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A new generalized Voronoi diagram is defined on the surface of a river with uniform flow; a point belongs to the territory of a site if and only if a boat starting from the site can reach the point faster than a boat starting from any other site. If the river runs slower than the boat, the Voronoi diagram can be obtained from the ordinary Voronoi diagram by a certain transformation, whereas if the river runs faster than the boat, the Voronoi diagram can be constructed by a plane sweep method.

Voronoi Diagrams in a River

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Abstract

A new generalized Voronoi diagram is defined on the surface of a river with uniform flow; a point belongs to the territory of a site if and only if a boat starting from the site can reach the point faster than a boat starting from any other site. If the river runs slower than the boat, the Voronoi diagram can be obtained from the ordinary Voronoi diagram by a certain transformation, whereas if the river runs faster than the boat, the Voronoi diagram can be constructed by a plane sweep method.

1. Introduction

The Voronoi diagram is one of fundamental concepts in applied geometry [Lee and Preparata, 1984] [Preparata and Shamos, 1985] [Boots, 1986], [Edelsbrunner, 1987] [Aurenhammer, 1988] and has many fields of application, such as geography [McLain, 1976], operations research [Iri, 1986] and pattern recognition [DeFloriani, 1989], to mention a few. Also this concept has been generalized in various directions, including Voronoi diagrams with respect to weighted distance [Aurenhammer and Edelsbrunner, 1984] [Imai, Iri and Murota, 1985], Voronoi diagrams with respect to L_p -metric [Chew and Drysdale, 1985], Voronoi diagrams generated by lines and areas [Yap, 1987] [Srinivasa and Nackman, 1987], Voronoi diagrams generated by subsets of sites [Lee, 1982] [Edelsbrunner and Seidel, 1986] and Voronoi diagrams on nonplanar surface [Mount, 1985] [Dehne

and Klein, 1987].

In this paper, we present another natural extension of the Voronoi diagram, which we call a “Voronoi diagram in a river”. The surface of a river is divided into territories of sites according to time required to reach a surface point by a boat starting from a site. We also consider algorithms for constructing the Voronoi diagram in a river. If the speed of water flow is smaller than the speed of the boat, the Voronoi diagram in a river can be obtained from the ordinary Voronoi diagram by a simple transformation. Otherwise, the Voronoi diagram can be constructed by a plane sweep method.

2. Voronoi Diagram Based on a Boat-Sail Distance

Let $S = \{s_1, s_2, \dots, s_n\}$ be a set of distinct points, called *sites*, in the plane \mathbb{R}^2 , where \mathbb{R} is the set of real numbers. Let us fix an (x, y) Cartesian coordinate system, and let (x_i, y_i) be the coordinates of site s_i . Suppose that there is a constant flow of speed w in the positive direction of the x axis. Suppose also that each site has a boat that runs at constant speed v (that is, the boat moves distance v per unit time if there is no flow). We assume that $w \geq 0$ and $v > 0$. We define

$$\alpha = \frac{w}{v}, \tag{1}$$

and call α the *relative flow speed*.

For any point $p = (x, y)$, let $T_i(p)$ denote the shortest time required by the boat to travel from site s_i to p . In this paper we sometimes call $T_i(p)$ the “boat-sail distance” from s_i to p though $T_i(p)$ does not satisfy the distance axioms. Since the flow is uniform, the shortest time is attained when the boat keeps facing in a fixed direction, in which case the boat travels along a straight line, as shown in Fig. 1. Let θ be the angle between the positive direction of the x axis and the direction in which the boat faces while it travels from s_i to p . Then, we get

$$x - x_i = T_i(p)v \cos \theta + T_i(p)w, \tag{2}$$

$$y - y_i = T_i(p)v \sin \theta. \tag{3}$$

Eliminating θ from (2) and (3), we get $T_i(p)$ explicitly. The result is as follows.

Case 1. $\alpha < 1$.

$$T_i(p) = \frac{-\alpha(x - x_i) + \sqrt{(x - x_i)^2 + (1 - \alpha^2)(y - y_i)^2}}{v(1 - \alpha^2)}. \quad (4)$$

Case 2. $\alpha = 1$.

$$T_i(p) = \begin{cases} \frac{(x - x_i)^2 + (y - y_i)^2}{2v(x - x_i)} & \text{for } x > x_i, \\ 0 & \text{for } x = x_i \text{ and } y = y_i, \\ \infty & \text{for } x \leq x_i \text{ and } y \neq y_i. \end{cases} \quad (5)$$

Case 3. $\alpha > 1$.

$$T_i(p) = \begin{cases} \frac{-\alpha(x - x_i) + \sqrt{(x - x_i)^2 + (1 - \alpha^2)(y - y_i)^2}}{v(1 - \alpha^2)} & \text{for } x - x_i \geq |y - y_i|\sqrt{\alpha^2 - 1}, \\ \infty & \text{otherwise.} \end{cases} \quad (6)$$

Fig. 2 shows the contours defined by $T_i(p) = \text{const.}$ for all the three cases. Let us note that there is a critical difference between the case where $\alpha < 1$ and the case where $\alpha \geq 1$; $T_i(p)$ has a finite value for any $p \in \mathfrak{R}^2$ if $\alpha < 1$, whereas this is not true if $\alpha \geq 1$. This intuitively corresponds to the fact that if $\alpha \geq 1$, the boat cannot travel against the flow of the river. The region in which $T_i(p) = \infty$ is represented by shaded area in the figure. The region with finite values of $T_i(p)$ forms a fan-shape area whose angle is $2 \arcsin(1/\alpha)$. Note that for $\alpha = 1$ the boundary of this region except for the site itself does not belong to the region, whereas for $\alpha > 1$ the boundary of this region belongs to the region. That is, any point p on the vertical line passing through the site s_i in Fig. 2(b) satisfies $T_i(p) = \infty$ provided that $p \neq s_i$, whereas any point p on the half lines emanating from s_i in (c) satisfies $T_i(p) < \infty$.

For each $s_i \in S$, let us define

$$V_\alpha(s_i) = \{p \mid T_i(p) < T_j(p) \text{ for any } j \neq i\}, \quad (7)$$

and call it the *Voronoi region* for s_i . Let us also define $\overline{V_\alpha(s_i)}$ by

$$\overline{V_\alpha(s_i)} = \{p \mid T_i(p) \leq T_j(p) \text{ for any } j \neq i\}. \quad (8)$$

Note that $V_\alpha(s_i)$ depends on the relative flow speed α , but not on v itself though $T_i(p)$ explicitly contains the factor $1/v$. This is because this factor is canceled out when we compare $T_i(p)$ and $T_j(p)$. The collection

$$V_\alpha(S) = \{V_\alpha(s_1), V_\alpha(s_2), \dots, V_\alpha(s_n)\} \quad (9)$$

is called the *Voronoi diagram in a river* generated by S with respect to relative flow speed α . A line segment shared by the boundary of two Voronoi regions is called a *Voronoi edge* and a point shared by the boundaries of three or more Voronoi regions is called a *Voronoi point*.

If $\alpha < 1$, $T_i(p)$ has a finite value for any point p and hence the Voronoi regions and their boundaries cover the whole plane. If $\alpha \geq 1$, on the other hand, there exists a nonempty region in which $T_i(p) = \infty$ for all $s_i \in S$, and hence the Voronoi regions and their boundaries do not exhaust the plane.

In general a Voronoi edge is composed of two types of segments. A segment of one type consists of points that have the same finite boat-sail distance from two sites. The set of points equally far from s_i and s_j is represented by equations

$$T_i(p) = T_j(p) < \infty, \quad (10)$$

which is a hyperbola. A segment of the other type comes from the boundary of the region where the boat-sail distance is infinite. The boundary of the region whose boat-sail distance from s_i is infinite is represented by

$$x - x_i = |y - y_i| \sqrt{\alpha^2 - 1} \quad (11)$$

(see equations (5) and (6)). A segment of this type arises only for $\alpha \geq 1$.

If $\alpha = 0$, $T_i(p)$ coincides with the Euclidean distance between s_i and p (up to the factor $1/v$), and hence $V_0(S)$ is the ordinary Voronoi diagram generated by S .

Fig. 3 shows the Voronoi diagrams generated by two sites s_1 and s_2 for various value of α , where s_2 is in the downstream of s_1 . As shown in (a), if $\alpha = 0$,

the Voronoi diagram in the river coincides with the ordinary Voronoi diagram and consequently the boundary of the two regions is the perpendicular bisector between the two sites. If $0 < \alpha \leq 1$, the boundary is a hyperbola, as in (b). If $\alpha = 1$, then as shown in (c), the boundary hyperbola contains the right site s_2 as its leftmost point, and also the diagram has the boundary between the Voronoi region of s_1 and the region to which the boat cannot reach; see the vertical line passing through s_1 .

Next consider the case where $\alpha > 1$ and $T_1(s_2) < \infty$, that is, $\alpha > 1$ but α is not very large and hence s_2 is reachable by a boat starting at s_1 . Then, as shown in (d), the slope of the boundary of a Voronoi region has discontinuity at the site. The boundary of $V_\alpha(s_1)$ consists of two half lines. The upper half of the boundary of $V_\alpha(s_2)$ consists of a segment of a straight line (segment s_2p) and a segment of a hyperbola (the curved line to the right of p). The lower half of the boundary of $V_\alpha(s_2)$ also consists of a straight line segment and a part of the same hyperbola, though the latter is out of the figure region. Note that the straight line segment s_2p belongs to $V_\alpha(s_2)$ but does not belong to $V_\alpha(s_1)$, because points in the line segment have smaller boat-sail distances from s_2 than from s_1 . On the other hand, the segments of the hyperbola belong to neither $V_\alpha(s_1)$ and $V_\alpha(s_2)$, because points in these segments have equal boat-sail distances from s_1 and from s_2 . If α becomes still larger so that $T_1(s_2) = \infty$, then, as shown in (e), both of the Voronoi regions $V_\alpha(s_1)$ and $V_\alpha(s_2)$ are adjacent to the leftmost area which is unreachable by any boat. In this case, the boundary of $V_\alpha(s_1)$ consists of a half line and a straight line segment (the segment s_1q in (e)) and the boundary of $V_\alpha(s_2)$ consists of a half line, a straight line segment (the segment s_2p in (e)) and part of a hyperbola.

