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Abstract

In this paper we propose a von Mises approximation of the critical value of a test and a saddlepoint approximation of it. They are specially useful to compute quantiles of complicated test statistics with a complicated distribution function, which is a very common situation in robustness studies. We also obtain the influence function of the critical value as an alternative way to analyse the robustness of a test.

Key Words: Robustness in hypotheses testing, von Mises expansion, influence function, tail area influence function, saddlepoint approximation.

AMS subject classification: Primary 62F35, secondary 62E17, 62F03

1 Introduction

The von Mises expansion of a functional has been used for several purposes since its introduction by von Mises (1947). Fernholz (1983), Filippova (1961) or Reeds (1976) used it to analyse the asymptotic behaviour of some statistics; Sen (1988) in relation with the jackknife; Hampel (1968, 1974) to define his influence function, one of the central concepts of robust statistics, and recently Fernholz (2001) in a multivariate setting.

In this paper we propose a general method to obtain an approximation of k_n^F, the critical value of a test based on a sample of size n of a random variable with distribution function F, which consists in considering k_n^F as a functional of the model distribution function F and using the first terms of its von Mises expansion.

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This von Mises expansion depends on the critical value of the test under another model, G, that we choose so that k_n^G is known, plus some other terms that essentially depend on $\overset{\bullet}{k}{}_n^G$, the influence function of the critical value under G.

We obtain a von Mises approximation of k_n^F truncating its von Mises expansion at a convenient point (usually up to the first or second order).

The von Mises approximation, obtained in this way, is more accurate when the distributions F and G are closer. This is specially useful in robustness studies where distribution F is frequently a slight deviation from a known model G, but complicated enough to render impossible an exact calculation of k_n^F.

Moreover, in most of the cases, distribution G will be the normal distribution, under which the critical value k_n^G is known from the classical statistics, being in this case the rest of the terms of the von Mises approximation easy to manage.

With these aims, in Section 2, we establish the preliminaries of the problem, obtaining in Section 3 the von Mises expansion of the critical value, which depends on its influence function.

For this reason, in Section 4 we study the influence function of the critical value, $\overset{\bullet}{k}{}_n^G$, that can be considered, also, as an alternative way to analyse the robustness of a test. With $\overset{\bullet}{k}{}_n^G$ we can see the influence of the contamination on the critical value of a test, with a fixed significance level, instead of fixing the critical value of the test and seeing the influence of contamination on the level and on the power. We obtain an expression for $\overset{\bullet}{k}{}_n^G$ that involves the Tail Area Influence Function, defined by Field and Ronchetti (1985).

In Section 5 we finally obtain an explicit von Mises approximation of the critical value of a test and some applications to the location problem.

Since in some situations it is difficult to obtain a manageable expression for the tail probability of the test, in Section 6 we use the Lugannani and Rice (1980) formula to obtain a saddlepoint approximation to the von Mises approximation of the critical value of a M-test, including some examples and a simulation study which confirms the accuracy of the proposed method. (For a general review of the saddlepoint approximation techniques see, for instance, Field and Ronchetti, 1990; Jensen, 1995)

With the approximations that we propose in this paper, we give a straightforward solution to the computation of the quantiles of the, usually complicated, distribution function of a robust test statistic. This distribution function is very hard to obtain (and so, to invert) firstly, because the test statistic often has an odd form and, secondly, because the distribution model (although it is a slight deviation of a classical model) makes very hard the exact computation of the distribution function of the test statistic, two situations which are very common in robustness studies.

2 Preliminaries

Although the method that we are going to expose in this paper can be extended to a more general setting, we shall consider here the one parameter situation, in which, the model distribution of the observable random variable X (with values on a sample space $\mathcal{X}$) belongs to a family of distributions $\mathcal{F} = \{F_\theta; \theta \in \Theta\}$, where Θ is a real interval.

Using a random sample $X_1, ..., X_n$ of X we are interested in testing the null hypothesis $H_0 : \theta = \theta_0$ using a test statistic $T_n = T_n(X_1, ..., X_n)$ that rejects H_0 when T_n is larger than the critical value k_n^F.

If we represent by $F_{n;\theta}$ the cumulative distribution function of the test statistic T_n when the model distribution of X is F_θ, we shall consider the critical value for the previous level-α test,

$$k_n^F = F_{n;\theta_0}^{-1}(1-\alpha)$$

as a functional of the model F_{θ_0} where, throughout this paper, the inverse of any distribution function G is defined, as usual, by $G^{-1}(s) = \inf\{y|G(y) \geq s\}$, $0 < s < 1$.

For instance, if $T_n = M$ is the sample median, then

$$k_n^F = F_{\theta_0}^{-1}(B^{-1}(1-\alpha)),$$

where B is the cumulative distribution function of a beta $\beta(\,(n+1)/2\,,\,(n+1)/2\,)$.

If $T_n = \overline{x}$ is the sample mean and $F_{\theta_0} \equiv \Phi_{\theta_0}$ the normal distribution

$N(\theta_0, \sigma)$, it will be

$$k_n^F = \frac{1}{\sqrt{n}} \left(\Phi_{\theta_0}^{-1}(1-\alpha) + \theta_0 \left(\sqrt{n} - 1\right) \right).$$

We shall usually drop off the subscript θ_0 to simplify the notation, but obviously, in the functional $k_n^{(\cdot)}$ all the parametric models, F, G,... are considered under the null hypothesis θ_0.

Let us observe that it does not matter that the functional k_n^F depends on n because we are not interested in the asymptotic (in n) distribution properties of k_n^F, except perhaps in Section 4.2 which is not in the main line of reasoning of the paper. n is in k_n^F what Reeds (1976, p. 39), calls an *auxiliary parameter*.

3 A von Mises expansion of the critical value

In this section we shall obtain an approximation of the critical value of the test considered before, using the von Mises expansion of a functional.

3.1 von Mises expansion of a functional

Let T be a functional defined on a convex set $\mathcal{F}$ of distribution functions and with range the set of the real numbers. For the situation we are considering in this paper this framework is enough; nevertheless in a general setting we should require the mathematical conditions for the Hadamard (or compact) differentiability of T (see Fernholz, 1983 or Sen, 1988).

If F and G are two members of $\mathcal{F}$ and $t \in [0,1]$ is a real number, let us define the function A of the real variable t by

$$A(t) = T\left((1-t)G + tF\right) = T\left(G + t(F-G)\right).$$

For our purpose, we shall consider the viewpoint adopted by Filippova (1961) and that Reeds (1976, p. 29) calls the *low-brow way* of the von Mises expansion of a functional T, which is just the ordinary Taylor expansion of the real function $A(t)$, assuming that A satisfies the usual conditions for a Taylor expansion to be valid if $t \in [0,1]$ (see, for instance, Serfling, 1980, pp. 43, theorem 1.12.1A). See also Fernholz (2001) for a good review of the von Mises calculations of higher order for statistical functionals.

