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VOLUME GROWTH AND POSITIVE SCALAR CURVATURE

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1. INTRODUCTION

Gromov conjectures that a uniformly contractible complete Riemannian manifold can not have uniformly positive scalar curvature (recall that a Riemannian manifold M is called uniformly contractible if for every $r > 0$, there exists $R > r$ such that every ball with radius r can be contracted to a point in the ball with radius R). This is a special case of a more general principle stating that a macroscopically large Riemannian manifold can not be microscopically small [6]. The purpose of this paper is to prove a version of the above principle when the volume of the Riemannian manifold grows subexponentially. In particular, we prove Gromov's conjecture that a uniformly contractible complete Riemannian manifold can not have uniformly positive scalar curvature when the Riemannian manifold has subexponential volume growth.

Our result on positive scalar curvature is obtained by computing the higher index of the Dirac operator when the Riemannian manifold has subexponential volume growth. The computation of higher index follows from our proof of the coarse Baum–Connes conjecture for spaces with subexponential growth. A local version of Gromov's notion of uniform embedding into Hilbert space [7], called local uniform embedding into Hilbert space, plays a key role in the proof. The main technical result used in the proof is the construction of a local uniform embedding of discrete metric spaces with subexponential growth into Hilbert space (cf. Section 4). Such a construction is based on certain probabilistic ideas. A finitely generated group with subexponential growth (with respect to a word length

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metric) is amenable and therefore admits a uniform embedding into Hilbert space by the Bekka-Cherix-Vallete Theorem [2]. However, in general it is not clear if a metric space with subexponential growth has property A due to the lack of homogeneity of the space (recall that property A is a weak amenability condition which guarantees uniform embedding into Hilbert space [19]).

Our result on the coarse Baum–Connes conjecture also implies the zero-in-the-spectrum conjecture stating that the spectrum of the Laplacian acting on the space of L^2-forms on a uniformly contractible Riemannian manifold contains zero if the Riemannian manifold has subexponential volume growth (cf. [13] for an excellent survey on the zero-in-the-spectrum conjecture).

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2. Positive Scalar Curvature and Macroscopically Large Riemannian Manifold

In this section we shall formulate a version of the general principle that a macroscopically large Riemannian manifold can not be microscopically small.

To motivate our discussion, we recall the following formula in Riemannian geometry:

$$\frac{\text{Volume}\, B_r(M,p)}{\text{Volume}\, B_r(\mathbb{R}^n,0)} = 1 - \frac{k(p)}{6(n+2)}r^2 + o(r^2),$$

where M is a Riemannian manifold of dimension n, $p \in M$, $B_r(M,p)$ is the open ball with radius r and center p, $\mathbb{R}^n$ is given the standard Euclidean metric, and $k(p)$ is the scalar curvature of M at p. This formula implies that if a Riemannian manifold has positive scalar curvature, then the volume of small balls is smaller than the volume of Euclidean balls of the same radius, i.e. the Riemannian manifold is microscopically small.

To formulate a notion of macroscopically large Riemannian manifold, we need to recall Roe's coarse homology theory [15].

Let Γ be a discrete metric space. Assume that Γ is locally finite in the sense that every ball in Γ has finitely many elements.

Definition 2.1. For each $d \geq 0$, the Rips complex $P_d(\Gamma)$ is the simplicial polyhedron where the set of vertices is Γ, and a finite subset $\{x_0, x_1, \ldots, x_n\} \subseteq \Gamma$ spans a simplex iff $d(x_i, x_j) \leq d$ for all $0 \leq i, j \leq n$.

Definition 2.2 [15]. For each non-negative integer i, the coarse homology $HX_i(\Gamma)$ is defined to be $\lim_{d\to\infty} H_i^{\ell f}(P_d(\Gamma))$, where $H_i^{\ell f}(P_d(\Gamma))$ is the locally finite simplicial homology of $P_d(\Gamma)$.

Given a Riemannian manifold M, let Γ be a net in M, i.e. a discrete subspace $\Gamma \subseteq M$ such that there exist $\delta > 0$, $c > 0$ for which $d(x, y) \geq \delta$ for all $x, y \in \Gamma$ satisfying $x \neq y$, and $d(p, \Gamma) \leq c$ for all $p \in M$. The coarse homology $HX_i(M)$ is defined to be $HX_i(\Gamma)$. It is not difficult to see that $HX_i(M)$ is independent of the choice of Γ.