As shown by this example, there is a great difference between the Voronoi diagram for $0 < \alpha < 1$ and that for $\alpha > 1$. If $\alpha < 1$, a point on a Voronoi edge has the equal boat-sail distance from two sites, and a Voronoi point has the equal boat-sail distance from three or more sites. If $\alpha > 1$, on the other hand, a point on a Voronoi edge does not necessarily have the equal boat-sail distance from two

sites. Indeed, if a point is on a straight-line portion of the boundary of a Voronoi region, there is only one site that has the shortest boat-sail distance to the point. For example, point p in Fig. 4(a) is equally far from the three sites s_1 , s_2 and s_3 , whereas point q in (b) is closest to s_3 only. Hence, in particular, $\overline{V_\alpha(s_i)}$ is a closed region if $0 \leq \alpha < 1$, but it is not necessarily closed if $\alpha \geq 1$.

3. Basic Properties

Let $C(x, y; r)$ denote the circle with radius r centered at (x, y) . For any α (≥ 0), point $(x + \alpha r, y)$ is called the α -center of the circle $C(x, y; r)$. If point p is the α -center of circle C , we say that C is a circle α -centered at p .

The α -center has the following physical meaning. Suppose that a boat which is initially on circle $C(x, y; r)$ moves toward the center at speed v . If there is no flow, the boat reaches the center in time r/v . If there is a flow of speed w , on the other hand, the boat is carried by $(r/v)w = \alpha r$ from the center of the circle in the direction of the flow. Thus, the α -center of $C(x, y; r)$ represents the point reached by a boat that starts from a point on the circle, initially facing toward the center. Moreover, r/v , the time required for the boat to reach the α -center, does not depend on the actual start point on the circle. Hence, the α -center p of circle $C(x, y; r)$ has the same value of $T_i(p)$ for any site s_i on $C(x, y; r)$. Thus, the α -center of a circle is the point equally “far” from any point on the circle in the sense of the boat-sail distance. This is why we chose the name “ α -center”.

Fig. 5 shows circles with a common α -center for the three cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$. It is suggested by this figure, and can be proved easily, that if $\alpha \leq 1$, a circle is contained in any larger circle with the same α -center, whereas if $\alpha > 1$, circles with a common α -center have a pair of common tangent lines that pass through the α -center.

A circle is called a *hitting circle* if it passes through at least one site in S . For any point $p \in \mathbb{R}^2$, let $F(p)$ be the smallest hitting circle centered at p , and let $f_\alpha(p)$ be the α -center of $F(p)$. Conversely, for any $q \in \mathbb{R}^2$, let $G_\alpha(q)$ be the smallest hitting circle α -centered at q , if such a circle exists, and let $g_\alpha(q)$ be the

center of $G_\alpha(q)$.

Fig. 6 shows examples of the smallest hitting circles for the case where $\alpha > 1$. (a) shows the case where there is only one site s_1 . Let us fix our attention on point p in the figure. Then, we get $F(p)$, the smallest hitting circle centered at p , $q = f_\alpha(p)$, the α -center of $F(p)$, $G_\alpha(q)$, the smallest hitting circle α -centered at q , and $p' = g_\alpha(q)$, the center of $G_\alpha(q)$, as shown in the figure. $F(p)$ and $G_\alpha(q)$ are circumscribed by the pair of half lines emanating from q forming apex angle $2 \arcsin(1/\alpha)$. In this example we see $F(p) \neq G_\alpha(f_\alpha(p))$ but $F(p') = G_\alpha(g_\alpha(p'))$. Indeed, we can see that $F(p) = G_\alpha(f_\alpha(p))$ holds if and only if $T_1(p) < \infty$, and hence $F(p) \neq G_\alpha(f_\alpha(p))$ for any point p in the shaded area in the figure.

The situation is more complicated if there are two or more sites. Fig. 6(b) shows the case where there are two sites s_1 and s_2 . Let us choose point p as shown here. Then, we get $F(p)$, $q = f_\alpha(p)$ and $G_\alpha(q)$ as illustrated. Moreover, we have two more hitting circles, C and C' , α -centered at q . In this case, the circle $F(p)$ hits s_1 while the circle $G_\alpha(q)$ hits s_2 . Thus, $F(p)$ and $G_\alpha(f_\alpha(p))$ does not necessarily coincide with each other even if $T_1(p) < \infty$.

$F(p)$ and $f_\alpha(p)$ are defined for any point p in the plane (unless S is empty). On the other hand, $G_\alpha(q)$ and $g_\alpha(q)$ are not necessarily defined; they are defined if and only if there exists at least one site s_i from which the point q is "reachable" in the sense that $T_i(q) \neq \infty$. Hence, in particular, if $\alpha < 1$, $G_\alpha(q)$ and $g_\alpha(q)$ are defined for any point q in the plane.

A circle is called an *empty circle* if it contains no site in its interior. $F(p)$ and $G_\alpha(q)$ are both empty circles, because they are the hitting circles we obtain for the first time when we continuously increase the radius of the circle from 0 to ∞ while fixing the center or the α -center, respectively, of the circle.

If the smallest hitting circle centered at p passes through site s_i , s_i is the nearest site from p in the sense of Euclidean distance and hence p is in the region $\overline{V_0(s_i)}$ in the ordinary Voronoi diagram (s_i is in $V_0(s_i)$ if s_i is the only site on the circle, while s_i is on the boundary of $V_0(s_i)$ if another site is also on the circle). Similarly, if the smallest hitting circle α -centered at p passes through site s_i , s_i

is the nearest site from p in the sense of the boat-sail distance and hence p is in the region $\overline{V_\alpha(s_i)}$ in the Voronoi diagram in the river.

Proposition 1. Suppose that $p, q \in \mathbb{R}^2$ and $F(p) = G_\alpha(q)$. Then, $p \in \overline{V_0(s_i)}$ if and only if $q \in \overline{V_\alpha(s_i)}$.

Proof. $p \in \overline{V_0(s_i)}$ is equivalent to the fact that s_i is on $F(p)$, the smallest hitting circle centered at p . Similarly, $q \in \overline{V_\alpha(s_i)}$ is equivalent to the fact that s_i is on $G_\alpha(q)$, the smallest hitting circle α -centered at q . Since $F(p) = G_\alpha(q)$, $p \in \overline{V_0(s_i)}$ is equivalent to $q \in \overline{V_\alpha(s_i)}$.

□

Proposition 1 tells that the nearest site to p in the sense of Euclidean distance coincides with the nearest site to q in the sense of the boat-sail distance, if and only if the smallest hitting circle centered at p coincides with the smallest hitting circle α -centered at q . Hence, if we find the correspondence between such a pair of points p and q , we can reduce the construction of $\mathcal{V}_\alpha(S)$ to the construction of $\mathcal{V}_0(S)$. So, our next question is when $F(p) = G_\alpha(q)$ holds.

Proposition 2. For any $q \in \mathbb{R}^2$ for which $g_\alpha(q)$ is defined, $G_\alpha(q) = F(g_\alpha(q))$ holds.

Proof. Let $p = g_\alpha(q)$. By the definition, $G_\alpha(q)$ is the smallest hitting circle α -centered at q , and p is its center. Hence, $G_\alpha(q)$ is a hitting circle centered at p . Assume that $G_\alpha(q) \neq F(p)$. Since $F(p)$ is a hitting circle, there exists at least one site, say s_i , on $F(p)$. On the other hand, $F(p)$ and $G_\alpha(q)$ has the common center p , and hence $F(p)$ is contained in the interior of $G_\alpha(q)$, and so is s_i . However, this is impossible because $G_\alpha(q)$ is the smallest hitting circle. □

Proposition 3. Suppose that $\alpha < 1$. Then, for any $p \in \mathbb{R}^2$, $F(p) = G_\alpha(f_\alpha(p))$.

Proof. Let $q = f_\alpha(p)$. The circle $F(p)$ is a hitting circle α -centered at q , but not necessarily smallest in general. Assume that $F(p) \neq G_\alpha(q)$. Since $G_\alpha(q)$ is

a hitting circle, there exists at least one site, say s_i , on $G_\alpha(q)$. Since $\alpha < 1$, a circle α -centered at q is contained in the interior of any larger circle α -centered at q . Hence, $G_\alpha(q)$ is contained in $F(p)$, and consequently so is s_i . However, this is impossible because $F(p)$ is the smallest hitting circle. \square

proposition 4. Suppose that $\alpha = 1$. Then, for any $p \in \mathbb{R}^2$ such that $f_\alpha(p)$ does not coincide with any site, $F(p) = G_\alpha(f_\alpha(p))$ holds.

Proof. Let $q = f_\alpha(p)$. Since $\alpha = 1$, q is the rightmost point on the circle $F(p)$. Assume that $F(p) \neq G_\alpha(q)$. By the definition $G_\alpha(q)$ is the smallest hitting circle α -centered at q , the circle $F(p)$ should be larger than $G_\alpha(q)$. Since $G_\alpha(q)$ is a hitting circle, there exists at least one site, say s_i , on $G_\alpha(q)$. On the other hand, $F(p)$ is an empty circle and hence should not contain s_i in its interior. This implies that s_i is on both the circles $F(p)$ and $G_\alpha(q)$; this is possible only when $s_i = f_\alpha(p)$. Thus we get the proposition. \square

Combining Proposition 1 with Proposition 2, 3 and 4, we immediately get the following propositions.

Proposition 5. For any point $q \in \mathbb{R}^2$ for which $g_\alpha(q)$ is defined, $q \in \overline{V_\alpha(s_i)}$ if and only if $g_\alpha(q) \in \overline{V_0(s_i)}$.

Proof. Suppose that $q \in \mathbb{R}^2$ is a point such that $g_\alpha(q)$ is defined, and let $p = g_\alpha(q)$. Then, from Proposition 2, we get $G_\alpha(q) = F(p)$ and consequently $q = f_\alpha(p)$. Hence, the proposition follows from Proposition 1. \square

Proposition 6. Suppose that $\alpha \leq 1$. Then for any $p \in \mathbb{R}^2$, $p \in \overline{V_0(s_i)}$ if and only if $f_\alpha(p) \in \overline{V_\alpha(s_i)}$.