If $A(t)$ can be expanded about $t = 0$, it will be

$$A(t) = A(0) + \sum_{k=1}^{m} \frac{A^{k)}(0)}{k!}\, t^k + Rem_m, \tag{3.1}$$

where $A^{k)}(0)$ is the ordinary kth derivative of A at the point 0, with respect to t, with $k = 1, \ldots, m$, and where the remainder term Rem_m depends on F and G, and on the $(m+1)$th derivative of A.

Evaluated at $t = 1$, (3.1) gives

$$T(F) = T(G) + \sum_{k=1}^{m} \frac{A^{k)}(0)}{k!} + Rem_m,$$

because $A(1) = T(G + F - G) = T(F)$ and $A(0) = T(G)$.

3.2 von Mises expansion of the critical value

Now, let us consider the functional $T(G_{\theta_0}) = k_n^G$. If the corresponding $A(t) = T\,(G + t(F - G))$ has a finite third derivative everywhere in the interval $(0, 1)$ and $A^{2)}(t)$ is continuous in $[0, 1]$, we have the *second-order expansion*

$$k_n^F = k_n^G + A^{1)}(0) + \frac{1}{2}\, A^{2)}(0) + Rem_2, \tag{3.2}$$

and the *first-order expansion*

$$k_n^F = k_n^G + A^{1)}(0) + Rem_1 \tag{3.3}$$

of the functional k_n^F, having the first one a higher degree of accuracy than the second one. (Obviously, for the first-order expansion to exist we need only to assume that $A^{2)}(t)$ is finite everywhere in $(0, 1)$ and that $A^{1)}(t)$ is continuous in $[0, 1]$.)

If there exist two *kernels* functions a_1^G and a_2^G (also called first- and second-order compact derivatives of the functional $k_n^{(\cdot)}$ at G), such that

$$A^{1)}(0) = \int a_1^G(x)\, d(F_{\theta_0} - G_{\theta_0})(x) \tag{3.4}$$

and

$$A^{2)}(0) = \int\int a_2^G(x,y)\, d(F_{\theta_0} - G_{\theta_0})(x)\, d(F_{\theta_0} - G_{\theta_0})(y),$$

where $a_2^G(x,y) = a_2^G(y,x) \;\; \forall\, x, y$, k_n^G is said to be a von Mises functional (Reeds, 1976).

When these kernels exist, they are only defined up to additive constants. We shall make them unique by imposing the usual conditions (constraint (2.3) of Withers, 1983, or (2.4) and (2.5) of Sen, 1988)

$$\int a_1^G(x)\, dG_{\theta_0}(x) = 0, \tag{3.5}$$

$$\int a_2^G(x,y)\, dG_{\theta_0}(x) = 0,$$

and

$$\int a_2^G(x,y)\, dG_{\theta_0}(y) = 0.$$

Then, the expansions (3.2) and (3.3) will be,

$$k_n^F = k_n^G + \int a_1^G(x)\, dF_{\theta_0}(x) + \frac{1}{2}\int\int a_2^G(x,y)\, dF_{\theta_0}(x)\, dF_{\theta_0}(y) + Rem_2$$

and

$$k_n^F = k_n^G + \int a_1^G(x)\, dF_{\theta_0}(x) + Rem_1.$$

Moreover, $a_1^G(x)$ is then, the Hampel's influence function of the functional k_n^G, which we shall represent throughout this paper by $\dot{k}_n^G(x)$ or just by $IF(x; k_n^G, G_{\theta_0})$, and the kernel $a_2^G(x,y)$ is related with the influence function $\dot{k}_n^G(x)$ through the expression (see Withers, 1983 Theorem 2.1, pag. 578, or Gatto and Ronchetti, 1996, pag. 667)

$$a_2^G(x,y) = \frac{\partial}{\partial\epsilon} IF(x; k_n^G, G_{\epsilon,y;\theta_0})\Big|_{\epsilon=0} + IF(y; k_n^G, G_{\theta_0}),$$

where $G_{\epsilon,y;\theta_0} := (1-\epsilon)G_{\theta_0} + \epsilon\delta_y$ is the contaminated model, and δ_y the distribution which puts mass 1 at $y \in I\!R$.

Let us observe that kernels a_1^G and a_2^G exist when the influence function $IF(x; k_n^G, G_{\theta_0})$ and its derivative exist. In this case, the first- and second-order expansions are, respectively,

$$k_n^F = k_n^G + \int IF(x; k_n^G, G_{\theta_0})\, dF_{\theta_0}(x) + Rem_1 \tag{3.6}$$

and

$$\begin{aligned} k_n^F &= k_n^G + \int IF(x; k_n^G, G_{\theta_0})\, dF_{\theta_0}(x) \\ &+ \frac{1}{2} \int \int \frac{\partial}{\partial \epsilon} IF(x; k_n^G, G_{\epsilon, y; \theta_0}) \Big|_{\epsilon=0} dF_{\theta_0}(x)\, dF_{\theta_0}(y) \\ &+ \frac{1}{2} \int IF(y; k_n^G, G_{\theta_0})\, dF_{\theta_0}(y) + Rem_2. \end{aligned} \tag{3.7}$$

The remainder term in one of these r-order expansions is an integral of the $(r+1)$-kernel (which is the element that depends on n) with respect to $\prod_{i=1}^{r+1} d(F_{\theta_0} - G_{\theta_0})(x_i)$, divided by $(r+1)!$ (see Withers, 1983, pag. 578). Thus, the error term will be smaller when F_{θ_0} and G_{θ_0} are closer. More exactly, $Rem_2 = o(||F - G||^2)$ uniformly in $F \in \mathcal{F}$ where $||F - G||$ refers to the usual sup-norm (i.e., $\sup_x |F_{\theta_0}(x) - G_{\theta_0}(x)|$).

Expressions (3.6) and (3.7) are important because, if the functional k_n^F depends on F_{θ_0} explicitly, we can use the usual Taylor expansion for the function $A(t)$ to obtain the desired approximations. Nevertheless, in most of the situations, this will not be the case and we have to use these expressions to obtain the first- and second-order expansions.

4 The influence function of the critical value

To obtain more explicit expressions for the von Mises expansions, we need to study the influence function of the critical value, obtaining also an alternative way to analyse the robustness of a test: To fix the level of the test and see how the critical value changes when the underlying distribution of the observations does not belong to the model but coincides with a distribution in a neighborhood of it. This is a different viewpoint of the usual robust approach to testing, which fixes the critical value under the model and then, investigates the change of the level and the power of the test under contamination.

For these purposes, we shall use the *Tail Area Influence Function* (TAIF) of T_n at G_θ, defined by Field and Ronchetti (1985) as

$$\mathrm{TAIF}(x;t;T_n,G_\theta) = \frac{\partial}{\partial\epsilon} P_{G_{\epsilon,x;\theta}}\{T_n > t\}\Big|_{\epsilon=0},$$

for all $x \in I\!R$ where the right hand side exists, being $G_{\epsilon,x;\theta} := (1-\epsilon)G_\theta + \epsilon\delta_x$ the contaminated model, and δ_x the distribution which puts mass 1 at $x \in I\!R$.