If a complete Riemannian manifold M has bounded geometry, (i.e. M has bounded sectional curvature and positive injectivity radius), then there exists a triangulation of M such that the set of diameters of all simplices in the triangulation is bounded and there exists $\delta > 0$ for which $d(x, y) \geq \delta$ if x and y are distinct vertices of the triangulation. For technical convenience, we choose the set of all vertices of the triangulation, denoted by Γ, to be the net for M in the definition of $HX_i(M)$ throughout this paper. There exists $d_0 > 0$ such that $M \subseteq P_{d_0}(\Gamma)$. Let $j : M \to P_{d_0}(\Gamma)$, be the inclusion map. j induces a homomorphism $j_* : H_i^{\ell f}(M) \to HX_i(M)$, where $H_i^{\ell f}(M)$ is the locally finite simplicial homology of M with respect to the triangulation.

Definition 2.3. Let M be a complete Riemannian oriented manifold with bounded geometry, let $[M]$ be the fundamental class in $H_n^{\ell f}(M)$ (n is the dimension of M). M is said to be macroscopically large if $j_*([M]) \neq 0$ in $HX_n(M)$.

Note that a uniformly contractible Riemannian manifold with bounded geometry is macroscopically large. However, $X \times S^2$ is not macroscopically large if X is a complete Riemannian oriented manifold with bounded geometry, S^2 is given a positive scalar multiple of the standard metric, and $X \times S^2$ is given the product Riemannian metric (note that $X \times S^2$ has uniformly positive scalar curvature if the positive scalar is chosen to be sufficiently small).

We remark that Gromov introduced several concepts of large Riemannian manifold in [6] (the concept of uniformly contractible Riemannian manifold is one of them). Our concept of a macroscopically large Riemannian manifold is closely related to Gromov's concepts.

Conjecture 2.4. *Let M be a complete Riemannian oriented manifold with bounded geometry. If M is macroscopically large, then M can not have uniformly positive scalar curvature.*

The above conjecture implies Gromov's conjecture that a uniformly contractible complete Riemannian manifold can not have uniformly positive scalar curvature. It should be pointed out that Conjecture 2.4 is a consequence of another conjecture of Gromov stating that if a complete Riemannian manifold M has uniformly positive scalar curvature, then the macroscopic dimension of M is at most $n-2$, where n is the dimension of M (cf. [5] for more details about the conjecture of Gromov).

3. The Coarse Baum–Connes Conjecture and Local Uniform Embedding into Hilbert Space

In this section, we briefly recall the coarse Baum–Connes conjecture and its application to positive scalar curvature. We shall also introduce the concept of local uniform embedding into Hilbert space and show that the coarse Baum–Connes conjecture holds for spaces admitting a local uniform embedding into Hilbert space.

Let Γ be a locally finite discrete metric space, let H be a separable and infinite dimensional Hilbert space. Define $H_\Gamma = \ell^2(\Gamma) \otimes H$. We have $H_\Gamma = \bigoplus_{x\in\Gamma}(\delta_x \otimes H)$, where δ_x is the Dirac function at x. For each bounded linear operator $T : H_\Gamma \to H_\Gamma$, there exists a matrix representation of T corresponding to the above decomposition: $T = (T_{x,y})_{x,y\in\Gamma}$, where $T_{x,y} : \delta_y \otimes H \to \delta_x \otimes H$, is a bounded linear operator for all $x, y \in \Gamma$.

A bounded linear operator $T : H_\Gamma \to H_\Gamma$ is called locally compact if $T_{x,y}$ is compact for every pair $x, y \in \Gamma$. T is said to have finite propagation if there exist r such that $T_{x,y} = 0$ whenever $d(x,y) \geq r$.

Definition 3.1 ([15]). The Roe algebra $C^*(\Gamma)$ is defined to be the C^*-algebra generated by all locally compact bounded linear operators acting on H_Γ with finite propagation.

If M is a complete Riemannian manifold and Γ is a net in M, then the K-theory of $C^*(\Gamma)$ is the receptacle of higher index of elliptic differential operators on M [15]. Higher index is an obstruction to invertibility of the elliptic differential operator (cf. Proposition 4.33 in [15]). In particular, by the Lichnerowicz formula, a complete Riemannian manifold can not have uniformly positive scalar curvature if the higher index of the Dirac operator is non-zero (cf. [16] [15]).

