

VORONOI TILINGS HIDDEN IN CRYSTALS —THE CASE OF MAXIMAL ABELIAN COVERINGS—

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Abstract. Consider a finite connected graph possibly with multiple edges and loops. In discrete geometric analysis, Kotani and Sunada constructed the crystal associated to the graph as a standard realization of the maximal abelian covering of the graph. As an application of what the author showed in an earlier paper with Seshadri as a by-product of Geometric Invariant Theory, he shows that the Voronoi tiling (also known as the Wigner-Seitz tiling) is hidden in the crystal, that is, the crystal does not intrude the interiors of the top-dimensional Voronoi cells. The result turns out to be closely related to the tropical Abel-Jacobi map of the associated compact tropical curve.

Introduction. Let Γ be a finite connected graph possibly with multiple edges and loops. With an orientation for each edge of Γ , we can define

$$C := C_1(\Gamma, \mathbf{R}) \supset C_1(\Gamma, \mathbf{Z}) =: \Lambda \quad \text{with the standard Euclidean metric}$$

$$H := H_1(\Gamma, \mathbf{R}) \supset H_1(\Gamma, \mathbf{Z}) =: H_{\mathbf{Z}}$$

with the orthogonal projection $\pi : C \rightarrow H$.

In discrete geometric analysis, Kotani-Sunada [K-S1] constructed, for the maximal abelian covering $\Gamma^{\text{ab}} \rightarrow \Gamma$ and a choice of a base vertex of Γ , a *standard realization*

$$\text{sr} : \Gamma^{\text{ab}} \rightarrow \text{Crystal}(\Gamma) \subset H,$$

where $\text{Crystal}(\Gamma)$ is a one-dimensional complex of line segments connecting lattice points in $\pi(\Lambda)$. Moreover, sr is equivariant with respect to the action of $H_{\mathbf{Z}}$ on Γ^{ab} and its translation action on H .

The standard realization of Kotani-Sunada collapses the bridges in the graph Γ so that

$$(\text{Crystal}(\Gamma))/H_1(\Gamma, \mathbf{Z}) \cong \overline{\Gamma},$$

where $\overline{\Gamma}$ is the graph obtained by the collapse of the bridges.

In this paper (under the obvious assumption $\dim H \geq 2$) we show that $\text{Crystal}(\Gamma)$ does not intrude the interiors of the top-dimensional Voronoi cells (also known as Wigner-Seitz cells) in the Voronoi tiling

$$\text{Vor}(H, \xi_0 + H_{\mathbf{Z}})$$

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for a suitable $\xi_0 \in H$, that is, the Voronoi tiling $\text{Vor}(H, \xi_0 + H_{\mathbf{Z}})$ is “hidden” in the crystal $\text{Crystal}(\Gamma)$.

For example, a rhombic dodecahedral tiling is hidden in the diamond crystal (cf. Example 22), while a truncated octahedral tiling is hidden in the K_4 crystal (cf. Example 23).

For the proof, we may assume without loss of generality that the graph Γ is bridgeless. In fact, we can specify ξ_0 (hence the centers $\xi_0 + H_{\mathbf{Z}}$ of the top-dimensional Voronoi cells) when Γ is endowed with a “strongly connected” orientation that is possible under the bridgelessness assumption thanks to a result due to Robbins [R].

A crucial role is played by Proposition 10, which describes the Voronoi cells in $\text{Vor}(H, \xi_0 + H_{\mathbf{Z}})$ arising out of graphs. The proposition not only describes the facets of the Voronoi cells but gives a recipe for computing the vertices of the Voronoi cells. Hence it is an improvement of Oda-Seshadri [O-S, Prop. 5.2, (1)], which was a by-product of an attempt to apply the notion of stability in Geometric Invariant Theory to the compactifications of the generalized Jacobian varieties of algebraic curves with nodes. (See Alexeev [A] for later developments.)

More generally, Kotani-Sunada [K-S1] considered the standard realization of free abelian coverings $\tilde{\Gamma} \rightarrow \Gamma$ that are not necessarily maximal abelian. Denote

$$L := \text{Image}(H_1(\tilde{\Gamma}, \mathbf{Z}) \rightarrow H_1(\Gamma, \mathbf{Z})) \subset H_{\mathbf{Z}},$$

which Sunada [S2] calls the vanishing subgroup, and let $E' \subset H$ be the orthogonal complement in H of the subspace spanned by L

$$E' := \{x \in H; (x, L) = 0\} \subset H \subset C$$

$$\text{with the orthogonal projection } \pi': C \longrightarrow E'.$$

$H_{\mathbf{Z}}/L$ is the free abelian covering group of $\tilde{\Gamma} \rightarrow \Gamma$. Then the standard realization in this case is

$$\tilde{\text{sr}}: \tilde{\Gamma} \longrightarrow \pi'(\text{Crystal}(\Gamma)) \subset E' \cong (H_{\mathbf{Z}}/L) \otimes_{\mathbf{Z}} \mathbf{R}.$$

Even in this case, we expect that $\pi'(\text{Crystal}(\Gamma))$ does not intrude the interiors of the top-dimensional cells in a $\pi'(H_{\mathbf{Z}})$ -periodic convex polyhedral tiling of E' (such as the Voronoi tiling

$$\text{Vor}(E', \xi'_0 + \pi'(H_{\mathbf{Z}}))$$

for a suitable choice of $\xi'_0 \in E'$).

For instance, the Lonsdaleite crystal is the orthogonal projection onto the 3-space of the standard realization in the 5-space of the maximal abelian covering. A tiling by regular hexagonal cylinders, which is a Voronoi tiling $\text{Vor}(E', \xi'_0 + \pi'(H_{\mathbf{Z}}))$ for a suitable $\xi_0 \in E'$, turns out to be hidden in the Lonsdaleite crystal (cf. Example 24).

If Γ is endowed with a strongly connected orientation and $\dim H \geq 2$, then we can show the existence of a *nondegenerate* $H_{\mathbf{Z}}$ -periodic subdivision \diamond of $\text{Vor}(H, \xi_0 + H_{\mathbf{Z}})$ in the sense of Oda-Seshadri [O-S, Prop. 7.6 and Thm. 7.7] so that

$$\text{Crystal}(\Gamma) \subset \text{Sk}^1(\diamond), \quad \text{and} \quad \pi(\Lambda) = \text{Sk}^0(\diamond).$$

\diamond is obtained as one of the *Namikawa tilings* (which we called Namikawa decompositions in [O-S]). The details will be explained elsewhere (cf. [O]).

The standard realization sr modulo $H_{\mathbf{Z}}$ turns out to be closely related to the tropical Abel-Jacobi map

$$\mu: \Gamma \longrightarrow \text{Jac}(\Gamma) := H/H_{\mathbf{Z}}$$

in the sense of Mikhalkin-Zharkov [M-Z], where Γ with weight 1 for each edge is regarded as a compact tropical curve. Our result implies that when the genus is greater than or equal to 2, the image of the tropical Abel-Jacobi map μ lies in a translate of the tropical theta divisor in the tropical Jacobian variety $\text{Jac}(\Gamma)$.

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1. Graphs and their associated Voronoi tilings. A *graph* Γ in this paper is a *finite connected* graph possibly with multiple edges and loops (sometimes called a finite connected multigraph possibly with loops), that is, $\Gamma = (\{v_i\}_{i \in I}, \{e_j\}_{j \in J})$ is a pair consisting of a finite set $\{v_i\}_{i \in I}$ of vertices and a finite set $\{e_j\}_{j \in J}$ of edges. Each edge e_j joins two vertices v_i and $v_{i'}$ for some $i, i' \in I$. We call e_j a *loop* when $i = i'$.

We assign and fix an *orientation* of each edge, so that e_j is regarded as an arrow from v_i to $v_{i'}$, and we denote

$$\text{source}(e_j) := v_i, \quad \text{target}(e_j) := v_{i'}.$$

The orientation enables us to define the homology and cohomology groups of Γ :

$$C_0(\Gamma, \mathbf{Z}) := \bigoplus_{i \in I} \mathbf{Z}v_i, \quad C_1(\Gamma, \mathbf{Z}) := \bigoplus_{j \in J} \mathbf{Z}e_j$$

with the boundary and coboundary maps

$$\partial: C_1(\Gamma, \mathbf{Z}) \longrightarrow C_0(\Gamma, \mathbf{Z}), \quad \delta: C_0(\Gamma, \mathbf{Z}) \longrightarrow C_1(\Gamma, \mathbf{Z})$$

defined by

$$\begin{aligned} \partial(e_j) &= \text{source}(e_j) - \text{target}(e_j), \\ \delta(v_i) &= \sum_{\substack{j \\ \text{source}(e_j)=v_i}} e_j - \sum_{\substack{j \\ \text{target}(e_j)=v_i}} e_j. \end{aligned}$$

We also consider their base extensions (e.g., $\partial_{\mathbf{R}}$ and $\delta_{\mathbf{R}}$) to the real numbers \mathbf{R} . We thus get the first homology space $H_1(\Gamma, \mathbf{R}) := \ker(\partial_{\mathbf{R}})$ and its lattice $H_1(\Gamma, \mathbf{Z}) := \ker(\partial)$, while $H^1(\Gamma, \mathbf{R}) := \text{coker}(\delta_{\mathbf{R}})$ and $H^1(\Gamma, \mathbf{Z}) := \text{coker}(\delta)$, although we do not use the cohomology space and group below. From now on, we denote $\partial_{\mathbf{R}}$ and $\delta_{\mathbf{R}}$ without the subscript \mathbf{R} .

We introduce the inner products on $C_0(\Gamma, \mathbf{R})$ and $C_1(\Gamma, \mathbf{R})$ by

$$[v_i, v_{i'}] := \begin{cases} 1 & i = i' \\ 0 & i \neq i' \end{cases} \quad \text{for } v_i, v_{i'} \in C_0(\Gamma, \mathbf{R}),$$

$$(e_j, e_{j'}) := \begin{cases} 1 & j = j' \\ 0 & j \neq j' \end{cases} \quad \text{for } e_j, e_{j'} \in C_1(\Gamma, \mathbf{R}).$$

The inner product (\cdot, \cdot) induces on $H_1(\Gamma, \mathbf{R})$ a Euclidean metric. Moreover, it is easy to see that ∂ and δ are adjoint to each other, i.e.,

$$(\delta y, x) = [y, \partial x], \quad \text{for all } y \in C_0(\Gamma, \mathbf{R}) \text{ and } x \in C_1(\Gamma, \mathbf{R}),$$

hence we get an orthogonal decomposition

$$C_1(\Gamma, \mathbf{R}) = H_1(\Gamma, \mathbf{R}) \oplus \delta C_0(\Gamma, \mathbf{R}) \quad (\text{orthogonal}).$$

For simplicity, we introduce the following notation:

- $C := C_1(\Gamma, \mathbf{R}) \supset C_1(\Gamma, \mathbf{Z}) =: \Lambda$
- $H := H_1(\Gamma, \mathbf{R}) \supset H_1(\Gamma, \mathbf{Z}) =: H_{\mathbf{Z}} = \Lambda \cap H$
- $\pi : C \rightarrow H$ the orthogonal projection

In this paper, we adopt the following convention of Bondy [B]:

DEFINITION 1. Let Γ be an oriented graph.