4.1 Influence function of the critical value

Let us suppose that $G_{n;\theta_0}$ has a density $g_{n;\theta_0}$ with respect to the Lebesgue measure. Let us represent by

$$k^G_{n;\epsilon} = G^{-1}_{n;\epsilon,x;\theta_0}(1-\alpha)$$

the contaminated critical value, i.e., the critical value when we suppose, as model for X, the contaminated distribution $G_{\epsilon,x;\theta_0} := (1-\epsilon)G_{\theta_0} + \epsilon\delta_x$, $0 \leq \epsilon \leq 1$.

Let us consider, as level α, a level which is achieved by the test, at least for all ϵ small enough, i.e., let us suppose that it is $G_{n;\epsilon,x;\theta_0}(k^G_{n;\epsilon}) = 1-\alpha$, or, equivalently, that it is

$$P_{G_{\epsilon,x;\theta_0}}\{T_n > k^G_{n;\epsilon}\} = \alpha. \tag{4.1}$$

This is not a serious restriction since the set of levels which satisfy that

$$G_{n;\epsilon,x;\theta_0}(k^G_{n;\epsilon}-) < 1-\alpha < G_{n;\epsilon,x;\theta_0}(k^G_{n;\epsilon})$$

will be the empty set when $\epsilon \downarrow 0$ because of the continuity of $G_{n;\theta_0}$ (which is the limit of ϵ at which we shall calculate the derivatives).

Then, using the chain rule in (4.1) to obtain the derivative with respect to ϵ at $\epsilon = 0$, we have

$$\mathrm{TAIF}(x;k^G_n;T_n,G_{\theta_0}) - g_{n;\theta_0}(k^G_n)\cdot \overset{\bullet}{k}{}^G_n = 0,$$

where

$$\overset{\bullet}{k}{}_n^G = \frac{\partial}{\partial \epsilon} k_{n;\epsilon}^G \Big|_{\epsilon=0}$$

is the influence function of the critical value.

Then, if $g_{n;\theta_0}(k_n^G) \neq 0$, it will be

$$\overset{\bullet}{k}{}_n^G = \frac{\text{TAIF}(x; k_n^G; T_n, G_{\theta_0})}{g_{n;\theta_0}(k_n^G)}.$$

4.2 Asymptotic behaviour of the influence function

Although this section is not in the main line of the paper, we will analyse here the robustness properties of the critical value. For this, we need first to obtain the limit of $\overset{\bullet}{k}{}_n^G$ as $n \to \infty$.

We shall restrict our attention to M-tests, i.e., to tests based on the statistic T_n solution of the equation

$$\sum_{i=1}^{n} \psi(x_i; T_n) = 0, \tag{4.2}$$

where $\psi : \mathcal{X} \times \Theta \longrightarrow I\!R$ is a given function and $\mathcal{X}$ is the sample space. In this situation, Field and Ronchetti (1985) obtained the limit behaviour of TAIF using three conditions that they call (A1), (A2) and (A3) and that essentially correspond to the situation of an M-estimator (unique) solution of (4.2), with $\psi(x;\theta)$ strictly monotonic in θ for all $x \in \mathcal{X}$ and such that $E[\psi(X;\theta)] = 0; \ \ \forall \theta \in \Theta$.

They proved that

$$\lim_{n\to\infty} n^{-1/2}\, \text{TAIF}(x; k_n^G; T_n, G_{\theta_0}) = \text{LIF}(x; T_n, G_{\theta_0}),$$

where $\text{LIF}(x; T_n, G_{\theta_0})$ is the level influence function defined by Rousseeuw and Ronchetti (1979, 1981).

Thus, from the asymptotic normality of the M-estimator T_n, we have

$$\lim_{n\to\infty} n^{-1/2} g_{n;\theta_0}(k_n^G) = \frac{1}{\sigma_0}\, \phi(z_\alpha)$$

where ϕ is, throughout this paper, the standard normal density, z_α the $(1-\alpha)$-quantile of the standard normal distribution and $\sigma_0^2 = E_{\theta_0}[\psi^2]/(-E_{\theta_0}[\psi'])^2$.

Then,

$$\begin{aligned}\lim_{n\to\infty} \overset{\bullet}{k}{}_n^G &= \lim_{n\to\infty} \frac{n^{-1/2}\,\mathrm{TAIF}(x;k_n^G;T_n,G_{\theta_0})}{n^{-1/2}\,g_{n;\theta_0}(k_n^G)} \\ &= \frac{\phi(z_\alpha)/\sigma_0 \cdot IF_{test}\,(x;T_n,G_{\theta_0})}{\phi(z_\alpha)/\sigma_0} \\ &= IF_{test}\,(x;T_n,G_{\theta_0}),\end{aligned}$$

because (see Hampel et al., 1986, pag. 199) it is

$$\mathrm{LIF}(x;T_n,G_{\theta_0}) = \frac{1}{\sigma_0}\cdot\phi(z_\alpha)\cdot IF_{test}\,(x;T_n,G_{\theta_0}).$$

Thus, we can think of $\overset{\bullet}{k}{}_n^G$ as a finite sample version of the influence function of the statistical test and then, with the same robustness properties, which can be used to evaluate finite sample behaviour of tests.

5 von Mises approximation of the critical value

From the first- and second-order von Mises expansions (3.6) and (3.7), and the expression obtained in the previous section for the influence function $\overset{\bullet}{k}{}_n^G\,(x) = IF(x;k_n^G,G)$, we can define the *first-order von Mises* (VOM) *approximation of k_n^F by k_n^G* as any of the right hand side members in the following equations

$$\begin{aligned}k_n^F &\simeq k_n^G + A^{1)}(0) \\ &= k_n^G + \int \overset{\bullet}{k}{}_n^G\,(x)\,dF_{\theta_0}(x) \\ &= k_n^G + \frac{1}{g_{n;\theta_0}(k_n^G)}\int \mathrm{TAIF}(x;k_n^G;T_n,G_{\theta_0})\,dF_{\theta_0}(x),\end{aligned}$$

where $g_{n;\theta_0}$ is the density of T_n when distribution function for X is G_{θ_0}.

Also we define the *second-order VOM approximation of* k_n^F *by* k_n^G as any of the right handed side members in the following equations

$$\begin{aligned} k_n^F &\simeq k_n^G + A^{1)}(0) + \frac{1}{2}A^{2)}(0) \\ &= k_n^G + \frac{3}{2}\int IF(x; k_n^G, G_{\theta_0}) dF_{\theta_0}(x) \\ &\quad + \frac{1}{2}\int\int \frac{\partial}{\partial\epsilon} IF(x; k_n^G, G_{\epsilon,y;\theta_0})\Big|_{\epsilon=0} dF_{\theta_0}(x)\, dF_{\theta_0}(y). \end{aligned}$$

The distribution G_{θ_0} is called a *pivotal distribution* and k_n^G a *pivotal critical value.*

As we mentioned before, these approximations will be more accurate when F_{θ_0} and G_{θ_0} are closer.