The following conjecture provides a way of computing higher index of elliptic differential operators.

Conjecture 3.2 (The Coarse Baum–Connes Conjecture). *Let Γ be a discrete metric space with bounded geometry. The (higher) index map ind*: $\lim_{d\to\infty} K_*(P_d(\Gamma)) \to K_*(C^*(\Gamma))$ *is an isomorphism, where $K_*(P_d(\Gamma)) = KK_*(C_0(P_d(\Gamma)), \mathbb{C})$ is the locally finite K-homology group for $P_d(\Gamma)$.*

Recall that a discrete metric space Γ is said to have bounded geometry if for each $r > 0$, there exists $N(r)$ such that

$$|B_r(x)| \leq N(r) \quad \text{for all} \quad x \in \Gamma,$$

where $B_r(x) = \{y \in \Gamma, d(y,x) < r\}$ and $|B_r(x)|$ is the number of elements in $B_r(x)$. The coarse Baum-Connes conjecture is false if the bounded geometry condition is dropped [20].

The following concept is a local version of Gromov's notion of uniform embedding into Hilbert space [7].

Definition 3.3. Let X be a metric space, let H be a separable and infinite dimensional Hilbert space. X is said to admit a local uniform embedding into H if there exist two non-decreasing functions ρ_1 and ρ_2 on $[0,\infty)$ satisfying $\lim_{r\to\infty} \rho_i(r) = +\infty$ $(i = 1, 2)$ such that for every finite subset $F \subseteq X$, there exists a map $f : F \to H$ satisfying

$$\rho_1(d(x,y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x,y))$$

for all $x, y \in F$.

The main result of this section is the following:

Theorem 3.4. *Let Γ be a discrete metric space with bounded geometry. If Γ admits a local uniform embedding into Hilbert space, then the coarse Baum–Connes conjecture holds for Γ.*

Theorem 3.4 is a refinement of the following result.

Theorem 3.5 ([19]). *Let Γ be a discrete metric space with bounded geometry. If Γ admits a uniform embedding into Hilbert space, then the coarse Baum–Connes conjecture holds for Γ.*

Recall that a map f from Γ to a separable and infinite dimensional Hilbert space H is said to be a uniform embedding if there exist two non-decreasing functions ρ_1 and ρ_2 on $\mathbb{R}_+ = [0, +\infty)$ such that $\lim_{r\to+\infty} \rho_i(r) = +\infty$ and

$$\rho_1(d(x,y)) \leq \|f(x) - f(y)| \leq \rho_2(d(x,y))$$

for all $x, y \in \Gamma$, [7].

Lemma 3.6. *Let Γ be a discrete metric space, let Γ' be a metric subspace of Γ such that $\Gamma' = \cup_{i=1}^{\infty} F_i$ for which each F_i is finite, and there exists a sequence of positive numbers $\{c_k\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty} c_k = \infty$ and $d(F_i, F_j) \geq c_{i+j}$ for all $i \neq j$ If Γ admits a local uniform embedding into Hilbert space, then Γ' admits a uniform embedding into Hilbert space.*

Proof. Let H be a separable and infinite dimensional Hilbert space. By assumption there exist non-decreasing functions ρ_1 and ρ_2 on $\mathbb{R}_+$ satisfying $\lim_{r\to\infty} \rho_i(r) = \infty$ $(i = 1, 2)$ such that, for each i, there exists a map $f_i : F_i \to H$, satisfying $\rho_1(d(x,y)) \leq ||f_i(x) - f_i(y)|| \leq \rho_2(d(x,y))$ for all $x, y \in F_i$. Using translations, we can require that $d(f_i(F_i), f_j(F_j)) \geq c_{i+j}$ for all $i \neq j$. Define a map $f : \Gamma' \to H$, by $f|_{F_i} = f_i$ for all i. It is not difficult to verify that $f : \Gamma' \to H$, is a uniform embedding. QED

Proof of Theorem 3.4. Fix $x_0 \in \Gamma$. Let

$$\Gamma_1 = \bigcup_{n=0}^{\infty} \{x \in \Gamma, (2n)^2 \leq d(x, x_0) \leq (2n+1)^2\},$$

$$\Gamma_2 = \bigcup_{n=0}^{\infty} \{x \in \Gamma, (2n+1)^2 \leq d(x, x_0) \leq (2n+2)^2\}.$$

For each $r \geq 0$, $Y \subseteq \Gamma$, let

$$B_r(Y) = \{x \in \Gamma : d(x, Y) < r\}.$$

Let $KX_i(Y) = \lim_{d\to\infty} K_*(P_d(Y))$.

