

As an illustration, Fig. 2.4 shows the load power P, the voltage V and the current magnitude I as a function of R_ℓ. An infinite R_ℓ corresponds to open-circuit conditions. As R_ℓ decreases, V drops while I increases. As long as R_ℓ remains larger than $R_{\ell maxP}$, the increase in I^2 gains over the decrease in R_ℓ and hence P increases. When R_ℓ becomes smaller than $R_{\ell maxP}$ the reverse holds true. Finally, $R_\ell = 0$ corresponds to short-circuit conditions.

Lossless transmission

Let us come back to the case where $R = 0$. The optimal load resistance under constant power factor is, according to (2.5):

$$R_{\ell maxP} = X\cos\phi$$

Substituting in (2.4) yields the maximum active power:

$$P_{max} = \frac{\cos\phi}{1+\sin\phi}\frac{E^2}{2X} \tag{2.6}$$

with the corresponding reactive power:

$$Q_{maxP} = \frac{\sin\phi}{1+\sin\phi}\frac{E^2}{2X} \tag{2.7}$$

and receiving-end voltage:

$$V_{maxP} = \frac{E}{\sqrt{2}\sqrt{1+\sin\phi}} \tag{2.8}$$

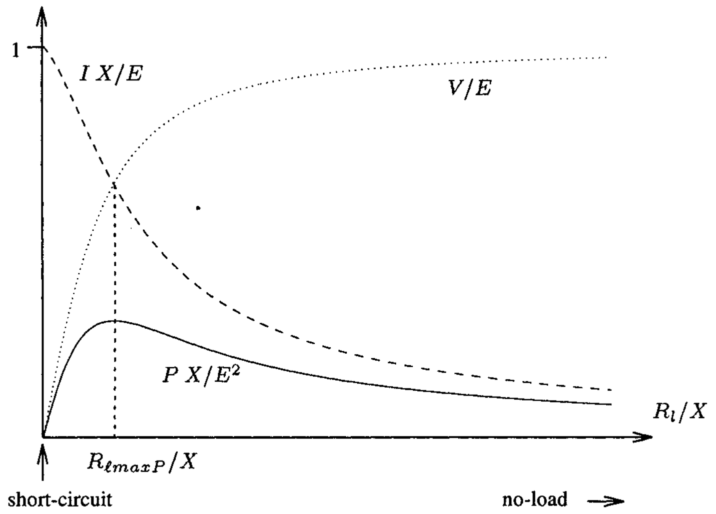

Figure 2.4 P, V and I as a function of R_ℓ, for a lossless system ($R = 0$) and under constant power factor ($\tan\phi = 0.2$)

Lossless transmission and unity power factor

If we assume furthermore that the load is perfectly compensated, so that $\cos\phi = 1$, the optimal resistance, maximum power and receiving-end voltage become respectively:

$$\begin{aligned} R_{\ell maxP} &= X \\ P_{max} &= \frac{E^2}{2X} \\ V_{maxP} &= \frac{E}{\sqrt{2}} \simeq 0.707E \end{aligned}$$

Extensions to multiport systems

Some generalizations of the above results to multiport systems are given in [Cal83]. Let a multiport circuit be characterized by

$$\bar{\mathbf{V}} = \bar{\mathbf{E}} + \mathbf{Z}\bar{\mathbf{I}}$$

where $\bar{\mathbf{V}}$ is the vector of terminal voltages, $\bar{\mathbf{E}}$ the vector of open-circuit voltages, $\bar{\mathbf{I}}$ the vector of injected currents and $\mathbf{Z}$ the (short-circuit) impedance matrix.

If the circuit is purely reactive, and characterized by $\mathbf{Z} = j\mathbf{X}$, it can be shown that the total active power delivered is maximized when a purely *resistive* network with impedance matrix $\mathbf{Z}_\ell = \mathbf{X}$ is connected to the multiport. The corresponding maximum power is easily obtained.

Furthermore if all the elements of the multiport matrix $\mathbf{Z}$ have the same argument ζ and all the elements of the loading matrix $\mathbf{Z}_\ell$ the same argument ϕ, i.e.

$$\mathbf{Z} = \mathbf{N}e^{j\,\zeta} \qquad \text{and} \qquad \mathbf{Z}_\ell = \mathbf{L}e^{j\,\phi}$$

the total active power is maximum when $\mathbf{N} = \mathbf{L}$.

Note that the individual load powers are not constrained with respect to each other in this derivation. If a pattern of load increase is specified, the maximum power delivered will be smaller. These aspects will be further discussed in Chapters 7 and 9.

Remark on load characteristics

Note that the maximum deliverable power given by either (2.3) or (2.6) depends only on the network parameters (R, X) and is independent of the load characteristic which was assumed to be that of an impedance for simplicity. This will be verified in the sequel, where no assumption will be made as to the nature of the load. For this purpose we now adopt a formulation in terms of powers.