If we choose as pivotal distribution the normal distribution Φ_{θ_0} we can calculate the critical value of the test under other model F, using the distribution of the statistic under the normal distribution, which is usually known from the classical statistics. Then, for instance, the first-order VOM approximation will be

$$k_n^F \simeq k_n^\Phi + \frac{1}{\phi_{n;\theta_0}(k_n^\Phi)}\int \mathrm{TAIF}(x; k_n^\Phi; T_n, \Phi_{\theta_0})\, dF_{\theta_0}(x), \tag{5.1}$$

where $\phi_{n;\theta_0}$ is the density function of T_n when Φ_{θ_0} is the model distribution of X. Of course, it is possible to change in (5.1) the normal distribution by another one, if we get a simplification in the computations, or if we improve the approximation because this new distribution is closer than Φ to F.

The VOM approximation is specially useful if the distribution $F_{n;\theta_0}$ of T_n is too complicated to obtain exactly k_n^F, but the model F is smooth enough to integrate the TAIF of Φ_{θ_0}.

Also, if the TAIF assumes only a small number of different values (namely two) we shall obtain the VOM approximation in an straightforward way.

Finally, let us observe that if F_{θ_0} and G_{θ_0} are not close enough and H_{θ_0} is another distribution between them, we can use the VOM approximations sequentially, approximating first k_n^H by the known critical value k_n^G and then, k_n^F by k_n^H.

5.1 The one sample median test

If we construct a test with a robust test statistic T_n, we shall usually arrive at a situation in which the exact distribution of T_n is too complicated (not allowing a direct computation of the critical value), but for which we can easily integrate the TAIF.

We shall describe in this section one of these situations in which it is also possible, for some models, to compare the exact results with the VOM approximation.

Namely, let us consider the *one sample median test* (with n odd and $\alpha < 0'5$) for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$, using a sample of the absolutely continuous random variable X with distribution function F_θ. This test is defined by

$$\varphi(M) = \begin{cases} 1 & \text{if } M > k_n^F \\ 0 & \text{otherwise,} \end{cases}$$

where M is the sample median and k_n^F the critical value defined through

$$F_{n;\theta_0}(k_n^F) = P_{\theta_0}\left\{M \leq k_n^F\right\} = 1 - \alpha,$$

where $F_{n;\theta_0}$ is the cumulative distribution of M under H_0.

Because the contaminated tail probability for the sample median is

$$P_{G_{\epsilon,x;\theta}}\{M > k_n^G\} = \frac{n!}{[(\frac{n-1}{2})!]^2} \int_0^{1-G_{\epsilon,x;\theta}(k_n^G)} y^{(n-1)/2}(1-y)^{(n-1)/2}\, dy$$

and the TAIF of this test will be

$$\begin{aligned} \text{TAIF}(x; k_n^G; M, G_{\theta_0}) &= \frac{\partial}{\partial\epsilon} P_{G_{\epsilon,x;\theta_0}}\{M > k_n^G\}\Big|_{\epsilon=0} \\ &= \frac{n!}{[(\frac{n-1}{2})!]^2} [1 - G_{\theta_0}(k_n^G)]^{(n-1)/2} [G_{\theta_0}(k_n^G)]^{(n-1)/2} \\ &\quad [G_{\theta_0}(k_n^G) - \delta_x(k_n^G)]. \end{aligned}$$

Since the density of M in k_n^G is

$$g_{n;\theta_0}(k_n^G) = \frac{n!}{[(\frac{n-1}{2})!]^2} [1 - G_{\theta_0}(k_n^G)]^{(n-1)/2} \; [G_{\theta_0}(k_n^G)]^{(n-1)/2} \; g_{\theta_0}(k_n^G),$$

the influence function of the critical value at G_{θ_0} will be

$$\overset{\bullet}{k}{}_n^G = \frac{\text{TAIF}(x; k_n^G; M, G_{\theta_0})}{g_{n;\theta_0}(k_n^G)} = \frac{G_{\theta_0}(k_n^G) - \delta_x(k_n^G)}{g_{\theta_0}(k_n^G)} \tag{5.2}$$

and we obtain the following first-order VOM approximation of k_n^F

$$\begin{aligned} k_n^F &\simeq k_n^G + \frac{1}{g_{n;\theta_0}(k_n^G)} \int \text{TAIF}(x; k_n^G; T_n, G_{\theta_0})\, dF_{\theta_0}(x) \\ &= k_n^G + \frac{G_{\theta_0}(k_n^G) - F_{\theta_0}(k_n^G)}{g_{\theta_0}(k_n^G)}. \end{aligned}$$

To stand out the other elements involved in this approximation, which have been considered before, let us observe that the functional we are considering here (restricted to the absolutely continuous distributions) is

$$T(G_{\theta_0}) = k_n^G = G_{\theta_0}^{-1}\left(B^{-1}(1-\alpha)\right)$$

and the function $A(t) = T\left(G_{\theta_0} + t(F_{\theta_0} - G_{\theta_0})\right)$ solves the equation

$$\left(G_{\theta_0} + t(F_{\theta_0} - G_{\theta_0})\right)(A(t)) = B^{-1}(1-\alpha)$$

i.e.,

$$G_{\theta_0}(A(t)) + t\, F_{\theta_0}(A(t)) - t\, G_{\theta_0}(A(t)) = B^{-1}(1-\alpha),$$

thus, differentiating this equation with respect to t at $t = 0$, we obtain the expression

$$g_{\theta_0}(k_n^G) \cdot A^{1)}(0) + F_{\theta_0}(k_n^G) - G_{\theta_0}(k_n^G) = 0,$$

because $A(0) = T(G_{\theta_0}) = k_n^G$ and g_{θ_0} and f_{θ_0} are the densities of G_{θ_0} and F_{θ_0}.

Then, it will be

$$A^{1)}(0) = \frac{G_{\theta_0}(k_n^G) - F_{\theta_0}(k_n^G)}{g_{\theta_0}(k_n^G)}.$$

Some easy computations prove that, if k_n^G is given by (5.2), then

$$\int \overset{\bullet}{k}{}_n^G(x)\, dG_{\theta_0}(x) = 0,$$

and

$$\begin{aligned} \int \overset{\bullet}{k_n^G}(x)\, d(F_{\theta_0} - G_{\theta_0})(x) &= \int \overset{\bullet}{k_n^G}(x)\, dF_{\theta_0}(x) \\ &= \frac{G_{\theta_0}(k_n^G) - F_{\theta_0}(k_n^G)}{g_{\theta_0}(k_n^G)} = A^{1)}(0), \end{aligned}$$

as it is required in (3.5) and (3.4).

5.2 Location-scale families

In some situations, the distribution of X depends not only on the parameter θ being tested but also on certain additional nuisance parameter, as it happens with the location and scale families.