- An edge e is itself with the given orientation, while $-e$ is the edge with the orientation opposite to the given one. Thus

$$\text{source}(-e) = \text{target}(e), \quad \text{target}(-e) = \text{source}(e).$$

- A walk w on Γ is a finite sequence

$$w: v_{i(0)} \xrightarrow{\varepsilon_{j(1)}e_{j(1)}} v_{i(1)} \xrightarrow{\varepsilon_{j(2)}e_{j(2)}} v_{i(2)} \longrightarrow \dots \xrightarrow{\varepsilon_{j(l)}e_{j(l)}} v_{i(l)} \longrightarrow \dots \xrightarrow{\varepsilon_{j(m)}e_{j(m)}} v_{i(m)}$$

with $\varepsilon_{j(l)} = \pm 1$ for all $1 \leq l \leq m$ and

$\text{target}(\varepsilon_{j(l)}e_{j(l)}) = v_{i(l)} = \text{source}(\varepsilon_{j(l+1)}e_{j(l+1)})$, for all l with $1 \leq l \leq m-1$.

- A *path* is a walk with all the vertices $v_{i(l)}$'s distinct (hence the edges $e_{j(l)}$'s are distinct as well). A *directed path* is a path with $\varepsilon_{j(l)} = 1$ for all l .
- A *trail* is a walk with all the edges $e_{j(l)}$'s distinct. A *directed trail* is a trail with $\varepsilon_{j(l)} = 1$ for all l .
- A *circuit* is a path coming back to the starting vertex, i.e., with $v_{i(0)} = v_{i(m)}$. A *directed circuit* is a directed path with $v_{i(0)} = v_{i(m)}$.

LEMMA 2. For an edge e of a (connected) graph Γ , the following are equivalent:

- (i) e is a bridge, i.e., its removal disconnects the graph Γ .
- (ii) $e \in \delta C_0(\Gamma, \mathbf{R})$.

(iii) $\pi(e) = 0$.

REMARK 3. A bridge is also called an isthmus, a cut edge, and a separating edge.

PROOF. (ii) and (iii) are obviously equivalent.

(i) implies (ii). Indeed, suppose e is a bridge. After its removal, denote by $I' \subset I$ the set of vertices of the connected component containing $\text{source}(e)$. Then obviously $e = \delta(\sum_{i \in I'} v_i)$.

(iii) implies (i). Indeed, if e is not a bridge, then there exists a circuit γ containing e . Then $(\gamma, \pi(e)) = (\gamma, e) \neq 0$ with γ regarded as an element of $H_Z = H_1(\Gamma, \mathbf{Z})$ in an obvious way (cf. Definition 13 below), hence $\pi(e) \neq 0$. \square

PROPOSITION 4. *For a graph Γ , the following are equivalent:*

- (1) Γ is bridgeless, i.e., Γ remains connected after the removal of any edge.
- (2) Γ is 2-edge connected, i.e., any pair of vertices can be connected by two paths without common edges.
- (3) Any edge is contained in a circuit.
- (4) Γ has a strongly connected orientation, i.e., it can be so oriented that there exists a directed path from any vertex to another.

The equivalence of (1), (2) and (3) is well known and easy to prove. (cf., e.g., Frank [F, Prop. 2.10]). The equivalence of (1) and (4) is due to Robbins [R]. (See also Frank [F, Cor. 2.13].) Here is an apt description by Robbins [R] of the equivalence:

Let us suppose that week-day traffic in our city is not particularly heavy, so that all streets are two-way, but that we wish to be able to repair any one street at a time and still detour traffic around it so that any point in the city may be reached from any other point. On week-ends no repairing is done, so that all streets are available, but due to heavy traffic (perhaps it is a college town with a noted football team) we wish to make all streets one-way and still be able to get from any point to any other without violating the law. Then the theorem states that if our street-system is suitable for week-day traffic it is also suitable for week-end traffic and conversely.

As we see in Remark 11, (3) below, we can describe all the possible strongly connected orientations on a bridgeless graph by Kotani-Sunada [K-S2].

DEFINITION 5. In a Euclidan space $E \cong \mathbf{R}^n$, consider a lattice $M \subset E$ and a coset $\mathcal{E} := \xi_0 + M \subset E$. For each $\xi \in \mathcal{E}$, define its Voronoi cell (also known as Wigner-Seitz cell) centered at ξ to be

$$V(\xi) := \{x \in E; \|x - \xi\| \leq \|x - \eta\|, \text{ for all } \eta \in \mathcal{E}\},$$

which is a convex polytope. Different Voronoi cells do not intersect one another in their relative interiors, and the union of the relative interiors of the members of

$$\text{Vor}(E, \xi_0 + M) := \{V(\xi) \mid \xi \in \xi_0 + M\} \text{ and their faces}$$

fills up the entire space E , giving rise to an M -periodic facet-to-facet convex polyhedral tiling called the *Voronoi tiling* of E . More intuitively, the M -translates of the Voronoi cell $V(\xi_0)$ centered at ξ_0 form a facet-to-facet tiling of E . In particular,

$$E = V(\xi_0) + M$$

so that $V(\xi_0)$ is a “fundamental domain” with respect to the translation action of M on E .

DEFINITION 6.

- For any subset $J' \subset J$, we denote

$$e(J') := \sum_{j \in J'} e_j \in C.$$

- In the Voronoi tiling $\text{Vor}(C, \Lambda)$, the Voronoi cell centered at the origin is denoted

$$\begin{aligned} V_J &:= \sum_{j \in J} \left[-\frac{1}{2}, \frac{1}{2} \right] e_j \\ &= \left\{ \sum_{j \in J} s_j e_j; -\frac{1}{2} \leq s_j \leq \frac{1}{2}, \text{ for all } j \in J \right\} \subset C. \end{aligned}$$

- In the Voronoi tiling $\text{Vor}(C, (e(J)/2) + \Lambda)$, the Voronoi cell centered at $e(J)/2$ is

$$\begin{aligned} D_J &:= \sum_{j \in J} [0, 1] e_j \\ &= \frac{e(J)}{2} + V_J \\ &= \left\{ \sum_{j \in J} s_j e_j; 0 \leq s_j \leq 1 \right\} \subset C. \end{aligned}$$

REMARK 7. For a subset $J_0 \subset J$, we have

$$D_J = e(J_0) + \sum_{j \in J_0} [0, 1](-e_j) + \sum_{j \in J \setminus J_0} [0, 1]e_j,$$

so that the re-orientation of the edges $\{e_j; j \in J_0\}$ corresponds to a translation of the Voronoi tiling.

DEFINITION 8. A graph-theoretical cycle (which we simply call a *cycle* from now on) is an element of $H_Z := H_1(\Gamma, \mathbf{Z})$ such that $(\gamma, e_j) = 0, \pm 1$ for all $j \in J$. For any cycle γ , let us denote for simplicity:

$$\begin{aligned} \gamma^+ &:= \{e_j; j \in J, (\gamma, e_j) = +1\} \\ \gamma^0 &:= \{e_j; j \in J, (\gamma, e_j) = 0\} \\ \gamma^- &:= \{e_j; j \in J, (\gamma, e_j) = -1\}. \end{aligned}$$

We denote their cardinalities by $|\gamma^+|$, $|\gamma^0|$ and $|\gamma^-|$.

For a subset $J' \subset J$, let us denote simply by J' or by $\{e_{j'}\}_{j' \in J'}$ the spanning subgraph of Γ with the vertex set $\{v_i\}_{i \in I}$ and the edge set $\{e_{j'}\}_{j' \in J'}$. (We follow the custom in graph theory so that a *spanning* subgraph is a subgraph with the same vertex set as that of the whole graph Γ .)

LEMMA-DEFINITION 9 (Elementary cycles, cf. [O-S, p. 21 and Lem. 4.6]). A cycle γ is said to be an *elementary cycle*, if the following equivalent conditions are satisfied:

- $\gamma \neq 0$ and it cannot be written in the form $\gamma = \gamma_1 + \gamma_2$ with nonzero cycles $\gamma_1, \gamma_2 \in H_1(\Gamma, \mathbf{Z})$ with no common e_j appearing in both γ_1 and γ_2 with nonzero coefficients.
- There exists a spanning tree T and $j \in J \setminus T$ such that either γ or $-\gamma$ is of the form

$$e_j + \sum_{t \in T} \varepsilon_t e_t, \quad \text{for } \varepsilon_t = 0, \pm 1.$$

- $H_1(\{I, \gamma^+ \cup \gamma^-\}, \mathbf{R})$ is one-dimensional ($= \mathbf{R}\gamma$).
- $\{\pi(e_j); j \in \gamma^0\}$ spans a codimension one subspace of $H := H_1(\Gamma, \mathbf{R})$, so that $\gamma^\perp = \sum_{j \in \gamma^0} \mathbf{R}\pi(e_j)$.

The proof is easy. Note that if γ is an elementary cycle, then so is $-\gamma$. Note also that $\gamma^+ \cup \gamma^-$ (if appropriately ordered) of an elementary cycle γ is a *circuit* in the sense of Definition 1.

The following proposition plays a central role in this paper:

PROPOSITION 10 (Associated Voronoi tiling). (1) Let $H := H_1(\Gamma, \mathbf{R})$ with its lattice $H_{\mathbf{Z}} := H_1(\Gamma, \mathbf{Z})$ and the orthogonal projection $\pi : C \rightarrow H$. Then $\pi(D_J)$ is the Voronoi cell centered at $\pi(e(J)/2)$ in $\text{Vor}(H, \pi(e(J)/2) + H_{\mathbf{Z}})$. Hence the $H_{\mathbf{Z}}$ -translates of $\pi(D_J)$ form a facet-to-facet tiling of H .

(2) We have

$$\pi(D_J) = \{x \in H; (x, \gamma) \leq |\gamma^+|, \text{ for all elementary cycles } \gamma\},$$

where the above system of defining inequalities are irredundant, i.e., the facets of $\pi(D_J)$ are exactly

$$\{x \in H; (x, \gamma) = |\gamma^+|\} \cap \pi(D_J) \quad \text{for elementary cycles } \gamma.$$

(3) For each elementary cycle γ , the vertices of $\pi(D_J)$ lying on the facet

$$\{x \in H; (x, \gamma) = |\gamma^+|\} \cap \pi(D_J)$$

are among

$$\{\pi(e(Q)); \gamma^+ \subset Q \subset \gamma^+ \cup \gamma^0\}.$$

REMARK 11. $\{e(J'); J' \subset J\}$ are the vertices of D_J . Hence

$$\pi(e(J')) \in \pi(D_J) \quad \text{for all } J' \subset J.$$

In particular, $0 = \pi(e(\emptyset)) \in \pi(D_J)$. By (2), we see that 0 is on the boundary of $\pi(D_J)$ if and only if $|\gamma^+| = 0$ for an elementary cycle γ .

The above Proposition 10 is an improvement of Oda-Seshadri [O-S, Prop. 5.2, (1)], since it not only describes the Voronoi cells but (3) also gives a recipe for computing the vertices of the Voronoi cells. In fact, we have more:

(1) We are in a situation of “space tiling zonotopes” dealt with by Venkov [V], McMullen [Mc1], [Mc2], [Mc3], Erdahl-Ryshkov [E-R], and Erdahl [E]. Namely,

$$\pi(D_J) = \sum_{j \in J} [0, 1] \pi(e_j)$$

is a *zonotope* whose $H_{\mathbf{Z}}$ -translates give rise to a facet-to-facet tiling of H . The family of hyperplanes

$$\mathcal{H}: \quad \{x \in H; (\pi(e_j), x) = m_j\}, \quad j \in J, \quad m_j \in \mathbf{Z}$$

gives a *dicing* in the sense of Erdahl-Ryshkov [E-R] and Erdahl [E] with $H_{\mathbf{Z}}$ as the lattice and with the elementary cycles as the *edge vectors*. Moreover, the arrangement of the hyperplanes \mathcal{H} coincides with the Delaunay tiling of H dual to $\text{Vor}(H, H_{\mathbf{Z}})$.