We have the following commuting diagram:

$$\begin{array}{ccccc}
\longrightarrow \lim\limits_{r\to\infty} KX_i(B_r(\Gamma_1) \cap B_r(\Gamma_2)) & \longrightarrow & \lim\limits_{r\to\infty} (KX_i(B_r(\Gamma_1)) \oplus KX_i(B_r(\Gamma_2))) & \longrightarrow & KX_i(\Gamma) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\longrightarrow \lim\limits_{r\to\infty} K_i(C^*(B_r(\Gamma_1) \cap B_r(\Gamma_2))) & \longrightarrow & \lim\limits_{r\to\infty} (K_i(C^*(B_r(\Gamma_1))) \oplus K_i(C^*(B_r(\Gamma_2)))) & \longrightarrow & K_i(C^*(\Gamma))
\end{array}$$

where the horizontal sequences are exact by [11], the vertical maps are index maps.

By the assumption of Theorem 3.4 and Lemma 3.6, $B_r(\Gamma_1) \cap B_r(\Gamma_2)$, $B_r(\Gamma_1)$, and $B_r(\Gamma_2)$ admit a uniform embedding into Hilbert space. by Theorem 3.5, the coarse Baum–Connes conjecture holds for $B_r(\Gamma_1) \cap B_r(\Gamma_2)$, $B_r(\Gamma_1)$ and $B_r(\Gamma_2)$. This, together with the above commuting diagram and the five Lemma, implies Theorem 3.4. QED

4. Local Uniform Embedding of Spaces with Subexponential Growth into Hilbert Space

In this section, we prove that a discrete metric space with subexponential growth admits a local uniform embedding into Hilbert space.

The following elementary probability result plays an important role in the proof of the main result of this section.

Lemma 4.1. *There exists $c > 0$ such that for every finite set X, every pair of natural numbers K and L satisfying $\frac{|X|}{20} \leq K \cdot L \leq \frac{|X|}{10}$, every pair of subsets $Y \subseteq X$, $Z \subseteq X$ satisfying $|Y| \leq 2K$, $|Z| \geq K$, and $Y \cap Z = \emptyset$, we have*

$$\frac{|\{A : A \subseteq X, |A| = L, A \cap Y = \emptyset, A \cap Z \neq \emptyset\}|}{|\{A : A \subseteq X, |A| = L\}|} > c,$$

where for each finite set F, $|F|$ is the number of elements in F.

Proof. Without loss of generality, we can assume that $|Y| = 2K$, $|Z| = K$. Let $K \cdot L = \frac{N}{a}$ for some a satisfying $10 \leq a \leq 20$, where $N = |X|$. We have

$$\begin{aligned}
&\frac{|\{A : A \subseteq X, A \cap Y = \emptyset, |A| = L\}|}{|\{A : A \subseteq X, |A| = L\}|} \\
&= \left(1 - \frac{2K}{N}\right)\left(1 - \frac{2K}{N-1}\right)\cdots\left(1 - \frac{2K}{N-(L-1)}\right) \\
&\geq \left(1 - \frac{2K}{N-L}\right)^L \geq \left(1 - \frac{2K}{(1-\frac{1}{a})N}\right)^{\frac{N}{aK}} \\
&= \left(\left(1 - \frac{2K}{(1-\frac{1}{a})N}\right)^{\frac{(1-\frac{1}{a})N}{2K}}\right)^{\frac{2}{a-1}} \\
&\geq e^{-\frac{2}{a-1}}.
\end{aligned} \tag{1}$$

Similarly, we have

$$\frac{|\{A : A \subseteq X, A \cap Y = A \cap Z = \emptyset, |A| = L\}|}{|\{A : A \subseteq X, |A| = L\}|}$$

$$= \left(1 - \frac{3K}{N}\right)\left(1 - \frac{3K}{N-1}\right)\cdots\left(1 - \frac{3K}{N-(L-1)}\right)$$