In the context of robustness for hypotheses testing, García-Pérez (1993, 1996) analysed this problem considering the tail ordering $<_t$ defined by Loh (1984) for which, if $F <_t G$ and we match the distributions at the location parameter, i.e., $f_{\theta_0}(\theta_0) = g_{\theta_0}(\theta_0)$, it is (if $\alpha < 0'5$) $k_n^F \leq k_n^G$ and $F_{\theta_0}(k_n^G) \geq G_{\theta_0}(k_n^G)$, so the first-order VOM approximation

$$k_n^F \simeq k_n^G - \left(F_{\theta_0}(k_n^G) - G_{\theta_0}(k_n^G)\right) / g_{\theta_0}(k_n^G)$$

shows explicitly the quality of the approximation: As G approaches to F, the right term approaches to k_n^F.

By this reason, when there exists an additional nuisance parameter, we shall match the densities as before.

Example 5.1 (One sample median test). Let us test $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$, using the one sample median test

$$\varphi(M) = \begin{cases} 1 & \text{if } M > k_n^F \\ 0 & \text{otherwise.} \end{cases}$$

If we suppose a normal distribution $\Phi_\theta \equiv N(\theta, \sigma)$ as model for X, we have, as exact critical value

$$k_n^\Phi = \theta_0 + \sigma\, \Phi^{-1}(B^{-1}(1 - \alpha)),$$

where Φ is, throughout this paper, the standard normal cumulative distribution function and B the distribution function of a beta $\beta((n+1)/2,(n+1)/2)$.

Thus, using this normal distribution as the pivotal distribution, the first-order VOM approximation for the critical value is

$$\begin{aligned} k_n^F &\simeq k_n^\Phi + \left(\Phi_{\theta_0}(k_n^\Phi) - F_{\theta_0}(k_n^\Phi)\right)/\phi_{\theta_0}(k_n^\Phi) \\ &= k_n^\Phi + \left(B^{-1}(1-\alpha) - F_{\theta_0}(k_n^\Phi)\right)/\phi_{\theta_0}(k_n^\Phi). \end{aligned}$$

If F_{θ_0} is a distribution model for which it is possible to obtain the quantiles, the exact critical value is

$$k_n^F = F_{\theta_0}^{-1}(B^{-1}(1-\alpha)).$$

Finally, using the asymptotic normality of the M-estimator sample median M, i.e., the fact that

$$\sqrt{n}(M-\theta_0) \longrightarrow N\left(0\,,\,\frac{1}{2\,f_{\theta_0}(\theta_0)}\right),$$

the normal approximation of the critical value will be

$$k_n^F \simeq \theta_0 + \frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}}\,\frac{1}{2\,f_{\theta_0}(\theta_0)}.$$

For instance, if F_{θ_0} is a logistic distribution $L(\theta_0,1)$, since the normal distribution is close to it (with respect to the tail ordering $<_t$) we shall choose as pivotal distribution a $N(\theta_0,\sigma)$ where σ is obtained from the side condition

$$f_{\theta_0}(\theta_0) = \frac{1}{4} = \phi_{\theta_0}(\theta_0) = \frac{1}{\sigma\sqrt{2\pi}}$$

i.e., $\sigma = \sqrt{2/\pi}$. Thus, for testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$, the exact critical value is

$$k_n^F = F_0^{-1}(B^{-1}(1-\alpha)),$$

the asymptotic normal approximation gives

$$k_n^F \simeq \Phi^{-1}(1-\alpha)\,\frac{2}{\sqrt{n}}$$

and the first-order VOM approximation states that

$$k_n^F \simeq k_n^\Phi + \frac{B^{-1}(1-\alpha) - F_0(k_n^\Phi)}{\phi_0(k_n^\Phi)},$$

where $k_n^\Phi = 4\,\Phi^{-1}(B^{-1}(1-\alpha))/\sqrt{2\pi}$ and ϕ_0 is the density of a $N(0, 4/\sqrt{2\pi})$. Table 1 shows, for different significance levels and a sample size $n = 1$, the critical values (exact, first-order VOM approximation and asymptotic normal approximation), and the relative errors in percentage for the first-order VOM approximation and for the asymptotic normal approximation.

	critical values			relative errors	
α	exact	VOM	normal	VOM	normal
0.005	5.2933	5.3394	5.1517	0.8709	2.6760
0.010	4.5951	4.5409	4.6527	1.1800	1.2530
0.025	3.6636	3.5913	3.9199	1.9726	6.9977
0.050	2.9444	2.8965	3.2897	1.6886	11.7261
0.100	2.1972	2.1774	2.5631	0.9030	16.6518
0.150	1.7346	1.7263	2.0729	0.4769	19.5011

Table 1: Critical values and relative errors of the first-order VOM approximation and the usual asymptotic normal approximation. $n = 1$

Figure 1 shows these relative errors for $0 < \alpha < 0'5$. The dotted line is for the first-order VOM approximation and the solid line for the asymptotic normal approximation. We see that the first one is very low (almost zero) and lower than the normal one except for some very low significance levels where the difference between both approximations is negligible.

Table 2 and Figure 2 show the values and the same conclusions for $n = 3$. As far as n increases, the first-order VOM approximation is even better, converging, when n goes to ∞, to the exact value.

□

6 Saddlepoint approximation

In some situations, the tail probability of the test statistic is too complicated to obtain the TAIF exactly. Here we shall use the Lugannani and Rice

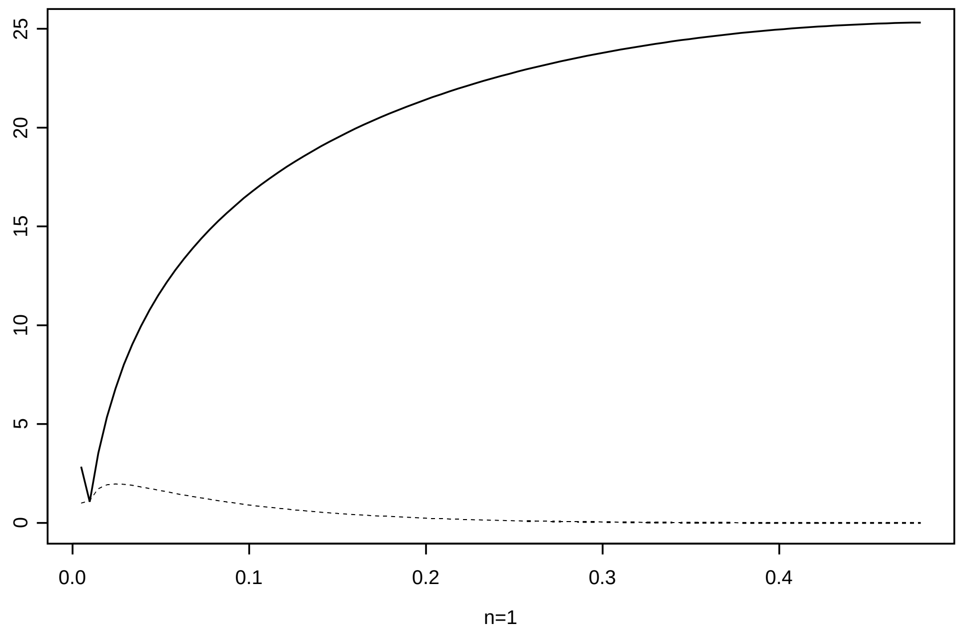

Figure 1: *Relative errors of the first-order VOM approximation (dotted line) and the usual asymptotic normal approximation (solid line)*

	critical values			relative errors	
α	exact	VOM	normal	VOM	normal
0.005	3.1422	3.0863	2.9743	1.7772	5.3427
0.010	2.7712	2.7302	2.6862	1.4760	3.0643
0.025	2.2622	2.2403	2.2632	0.9687	0.0414
0.050	1.8545	1.8437	1.8993	0.5777	2.4188
0.100	1.4128	1.4092	1.4798	0.2527	4.7464
0.150	1.1287	1.1274	1.1968	0.1186	6.0315