(2) Define the *support function* $s_D: H \rightarrow \mathbf{R}$ of $\pi(D_J)$ by sending $u \in H$ to

$$\begin{aligned} s_D(u) &:= \max_{y \in \pi(D_J)} (u, y) = \max_{x \in D_J} (u, x) \\ &= \max_{\substack{0 \leq s_j \leq 1 \\ j \in J}} (u, \sum_{j \in J} s_j e_j) \\ &= \max_{\substack{0 \leq s_j \leq 1 \\ j \in J}} \sum_{j \in J} s_j (u, e_j) = \sum_{j \in J} (u, e_j)_+, \end{aligned}$$

since $(u, \pi(x)) = (u, x)$ for all $u \in H$ and $x \in C$, where

$$(u, e_j)_+ := \begin{cases} (u, e_j) & \text{if } (u, e_j) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\pi(D_J) = \{x \in H; (u, x) \leq s_D(u), \text{ for all } u \in H\}.$$

s_D is *strictly convex* and piecewise linear with respect to the *normal fan* Σ of $\pi(D_J)$, which obviously is determined by the arrangement of the hyperplanes

$$\mathcal{H}_0: \quad (\pi(e_j))^\perp = \{u \in H; (u, \pi(e_j)) = 0\}, \quad \text{for } j \in J$$

passing through the origin. Note that $\pi(e_j) \neq 0$ for all $j \in J$ if Γ is bridgeless by Lemma 2.

- The set $\Sigma(1)$ of one-dimensional cones in Σ is in one-to-one correspondence with the set of facets of $\pi(D_J)$, hence with the set of elementary cycles so that

$$\Sigma(1) = \{\mathbf{R}_{\geq 0}\gamma; \gamma \text{ elementary cycles}\}.$$

For an elementary cycle γ , we have

$$H_1(\{I, \gamma^+ \cup \gamma^-\}, \mathbf{R}) = \bigcap_{j \in \gamma^0} (\pi(e_j))^\perp = \mathbf{R}_{\geq 0}\gamma \cup \mathbf{R}_{\geq 0}(-\gamma).$$

- Denote $g := \dim H$. Then the set $\Sigma(g)$ of top-dimensional cones in Σ is in one-to-one correspondence with the set of vertices of $\pi(D_J)$. In fact, each cone $\sigma \in \Sigma(g)$ is the closure of a connected component of

$$H \setminus \bigcup_{j \in J} (\pi(e_j))^\perp.$$

Let

$$Q := \{j \in J; (\sigma, \pi(e_j)) \subset \mathbf{R}_{\geq 0}\}.$$

Then for $u \in \sigma$ we have

$$s_D(u) = \sum_{j \in Q} (u, \pi(e_j)) = (u, \pi(e(Q)))$$

so that $\pi(e(Q))$ is the vertex of $\pi(D_J)$ corresponding to σ .

If $\mathbf{R}_{\geq 0}\gamma_1, \dots, \mathbf{R}_{\geq 0}\gamma_l$ are the one-dimensional faces of the g -dimensional cone σ , and if Γ is bridgeless so that $\pi(e_j) \neq 0$ for all $j \in J$ (cf. Lemma 2), then

$$\begin{aligned} \sigma &= \mathbf{R}_{\geq 0}\gamma_1 + \dots + \mathbf{R}_{\geq 0}\gamma_l \\ Q &= \bigcup_{1 \leq k \leq l} \gamma_k^+ = J \setminus \left(\bigcup_{1 \leq k \leq l} \gamma_k^- \right). \end{aligned}$$

(3) In fact, we can say more thanks to Kotani-Sunada [K-S2, Theorem 1.3, (3)]: Suppose Γ is bridgeless in the sense of Proposition 4. Then $\Sigma(g)$ is in one-to-one correspondence with the set of strongly connected orientations of Γ . Given a strongly connected orientation of Γ , the corresponding $\sigma \in \Sigma(g)$ is of the form

$$\sigma = \mathbf{R}_{\geq 0}\gamma_1 + \dots + \mathbf{R}_{\geq 0}\gamma_l,$$

where $\{\gamma_1, \dots, \gamma_l\}$ is the set of elementary cycles γ such that

$$\gamma^- = \emptyset.$$

(We can also show that $\pi(e(J))$ and 0 are among the vertices of $\pi(D_J)$.)

Indeed, in connection with the *homological directions of random walks* on Γ , Kotani-Sunada [K-S2] considers in H the unit ball

$$\overline{\mathcal{D}_0} := \{x \in H; \|x\|_1 \leq 1\}$$

with respect to the l^1 -norm of H induced by the l^1 -norm

$$\left\| \sum_{j \in J} c_j e_j \right\|_1 := \sum_{j \in J} |c_j|$$

of C . Among others, Kotani and Sunada show that $\overline{\mathcal{D}_0}$ is a convex polytope with the set of vertices

$$\left\{ \frac{\gamma}{\|\gamma\|_1}; \gamma \text{ elementary cycles} \right\}$$

and that the *facets* of $\overline{\mathcal{D}_0}$ are in one-to-one correspondence with the strongly connected orientations of Γ . Since

$$\|\gamma\|_1 = (\gamma, \gamma)$$

for elementary cycles γ , we see that

$$2\pi(V_J) = \{x \in H; (x, \gamma) \leq (\gamma, \gamma), \text{ for all elementary cycles } \gamma\}$$

is the polytope dual to $\overline{\mathcal{D}_0}$. Thus the facets of $\overline{\mathcal{D}_0}$ are in one-to-one correspondence with the vertices of $2\pi(V_J)$, hence with $\Sigma(g)$, since Σ is the normal fan of $2\pi(V_J)$ as well. Σ is the fan consisting of the cones joining 0 and the proper faces of $\overline{\mathcal{D}_0}$.

PROOF OF PROPOSITION 10. (1) and (2) are consequences of [O-S, Prop. 5.2, (1)] (in different notation) to the effect that

$$\pi(V_J) = \left\{ x \in H; (x, \gamma) \leq \frac{(\gamma, \gamma)}{2}, \text{ for all elementary cycles } \gamma \right\}$$

with the defining inequalities above being irredundant so that $\pi(V_J)$ is the Voronoi cell centered at 0 in $\text{Vor}(H, H_Z)$.

Indeed, for each elementary cycle γ , we have

$$(\gamma, \gamma) = |\gamma^+| + |\gamma^-|$$

and

$$(\pi(e(J)), \gamma) = (e(J), \gamma) = |\gamma^+| - |\gamma^-|$$

(since $\gamma \in H$) so that $(x, \gamma) \leq (\gamma, \gamma)/2$ is equivalent to

$$\begin{aligned} \left(x + \frac{\pi(e(J))}{2}, \gamma \right) &\leq \left(\frac{\pi(e(J))}{2}, \gamma \right) + \frac{(\gamma, \gamma)}{2} \\ &= \frac{|\gamma^+| - |\gamma^-|}{2} + \frac{|\gamma^+| + |\gamma^-|}{2} = |\gamma^+|. \end{aligned}$$

Here is another direct proof of (2), which gives the proof of (3) as well:

If γ is a cycle, then for $0 \leq s_j \leq 1$ for all $j \in J$, we have

$$\begin{aligned} \left(\gamma, \pi \left(\sum_{j \in J} s_j e_j \right) \right) &= \left(\gamma, \sum_{j \in J} s_j e_j \right) \\ &= \sum_{j \in J} s_j (\gamma, e_j) \leq \sum_{\substack{j \in J \\ (\gamma, e_j) > 0}} (\gamma, e_j) = |\gamma^+|. \end{aligned}$$

Hence

$$\pi(D_J) \subset \{x \in H; (x, \gamma) \leq |\gamma^+|, \text{ for all cycle } \gamma\}.$$

Suppose $H \ni \alpha \neq 0$ and $c \in \mathbf{R}$ determine a facet $\{x \in H; (\alpha, x) = c\} \cap \pi(D_J)$ of $\pi(D_J)$. Let $\pi(e(Q_1)), \dots, \pi(e(Q_r))$ for subsets $Q_1, \dots, Q_r \subset J$ be the vertices of $\pi(D_J)$ on this facet, so that

$$(\alpha, \pi(e(Q))) \leq c, \quad \text{for all } Q \subset J$$

with the equality holding if $Q = Q_1, \dots, Q_r$. Denote

$$\begin{aligned}\alpha^+ &:= \{j \in J; (\alpha, e_j) > 0\} \\ \alpha^0 &:= \{j \in J; (\alpha, e_j) = 0\} \\ \alpha^- &:= \{j \in J; (\alpha, e_j) < 0\}.\end{aligned}$$

Hence $\alpha \in H_1(\{I, \alpha^+ \cup \alpha^-\}, \mathbf{R})$ and is uniquely determined by the condition

$$(\alpha, \pi(e(Q))) = c, \quad \text{for } Q = Q_1, \dots, Q_r.$$

Hence $\dim H_1(\{I, \alpha^+ \cup \alpha^-\}, \mathbf{R}) = 1$ and α is a positive scalar multiple of an elementary cycle γ by Lemma-Definition 9. Obviously,

$$(\gamma, \pi(e(Q))) \leq |\gamma^+|, \quad \text{for all } Q \subset J$$

with the equality holding if and only if $\gamma^+ \subset Q \subset \gamma^+ \cup \gamma^0$. In particular,

$$\{Q_1, \dots, Q_r\} \subset \{Q \subset J; \gamma^+ \subset Q \subset \gamma^+ \cup \gamma^0\}.$$

□

The following is a special case of a well-known result valid in the more general setting where the inner product induced on Λ is *integral* and *unimodular*. Thanks are due to Mathieu Dutour-Sikirić for pointing this out to the author. In the following case of a graph, however, we have a recipe for constructing mutually dual \mathbf{Z} -bases as well.

LEMMA 12. *$\pi(\Lambda)$ and $H_{\mathbf{Z}} := H_1(\Gamma, \mathbf{Z}) = \Lambda \cap H$ are mutually dual lattices in $H := H_1(\Gamma, \mathbf{R})$. Out of each spanning tree T of Γ , mutually dual \mathbf{Z} -bases can be constructed.*

PROOF. Obviously,

$$(\pi(\lambda), H_{\mathbf{Z}}) = (\lambda, H_{\mathbf{Z}}) \subset \mathbf{Z},$$

for all $\lambda \in \Lambda$ so that $\pi(\Lambda)$ is contained in the dual lattice of $H_{\mathbf{Z}}$. Fix a spanning tree $T \subset J$. Then clearly

$$C_1(\Gamma, \mathbf{Z}) = C_1(J \setminus T, \mathbf{Z}) + \delta C_0(\Gamma, \mathbf{Z}).$$

Moreover for each $j \in J \setminus T$, there exists an elementary cycle $\gamma_j \in H_{\mathbf{Z}}$ with $\gamma_j - e_j \in C_1(T, \mathbf{Z})$. We are done, since $\{\gamma_j\}_{j \in J \setminus T}$ is a \mathbf{Z} -basis of $H_{\mathbf{Z}}$ (cf., e.g., [O-S, pp. 21–22]) and

$$(\gamma_j, \pi(e_{j'})) = (\gamma_j, e_{j'}) = \begin{cases} 1 & j = j' \\ 0 & j \neq j'. \end{cases}$$

□

2. Crystals. We now briefly recall the standard realization and crystals due to Kotani-Sunada [K-S1].

Regard a (connected) graph Γ as a one-dimensional cell complex in an obvious way, and fix a base vertex v_0 . Denote by $\Omega(\Gamma)$ the universal covering space of Γ regarded as consisting of the homotopy classes of “curves” on Γ starting from v_0 , while $\pi_1(\Gamma, v_0)$ is the fundamental group of Γ regarded as consisting of the homotopy classes of “curves” on Γ

starting from v_0 and ending at v_0 . (We here use “curve” for customarily used “path” to avoid confusion with that defined in Definition 1.)