Proof. For any p , let $q = g_\alpha(p)$. Case 1. First suppose that $\alpha < 1$. Then, from Proposition 3 we get $F(p) = G_\alpha(q)$, and hence it follows from Proposition 1 that $p \in \overline{V_0(s_i)}$ if and only if $q \in \overline{V_\alpha(s_i)}$. Case 2. Next suppose that $\alpha = 1$. Case 2a. If $f_\alpha(p) \neq s_i$ for any site s_i , it follows from Proposition 4 that $F(p) = G_\alpha(f_\alpha(p))$,

and hence from Proposition 1 we get that $p \in \overline{V_0(s_i)}$ if and only if $q \in \overline{V_\alpha(s_i)}$. Case 2b. If $f_\alpha(p) = s_i$ for some site s_i , we get directly from the definition of the Voronoi diagram in the river that $p \in \overline{V_0(s_i)}$ and $q \in \overline{V_\alpha(s_i)}$. If, beside s_i , there is also another site s_j on the circle $F(p)$, p is the midpoint of the line segment connecting s_i and s_j , and hence $p \in \overline{V_0(s_j)}$ and $q \in \overline{V_\alpha(s_j)}$. Thus, we get the proposition. \square

Proposition 6 says that if $\alpha \leq 1$, the mapping f_α gives the one-to-one correspondence between the Voronoi diagram in a river and the ordinary Voronoi diagram, and hence, in particular, the topological structure of the Voronoi diagram in a river is isomorphic to the topological structure of the ordinary Voronoi diagram. If $\alpha = 0$, the Voronoi diagram in a river coincides with the ordinary Voronoi diagram. For positive α , the Voronoi diagram is deformed by the stream of water, but if α is small, the deformation is also small, so that we can expect that the resultant diagram does not differ much from the ordinary Voronoi diagram. Proposition 6 states this intuition more explicitly; if $\alpha \leq 1$, the deformation does not give any topological change.

On the other hand, Proposition 5 says that if $\alpha > 1$, the Voronoi diagram in a river reflects the ordinary Voronoi diagram only in the area at which g_α is defined.

4. Interpretation as a Forest of Cones

The ordinary Voronoi diagram can be considered as an orthographic projection of a three-dimensional scene composed of cones. In this section we will see that this way of interpretation can be generalized for any α .

Consider (x, y, z) Cartesian coordinate system fixed to the three-dimensional space, and suppose that the sites s_1, s_2, \dots, s_n are given in the x - y plane. For each site s_i , we consider the cone defined by

$$z = -\sqrt{(x - x_i)^2 + (y - y_i)^2}. \quad (12)$$

As shown in Fig. 7(a), this cone has the apex at s_i , has apex angle $\pi/2$ and is

open downward. The intersection of this cone with plane $z = -t$, where t is a positive constant, forms a circle, and the orthographic projection of this circle onto the x - y plane can be considered as the front of the wave at time t created at s_i at time 0 and growing at the unit speed in every direction.

The cones associated with all the sites intersect each other and altogether form a “forest of cones”. Assume that the cones are made of opaque surface. Then, the ordinary Voronoi diagram coincides with the picture obtained by seeing this forest of cones from the view point at infinity in the positive direction of the z axis. This is because the cones can be considered as the fronts of the waves created at all the sites simultaneously, and a Voronoi edge is a set of points at which two wave fronts meet.

Recall that if $\alpha > 0$, the front of the wave created at s_i still forms a circle but the center is displaced from s_i . If the radius of the circle is r , the center is at $(x_i + \alpha r, y_i)$. Hence, as shown in Fig. 7(b), the front of the wave at various times can be obtained by viewing the cone in the direction

$$\mathbf{d}_\alpha = \left(0, \frac{-\alpha}{\sqrt{1 + \alpha^2}}, \frac{-1}{\sqrt{1 + \alpha^2}}\right). \quad (13)$$

In particular, if $\alpha = 0$, $\mathbf{d}_\alpha = (0, 0, -1)$, which coincides with the direction of the orthographic projection, and if $\alpha = 1$, $\mathbf{d}_\alpha = (1, -1/\sqrt{2}, -1/\sqrt{2})$, which is parallel to the y - z plane and forms angle $\pi/4$ with both the y axis and the z axis. Hence, the Voronoi diagram with respect to the relative flow speed α can be obtained by projecting the visible portion of the forest of cones in the direction parallel to \mathbf{d}_α onto the x - y plane.

If $0 \leq \alpha < 1$, the surface of the cone is entirely visible from the viewer who sees the cone in the direction \mathbf{d}_α unless the surface is intersected by other cones. Hence, the topological structure of the diagram does not change while the view direction \mathbf{d}_α is altered within $0 \leq \alpha < 1$. If $\alpha > 1$, on the other hand, a portion of the surface is occluded by the cone itself, and consequently the topological structure of the resultant diagram differs from the ordinary Voronoi diagram. This is another way of intuitively understanding Proposition 6.

5. Algorithms

On the basis of our observations, here we design algorithms for constructing the Voronoi diagrams in the river. First, let us consider the case where $\alpha \leq 1$. Proposition 6 tells us that in this case a Voronoi region in the Voronoi diagram in the river is obtained from the Voronoi region for the same site in the ordinary Voronoi diagram by mapping f_α , that is, $\overline{V_\alpha(s_i)} = f_\alpha(\overline{V_0(s_i)})$. Hence we get the next algorithm.

Algorithm 1 (Voronoi diagram with relative flow speed $\alpha \leq 1$)

Input: Set $S = \{s_1, s_2, \dots, s_n\}$ of sites, and α ($0 < \alpha \leq 1$).

Output: Voronoi diagram $\mathcal{V}_\alpha(S)$.

Procedure:

Step 1. Construct the ordinary Voronoi diagram $\mathcal{V}_0(S)$.

Step 2. Transform Voronoi edges and Voronoi points in $\mathcal{V}_0(S)$ by mapping f_α , and return the resulting diagram.

For Step 1 we can employ existing algorithms for ordinary Voronoi diagrams: for example, an incremental algorithm [Ohya, Iri and Murota, 1984] which runs in $O(n)$ time on the average for uniformly distributed sites, its robust version [Sugihara and Iri, 1989a, 1989b], a divide-and-conquer algorithm [Shamos and Hoey, 1975] [Guibas and Stolfi, 1985] which runs in $O(n \log n)$ time both in the worst-case sense and in the average-case sense, its improved version [Katajainen and Koppinen, 1988] which runs in $O(n \log n)$ time in the worst case and in $O(n)$ time on the average, and its robust version [Ooishi, 1990] [Sugihara, Ooishi and Imai, 1990]. The number of Voronoi edges and Voronoi vertices in the ordinary Voronoi diagram is of $O(n)$, and hence if we assume that each Voronoi edge is mapped by f_α in constant time, Step 2 can be done in $O(n)$ time. Thus, Algorithm 1 runs in $O(n \log n)$ time in the worst case and in $O(n)$ time on the average.

Next let us consider the case where $\alpha > 1$. If $\alpha \geq 1$, the Voronoi diagram

in a river has the remarkable property that any site s_i is the leftmost point of the corresponding region $\overline{V_\alpha(s_i)}$. This can be understood easily if we note that all the points reachable from site s_i form a fan-shape area open rightward from the apex at s_i , as shown in Fig. 2(b) and (c), and that the region $\overline{V_\alpha(s_i)}$ is contained in this fan-shape area. This property implies that the plane sweep method, one of fundamental techniques in computational geometry, can be used for the construction of the Voronoi diagram.

In the plane sweep method the plane is swept by a vertical line, called a sweepline, from left to right, and every time a new event happens a problem at hand is solved partially along the sweepline. Hence, by this method we can reduce a two-dimensional problem to a set of “almost one-dimensional” problems. Since site s_i is the leftmost point of the region $\overline{V_\alpha(s_i)}$, all the events related with this region happen only after the sweepline hits s_i . Thus, it is natural to apply the plane sweep method to the construction the Voronoi diagram.

Indeed, the plane sweep method for constructing ordinary Voronoi diagrams proposed by Fortune (1986, 1987) is essentially a method for constructing the Voronoi diagram in the river for $\alpha = 1$. In his method, first the Voronoi diagram in the river is constructed by the plane sweep technique in $O(n \log n)$ time, and next it is transformed by the mapping g_α to get the ordinary Voronoi diagram, where $\alpha = 1$.

To construct the plane sweep method for the Voronoi diagram for $\alpha > 1$, let us consider the boundary of two Voronoi regions in more detail. Let s_i and s_j be two sites such that $x_i \geq x_j$. If there is no other sites, the boundary of Voronoi region of s_i can be divided at s_i into two portions, the portion extending in the right upper direction and the portion extending in the right lower direction. We call them the *upper wing* and the *lower wing*, respectively, of s_i with respect to s_j and denote them by $w^+(s_i, s_j)$ and by $w^-(s_i, s_j)$; see Fig. 8.

In general, the upper and lower wings consist of part of a straight line and part of a hyperbola. As shown in Fig. 8, let l^+ and l^- be two half lines emanating from s_i with angle $\arcsin(1/\alpha)$ and $-\arcsin(1/\alpha)$, respectively, with respect to

the x axis. As we saw in equation (11), the left portions of the wings coincide with those half lines. Let l_{ij} be the perpendicular bisector of s_i and s_j , and let p^+ and p^- be the points of intersection between l_{ij} and the half lines emanating from s_i with angle $\pi/2 + \arcsin(1/\alpha)$ and $3\pi/2 - \arcsin(1/\alpha)$, respectively. Furthermore, let q^+ and q^- be the points of intersection between l^+ and the horizontal line passing through p^+ and between l^- and the horizontal line passing through p^- , respectively. Then, from equations (5) and (6) we can see that the upper wing $w^+(s_i, s_j)$ is composed of the straight line segment $s_i q^+$ and part of the hyperbola defined by equation (10), and similarly the lower wing $w^-(s_i, s_j)$ is composed of the straight line segment $s_i q^-$ and part of the same hyperbola. Another way to understand this is to interpret the diagram as the picture of the forest of cones. If we consider a picture of the cone seen from above, the triangle $s_i p^+ p^-$ corresponds to the picture of the surface of the cone that are invisible from the viewer with view direction \mathbf{d}_α , and hence the line segment $s_i p^+$ and $s_i p^-$ (when considered as the line segments on the surface of the cone) correspond to the silhouette of the cone for that viewer. Thus, if the cone is projected in the direction parallel to \mathbf{d}_α , the line segments $s_i q^+$ and $s_i q^-$ are the images of the silhouette of the cone and the other parts of the wings are the images of the line of intersection of the two cones associated with s_i and s_j .

Fig. 8 illustrates the case where $T_j(s_i) < \infty$. If $T_j(s_i) = \infty$ and $y_i < y_j$ (this happens when the slope of l_{ij} is positive and smaller than the slope of l^+), q^- goes to infinity and $w^-(s_i, s_j)$ becomes the half line l^- . Similarly, if $T_j(s_i) = \infty$ and $y_i > y_j$, q^+ goes to infinity and $w^+(s_i, s_j)$ becomes the half line l^+ .