Table 2: *Critical values and relative errors of the first-order VOM approximation and the usual asymptotic normal approximation.* $n = 3$

(1980) formula to obtain, first, a saddlepoint approximation to the TAIF of the M-statistic and finally to the critical value k_n^F this result except for the paper by Maesono and Penev (1998), is one of the first saddlepoint approximations to the critical value of a test (i.e., to the quantile of the

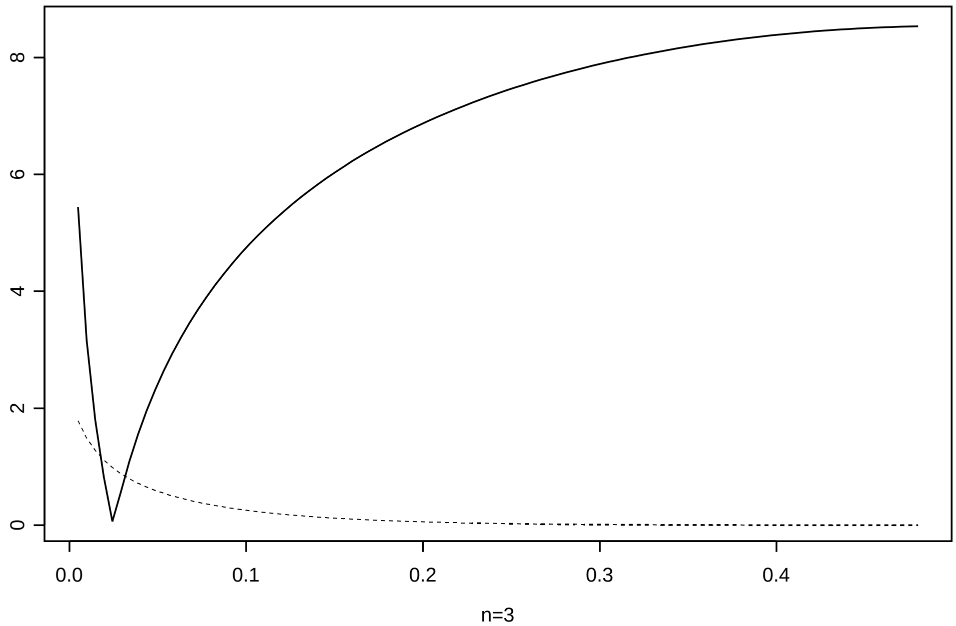

Figure 2: *Relative errors of the first-order VOM approximation (dotted line) and the usual asymptotic normal approximation (solid line)*

distribution of a statistic).

Let us denote by $K(\lambda, t)$ the function

$$K(\lambda, t) = \log \int_{-\infty}^{+\infty} e^{\lambda\psi(y,t)}\, dG_{\theta_0}(y).$$

By the Lugannani and Rice (1980) formula (see Daniels, 1983) it is

$$P_{G_{\theta_0}}\{T_n > t\} = 1 - \Phi(s) + \phi(s)\left[\frac{1}{r} - \frac{1}{s} + O(n^{-3/2})\right], \qquad (6.1)$$

where s and r are the functionals

$$s = \sqrt{-2nK(z_0, t)} := \sqrt{n}\, s_1,$$

$$r = z_0\sqrt{nK''(z_0, t)} := \sqrt{n}\, r_1,$$

where $K''(\lambda, t)$ denotes the second partial derivative of $K(\lambda, t)$ with respect to the first argument, and z_0 is the saddlepoint, i.e., the solution of the equation

$$K'(z_0,t) = \int_{-\infty}^{+\infty} e^{z_0\psi(y,t)}\, \psi(y,t)\, dG_{\theta_0}(y) = 0, \qquad (6.2)$$

which depends on G_{θ_0}; the same happens with the functionals K, s and r.

Using a dot over a generic functional T to represent its influence function,

$$\overset{\bullet}{T}= \frac{\partial}{\partial\epsilon} T(G_{\epsilon,x})\Big|_{\epsilon=0},$$

the saddlepoint approximation to the TAIF, from (6.1),

$$\begin{aligned}
\text{TAIF}(x;t;T_n,G_{\theta_0}) &= \frac{\partial}{\partial\epsilon} P_{G_{\epsilon,x;\theta_0}}\{T_n > t\}\Big|_{\epsilon=0} \\
&= \frac{\phi(s)}{r_1}\left\{-s_1\ \overset{\bullet}{s}_1\ n^{1/2} - \frac{\overset{\bullet}{r}_1}{r_1} n^{-1/2} + \frac{\overset{\bullet}{s}_1\ r_1}{s_1^2} n^{-1/2}\right\} + O(n^{-1/2}) \\
&= \frac{\phi(s)}{r_1}\left\{\overset{\bullet}{K}\ n^{1/2} + \left[-\frac{\overset{\bullet}{z}_0}{z_0} - \frac{\overset{\bullet}{K''}}{2K''} - \frac{\overset{\bullet}{K}\ z_0\sqrt{K''}}{(-2K)^{3/2}}\right] n^{-1/2}\right\} + O(n^{-1/2}). \qquad (6.3)
\end{aligned}$$

From (6.2) we have, after contaminating z_0 and G_{θ_0},

$$\overset{\bullet}{z}_0 = -\frac{e^{z_0\psi(x,t)}\psi(x,t)}{\int \psi^2(y,t)\, e^{z_0\psi(y,t)}\, dG_{\theta_0}(y)}$$

and

$$\begin{aligned}
\overset{\bullet}{K} &= \frac{\partial}{\partial\epsilon} \log \int \exp\left\{z_0(G_{\epsilon,x;\theta_0})\,\psi(y,t)\right\}\, dG_{\epsilon,x;\theta_0}(y)\Big|_{\epsilon=0} \\
&= \frac{\partial}{\partial\epsilon} \log\Big\{(1-\epsilon)\int \exp\left\{z_0(G_{\epsilon,x;\theta_0})\,\psi(y,t)\right\}\, dG_{\theta_0}(y) \\
&\qquad + \epsilon\ \ e^{z_0(G_{\epsilon,x;\theta_0})\psi(x,t)}\Big\}\Big|_{\epsilon=0} \\
&= \frac{e^{z_0\psi(x,t)}}{\int e^{z_0\psi(y,t)} dG_{\theta_0}(y)} - 1.
\end{aligned}$$