(2)
$$\leq \left(1 - \frac{3K}{N}\right)^{L} = \left(\left(1 - \frac{3K}{N}\right)^{(\frac{N}{3K}+1)}\right)^{\frac{3N}{(N+3K)a}}$$

$$\leq e^{-\frac{3N}{(N+3K)a}} \leq e^{-\frac{3}{a+3}}.$$

Set $c = \min\{e^{-\frac{2}{a-1}} - e^{-\frac{3}{a+3}} : 10 \leq a \leq 20\}$. Combining (1) with (2), we obtain our desired inequality. QED

Lemma 4.2. *If $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, is a non-decreasing positive function satisfying $\lim_{t\to\infty} \alpha(t) = +\infty$, then there exists a sequence $0 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$ such that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and $\sum_{n=1}^{\infty} \frac{\alpha(\alpha(0)n)}{a_n}$ diverges, where $\mathbb{R}_+ = [0, \infty)$.*

Proof. Choose a sequence of non-negative integers $\{P_n\}_{n=1}^{\infty}$ satisfying $P_n - P_{n-1} > 10P_{n-1}$ and $\alpha(\alpha(0)P_n) > 2^{2n}$ for all $n \geq 2$, $P_1 = 0$. Define $a_k = 2^n(P_{n+1} - P_n)$ if $P_n < k \leq P_{n+1}$. It is easy to verify that $\{a_k\}_{k=1}^{\infty}$ satisfies the desired properties. QED

Lemma 4.3. *Let α be a non-decreasing positive function on $\mathbb{R}_+ = [0, \infty)$. If $\sup_{x\in\Gamma} |B_r(x)| \leq 2^{\frac{r}{\alpha(r)}}$ and $|B_r(x)| \geq 2^n$ for some $r \geq 0$ and some non-negative integer n, then $r \geq \alpha(\alpha(0)n)n$.*

Proof. We have $2^{\frac{r}{\alpha(r)}} \geq 2^n$. Hence $\frac{r}{\alpha(r)} \geq n$. It follows that $r \geq \alpha(r)n \geq \alpha(0)n$, where the second inequality is derived using the non-decreasing property of α. Again by the non-decreasing property of α, we obtain $r \geq \alpha(\alpha(0)n)n$. QED

Definition 4.4. A discrete metric space Γ is said to have subexponential growth if

$$\lim_{r\to+\infty} \frac{\ln(\sup_{x\in X} |B_r(x)|)}{r} = 0,$$

where $B_r(x) = \{y \in \Gamma : d(y, x) < r\}$.

Theorem 4.5. *Let H be a separable and infinite dimensional Hilbert space. If Γ is a discrete metric space with subexponential growth, then Γ admits a local uniform embedding into H.*

Proof. The subexponential growth condition of Γ implies that there exists a non-decreasing positive function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to\infty} \alpha(r) = +\infty$ and $|B_r(x)| \leq 2^{\frac{r}{\alpha(r)}}$ for all $r \geq 0$.

Without loss of generality, we can assume that Γ is an infinite set.

Fix $x_0 \in \Gamma$. Given a finite subset $F \subseteq \Gamma$, there exists $r_0 > 0$ such that $F \subseteq B_{r_0}(x_0)$. Let r_1 be a positive number such that $|B_{r_1}(x_0)| \geq 10^3 |B_{2r_0}(x_0)|$.

Let $N = |B_{r_1}(x_0)|$, let n be a positive integer such that

$$\frac{N}{20} \leq 2^n \leq \frac{N}{10}.$$

We have $n \geq 5$. For any non-negative integer p satisfying $0 \leq p \leq n$, define

$$\ell_p = |\{A \subseteq B_{r_1}(x_0), |A| = 2^p\}|.$$

Let $2^{B_{r_1}(x_0)}$ be the set of all subsets of $B_{r_1}(x_0)$.

We define $f : F \to \ell^2(2^{B_{r_1}(x_0)})$ by

$$(f(x))(A) = \begin{cases} \dfrac{d(x, A)}{\sqrt{\ell_p a_{n-p+1}}} & \text{if} \quad A \subset 2^{B_{r_1}(x_0)}, |A| = 2^p \ ' \text{ for some } \quad 1 \leq p \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

where a_{n-p+1} is as in Lemma 4.2.