We formally define $w^+(s_i, s_\infty) = l^+$ and $w^-(s_i, s_\infty) = l^-$. This intuitively corresponds to that we introduce new site s_∞ that is at infinity in the negative direction of the x axis and consider $w^+(s_i, s_\infty)$ and $w^-(s_i, s_\infty)$ as the wings of s_i with respect to s_∞ .

Now, we are ready to describe the plane sweep method. Suppose that we are given the set of sites. In the preprocessing stage, we first put the sites in priority queue Q , and next choose and delete from Q the site, say s_i , with the smallest x

coordinate, and generate list

$$L = (V_\alpha(s_\infty), w^-(s_i, s_\infty), V_\alpha(s_i), w^+(s_i, s_\infty), V_\alpha(s_\infty)),$$

which represents the alternating sequence of regions and wings which one meets as one travels from the bottom upward along the vertical sweepline that lies just to the right of s_i ; see vertical line L_1 in Fig. 9. In the main stage of the processing, while Q is not empty we choose and delete from Q the point p with the smallest x coordinate and modify the list L according to the change that happens when the sweepline hits p from left to right. That is, if p is a site, say s_j , we change the list L so that $V_\alpha(s_j)$ and its upper and lower wings are inserted in an appropriate position, and also we add to Q the points of intersection of the newly inserted wings with the wings adjacent in L if they exist (see sweepline L_2 in the figure). If p is a point of intersection of two wings, we change L so that one of Voronoi regions and its upper and lower boundaries are replaced by a new wing (see sweepline L_3 in the figure). During the processing, all the changes in L are stored and thus the Voronoi diagram is constructed.

We refer to Fortune (1986, 1987) for the formal description of the algorithm and its analysis, because the algorithm is the same except for the actual shape of the upper and lower wings. Thus, because of the same reason as in Fortune (1986, 1987), we can construct the Voronoi diagram with respect to the relative flow speed α ($\alpha > 1$) in $O(n \log n)$ time.

Finally, some examples of Voronoi diagrams in a river are shown in Figs. 10 and 11. Fig. 10 shows the Voronoi diagram for (a) $\alpha = 0$, (b) $\alpha = 0.5$, (c) $\alpha = 1.0$, and (d) $\alpha = 2.0$, generated by the set of 30 sites randomly located in a square region. Fig. 11 shows the Voronoi diagram for (a) $\alpha = 0.0$, (b) $\alpha = 0.8$, and (c) $\alpha = 1.5$ generated by 500 sites randomly located in a square.

6. Concluding Remarks

The concept of the Voronoi diagram is extended to that in a river, where each site generates its Voronoi region according to the boat-sail distance. The Voronoi diagram thus extended has one parameter α , which is the ratio of the speed of

the water stream to the speed of a boat.

For $\alpha = 0$, the Voronoi diagram in the river coincides with the ordinary Voronoi diagram. As the speed of the water stream increases, the Voronoi diagram in the river changes its shape gradually. However, while $\alpha \leq 1$, the change is not drastic in the sense that the topological structure of the diagram remains unchanged; consequently we can construct the Voronoi diagram in the river from the ordinary Voronoi diagram by a simple transformation. When the speed of the river stream is faster than that of the boat (i.e., $\alpha > 1$), the topological structure of the ordinary Voronoi diagram is no more preserved, and hence the Voronoi diagram cannot be constructed by a simple transformation from the ordinary Voronoi diagram. Fortunately, however, Voronoi regions are generated only to the downstream of the corresponding sites, and consequently the plane sweep method can be applied naturally for the construction of the Voronoi diagram.

For $\alpha = 1$, the Voronoi diagram coincides with the diagram used by Fortune (1986, 1977) in his plane sweep algorithm for constructing the ordinary Voronoi diagram. Indeed, the actual motivation of the present work was to give physical meaning to the transformation used in Fortune's plane sweep method. It seems interesting to note that among all possible values of the parameter α , $\alpha = 1$ is the only case where the topological structure of the ordinary Voronoi diagram is preserved and still the plane sweep method can be applied.

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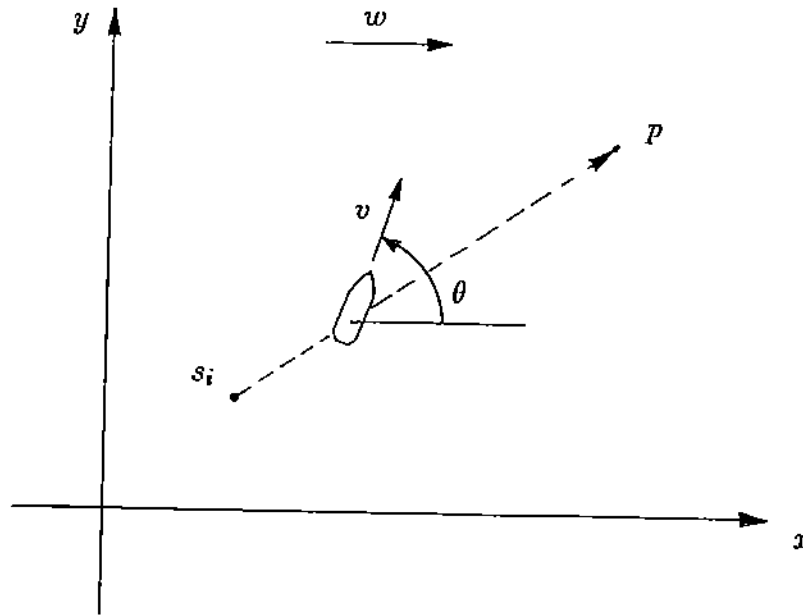


Fig. 1. Shortest path for a boat in a river.

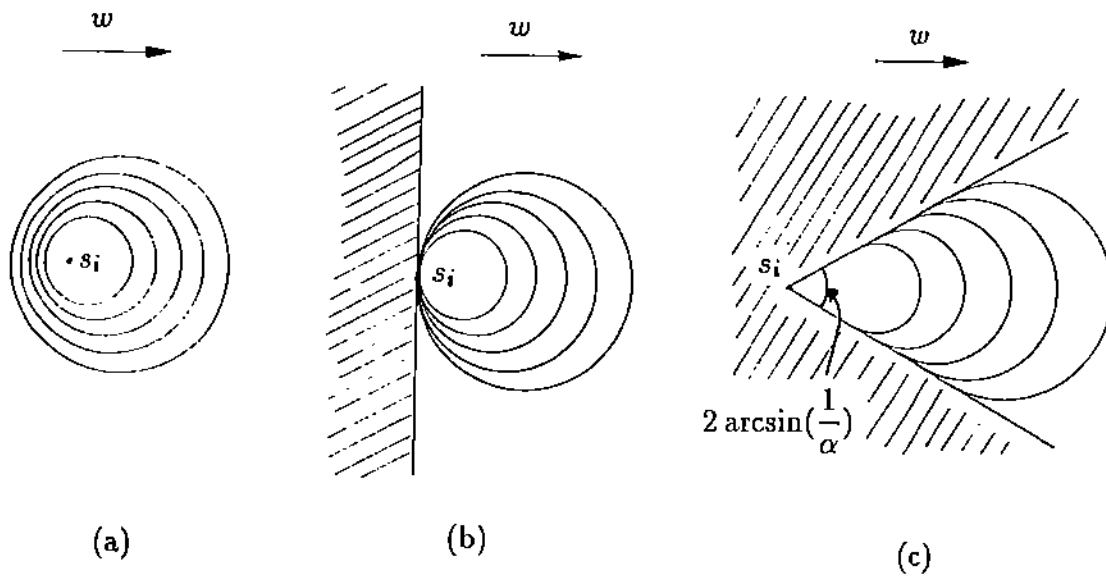


Fig. 2. Contour lines generated by $T_i(p) = \text{const.}$: (a) $0 \leq \alpha < 1$; (b) $\alpha = 1$; (c) $\alpha > 1$.

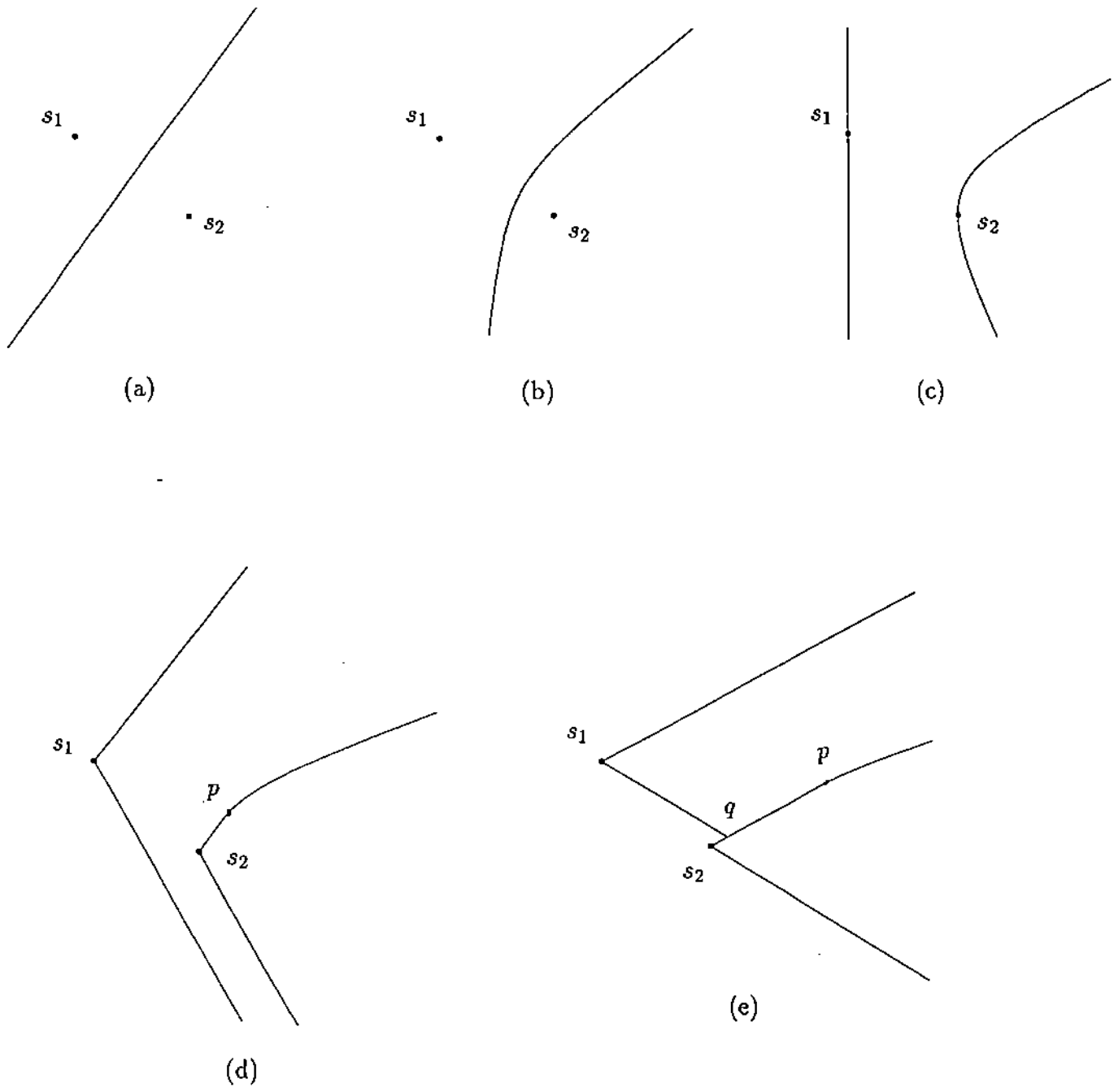


Fig. 3. Voronoi diagram in a river generated by two sites: (a) $\alpha = 0$; (b) $\alpha = 0.5$; (c) $\alpha = 1.0$; (d) $\alpha = 1.2$; (e) $\alpha = 2.0$.