Finally, since

$$K''(z_0,t) = \frac{\int \psi^2(y,t)\, e^{z_0\psi(y,t)}\, dG_{\theta_0}(y)}{\int e^{z_0\psi(y,t)}\, dG_{\theta_0}(y)},$$

it will be

$$\overset{\bullet}{K}''= e^{z_0\psi(x,t)}\left\{\frac{\psi^2(x,t)}{\int e^{z_0\psi}dG_{\theta_0}}-\frac{\psi(x,t)\int\psi^3 e^{z_0\psi}dG_{\theta_0}}{[\int e^{z_0\psi}dG_{\theta_0}][\int\psi^2 e^{z_0\psi}dG_{\theta_0}]}-\frac{\int\psi^2 e^{z_0\psi}dG_{\theta_0}}{[\int e^{z_0\psi}dG_{\theta_0}]^2}\right\}.$$

Then, replacing $\overset{\bullet}{z}_0$, $\overset{\bullet}{K}$ and $\overset{\bullet}{K}''$ in (6.3) we obtain the saddlepoint approximation to the TAIF that, if we consider only the terms of order $n^{1/2}$, will be, at $t=k_n^G$,

$$\text{TAIF}(x;k_n^G;T_n,G_{\theta_0})=\frac{\phi(s)}{r_1}\,n^{1/2}\left(\frac{e^{z_0\psi(x,k_n^G)}}{\int e^{z_0\psi(y,k_n^G)}\,dG_{\theta_0}(y)}-1\right)+O(n^{-1/2}).$$

Also in Daniels (1983) we find a saddlepoint approximation of the density function that, with the notation used here, is

$$g_{n;\theta_0}(k_n^G)= \left(\frac{n}{2\pi K''(z_0,k_n^G)}\right)^{1/2}\left(-\frac{K^{(1)}(z_0,k_n^G)}{z_0}\right)\exp\{nK(z_0,k_n^G)\}\left(1+O(n^{-1})\right),$$

where

$$K^{(1)}(z_0,k_n^G)=\left.\frac{\partial}{\partial t}K(z_0,t)\right|_{t=k_n^G}.$$

Thus, after some simplifications, the saddlepoint approximation to the influence function of the critical value is

$$\begin{aligned}\overset{\bullet}{k}{}_n^G(x) &= \frac{\text{TAIF}(x;k_n^G;T_n,G_{\theta_0})}{g_{n;\theta_0}(k_n^G)}\\ &\simeq \frac{-e^{z_0\psi(x,k_n^G)}+\int e^{z_0\psi(y,k_n^G)}\,dG_{\theta_0}(y)}{z_0\int e^{z_0\psi(y,k_n^G)}\,\psi'(y,k_n^G)\,dG_{\theta_0}(y)},\end{aligned}$$

where

$$\psi'(y,k_n^G)=\left.\frac{\partial}{\partial t}\psi(y,t)\right|_{t=k_n^G}.$$

Finally, the saddlepoint approximation of the VOM approximation (VOM+SAD approximation in the sequel) of the critical value of a M-test will be (denoting the saddlepoint by z_0^G to remark that it is the solution with respect the distribution G_{θ_0}),

$$\begin{aligned} k_n^F &\simeq k_n^G + \int \overset{\bullet}{k_n^G}(x) dF_{\theta_0}(x) \\ &\simeq k_n^G + \frac{-\int e^{z_0^G \psi(x,k_n^G)}\, dF_{\theta_0}(x) + \int e^{z_0^G \psi(y,k_n^G)}\, dG_{\theta_0}(y)}{z_0^G \int e^{z_0^G \psi(y,k_n^G)}\, \psi'(y,k_n^G)\, dG_{\theta_0}(y)}, \quad (6.4) \end{aligned}$$

where G_{θ_0} is a distribution close to F_{θ_0} in order to obtain a better approximation. In robustness studies, the distribution G will usually be the normal distribution (for which we know the critical value k_n^G) and F a slight deviation from this classical model.

Example 6.1 (Mean test). Let us consider the mean test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$, using a sample from the random variable X with distribution function F_θ, defined by

$$\varphi(\overline{x}) = \begin{cases} 1 & \text{if } \overline{x} > k_n^F \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{x}$ is the sample mean and $k_n^F$is defined through

$$F_{n;\theta_0}(k_n^F) = P_{\theta_0}\left\{\overline{x} \le k_n^F\right\} = 1 - \alpha.$$

If we choose, as pivotal distribution G, a normal $\Phi_\theta \equiv N(\theta, \sigma)$ where σ is known and θ unknown, the pivotal critical value is the usual

$$k_n^G \equiv k_n^\Phi = \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}.$$

Because $\overline{x}$ is an M-estimator with ψ-function $\psi(y,t) = y - t$, it is

$$\psi'(y, k_n^\Phi) = \frac{\partial}{\partial t}\psi(y,t)\bigg|_{t=k_n^\Phi} = -1,$$

and Equation (6.2), in this case is

$$\int_{-\infty}^{+\infty} e^{z_0^\Phi\,(y-k_n^\Phi)}\,(y - k_n^\Phi)\, d\Phi_{\theta_0}(y) = 0,$$

from which we obtain the saddlepoint

$$z_0^\Phi = \frac{k_n^\Phi - \theta_0}{\sigma^2} = \frac{z_\alpha}{\sigma\sqrt{n}}.$$

Since it is

$$\int e^{z_0^\Phi\,\psi(y,k_n^\Phi)}\,d\Phi_{\theta_0}(y) = \exp\left\{-\frac{1}{2\sigma^2}(k_n^\Phi-\theta_0)^2\right\},$$

from (6.4) we have

$$k_n^F \simeq \theta_0 + \frac{\sigma z_\alpha}{\sqrt{n}} - \frac{\sigma\sqrt{n}}{z_\alpha}\left(1 - \exp\left\{-\frac{z_\alpha^2}{2n} - \frac{z_\alpha\theta_0}{\sigma\sqrt{n}}\right\}\int e^{z_\alpha x/(\sigma\sqrt{n})}\,dF_{\theta_0}(x)\right). \tag{6.5}$$

The approximation (6.5) has numerous applications (even in goodness-of-fit tests; see García-Pérez, 2000).

Let us suppose now that the distribution F_θ of X is the contaminated normal

$$F_\theta(y) = (1-\epsilon)\,\Phi_{\theta,\sigma}(y) + \epsilon\,\Phi_{\theta,\sqrt{k}\sigma}(y).$$

The contaminated critical value, i.e., k_n^F, can be calculated from (6.5) obtaining

$$k_n^F \simeq \theta_0 + \frac{\sigma z_\alpha}{\sqrt{n}} - \frac{\sigma\sqrt{n}}{z_\alpha}\,\epsilon\left(1 - e^{z_\alpha^2(k-1)/(2n)}\right), \tag{6.6}$$

from which we see the effect (for finite sample sizes) on the traditional critical value $k_n^\Phi = \theta_0 + z_\alpha\sigma/\sqrt{n}$, of a proportion ϵ of contaminated data in the sample coming from a normal distribution with variance $k^2\sigma^2$: If $k>1$, it is $1 - e^{z_\alpha^2(k-1)/(2n)} < 0$ and $k_n^F > k_n^\Phi$, increasing the critical value a contamination with bigger variance. On the contrary, if $k<1$, it is $1 - e^{z_\alpha^2(k-1)/(2n)} > 0$ and $k_n^F < k_n^\Phi$, decreasing the critical value a contamination with smaller variance.