2.2.3 Maximum power derived from load flow equations

For the sake of simplicity, we neglect the transmission resistance R (see Fig. 2.2). We also take the ideal voltage source as the phase reference by setting $\bar{E} = E\angle 0$. We denote the load voltage magnitude and phase angle by V and θ respectively.

One easily obtains from Fig. 2.2:

$$\bar{V} = \bar{E} - jX\bar{I}$$

The complex power *absorbed* by the load is:

$$\begin{aligned} S &= P + jQ = \bar{V}\,\bar{I}^\star = \bar{V}\,\frac{\bar{E}^\star - \bar{V}^\star}{-jX} \\ &= \frac{j}{X}(EV\cos\theta + jEV\sin\theta - V^2) \end{aligned} \tag{2.9}$$

which decomposes into:

$$P = -\frac{EV}{X}\sin\theta \tag{2.10a}$$

$$Q = -\frac{V^2}{X} + \frac{EV}{X}\cos\theta \tag{2.10b}$$

Equations (2.10a,b) are the *power flow* or *load flow* equations of the lossless system. For a given load (P, Q), they have to be solved with respect to V and θ, from which all other variables can be computed. Let us determine for which values of (P, Q) there is one solution.

Eliminating θ from (2.10a,b) gives:

$$(V^2)^2 + (2QX - E^2)V^2 + X^2(P^2 + Q^2) = 0 \tag{2.11}$$

This is a second-order equation with respect to V^2. The condition to have at least one solution is:

$$(2QX - E^2)^2 - 4X^2(P^2 + Q^2) \geq 0$$

which can be simplified into:

$$-P^2 - \frac{E^2}{X}Q + (\frac{E^2}{2X})^2 \geq 0 \tag{2.12}$$

The equality in (2.12) corresponds to a parabola in the (P, Q) plane, as shown in Fig. 2.5. All points "inside" this parabola satisfy (2.12) and thus lead to two load flow solutions. Outside there is no solution while on the parabola there is a single solution.

This parabola is the locus of all maximum power points. Points with negative P correspond to a maximum generation while each point with positive P corresponds to the maximum load under a given power factor, as derived in the previous section.

The locus is symmetric with respect to the Q-axis (i.e. with respect to changing P into $-P$). In other words, the maximum power that can be injected at the load end is exactly equal to the maximum power that can be absorbed. However, this symmetry disappears if one takes into account the line resistance.

Setting $P = 0$ in (2.12) one obtains:

$$Q \leq \frac{E^2}{4X}$$

Noting that E^2/X is the short-circuit power at the load bus, i.e. the product of the no-load voltage E by the short-circuit current E/X, the maximum of purely reactive load is one fourth of the short-circuit power.

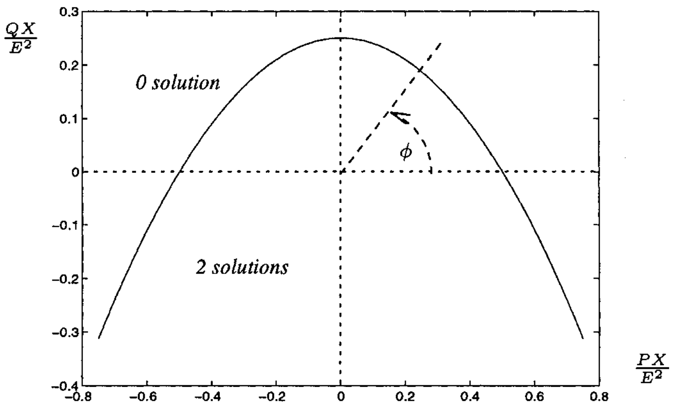

Figure 2.5 Domain of existence of a load flow solution

Similarly, by setting $Q = 0$ in (2.12) one gets:

$$P \leq \frac{E^2}{2X}$$

which is the same power limit we derived for a lossless line with unity power factor, and corresponds to half the short-circuit power.

As can be seen, there is a fundamental difference between the active and reactive powers: any active power can be consumed provided that enough reactive power is injected at the load bus ($Q < 0$), while the reactive load power can never exceed $E^2/4X$. This difference comes from the inductive nature of the transmission system and further illustrates the difficulty of transporting large amounts of reactive power. Note that in practice the large reactive support that is required for large active power will finally result in unacceptably high load bus voltage.

2.3 POWER-VOLTAGE RELATIONSHIPS

Assuming that condition (2.12) holds, the two solutions of (2.11) are given by:

$$V = \sqrt{\frac{E^2}{2} - QX \pm \sqrt{\frac{E^4}{4} - X^2P^2 - XE^2Q}} \qquad (2.13)$$