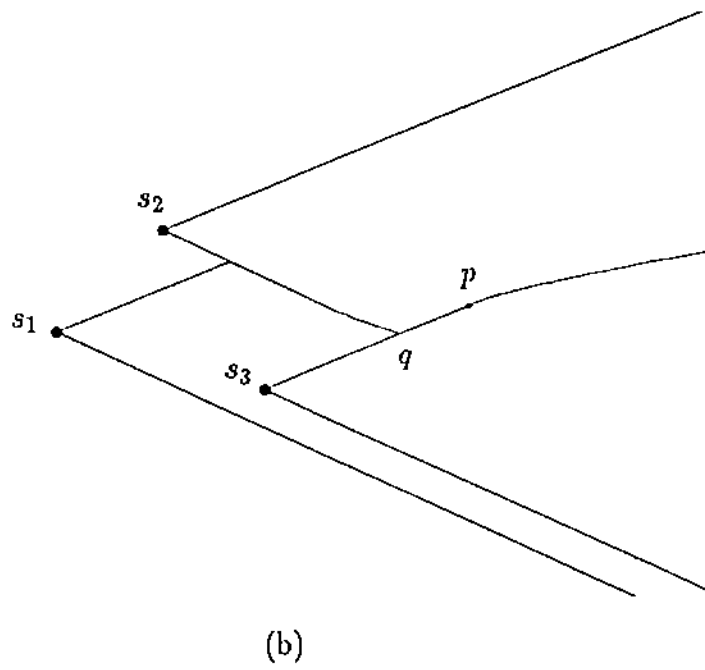
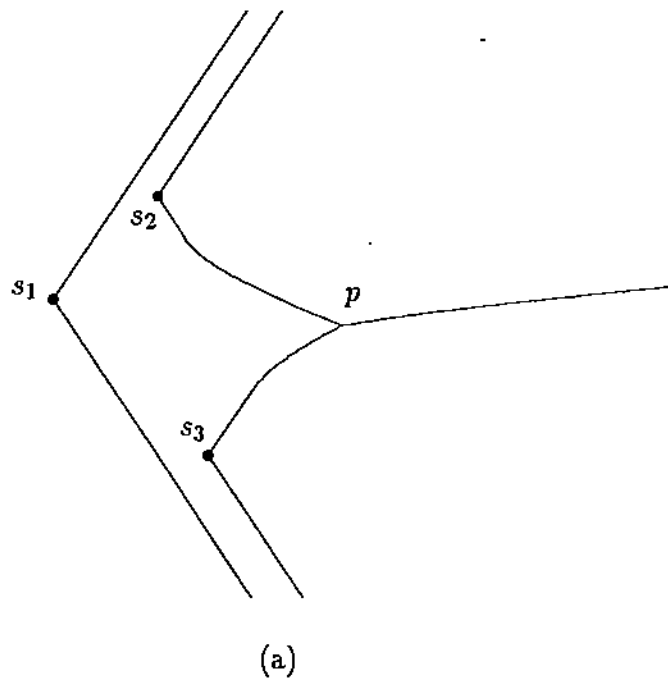


Fig. 4. Two types of Voronoi vertices for $\alpha > 1$.

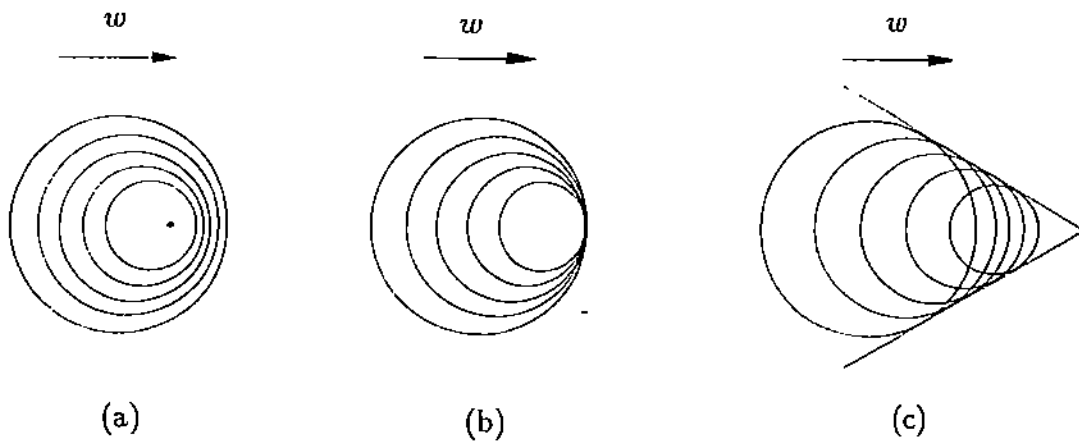


Fig. 5. Circles with a common α -center: (a) $0 \leq \alpha < 1$; (b) $\alpha = 1$; (c) $\alpha > 1$.

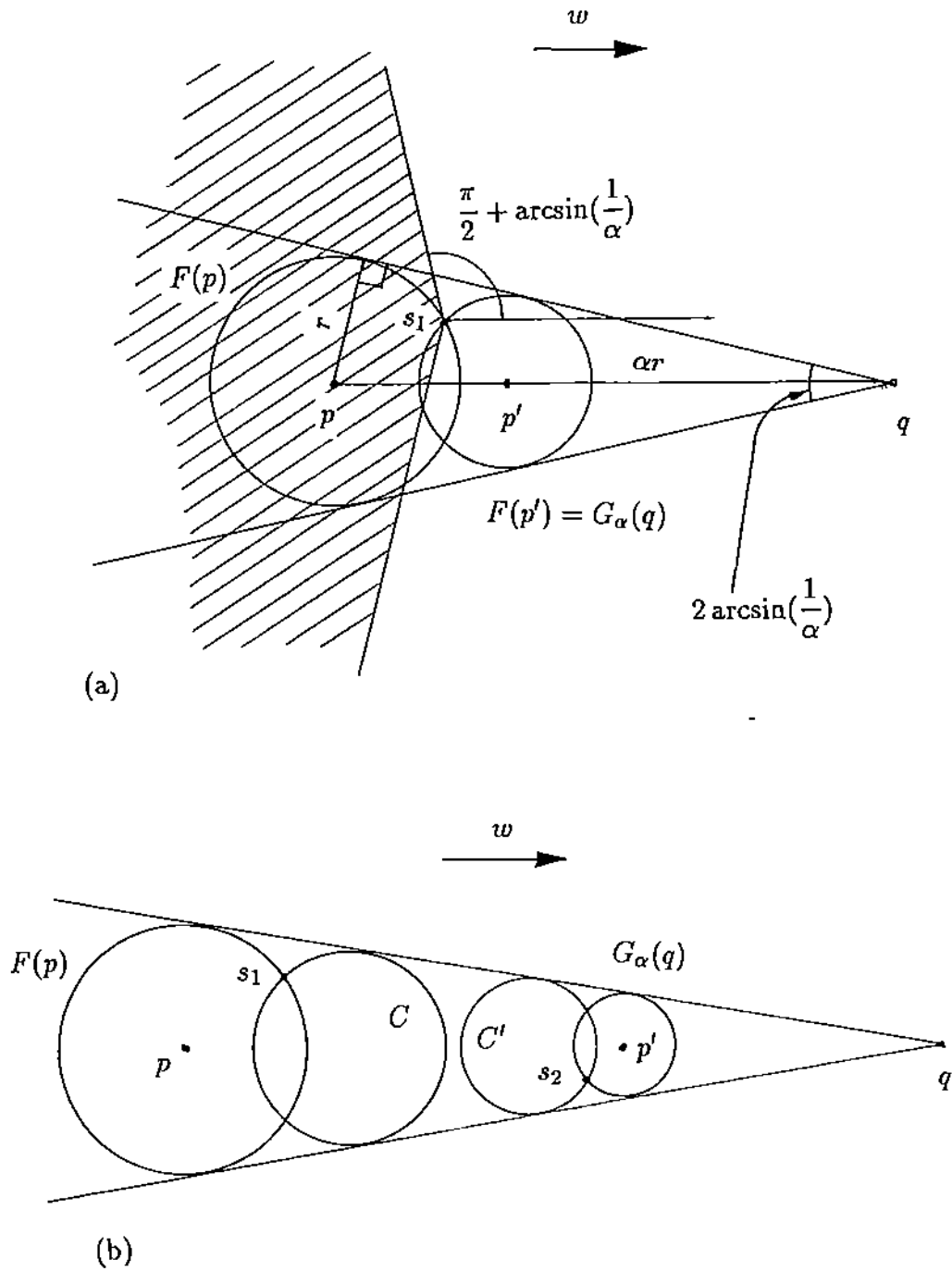


Fig. 6. Smallest hitting circle: (a) the case where there is only one site; (b) the case where there are two sites.