We have a canonical surjective map from $\Omega(\Gamma)$ to the maximal abelian covering Γ^{ab} of Γ . The concatenation action $\pi_1(\Gamma, v_0) \times \Omega(\Gamma) \rightarrow \Omega(\Gamma)$ induces the canonical action

$$H_1(\Gamma, \mathbf{Z}) \times \Gamma^{\text{ab}} \rightarrow \Gamma^{\text{ab}}.$$

A walk w on Γ from v_0 gives rise to a point on $\Omega(\Gamma)$ (and on Γ^{ab}) as well as a “curve” on $\Omega(\Gamma)$ (and on Γ^{ab}) ending at the point. For simplicity, we denote also by w the point as well as the “curve”. ($\Omega(\Gamma)$ and Γ^{ab} are also regarded as “infinite graphs” with these points as vertices.)

DEFINITION 13. For a walk w on Γ of the form

$$\begin{aligned} v_{i(0)} \xrightarrow{\varepsilon_{j(1)}e_{j(1)}} v_{i(1)} \xrightarrow{\varepsilon_{j(2)}e_{j(2)}} v_{i(2)} \longrightarrow \\ \dots \xrightarrow{\varepsilon_{j(l)}e_{j(l)}} v_{i(l)} \longrightarrow \dots \xrightarrow{\varepsilon_{j(m)}e_{j(m)}} v_{i(m)} \end{aligned}$$

we define the element $\lambda(w) \in \Lambda = C_1(\Gamma, \mathbf{Z})$ by

$$\lambda(w) := \begin{cases} 0 & \text{if } m = 0 \\ \sum_{1 \leq l \leq m} \varepsilon_{j(l)}e_{j(l)} & \text{if } m > 0. \end{cases}$$

When a walk w on Γ starts from $v_0 = v_{i(0)}$ in the above notation, we denote by w_l the part of w from v_0 to $v_{i(l)}$ for $l = 0, \dots, m$. Then we have a Λ -polygonal curve in $C := C_1(\Gamma, \mathbf{R})$

$$\text{pc}(w) := \bigcup_{1 \leq l \leq m} [\lambda(w_{j(l-1)}), \lambda(w_{j(l)})] \subset C,$$

where $[\lambda, \lambda']$ for $\lambda, \lambda' \in \Lambda$ is the Λ -line segment in C joining λ and λ' .

Applying the orthogonal projection $\pi : C \rightarrow H := H_1(\Gamma, \mathbf{R})$, we have a map from $\Omega(\Gamma)$ to the set of $\pi(\Lambda)$ -polygonal curves $\pi(\text{pc}(w))$ in H , which induces the *standard realization*

$$\text{sr} : \Gamma^{\text{ab}} \longrightarrow \text{Crystal}(\Gamma) := \bigcup_{w \text{ walks from } v_0} \pi(\text{pc}(w)) \subset H$$

of Kotani-Sunada [K-S1] (see also Sunada [S1], [S2]).

The standard realization sr is equivariant with respect to the concatenation action of $H_{\mathbf{Z}} := H_1(\Gamma, \mathbf{Z})$ on Γ^{ab} and the translation action of $H_{\mathbf{Z}}$ on H . The crystal $\text{Crystal}(\Gamma)$ is a one-dimensional complex in H of $\pi(\Lambda)$ -line segments joining lattice points in $\pi(\Lambda)$ and is $H_{\mathbf{Z}}$ -periodic, i.e., invariant under the translation action of the sublattice $H_{\mathbf{Z}}$.

The standard realization sr collapses the bridges in the graph Γ , and

$$\text{Crystal}(\Gamma)/H_1(\Gamma, \mathbf{Z}) \cong \overline{\Gamma},$$

where $\overline{\Gamma}$ is the graph obtained by the collapse of the bridges in Γ .

Thus to show that $\text{Vor}(H, \xi_0 + H_{\mathbf{Z}})$ for some $\xi_0 \in H$ is “hidden” in $\text{Crystal}(\Gamma)$, we may assume without loss of generality that the graph Γ is bridgeless.

More generally, Kotani-Sunada [K-S1] considered also the crystal for free abelian coverings $\tilde{\Gamma}$ of Γ that are not necessarily maximal abelian.

Let $L \subset H_{\mathbf{Z}}$ be the image of $H_1(\tilde{\Gamma}, \mathbf{Z}) \rightarrow H_1(\Gamma, \mathbf{Z})$ so that L is a subgroup such that $\Gamma^{\text{ab}}/L = \tilde{\Gamma}$ and that $H_{\mathbf{Z}}/L$ is the free abelian covering group for $\tilde{\Gamma} \rightarrow \Gamma$.

Denote

$$\begin{aligned} E' &:= \{x \in H; (x, L) = 0\} \subset H \\ &\cong (H_{\mathbf{Z}}/L) \otimes_{\mathbf{Z}} \mathbf{R} \end{aligned}$$

with $\pi': C \rightarrow E'$ the orthogonal projection.

E' has lattices

$$\Lambda \cap E' \subset \pi'(H_{\mathbf{Z}}) \subset \pi'(\Lambda).$$

$\Lambda \cap E'$ and $\pi'(\Lambda)$ are *mutually dual* lattices, since the inner product on C is integral and unimodular with respect to the orthonormal lattice Λ . The standard realization of Kotani-Sunada in this more general setting is

$$\tilde{\text{sr}}: \tilde{\Gamma} \longrightarrow \pi'(\text{Crystal}(\Gamma)) \subset E' \cong (H_{\mathbf{Z}}/L) \otimes_{\mathbf{Z}} \mathbf{R}.$$

$\pi'(\text{Crystal}(\Gamma))$ is a one-dimensional complex of $\pi'(\Lambda)$ -*line segments* joining lattice points in $\pi'(\Lambda)$, and is $\pi'(H_{\mathbf{Z}})$ -periodic, i.e., invariant under the translation action of the sublattice $\pi'(H_{\mathbf{Z}})$.

We have a commutative diagram

$$\begin{array}{ccc} \Gamma^{\text{ab}} & \xrightarrow{\text{sr}} & \text{Crystal}(\Gamma) \\ \downarrow & & \downarrow \pi' \\ \tilde{\Gamma} = \Gamma^{\text{ab}}/L & \xrightarrow{\tilde{\text{sr}}} & \pi'(\text{Crystal}(\Gamma)). \end{array}$$

$\tilde{\text{sr}}$ induces

$$\tilde{\Gamma}/(H_{\mathbf{Z}}/L) \longrightarrow \pi'(\text{Crystal}(\Gamma))/\pi'(H_{\mathbf{Z}}),$$

which need not be bijective, even if Γ is bridgeless so that

$$\Gamma^{\text{ab}}/H_{\mathbf{Z}} \longrightarrow \text{Crystal}(\Gamma)/H_{\mathbf{Z}}$$

is bijective.

3. Main theorem. We are now ready to prove the following theorem which was conjectured at the 2010 Annual Meeting of the Japan Society of Industrial and Applied Mathematics held at Meiji University in Tokyo on September 6, 2010:

THEOREM 14. *Let $\Gamma = \{I, J\}$ be a bridgeless graph with $\dim H_1(\Gamma, \mathbf{R}) \geq 2$. Then after a strongly connected re-orientation and a change of the base vertex v_0 if necessary, the crystal obtained as the standard realization of the maximal abelian covering Γ^{ab} does not intrude the interiors of the top-dimensional Voronoi cells in*

$$\text{Vor} \left(H_1(\Gamma, \mathbf{R}), \pi \left(\frac{\sum_{j \in J} e_j}{2} \right) + H_1(\Gamma, \mathbf{Z}) \right),$$

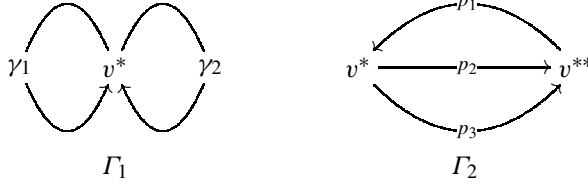
that is, for some $r < \dim H_1(\Gamma, \mathbf{R})$ we have:

$$\text{Crystal}(\Gamma) \subset \text{Sk}^r \left(\text{Vor} \left(H_1(\Gamma, \mathbf{R}), \pi \left(\frac{\sum_{j \in J} e_j}{2} \right) + H_1(\Gamma, \mathbf{Z}) \right) \right),$$

where Sk^r denotes the r -skeleton. Thus a Voronoi tiling is “hidden” in the crystal $\text{Crystal}(\Gamma)$.

LEMMA 15. *Let Γ be a strongly connected graph with $\dim H_1(\Gamma, \mathbf{R}) \geq 2$. Then Γ has one of the following (not necessarily spanning) subgraphs Γ_1 and Γ_2 :*

- Γ_1 consists of two directed circuits γ_1 and γ_2 meeting only at a vertex v^* .
- Γ_2 consists of three directed paths p_1, p_2, p_3 between two distinct vertices v^* and v^{**} with p_1, p_2, p_3 disjoint except at the end vertices v^* and v^{**} , where p_1 is a directed path from v^{**} to v^* , while p_2 and p_3 are directed paths from v^* to v^{**} .



PROOF. (i) If Γ has only one vertex, then Γ consists of at least two loops so that it has a subgraph Γ_1 .

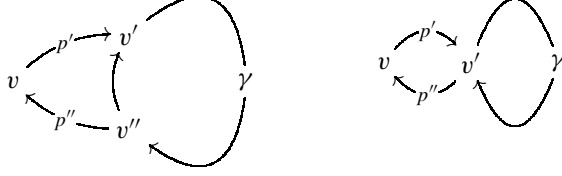
(ii) Suppose Γ has only two vertices. If there are two loops at a vertex, then they form a subgraph Γ_1 . If there exists only one loop at a vertex, then there exist two edges in opposite directions connecting the vertex with the other by the strong-connectivity assumption, and we obviously have a subgraph Γ_1 . If Γ has no loops, then there exist at least three edges connecting the two vertices with at least two edges in opposite directions by our assumptions. Thus Γ has a subgraph Γ_2 .

(iii) Suppose now that Γ has at least three vertices. By assumption, we can choose a directed circuit γ .

(iii-a) If γ is Hamiltonian (i.e., passes through all the vertices of Γ), then since $\dim H_1(\Gamma, \mathbf{R}) \geq 2$, there exists either a loop or an edge e connecting two distinct vertices. Obviously, we have a subgraph Γ_1 in the former case and Γ_2 in the latter case.