This agrees with the known fact that, under the contaminated model F_θ, the variance of $\overline{x}$ is

$$\frac{\sigma^2}{n}(1+\epsilon(k-1)),$$

i.e., bigger (smaller) than the variance of $\overline{x}$ without contamination if $k>1$ ($k<1$). □

Example 6.2 (Variance test). For testing $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma > \sigma_0$ under a normal model, i.e., if X follows a normal distribution $\Phi_{\mu,\sigma} \equiv N(\mu, \sigma)$ where μ is known and σ unknown, we use the test

$$\varphi(X_1, ..., X_n) = \begin{cases} 1 & \text{if} \quad \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(X_i - \mu)^2 > k_n^F \\ & \\ 0 & \text{otherwise,} \end{cases}$$

where the (pivotal) critical value is

$$\frac{n\, k_n^F}{\sigma_0^2} = \chi^2_{n;\alpha}.$$

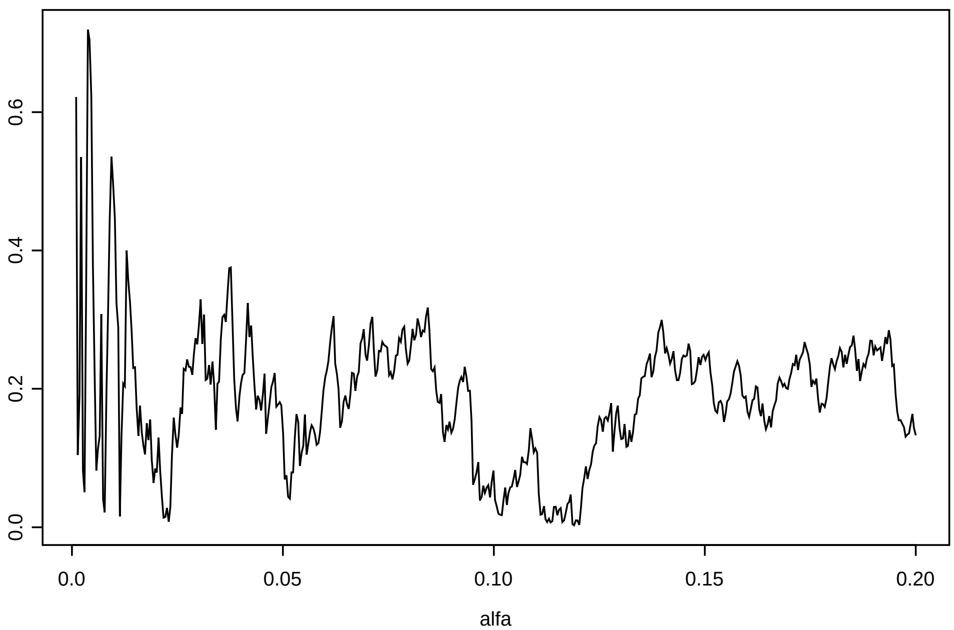

Figure 3: Simulation of the relative errors with a contaminated normal model and $n = 4$

If we consider, again, a contaminated normal distribution,

$$F_\theta(y) = (1 - \epsilon)\, \Phi_{\mu,\sigma}(y) + \epsilon\, \Phi_{\mu,\sqrt{k}\sigma}(y),$$

the VOM+SAD approximation of the critical value is

$$\frac{n\,k_n^F}{\sigma_0^2} \simeq \chi^2_{n;\alpha} + \frac{2\,n\,\chi^2_{n;\alpha}}{\chi^2_{n;\alpha} - n}\,\epsilon\left[\sqrt{\frac{n}{\chi^2_{n;\alpha}\,(1-k) + k\,n}} - 1\right],$$

from which we see the same effect as before, on the classical critical value, of a proportion ϵ of contaminated data in the sample: if $k > 1$ it is

$$\sqrt{n/(\chi^2_{n;\alpha}\,(1-k) + k\,n)} - 1 > 0.$$

□

6.1 Simulation results

We conclude the section evaluating the accuracy of the VOM+SAD approximation of the critical value (6.6), computing the relative errors for different significance levels (*alfa* in the figures), considering as model, first, the contaminated normal $0.9\,N(2,1) + 0.1\,N(2,\sqrt{1.5})$ and a sample size $n = 4$, (Figure 3), and then, the contaminated normal $0.95\,N(2,1) + 0.05\,N(2,\sqrt{3})$ and a sample size $n = 10$ (Figure 4). In both situations the exact critical value is simulated using 10.000 replications.

From these simulations we see the great accuracy of the approximations.

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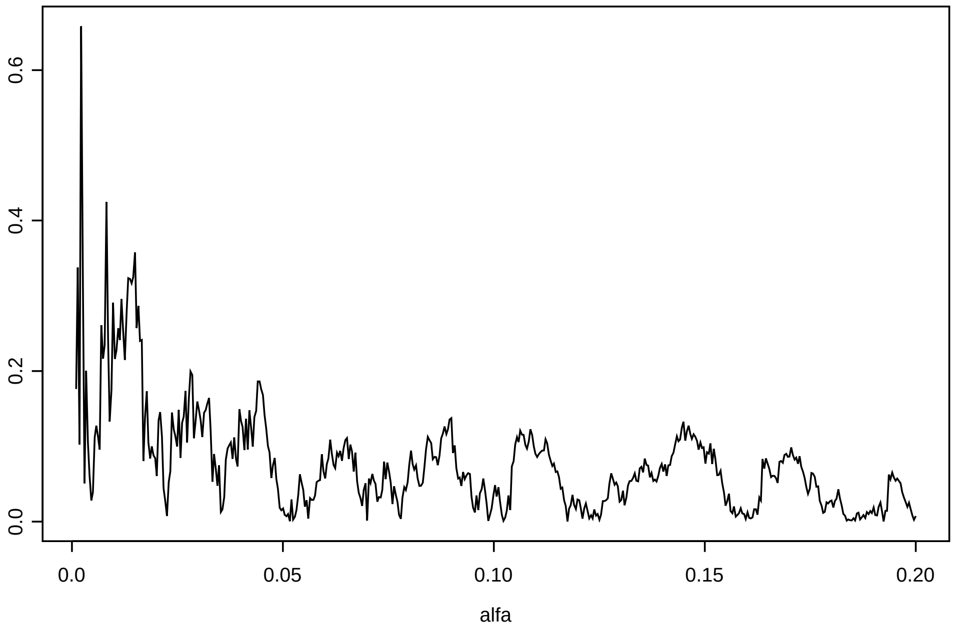

Figure 4: *Simulation of the relative errors with a contaminated normal model and* $n = 10$

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