Let

$$\rho_2(r) = \left(\sum_{i=1}^{\infty} \frac{1}{a_i} \right) r.$$

It is straightforward to verify

$$\|f(x) - f(y)\|_{\ell^2(2^{B_{r_1}(x_0)})} \leq \rho_2(d(x, y))$$

for all $x, y \in F$.

For any $0 \leq k \leq n$, we define

$$c_k(x,y) = \inf\{r > 0, |B_r(x)| \geq 2^k, |B_r(y)| \geq 2^k\}$$

for all $x, y \in F$. We have $c_0(x,y) = 0$ for all $x, y \in F$.

Assume that $x \neq y$. Let k_0 be the maximal integer k such that $c_k(x,y) < \frac{d(x,y)}{2}$. We have either

$$2^{k_0} \leq |B_{\frac{d(x,y)}{2}}(x)|,$$

or

$$2^{k_0} \leq |B_{\frac{d(x,y)}{2}}(y)|.$$

Without loss of generality, we assume that

$$2^{k_0} \leq |B_{\frac{d(x,y)}{2}}(x)|.$$

By the choice of r_1, we have $k_0 \leq n - 5$.

Claim 1. Let $n - k_0 + 1 \leq p \leq n$. We have

$$\sum_{A \subseteq B_{r_1}(x_0), |A| = 2^p} |(f(x))(A) - (f(y))(A)|^2 \geq c \frac{(c_{n-p+1}(x,y) - c_{n-p}(x,y))^2}{a_{n-p+1}},$$

where c is as in Lemma 4.1, x and y are distinct elements in F.

Proof of Claim 1. For $x, y \in F$, let

$$b_{x,k} = \lim_{r \to c_k(x,y)^-} |B_r(x)|,$$

$$b_{y,k} = \lim_{r \to c_k(x,y)^-} |B_r(y)|.$$

We have either $b_{x,k} \leq 2^k$ or $b_{y,k} \leq 2^k$ for each k. Without loss of generality, we can assume that $b_{x,n-p+1} \leq 2^{n-p+1}$.

For any $\varepsilon > 0$, by Lemma 4.1, we have

$$\frac{|\{A : A \subseteq B_{r_1}(x_0), |A| = 2^p, A \cap B_{c_{n-p+1}(x,y)-\varepsilon}(x) = \emptyset, A \cap B_{c_{n-p}(x,y)+\varepsilon}(y) \neq \emptyset\}|}{|\{A : A \subseteq B_{r_1}(x_0), |A| = 2^p\}|} > c$$

since

$$|B_{c_{n-p+1}(x,y)-\varepsilon}(x)| \leq 2^{n-p+1},$$

$$|B_{c_{n-p}(x,y)+\varepsilon}(y)| \geq 2^{n-p},$$

$$\frac{N}{20} \leq 2^{n-p} \cdot 2^p \leq \frac{N}{10},$$

$$B_{c_{n-p+1}(x,y)-\varepsilon}(x) \cap B_{c_{n-p}(x,y)+\varepsilon}(y) = \emptyset.$$

Note that if $A \cap B_{c_{n-p+1}(x,y)-\varepsilon}(x) = \emptyset$ and $A \cap B_{c_{n-p}(x,y)+\varepsilon}(y) \neq \emptyset$, then

$$|d(x, A) - d(y, A)| \geq c_{n-p+1}(x, y) - c_{n-p}(x, y) - 2\varepsilon.$$

Hence we have

$$\frac{|\{A : A \subseteq B_{r_1}(x_0), |A| = 2^p, |d(x, A) - d(y, A)| \geq c_{n-p+1}(x, y) - c_{n-p}(x, y) - 2\varepsilon\}|}{|\{A : A \subseteq B_{r_1}(x_0), |A| = 2^p\}|} > c$$

for any $\varepsilon > 0$. Now our claim follows from the above inequality.

Claim 2.

$$\sum_{A \subseteq B_r(x_0), |A| = 2^{n-k_0}} |(f(x))(A) - (f(y))(A)|^2 \geq c \frac{(\frac{d(x,y)}{2} - c_{k_0}(x, y))^2}{a_{k_0+1}},$$

where c is as in Lemma 4.1, x and y are distinct elements in F.