(iii-b) If γ is not Hamiltonian, take a vertex v not on γ . By assumption, there exists a directed path p' from v to a vertex v' on γ with p' and γ disjoint except at v' . There also exists a directed path p'' to v from a vertex v'' on γ with p'' and γ disjoint except at v'' . If $v' \neq v''$, then we may assume that p' and p'' are disjoint except at v . Indeed, otherwise, there certainly exists a vertex v''' that is closest to v' on p' as well as to v'' on p'' . We then replace v by v''' . In this case, Γ has a subgraph Γ_2 . If $v' = v''$, then we may again assume that p' and

p'' are disjoint except at v and v' . In this case Γ has a subgraph Γ_1 .



□

To prove Main Theorem 14, let us endow Γ with a strongly connected orientation, which is possible by Proposition 4. Since $\dim H_1(\Gamma, \mathbf{R}) \geq 2$, Lemma 15 guarantees the existence of a subgraph Γ_1 or Γ_2 . In either case, let $v_0 := v^*$ be the base vertex for the standard realization of the crystal. Recall our definition in Definition 13 of $\lambda(w)$ for a walk on Γ .

LEMMA 16. *Under the choice of the strongly connected orientation and the base vertex v_0 , for any walk w on Γ from v_0 there exists a directed path p from v_0 to the same end vertex as that of w such that $\lambda(w) - \lambda(p) \in H_1(\Gamma, \mathbf{Z})$. Moreover, for directed paths p and p' from v_0 to v , we have $\lambda(p) - \lambda(p') \in H_1(\Gamma, \mathbf{Z})$.*

PROOF. The second assertion is obvious.

As for the first, suppose the walk w starts out with a path p_1 from v_0 to a vertex v followed by an edge $-e$ from v to v' :

$$w = p_1 + (-e) + q,$$

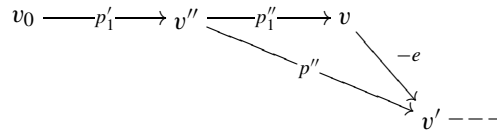
where q is the part of the walk w from v' onward. Since e is an edge from v' to v , there exists a directed path p' from v to v' by the strong connectivity.

If p' and p_1 are disjoint except at the vertex v , then replace the beginning of the walk w by $p_1 + p'$ and consider

$$w' := p_1 + p' + q.$$

Obviously, $\lambda(w) - \lambda(w') \in H_1(\Gamma, \mathbf{Z})$.

If p' and p_1 have common edges, there certainly exists a subpath p'' of p' (the ending part of the directed path p') from a vertex v'' on p_1 to v' such that p'' and p_1 are disjoint except at v'' . Denote by p'_1 (resp. p''_1) the subpath of p_1 from v_0 to v'' (resp. from v'' to v).



Then replace the beginning of the walk w by $p'_1 + p''$ and consider

$$w'' := p'_1 + p'' + q.$$

Obviously, $\lambda(w) - \lambda(w'') \in H_1(\Gamma, \mathbf{Z})$.

The rest of the proof follows by induction on the number of edges in w appearing with the minus sign. □

PROOF OF MAIN THEOREM 14. By Proposition 10, (1), the $H_{\mathbf{Z}}$ -translates of $\pi(D_J)$ form a facet-to-facet tiling of H so that

$$H = \pi(D_J) + H_{\mathbf{Z}},$$

that is, $\pi(D_J)$ is a “fundamental domain” with respect to the translation action of $H_{\mathbf{Z}}$ on H .

By Lemma 16, it suffices to consider only directed paths on Γ from v_0 . Since the images by π in C of the directed paths from v_0 are contained in D_J , their projections under π are contained in $\pi(D_J)$, whose facets are of the form

$$\{x \in H; (\gamma, x) = |\gamma^+|\} \cap \pi(D_J) \quad \text{for an elementary cycle } \gamma.$$

To show that the images under π of the directed paths from v_0 are on the boundary of $\pi(D_J)$, it suffices to show the following:

For any edge e and an appropriate choice of a directed path p from v_0 to $v := \text{source}(e)$,

$$p: \quad v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \longrightarrow \cdots \xrightarrow{e_m} v_m = v,$$

there exists an elementary cycle γ such that

$$\begin{aligned} (\gamma, \lambda(p)) &= |\gamma^+| \\ (\gamma, e) &= 0, \end{aligned}$$

where

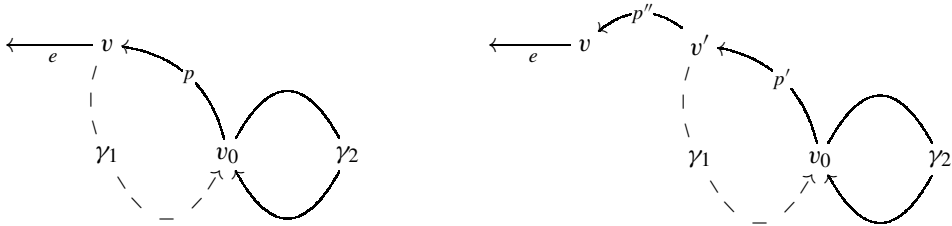
$$\lambda(p) := e_1 + e_2 + \cdots + e_m$$

as in Definition 13. By Lemma 15, we have two cases:

(Case 1) Γ contains a subgraph Γ_1 with two directed circuits γ_1 and γ_2 meeting only at the common vertex v_0 .

If $v := \text{source}(e)$ is on γ_1 , choose the directed path p from v_0 to v to be the directed path along γ_1 from v_0 to v . Let $\gamma := -\lambda(\gamma_2)$. Then we are done, since

$$\begin{aligned} (\gamma, \lambda(p)) &= 0 = |\gamma^+| \\ (\gamma, e) &= 0. \end{aligned}$$



If v is on γ_2 , we are done in a similar manner.

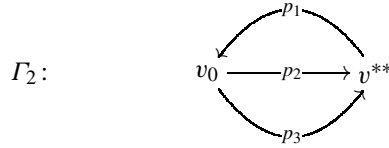
Suppose v is neither on γ_1 nor on γ_2 . Without loss of generality, we may assume the existence of a directed path p'' to v from a vertex v' on γ_1 such that p'' and Γ_1 are disjoint

except at v' . Denote by p' the directed path along γ_1 from v_0 to v' , and let $p := p' + p''$ be the direct path p' followed by p'' . Then $\gamma := -\lambda(\gamma_2)$ again satisfies

$$\begin{aligned}(\gamma, \lambda(p)) &= 0 = |\gamma^+| \\ (\gamma, e) &= 0,\end{aligned}$$

and we are done.

(Case 2) Γ contains a subgraph Γ_2 consisting of three paths p_1, p_2, p_3 , disjoint except at the end vertices, between v_0 and v^{**} with p_1 directed from v^{**} to v_0 , while p_2 and p_3 are directed from v_0 to v^{**} .

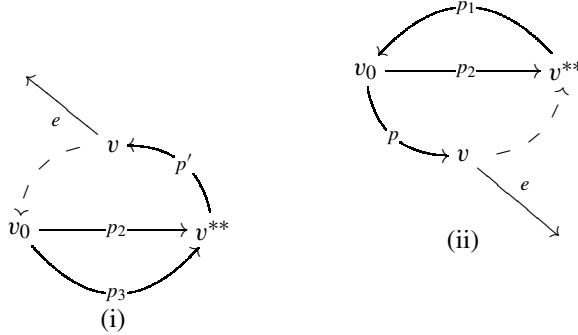


For an edge e with $v := \text{source}(e)$, we have four cases to consider:

(i) v is on p_1 . Let p' be the directed path along p_1 from v^{**} to v , and let $p := p_2 + p'$ be the directed path p_2 followed by p' . Then the elementary cycle $\gamma := \lambda(p_2) - \lambda(p_3)$ satisfies

$$\begin{aligned}(\gamma, \lambda(p)) &= (\text{length of } p_2) = |\gamma^+| \\ (\gamma, e) &= 0,\end{aligned}$$

and we are done.



(ii) v is on p_3 . Let p be the directed path along p_3 from v_0 to v . Then the elementary cycle $\gamma := -\lambda(p_1) - \lambda(p_2)$ satisfies

$$\begin{aligned}(\gamma, \lambda(p)) &= 0 = |\gamma^+| \\ (\gamma, e) &= 0,\end{aligned}$$

and we are done.

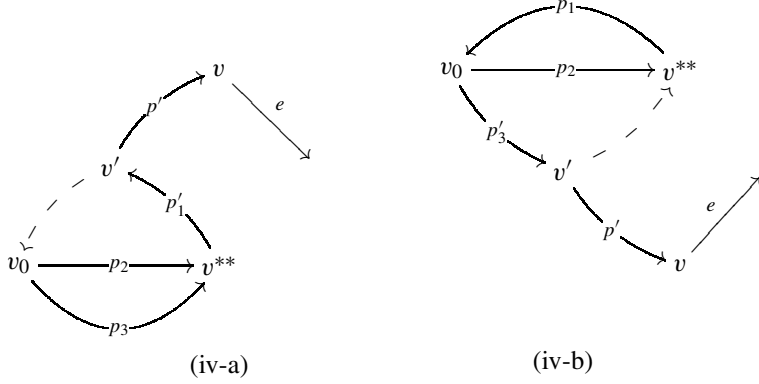
(iii) v is on p_2 . The proof is similar to that for (ii).

(iv) v is not on p_1, p_2, p_3 . There certainly exists a directed path p' from a vertex v' of Γ_2 to v such that p' and Γ_2 are disjoint except at v' . We have three cases to consider:

(iv-a) v' is on p_1 . Let p'_1 be the directed path along p_1 from v^{**} to v' and let the directed path $p := p_2 + p'_1 + p'$ from v_0 to v be p_2 followed by p'_1 and then by p' . Then the elementary cycle $\gamma := \lambda(p_2) - \lambda(p_3)$ satisfies

$$\begin{aligned} (\gamma, \lambda(p)) &= (\text{length of } p_2) = |\gamma^+| \\ (\gamma, e) &= 0, \end{aligned}$$

and we are done.



(iv-b) v' is on p_3 . Let p'_3 be the directed path along p_3 from v_0 to v' and let the directed path $p := p'_3 + p'$ from v_0 to v be p'_3 followed by p' . Then the elementary cycle $\gamma := -\lambda(p_1) - \lambda(p_2)$ satisfies

$$\begin{aligned} (\gamma, \lambda(p)) &= 0 = |\gamma^+| \\ (\gamma, e) &= 0, \end{aligned}$$

and we are done.

(iv-c) v' is on p_2 . The proof is similar to that for (iv-b). \square

REMARK 17. As the proof shows, $(\text{Crystal}(\Gamma)) \cap \pi(D_J)$ consists of the $\pi(\Lambda)$ -polygonal curves arising out of directed trails from v_0 .

REMARK 18. If Γ is endowed with a strongly connected orientation and $\dim H \geq 2$, then we can show the existence of a *nondegenerate* $H_{\mathbf{Z}}$ -periodic subdivision \diamond of $\text{Vor}(H, \pi(e(J)/2) + H_{\mathbf{Z}})$ in the sense of Oda-Seshadri [O-S, Prop. 7.6 and Thm. 7.7] so that

$$\begin{aligned} \text{Crystal}(\Gamma) &\subset \text{Sk}^1(\diamond) \\ \pi(\Lambda) &= \text{Sk}^0(\diamond). \end{aligned}$$

\diamond is obtained as one of the Namikawa tilings (which we called Namikawa decompositions in [O-S]). The details are given elsewhere (cf. [O]).