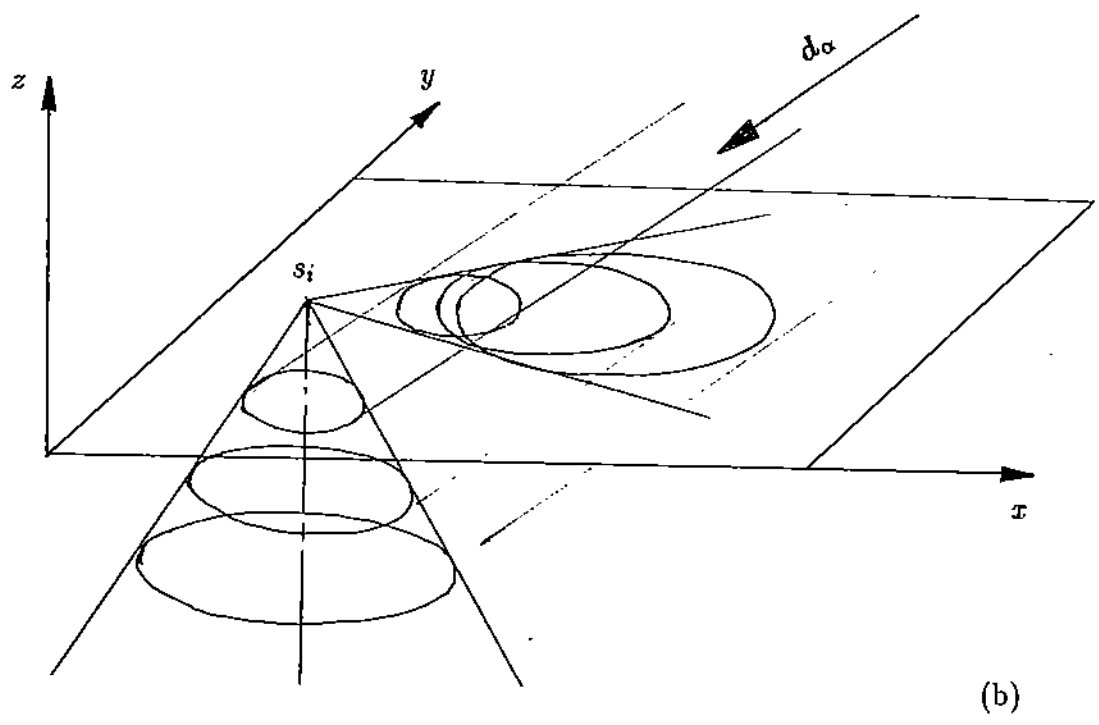
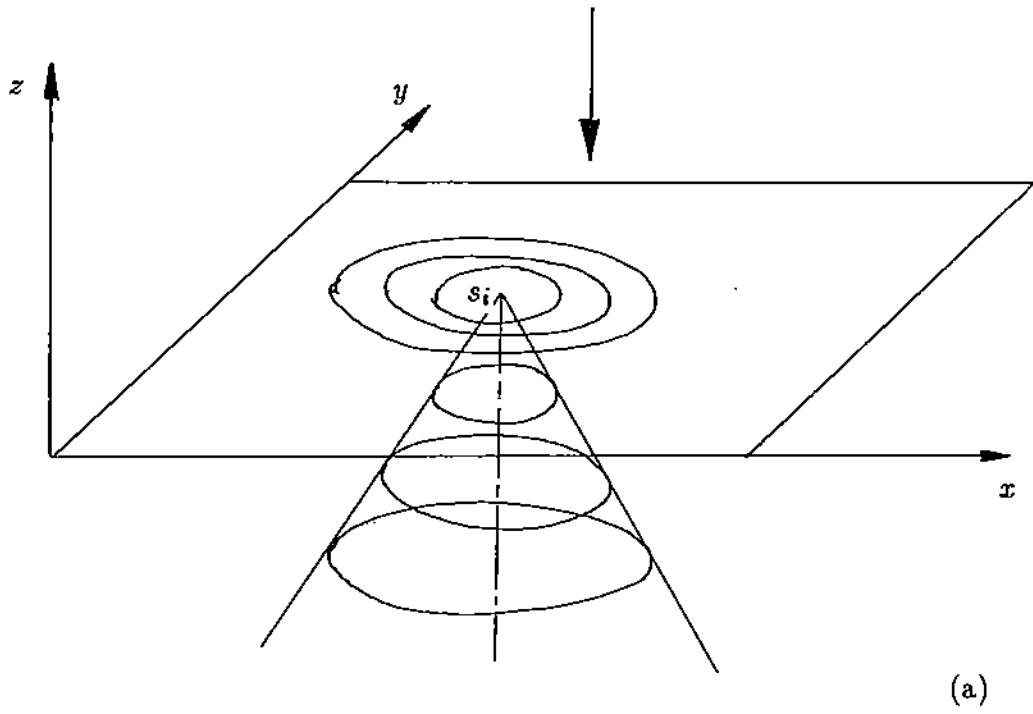


Fig. 7. Interpretation as a picture of a cone: (a) cone and its orthographic projection; (b) cone and its oblique projection.

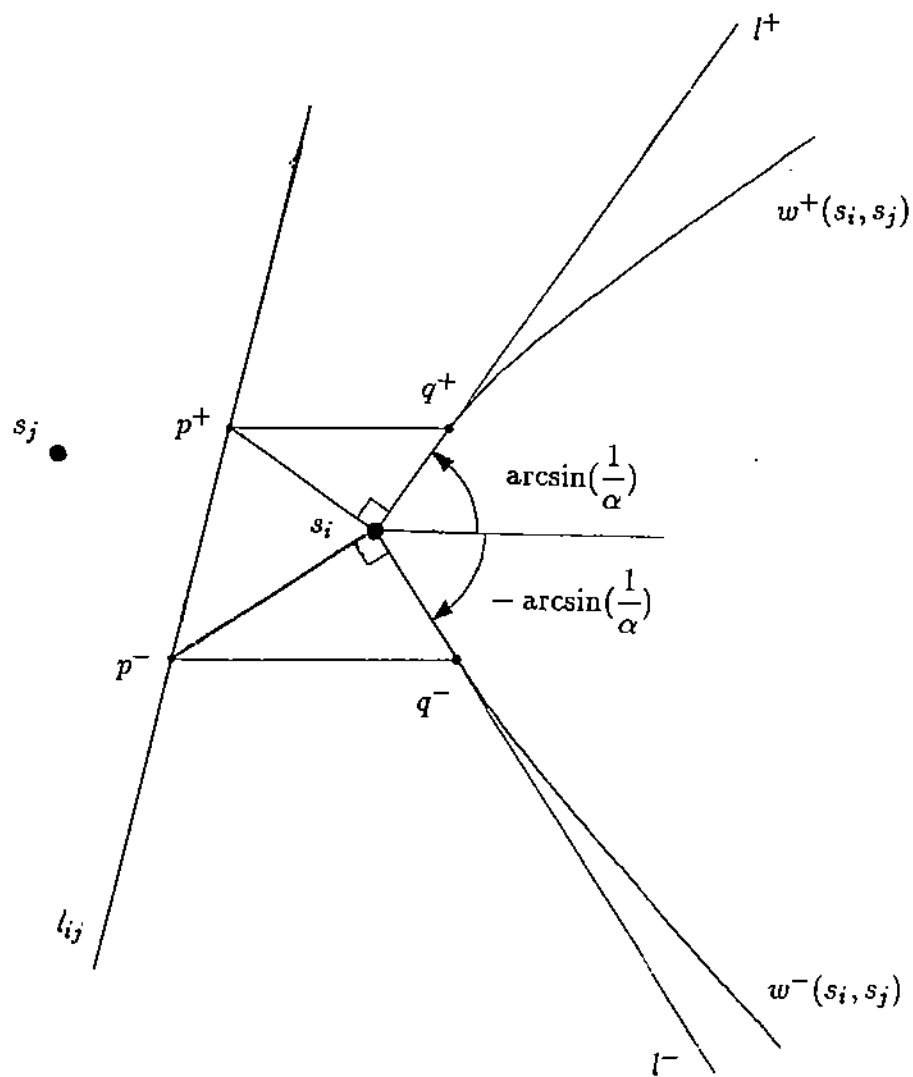


Fig. 8. Upper and lower wings constituting the boundary of a Voronoi region in a Voronoi diagram generated by two sites.

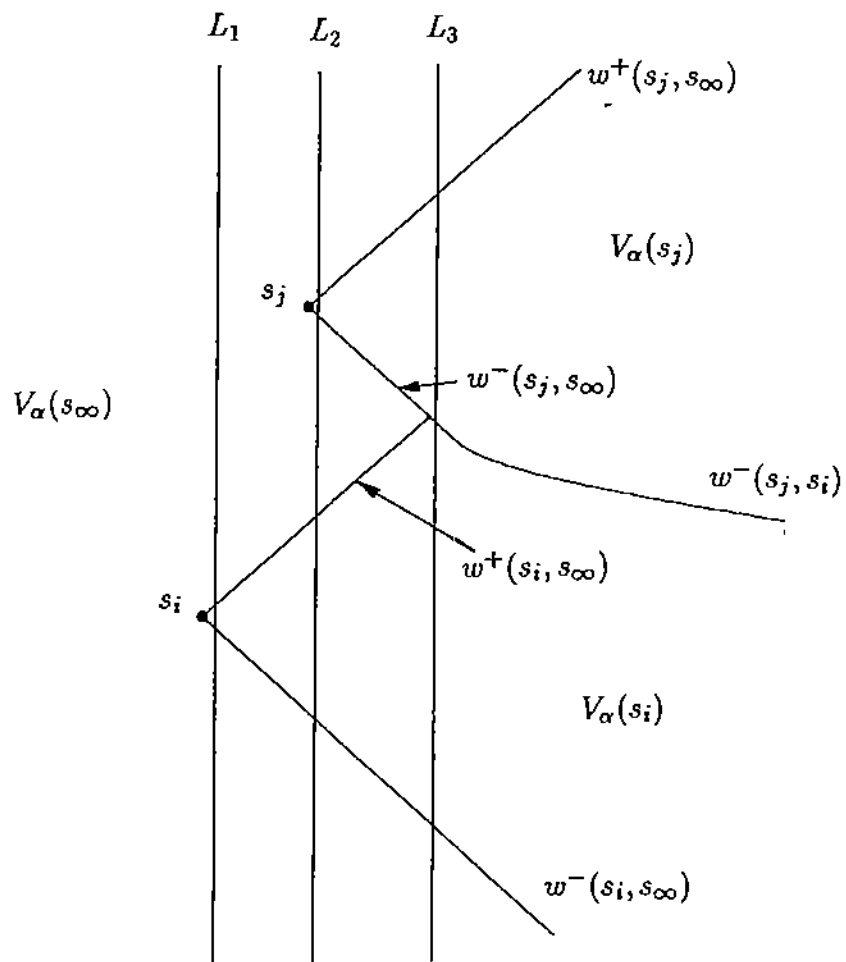


Fig. 9. Plane sweep method.