Proof of Claim 2. By definition of k_0, we have

$$c_{k_0+1}(x, y) \geq \frac{d(x, y)}{2}.$$

This implies that either

$$|B_{\frac{d(x,y)}{2}-\varepsilon}(x)| \leq 2^{k_0+1},$$

or

$$|B_{\frac{d(x,y)}{2}-\varepsilon}(y)| \leq 2^{k_0+1}.$$

Without loss of generality we can assume that

$$|B_{\frac{d(x,y)}{2}-\varepsilon}(x)| \leq 2^{k_0+1}.$$

Now the proof of Claim 2 is completely similar to that of Claim 1 (replace $B_{c_{n-p+1}(x,y)+\varepsilon}(x)$ by $B_{\frac{d(x,y)}{2}-\varepsilon}(x)$, and replace $B_{c_{n-p}(x,y)+\varepsilon}(y)$ by $B_{c_{k_0}(x,y)+\varepsilon}(y)$ in the proof of Claim 1).

Combining Claim 1 with Claim 2, we obtain the following inequality:

$$\text{(1)} \qquad \|f(x)-f(y)\|_{\ell^2}^2 \geq c\left(\sum_{i=1}^{k_0}\frac{(c_i(x,y)-c_{i-1}(x,y))^2}{a_i}+\frac{(\frac{d(x,y)}{2}-c_{k_0}(x,y))^2}{a_{k_0+1}}\right).$$

Let $s_i = c_i(x,y)-c_{i-1}(x,y)$, and $t_i = s_0^2+s_1^2+\cdots+s_i^2$ for all $1\leq i\leq k_0$. By Lemma 4.3 we have

$$t_k=\sum_{i=1}^{k}s_i^2 \geq \frac{(\sum_{i=1}^{k}s_i)^2}{k}=\frac{c_k^2(x,y)}{k}\geq k\alpha^2(\alpha(0)k).$$

It follows that

$$\begin{aligned}
&\|f(x)-f(y)\|_{\ell^2}^2 \geq c\sum_{i=1}^{k_0}\frac{t_i-t_{i-1}}{a_i}\\
&= t_1\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+t_2\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+\cdots+t_{k_0-1}\left(\frac{1}{a_{k_0-1}}-\frac{1}{a_{k_0}}\right)+t_{k_0}\frac{1}{a_{k_0}}\\
&\geq \alpha^2(\alpha(0))\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+2\alpha^2(\alpha(0)2)\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+\cdots\\
\text{(2)}\quad &\quad +(k_0-1)\alpha^2(\alpha(0)(k_0-1))\left(\frac{1}{a_{k_0-1}}-\frac{1}{a_{k_0}}\right)+k_0\alpha^2(\alpha(0)k_0)\frac{1}{a_{k_0}}\\
&= \sum_{i=1}^{k_0}\frac{i\alpha^2(\alpha(0)i)-(i-1)\alpha^2(\alpha(0)(i-1))}{a_i}\\
&\geq \sum_{i=1}^{k_0}\frac{\alpha^2(\alpha(0)i)}{a_i}.
\end{aligned}$$

Let

$$t_{k_0+1}=\sum_{i=1}^{k_0}s_i^2+\left(\frac{d(x,y)}{2}-c_{k_0}(x,y)\right)^2.$$

We have

$$t_{k_0+1} \geq \frac{(\sum_{i=1}^{k_0} s_i + \frac{d(x,y)}{2} - c_{k_0}(x,y))^2}{k_0+1} = \frac{d^2(x,y)}{4(k_0+1)}.$$

It follows that

$$\begin{aligned} \|f(x)-f(y)\|_{\ell^2}^2 &\geq c\sum_{i=1}^{k_0+1} \frac{t_i - t_{i-1}}{a_i} \\ &= t_1\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+\cdots+t_{k_0}\left(\frac{1}{a_{k_0}}-\frac{1}{a_{k_0+1}}\right)+t_{k_0+1}\frac{1}{a_{k_0+1}} \\ &\geq \frac{d^2(x,y)}{4(k_0+1)a_{k_0+1}}. \end{aligned} \tag{3}$$

Let $k(r)$ be the smallest integer such that

$$(k(r)+1)a_{k(r)+1} > r.$$

By Lemma 4.2, we know that

$$\lim_{r\to\infty} k(r) = +\infty.$$

Define

$$\rho_1(r) = \min\left\{\frac{\sqrt{r}}{2}, \sqrt{\sum_{i=1}^{k(r)} \frac{\alpha^2(\alpha(0)i)}{a_i}}\right\}.$$

By Lemma 4.2, $\rho_1(r)$ is a non-decreasing function on $\mathbb{R}_+$ satisfying $\lim_{r\to+\infty}\rho_1(r) = +\infty$.