Let $\tilde{\Gamma} \rightarrow \Gamma$ be a non-maximal abelian covering with the vanishing subgroup

$$L := \text{Image}(H_1(\tilde{\Gamma}, \mathbf{Z}) \rightarrow H_1(\Gamma, \mathbf{Z}))$$

and the standard realization

$$\tilde{\text{sr}}: \tilde{\Gamma} \longrightarrow \pi'(\text{Crystal}(\Gamma)) \subset E' \cong (H_{\mathbf{Z}}/L) \otimes_{\mathbf{Z}} \mathbf{R}.$$

CONJECTURE 19. *For the non-maximal abelian covering $\tilde{\Gamma} \rightarrow \Gamma$, the crystal $\pi'(\text{Crystal}(\Gamma))$ does not intrude the interiors of the top-dimensional tiles in a $\pi'(H_{\mathbf{Z}})$ -periodic convex polyhedral tiling of E' .*

For instance, the Lonsdaleite crystal is the orthogonal projection onto the 3-space of the standard realization in the 5-space of the maximal abelian covering. A tiling by regular hexagonal cylinders, which is a Voronoi tiling $\text{Vor}(E', \xi'_0 + \pi'(H_{\mathbf{Z}}))$ for a $\xi'_0 \in E'$, turns out to be hidden in the Lonsdaleite crystal (cf. Example 24).

Depending on L , however, convex polyhedral tilings other than Voronoi tilings may be needed. It would be of interest to characterize L for which $\text{Vor}(E', \xi'_0 + \pi'(H_{\mathbf{Z}}))$ for some ξ'_0 is the $\pi'(H_{\mathbf{Z}})$ -periodic convex polyhedral tiling in question.

REMARK 20. As in Mikhalkin-Zharkov [M-Z], our finite connected bridgeless graph Γ can be regarded as a *compact tropical curve* with the metric of length 1 for each edge. The real torus $\text{Jac}(\Gamma) := H/J_{\mathbf{Z}}$ turns out to be its *tropical Jacobian variety* with $g := \dim H$ being the *genus* of Γ .

Suppose $g \geq 2$ and endow Γ with a strongly connected orientation. Denote by $\varpi: H \rightarrow \text{Jac}(\Gamma)$ the canonical projection. Then the standard realization $\text{sr}: \Gamma^{\text{ab}} \rightarrow H$ induces modulo $H_{\mathbf{Z}}$ the *tropical Abel-Jacobi map* $\mu: \Gamma \rightarrow \text{Jac}(\Gamma)$ (with $\mu(v_0) = 0$) such that the following diagram is commutative:

$$\begin{array}{ccc} \Gamma^{\text{ab}} & \xrightarrow{\text{sr}} & H \\ \downarrow & & \downarrow \varpi \\ \Gamma & \xrightarrow{\mu} & \text{Jac}(\Gamma). \end{array}$$

The \mathbf{Z} -module $\text{Div}(\Gamma)$ of *tropical divisors* on Γ consists of finite formal \mathbf{Z} -linear combinations of points on Γ (including those on the edges) with the degree map

$$\text{deg}: \text{Div}(\Gamma) \longrightarrow \mathbf{Z}$$

given by the sum of the coefficients. For $m \in \mathbf{Z}$, denote by $\text{Div}^m(\Gamma)$ the set of divisors of degree m . The tropical Abel-Jacobi map μ determines an obvious map $\mu_m: \text{Div}^m(\Gamma) \rightarrow \text{Jac}(\Gamma)$. The boundary map ∂ induces

$$\partial: C_1(\Gamma, \mathbf{Z}) \longrightarrow \text{Div}^0(\Gamma), \quad \partial(e_j) = \text{source}(e_j) - \text{target}(e_j), \quad \text{for all } j \in J,$$

and a commutative diagram

$$\begin{array}{ccc} C_1(\Gamma, \mathbf{Z}) & \xrightarrow{\pi} & H \\ \downarrow \partial & & \downarrow \varpi \\ \text{Div}^0(\Gamma) & \xrightarrow{\mu_0} & \text{Jac}(\Gamma) \end{array}$$

with

$$\varpi(\pi(e_j)) = \mu_0(\partial e_j) = \mu(\text{source}(e_j)) - \mu(\text{target}(e_j)), \quad \text{for all } j \in J.$$

For each vertex v_i , let

$$\begin{aligned} \text{valency}_+(v_i) &:= \#\{j \in J; \text{source}(e_j) = v_i\} \\ \text{valency}_-(v_i) &:= \#\{j \in J; \text{target}(e_j) = v_i\} \\ \text{valency}(v_i) &:= \text{valency}_+(v_i) + \text{valency}_-(v_i). \end{aligned}$$

Following Mikhalkin-Zharkov [M-Z], denote

$$\begin{aligned} K_+ &:= \sum_{i \in I} (\text{valency}_+(v_i) - 1)v_i \in \text{Div}^{g-1}(\Gamma) \\ K_- &:= \sum_{i \in I} (\text{valency}_-(v_i) - 1)v_i \in \text{Div}^{g-1}(\Gamma) \\ K &:= K_+ + K_- = \sum_{i \in I} (\text{valency}(v_i) - 2)v_i \in \text{Div}^{2g-2}(\Gamma). \end{aligned}$$

K is a *tropical canonical divisor*. Since

$$\varpi(\pi(e_j)) = \mu_0(\partial e_j) = \mu(\text{source}(e_j)) - \mu(\text{target}(e_j)), \quad \text{for all } j \in J,$$

we easily see that

$$\varpi(\pi(e(J))) = \mu_0(K_+ - K_-).$$

As in [M-Z], we see that $[\Theta] := \varpi(\text{Sk}^{g-1}(\text{Vor}(H, H_Z))) \subset \text{Jac}(\Gamma)$ is the *tropical theta divisor*, while the image $W_{g-1} \subset \text{Jac}(\Gamma)$ of the obvious map

$$\mu_{g-1}: \underbrace{\Gamma \times \cdots \times \Gamma}_{(g-1)\text{-times}} \rightarrow \text{Jac}(\Gamma)$$

turns out to be

$$W_{g-1} = \varpi \left(\text{Sk}^{g-1} \left(\text{Vor} \left(H, \pi \left(\frac{e(J)}{2} \right) + H_Z \right) \right) \right),$$

hence we have *tropical Riemann's theorem* (cf. [M-Z, Cor. 8.6])

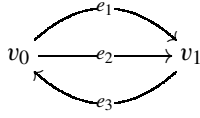
$$W_{g-1} = [\Theta] + \frac{1}{2}\mu_0(K_+ - K_-),$$

which obviously contains the image of the tropical Abel-Jacobi map μ .

However, it is more natural as in Alexeev [A] to consider $\text{Pic}^{g-1}(\Gamma)$, which is a principal homogeneous space under $\text{Pic}^0(\Gamma) = \text{Jac}(\Gamma)$. The canonical theta divisor $\Theta_{g-1} \subset \text{Pic}^{g-1}(\Gamma)$ is the image of the set of effective divisors on Γ of degree $g-1$.

4. Examples.

EXAMPLE 21 (Graphene crystal). The graphene is the unique 2-dimensional *strongly isotropic* crystal (cf. Sunada [S1]).



The elementary cycles:

$$\pm \left\{ \begin{array}{l} \gamma_1 := e_1 + e_3 \\ \gamma_2 := e_2 + e_3 \\ \gamma_1 - \gamma_2 = e_1 - e_2 \end{array} \right\}.$$

For simplicity, denote $\bar{e}_j := \pi(e_j)$ for all j . Then, $\delta v_0 = -\delta v_1 = e_1 + e_2 - e_3$ implies

$$\bar{e}_1 + \bar{e}_2 = \bar{e}_3,$$

and

$$\begin{cases} \gamma_1 = 2\bar{e}_1 + \bar{e}_2 \\ \gamma_2 = \bar{e}_1 + 2\bar{e}_2 \end{cases} \quad \text{hence} \quad \begin{cases} \bar{e}_1 = \frac{2\gamma_1 - \gamma_2}{3} \\ \bar{e}_2 = \frac{-\gamma_1 + 2\gamma_2}{3}. \end{cases}$$

In view of

$$(\gamma_1, \gamma_1) = (\gamma_2, \gamma_2) = 2, \quad (\gamma_1, \gamma_2) = 1,$$

we have

$$(\bar{e}_1, \bar{e}_1) = (\bar{e}_2, \bar{e}_2) = 2/3, \quad (\bar{e}_1, \bar{e}_2) = -1/3$$

and

$$(\gamma_1, \bar{e}_1) = (\gamma_2, \bar{e}_2) = 1 \quad (\gamma_1, \bar{e}_2) = (\gamma_2, \bar{e}_1) = 0,$$

hence $\{\gamma_1, \gamma_2\}$ and $\{\bar{e}_1, \bar{e}_2\}$ are mutually dual bases of $H_{\mathcal{Z}}$ and $\pi(\Lambda)$, respectively. The Voronoi cell $\pi(D_J)$ is a regular hexagon, and $(\text{Crystal}(\Gamma)) \cap \pi(D_J)$ is its circumference, that is (cf. Figure 1),

$$\begin{aligned} & [0, \bar{e}_1] \cup [\bar{e}_1, \bar{e}_1 + \bar{e}_3] \cup [\bar{e}_1 + \bar{e}_3, \bar{e}_1 + \bar{e}_3 + \bar{e}_2] \\ & \cup [0, \bar{e}_2] \cup [\bar{e}_2, \bar{e}_2 + \bar{e}_3] \cup [\bar{e}_2 + \bar{e}_3, \bar{e}_2 + \bar{e}_3 + \bar{e}_1]. \end{aligned}$$

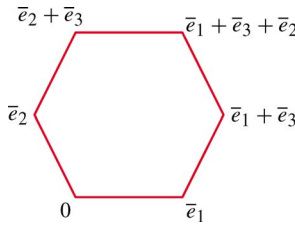
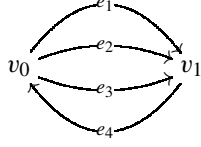


FIGURE 1. Graphene.

EXAMPLE 22 (Diamond crystal). The diamond is one of the two 3-dimensional *strongly isotropic* crystals (cf. Sunada [S1]).



The elementary cycles:

$$\pm \left\{ \begin{array}{l} \gamma_1 := e_1 + e_4 \\ \gamma_2 := e_2 + e_4 \\ \gamma_3 := e_3 + e_4 \\ \gamma_1 - \gamma_2 = e_1 - e_2 \\ \gamma_2 - \gamma_3 = e_2 - e_3 \\ \gamma_3 - \gamma_1 = -e_1 + e_3 \end{array} \right\}.$$

For simplicity, denote $\bar{e}_j := \pi(e_j)$ for all j . Then $\delta v_0 = -\delta v_1 = e_1 + e_2 + e_3 - e_4$ implies

$$\bar{e}_4 = \bar{e}_1 + \bar{e}_2 + \bar{e}_3.$$

Thus

$$\begin{cases} \gamma_1 = 2\bar{e}_1 + \bar{e}_2 + \bar{e}_3 \\ \gamma_2 = \bar{e}_1 + 2\bar{e}_2 + \bar{e}_3 \\ \gamma_3 = \bar{e}_1 + \bar{e}_2 + 2\bar{e}_3 \end{cases} \quad \text{hence} \quad \begin{cases} \bar{e}_1 = \frac{3\gamma_1 - \gamma_2 - \gamma_3}{4} \\ \bar{e}_2 = \frac{-\gamma_1 + 3\gamma_2 - \gamma_3}{4} \\ \bar{e}_3 = \frac{-\gamma_1 - \gamma_2 + 3\gamma_3}{4}. \end{cases}$$

In view of

$$(\gamma_1, \gamma_1) = (\gamma_2, \gamma_2) = (\gamma_3, \gamma_3) = 2, \quad (\gamma_1, \gamma_2) = (\gamma_2, \gamma_3) = (\gamma_1, \gamma_3) = 1,$$

we get

$$(\bar{e}_1, \bar{e}_1) = (\bar{e}_2, \bar{e}_2) = (\bar{e}_3, \bar{e}_3) = 1/2, \quad (\bar{e}_1, \bar{e}_2) = (\bar{e}_2, \bar{e}_3) = (\bar{e}_1, \bar{e}_3) = -1/4$$

with $\{\gamma_1, \gamma_2, \gamma_3\}$ and $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ being mutually dual bases of H_Z and $\pi(\Lambda)$, respectively.