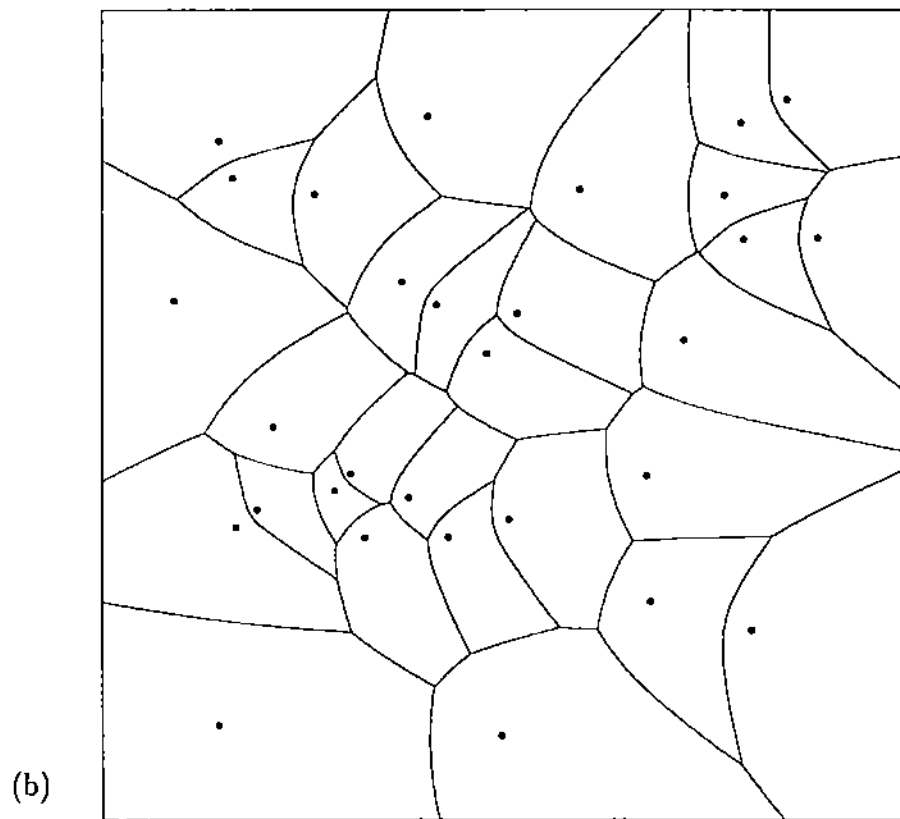
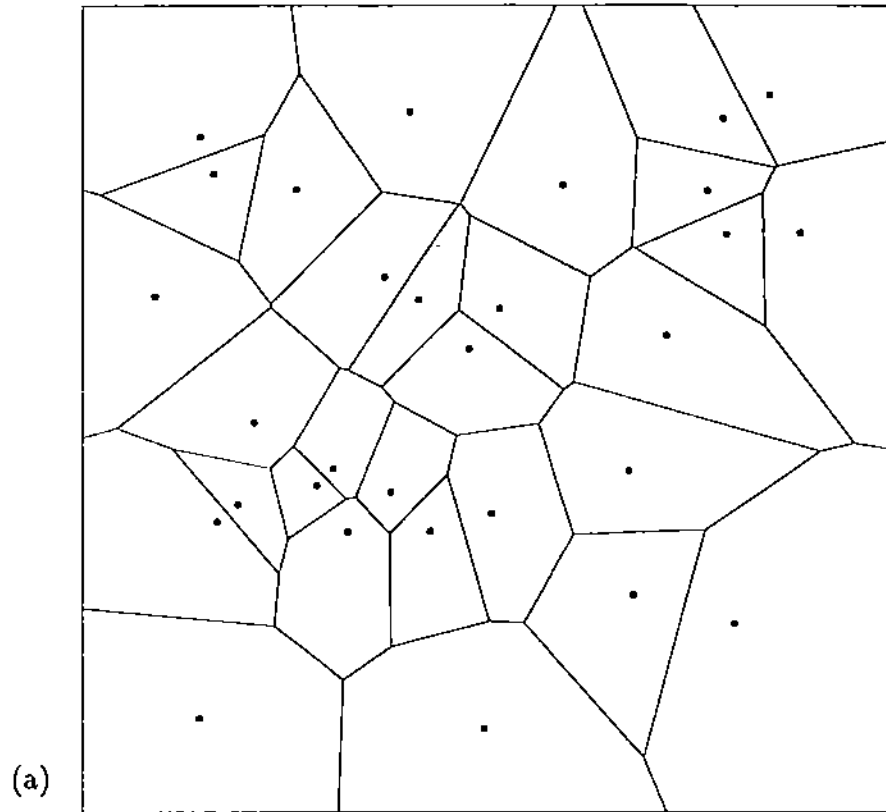
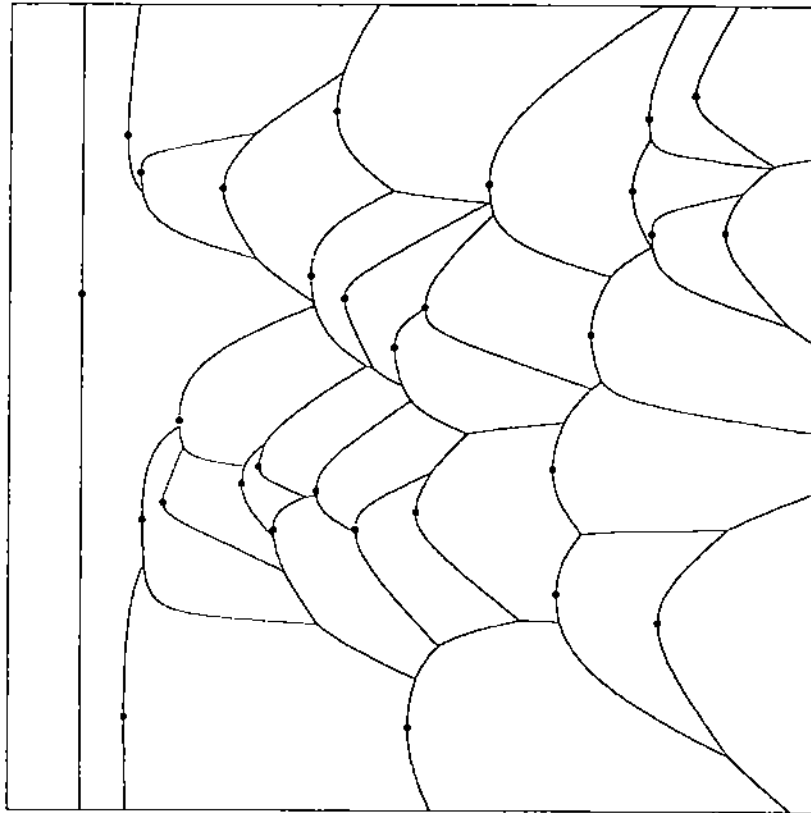


Fig. 10. Voronoi diagram in a river generated by 30 sites: (a) $\alpha = 0.0$; (b) $\alpha = 0.5$; (c) $\alpha \approx 1.0$; (d) $\alpha = 2.0$.

(c)



(d)