Finally, by inequalities (2) and (3), we have

$$\|f(x)-f(y)\|_{\ell^2} \geq \begin{cases} \frac{\sqrt{d(x,y)}}{2} & \text{if} \quad (k_0+1)a_{k_0+1} \leq d(x,y) \\ \sqrt{\sum_{i=1}^{k_0} \frac{\alpha^2(\alpha(0)i)}{a_i}} & \text{if} \quad (k_0+1)a_{k_0+1} > d(x,y) \end{cases}$$
$$\geq \rho_1(d(x,y)).$$

QED

We remark that, unlike the case of a finitely generated group with a word-length metric and subexponential growth, it is not clear if a discrete metric space with subexponential growth has property A due to the lack of homogeneity of the space.

5. Main Theorem

In this section, we prove the main theorem of this paper and discuss its application to positive scalar curvature.

Theorem 5.1. *If* Γ *is a discrete metric space with subexponential growth, then the coarse Baum–Connes conjecture holds for* Γ.

Proof. Theorem 5.1 follows from Theorem 3.4 and Theorem 4.5. QED

The following result verifies Conjecture 2.4 for Riemannian spin manifold with subexponential volume growth.

Definition 5.2. A Riemannian manifold M is said to have subexponential volume growth if

$$\lim_{r\to+\infty}\frac{\ln(\sup_{x\in M}\text{Volume}\,(B_r(x)))}{r}=0,$$

where $B_r(x)=\{y\in M : d(y,x)<r\}$.

Theorem 5.3. *Let* M *be a complete Riemannian spin manifold with bounded geometry. If* M *has subexponential volume growth and is macroscopically large, then* M *can not have uniformly positive scalar curvature.*

Proof. Let D be the Dirac operator on M, let Γ be a net in M chosen as in Section 2. Using the bounded geometry property of M, it is not difficult to see that subexponential volume growth of M implies subexponential growth of Γ. We have the following commuting diagram:

$$\begin{array}{ccc} & K_*(M) & \\ j_*\swarrow & & \searrow\text{ind} \\ \lim\limits_{d\to\infty} K_*(P_d(\Gamma)) & \xrightarrow{\text{ind}} & K_*(C^*(\Gamma)), \end{array}$$

where j_* is induced by the inclusion $j : M\to P_d(\Gamma)$, for sufficiently large d (cf. Section 2), and ind is the (higher) index map. Since M is macroscopically large, we have $j_*([D])\neq 0$.

It follows from the above commuting diagram and the coarse Baum–Connes conjecture for Γ that the higher index of D : ind $([D])$, is non-zero. This, together with Proposition 4.33 in [15] and the Lichnerowicz argument, implies that M can not have uniformly positive scalar curvature. QED

We should point out that the argument in the proof of Theorem 5.3 is well-known (cf. [15], [16]). The original idea of using higher index to prove non-existence of Riemannian metrics with positive scalar curvature is due to Rosenberg [16].

Corollary 5.4. *Let M be a uniformly contractible complete Riemannian manifold with bounded geometry. If M has subexponential volume growth, then M can not have uniformly positive scalar curvature.*

Corollary 5.4 follows from Theorem 5.3 and the fact that a uniformly contractible Riemannian manifold is macroscopically large. We remark that, if M is the universal cover of a compact aspherical manifold, then Corollary 5.4 follows from the remarkable work of Higson and Kasparov on the Baum-Connes conjecture [10]. Corollary 5.4 also greatly improves a result in [21] stating that a uniformly contractible complete Riemannian manifold with polynomial volume growth and polynomial contractibility radius growth can not have uniformly positive scalar curvature.

Theorem 5.5. *Let M be a macroscopically large complete Riemannian manifold with bounded geometry. If M has subexponential volume growth, then the zero-in-the-spectrum conjecture holds for M, i.e. the spectrum of the Laplacian acting on the space of L^2-forms of M contains zero.*

Proof. The proof is completely similar to that of Theorem 5.3 (replace the Dirac operator with $d + d^*$). QED

The interested reader should consult [13] for an excellent survey on the zero-in-the-spectrum conjecture.

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