Let

$$\begin{cases} u_1 := \bar{e}_2 + \bar{e}_3 \\ u_2 := \bar{e}_1 + \bar{e}_3 \\ u_3 := \bar{e}_1 + \bar{e}_2. \end{cases}$$

Then it is easy to see that $\{u_1, u_2, u_3\}$ is orthonormal and

$$H_Z = \mathbf{Z}\gamma_1 + \mathbf{Z}\gamma_2 + \mathbf{Z}\gamma_3 \subset \mathbf{Z}u_1 + \mathbf{Z}u_2 + \mathbf{Z}u_3 \subset \pi(\Lambda) = \mathbf{Z}\bar{e}_1 + \mathbf{Z}\bar{e}_2 + \mathbf{Z}\bar{e}_3$$

with

$$\begin{cases} \gamma_1 = u_2 + u_3 \\ \gamma_2 = u_1 + u_3 \\ \gamma_3 = u_1 + u_2 \end{cases}$$

and each lattice is a sublattice of index 2 of the one to the right. In particular, H_Z is the *face centered cubic* (fcc) lattice, which is also known as A_3 .

The Voronoi cell $\pi(D_J)$ is a rhombic dodecahedron (cf. Figure 2) with

$$(\text{Crystal}(\Gamma)) \cap \pi(D_J) =$$

$$[0, \bar{e}_1] \cup [\bar{e}_1, \bar{e}_1 + \bar{e}_4] \cup [\bar{e}_1 + \bar{e}_4, \bar{e}_1 + \bar{e}_4 + \bar{e}_2] \cup [\bar{e}_1 + \bar{e}_4, \bar{e}_1 + \bar{e}_4 + \bar{e}_3]$$

$$\begin{aligned} & \cup [0, \bar{e}_2] \cup [\bar{e}_2, \bar{e}_2 + \bar{e}_4] \cup [\bar{e}_2 + \bar{e}_4, \bar{e}_2 + \bar{e}_4 + \bar{e}_1] \cup [\bar{e}_2 + \bar{e}_4, \bar{e}_2 + \bar{e}_4 + \bar{e}_3] \\ & \cup [0, \bar{e}_3] \cup [\bar{e}_3, \bar{e}_3 + \bar{e}_4] \cup [\bar{e}_3 + \bar{e}_4, \bar{e}_3 + \bar{e}_4 + \bar{e}_1] \cup [\bar{e}_3 + \bar{e}_4, \bar{e}_3 + \bar{e}_4 + \bar{e}_2]. \end{aligned}$$

(cf. Figure 2).

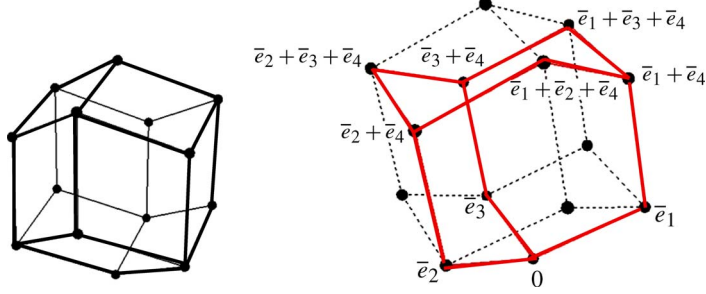
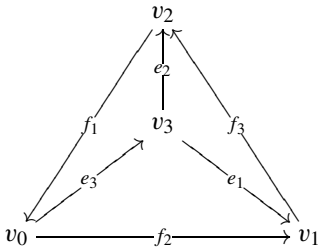


FIGURE 2. Rhombic dodecahedron and diamond crystal on its surface.

EXAMPLE 23 (K_4 crystal). The K_4 crystal was shown by Sunada [S1] to possess the “maximal symmetry” and “strong isotropy” properties, which it (together with its mirror image) shares in dimension three only with the diamond crystal.



The elementary cycles:

$$\pm \left\{ \begin{array}{l} \gamma_1 := e_2 + e_3 + f_1 \\ \gamma_2 := -e_1 - e_3 + f_2 \\ \gamma_3 := e_1 - e_2 + f_3 \\ \gamma'_1 := -e_2 - e_3 + f_2 + f_3 = \gamma_2 + \gamma_3 \\ \gamma'_2 := e_1 + e_3 + f_1 + f_3 = \gamma_1 + \gamma_3 \\ \gamma'_3 := -e_1 + e_2 + f_1 + f_2 = \gamma_1 + \gamma_2 \\ \gamma_0 := f_1 + f_2 + f_3 = \gamma_1 + \gamma_2 + \gamma_3 \end{array} \right\}.$$

For simplicity, denote $\bar{e}_j := \pi(e_j)$ and $\bar{f}_j := \pi(f_j)$ for all j . Then

$$\begin{aligned} \delta v_0 &= e_3 - f_1 + f_2 \\ \delta v_1 &= -e_1 - f_2 + f_3 \\ \delta v_2 &= -e_2 + f_1 - f_3 \\ \delta v_3 &= e_1 + e_2 - e_3 \end{aligned}$$

implies

$$\begin{cases} \bar{e}_1 = -\bar{f}_2 + \bar{f}_3 \\ \bar{e}_2 = \bar{f}_1 - \bar{f}_3 \\ \bar{e}_3 = \bar{f}_1 - \bar{f}_2. \end{cases}$$

Hence

$$\begin{cases} \gamma_1 = 3\bar{f}_1 - \bar{f}_2 - \bar{f}_3 \\ \gamma_2 = -\bar{f}_1 + 3\bar{f}_2 - \bar{f}_3 \\ \gamma_3 = -\bar{f}_1 - \bar{f}_2 + 3\bar{f}_3 \end{cases} \quad \begin{cases} \gamma_0 = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 \\ \gamma'_1 = 2(-\bar{f}_1 + \bar{f}_2 + \bar{f}_3) \\ \gamma'_2 = 2(\bar{f}_1 - \bar{f}_2 + \bar{f}_3) \\ \gamma'_3 = 2(\bar{f}_1 + \bar{f}_2 - \bar{f}_3). \end{cases}$$

Consequently,

$$\begin{cases} \bar{f}_1 = \frac{2\gamma_1 + \gamma_2 + \gamma_3}{4} \\ \bar{f}_2 = \frac{\gamma_1 + 2\gamma_2 + \gamma_3}{4} \\ \bar{f}_3 = \frac{\gamma_1 + \gamma_2 + 2\gamma_3}{4} \end{cases} \quad \begin{cases} \bar{e}_1 = \frac{-\gamma_2 + \gamma_3}{4} \\ \bar{e}_2 = \frac{\gamma_1 - \gamma_3}{4} \\ \bar{e}_3 = \frac{\gamma_1 - \gamma_2}{4}. \end{cases}$$

In view of

$$(\gamma_1, \gamma_1) = (\gamma_2, \gamma_2) = (\gamma_3, \gamma_3) = 3, \quad (\gamma_1, \gamma_2) = (\gamma_2, \gamma_3) = (\gamma_1, \gamma_3) = -1,$$

we have

$$(\bar{f}_1, \bar{f}_1) = (\bar{f}_2, \bar{f}_2) = (\bar{f}_3, \bar{f}_3) = 1/2, \quad (\bar{f}_1, \bar{f}_2) = (\bar{f}_2, \bar{f}_3) = (\bar{f}_1, \bar{f}_3) = 1/4$$

with $\{\gamma_1, \gamma_2, \gamma_3\}$ and $\{\bar{f}_1, \bar{f}_2, \bar{f}_3\}$ being mutually dual bases of H_Z and $\pi(\Lambda)$, respectively.

Let

$$\begin{cases} u_1 := -\bar{f}_1 + \bar{f}_2 + \bar{f}_3 \\ u_2 := \bar{f}_1 - \bar{f}_2 + \bar{f}_3 \\ u_3 := \bar{f}_1 + \bar{f}_2 - \bar{f}_3. \end{cases}$$

Then it is easy to see that $\{u_1, u_2, u_3\}$ is orthonormal and

$$H_Z = \mathbf{Z}\gamma_1 \oplus \mathbf{Z}\gamma_2 \oplus \mathbf{Z}\gamma_3 \subset \mathbf{Z}u_1 \oplus \mathbf{Z}u_2 \oplus \mathbf{Z}u_3 \subset \pi(\Lambda) = \mathbf{Z}\bar{f}_1 \oplus \mathbf{Z}\bar{f}_2 \oplus \mathbf{Z}\bar{f}_3$$

with

$$\begin{cases} \gamma_1 = -u_1 + u_2 + u_3 \\ \gamma_2 = u_1 - u_2 + u_3 \\ \gamma_3 = u_1 + u_2 - u_3 \end{cases}$$

and each lattice is a sublattice of index 4 of the one to the right. In particular, H_Z is the *body centered cubic* (bcc) lattice, which is also known as A_3^* . The Voronoi cell $\pi(D_J)$ is a truncated octahedron (also known as Kelvin polytope) (cf. Figure 3) such that

$$\begin{aligned} (\text{Crystal}(\Gamma)) \cap \pi(D_J) = \\ [0, \bar{e}_3] \cup [\bar{e}_3, \bar{e}_3 + \bar{e}_1] \cup [\bar{e}_3 + \bar{e}_1, \bar{e}_3 + \bar{e}_1 + \bar{f}_3] \end{aligned}$$

$$\begin{aligned}
 & \cup [\bar{e}_3 + \bar{e}_1 + \bar{f}_3, \bar{e}_3 + \bar{e}_1 + \bar{f}_3 + \bar{f}_1] \\
 & \cup [\bar{e}_3 + \bar{e}_1 + \bar{f}_3 + \bar{f}_1, \bar{e}_3 + \bar{e}_1 + \bar{f}_3 + \bar{f}_1 + \bar{f}_2] \\
 & \cup [\bar{e}_3, \bar{e}_3 + \bar{e}_2] \cup [\bar{e}_3 + \bar{e}_2, \bar{e}_3 + \bar{e}_2 + \bar{f}_1] \cup [\bar{e}_3 + \bar{e}_2 + \bar{f}_1, \bar{e}_3 + \bar{e}_2 + \bar{f}_1 + \bar{f}_2] \\
 & \cup [\bar{e}_3 + \bar{e}_2 + \bar{f}_1 + \bar{f}_2, \bar{e}_3 + \bar{e}_2 + \bar{f}_1 + \bar{f}_2 + \bar{f}_3] \\
 & \cup [0, \bar{f}_2] \cup [\bar{f}_2, \bar{f}_2 + \bar{f}_3] \cup [\bar{f}_2 + \bar{f}_3, \bar{f}_2 + \bar{f}_3 + \bar{f}_1] \\
 & \cup [\bar{f}_2 + \bar{f}_3 + \bar{f}_1, \bar{f}_2 + \bar{f}_3 + \bar{f}_1 + \bar{e}_3] \\
 & \cup [\bar{f}_2 + \bar{f}_3 + \bar{f}_1 + \bar{e}_3, \bar{f}_2 + \bar{f}_3 + \bar{f}_1 + \bar{e}_3 + \bar{e}_1] \\
 & \cup [\bar{f}_2 + \bar{f}_3 + \bar{f}_1 + \bar{e}_3, \bar{f}_2 + \bar{f}_3 + \bar{f}_1 + \bar{e}_3 + \bar{e}_2].
 \end{aligned}$$