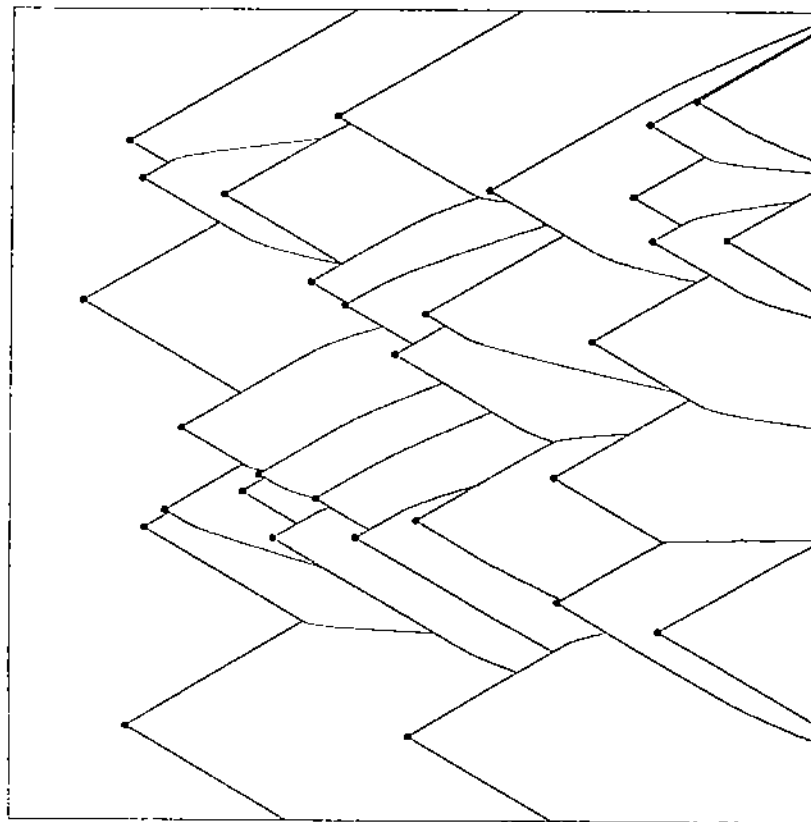
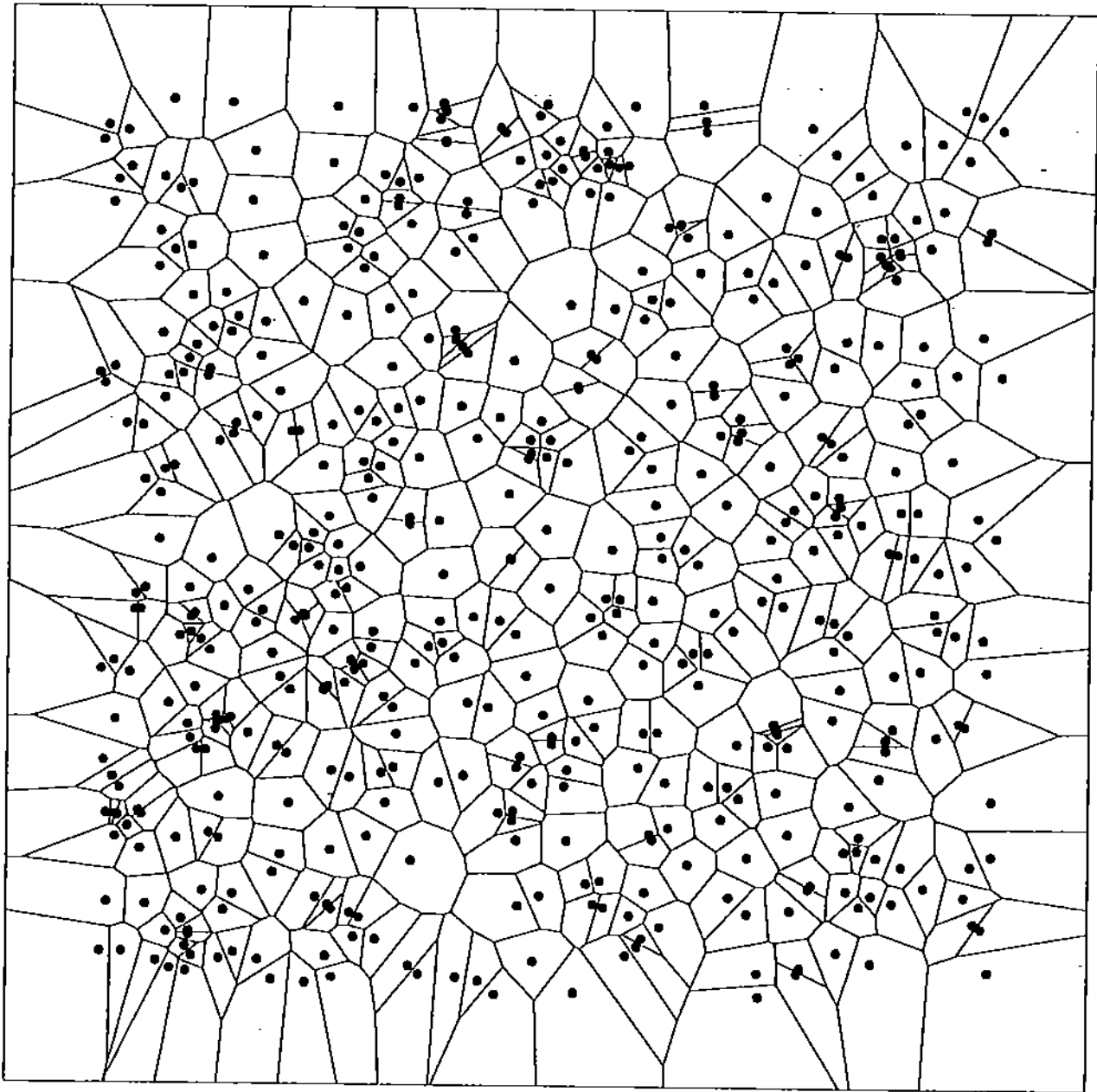
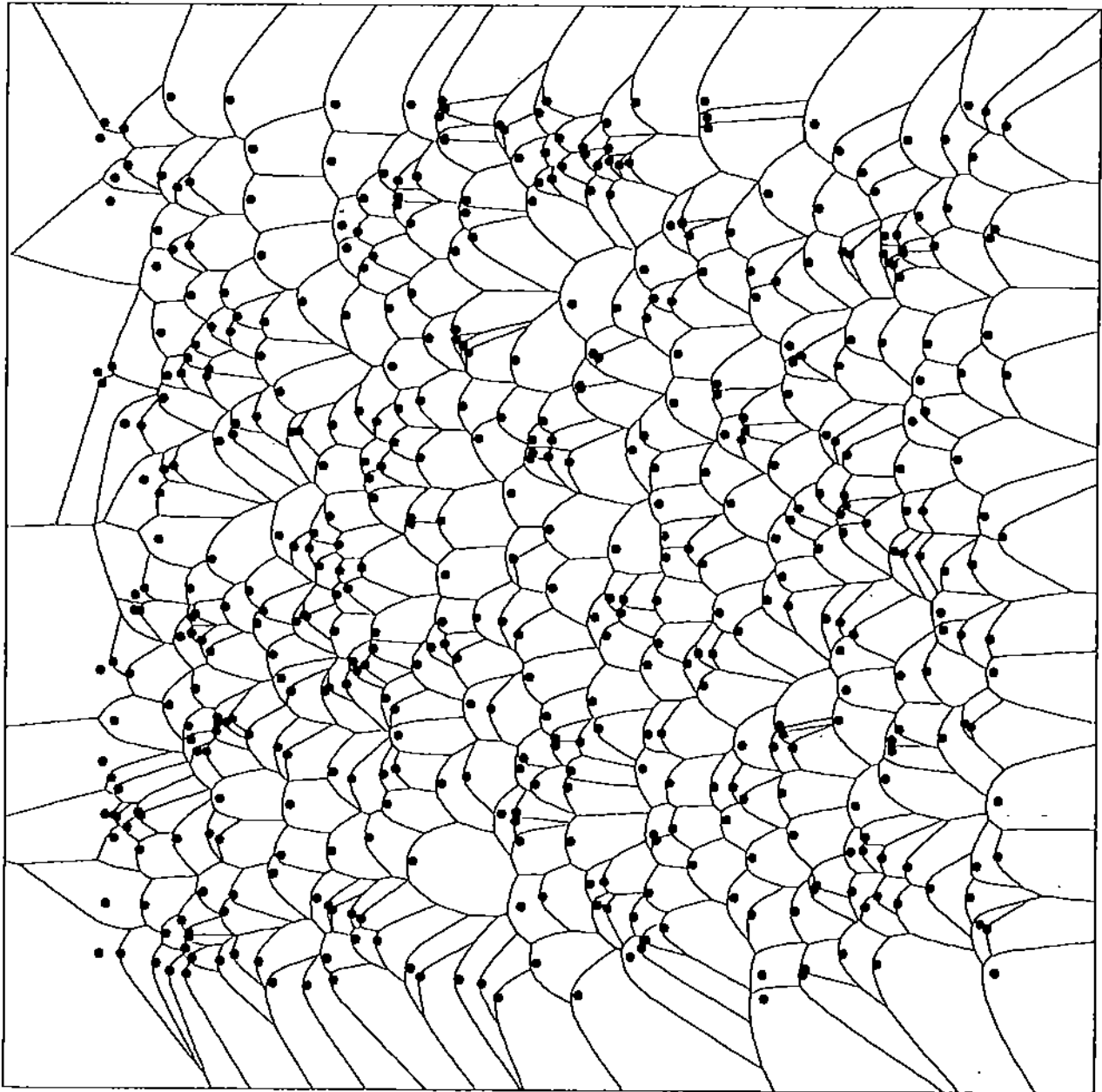


Fig. 10 (continued).



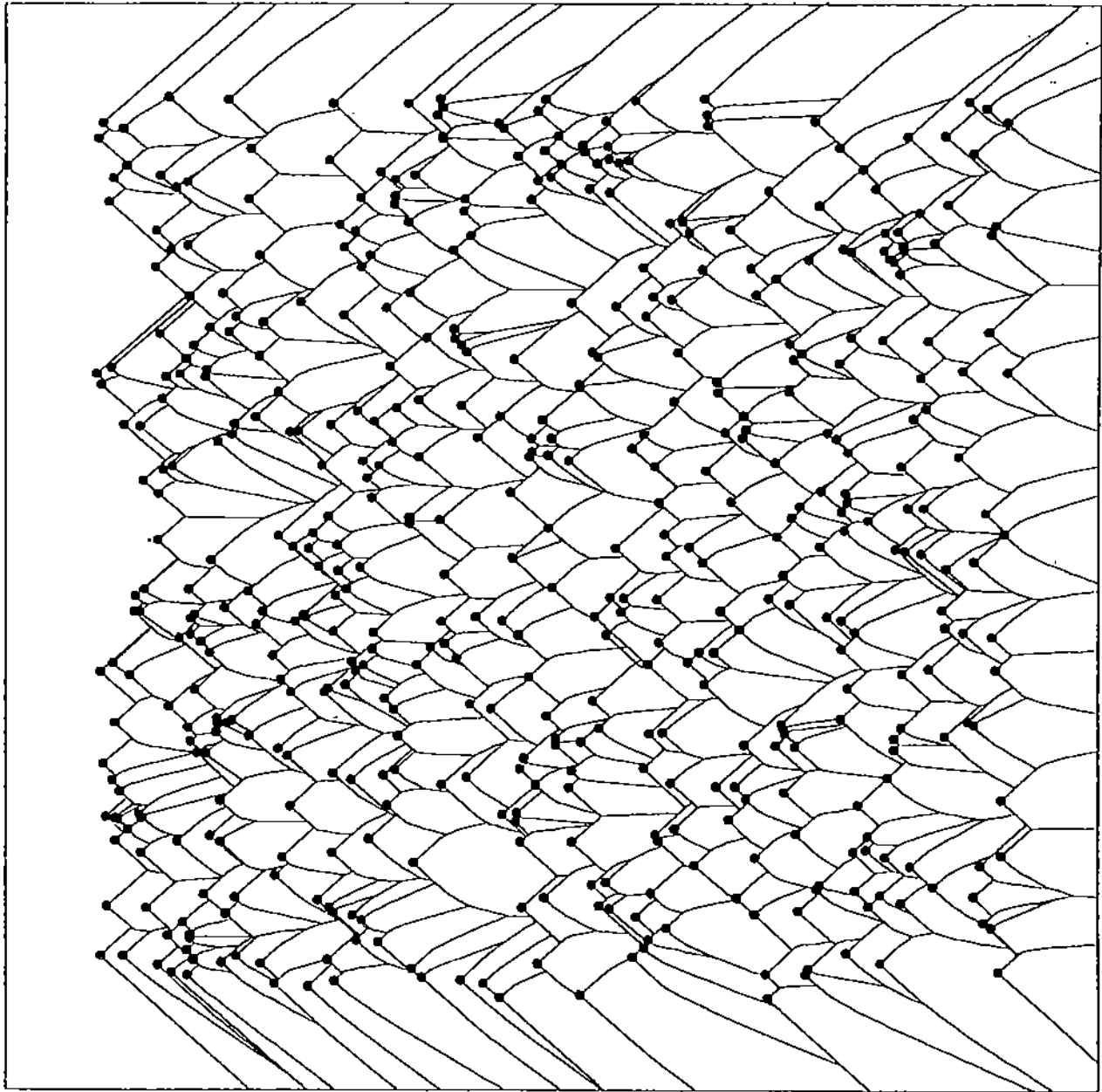
(a)

Fig. 11. Voronoi diagram in a river generated by 500 sites: (a) $\alpha = 0.0$; (b) $\alpha = 0.8$;
(c) $\alpha = 1.5$.



(b)

Fig. 11 (continued).



(c)

Fig. 11 (continued).