(cf. Figure 3)

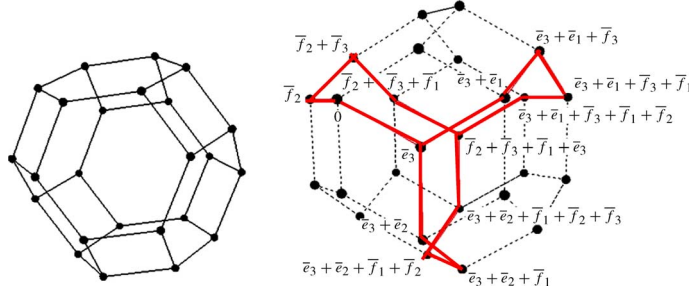
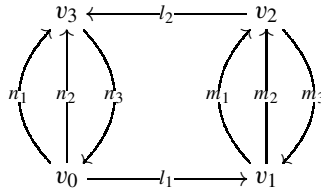


FIGURE 3. Truncated octahedron (also known as Kelvin polytope) and K_4 crystal on its surface.

EXAMPLE 24 (Lonsdaleite (also known as Hexagonal diamond) crystal).



We have

$$H_Z = \mathbf{Z}(l_1 - m_3 + l_2 + n_3) \oplus \mathbf{Z}(m_1 + m_3) \oplus \mathbf{Z}(m_2 + m_3) \oplus \mathbf{Z}(n_1 + n_3) \oplus \mathbf{Z}(n_2 + n_3).$$

Let

$$L := \mathbf{Z}(m_1 + m_3 - n_1 - n_3) \oplus \mathbf{Z}(m_2 + m_3 - n_2 - n_3) \subset H_Z$$

and consider the free abelian covering

$$\tilde{\Gamma} := \Gamma^{\text{ab}}/L \longrightarrow \Gamma.$$

The associated standard realization of $\tilde{\Gamma}$ in the 3-dimensional Euclidean space

$$E' := (H_Z/L) \otimes_{\mathbf{Z}} \mathbf{R}$$

turns out to be the Lonsdaleite crystal. Since

$$\begin{cases} \delta v_0 = n_1 + n_2 - n_3 + l_1 \\ \delta v_1 = m_1 + m_2 - m_3 - l_1 \\ \delta v_2 = -m_1 - m_2 + m_3 + l_2 \\ \delta v_3 = -n_1 - n_2 + n_3 - l_2, \end{cases}$$

we see that

$$C_1(\Gamma, \mathbf{R}) =: C \supset E' = \left\{ \begin{array}{l} n_1 + n_2 - n_3 + l_1, m_1 + m_2 - m_3 - l_1, \\ -m_1 - m_2 + m_3 + l_2, -n_1 - n_2 + n_3 - l_2, \\ m_1 + m_3 - n_1 - n_3, m_2 + m_3 - n_2 - n_3 \end{array} \right\}^\perp$$

with the orthogonal projection $\pi': C \rightarrow E'$. For simplicity denote

$$l'_j := \pi'(l_j), m'_j := \pi'(m_j), n'_j := \pi'(n_j), \quad \text{for all } j.$$

Then obviously

$$\begin{aligned} l'_1 &= m'_1 + m'_2 - m'_3 = -n'_1 - n'_2 + n'_3 \\ l'_2 &= m'_1 + m'_2 - m'_3 = -n'_1 - n'_2 + n'_3 \\ m'_1 + m'_3 &= n'_1 + n'_3 \\ m'_2 + m'_3 &= n'_2 + n'_3. \end{aligned}$$

Let

$$\begin{aligned} q_1 &:= -m'_2 + n'_3 = -n'_2 + m'_3 \\ q_2 &:= -m'_1 + n'_3 = -n'_1 + m'_3 \\ q_3 &:= q_1 + q_2 - m'_3 = -q_1 - q_2 + n'_3. \end{aligned}$$

Then easy computation shows that

$$\begin{aligned} m'_1 &= q_1 + q_3 \\ m'_2 &= q_2 + q_3 \\ m'_3 &= q_1 + q_2 - q_3 \\ n'_1 &= q_1 - q_3 \\ n'_2 &= q_2 - q_3 \\ n'_3 &= q_1 + q_2 + q_3 \\ l'_1 &= 3q_3 \\ l'_2 &= 3q_3. \end{aligned}$$

With $\Lambda := C_1(\Gamma, \mathbf{Z})$, we see easily that

$$\Lambda \cap E' = \mathbf{Z}\gamma_1 \oplus \mathbf{Z}\gamma_2 \oplus \mathbf{Z}\gamma_3,$$

where

$$\begin{aligned}\gamma_1 &:= m_1 + m_3 + n_1 + n_3 \\ \gamma_2 &:= m_2 + m_3 + n_2 + n_3 \\ \gamma_3 &:= m_1 + m_2 - m_3 - n_1 - n_2 + n_3 + 3l_1 + 3l_2.\end{aligned}$$

In E' we have a sequence of lattices

$$\Lambda \cap E' \subset \pi'(H_Z) \subset \pi'(\Lambda)$$

with

$$\begin{aligned}\pi'(\Lambda) &= \mathbf{Z}q_1 \oplus \mathbf{Z}q_2 \oplus \mathbf{Z}q_3 \\ \pi'(H_Z) &= \mathbf{Z}(2q_1 + q_2) \oplus \mathbf{Z}(q_1 + 2q_2) \oplus \mathbf{Z}(8q_3) \\ \Lambda \cap E' &= \mathbf{Z}\gamma_1 \oplus \mathbf{Z}\gamma_2 \oplus \mathbf{Z}\gamma_3.\end{aligned}$$

Since

$$\begin{cases} \gamma_1 = 4q_1 + 2q_2 \\ \gamma_2 = 2q_1 + 4q_2 \\ \gamma_3 = 24q_3 \end{cases} \quad \text{hence} \quad \begin{cases} q_1 = \frac{2\gamma_1 - \gamma_2}{6} \\ q_2 = \frac{-\gamma_1 + 2\gamma_2}{6} \\ q_3 = \frac{\gamma_3}{24} \end{cases}$$

with

$$(\gamma_1, \gamma_1) = (\gamma_2, \gamma_2) = 4, \quad (\gamma_3, \gamma_3) = 24, \quad (\gamma_1, \gamma_2) = 2, \quad (\gamma_1, \gamma_3) = (\gamma_2, \gamma_3) = 0,$$

we see that

$$(q_1, q_1) = (q_2, q_2) = 1/3, \quad (q_1, q_2) = -1/6, \quad (q_3, q_3) = 1/24, \quad (q_1, q_3) = (q_2, q_3) = 0$$

and that $\{\gamma_1, \gamma_2, \gamma_3\}$ and $\{q_1, q_2, q_3\}$ are mutually dual bases of $\Lambda \cap E'$ and $\pi'(\Lambda)$, respectively.

The Lonsdaleite crystal, which is obtained as the standard realization $\pi'(\text{Crystal}(\Gamma))$ of a non-maximal abelian covering, also turns out not to intrude the interiors of the top-dimensional Voronoi cells in

$$\text{Vor}(E', q_1 + q_2 + 3q_3 + \pi'(H_Z)).$$

The Voronoi cell

$$V(q_1 + q_2 + 3q_3) \in \text{Vor}(E', q_1 + q_2 + 3q_3 + \pi'(H_Z))$$

is a regular hexagonal cylinder (cf. Figure 4) with vertices

$$\begin{aligned}-q_3, q_1 - q_3, 2q_1 + q_2 - q_3, 2q_1 + 2q_2 - q_3, q_1 + 2q_2 - q_3, q_2 - q_3, \\ 7q_3, q_1 + 7q_3, 2q_1 + q_2 + 7q_3, 2q_1 + 2q_2 + 7q_3, q_1 + 2q_2 + 7q_3, q_2 + 7q_3.\end{aligned}$$

$V(q_1 + q_2 + 3q_3) \cap \pi'(\text{Crystal}(\Gamma))$ consists of the following (cf. Figure 4):

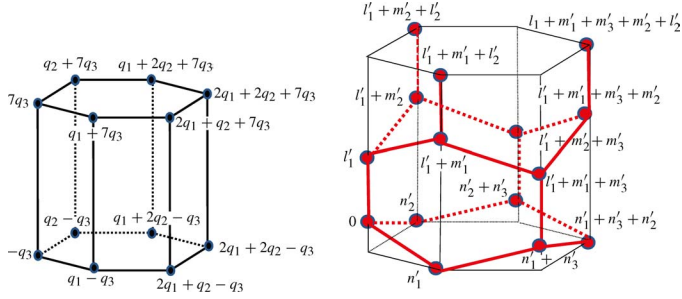


FIGURE 4. Regular hexagonal cylinder and Lonsdaleite crystal on its surface.

- the hexagonal closed curve joining

$$\begin{cases} 0 \\ n'_1 & = q_1 - q_3 \\ n'_1 + n'_3 & = 2q_1 + q_2 \\ n'_1 + n'_3 + n'_2 & = 2q_1 + 2q_2 - q_3 \\ n'_2 + n'_3 & = q_1 + 2q_2 \\ n'_2 & = q_2 - q_3 \\ 0 \end{cases}$$

in this order,

- the hexagonal closed curve joining

$$\begin{cases} l'_1 & = 3q_3 \\ l'_1 + m'_1 & = q_1 + 4q_3 \\ l'_1 + m'_1 + m'_3 & = 2q_1 + q_2 + 3q_3 \\ l'_1 + m'_1 + m'_3 + m'_2 & = 2q_1 + 2q_2 + 4q_3 \\ l'_1 + m'_2 + m'_3 & = q_1 + 2q_2 + 3q_3 \\ l'_1 + m'_2 & = q_2 + 4q_3 \\ l'_1 & = 3q_3 \end{cases}$$

in this order,

- the line segment joining

$$\begin{cases} 0 \\ l'_1 = 3q_3 \end{cases}$$

- the line segment joining

$$\begin{cases} n'_1 + n'_3 & = 2q_1 + q_2 \\ n'_1 + n'_3 + l'_1 & = 2q_1 + q_2 + 3q_3 \end{cases}$$

- the line segment joining

$$\begin{cases} n'_2 + n'_3 & = q_1 + 2q_2 \\ n'_2 + n'_3 + l'_1 & = q_1 + 2q_2 + 3q_3 \end{cases}$$

- the line segment joining

$$\begin{cases} l'_1 + m'_1 &= q_1 + 4q_3 \\ l'_1 + m'_1 + l'_2 &= q_1 + 7q_3 \end{cases}$$

- the line segment joining

$$\begin{cases} l'_1 + m'_1 + m'_3 + m'_2 &= 2q_1 + 2q_2 + 4q_3 \\ l'_1 + m'_1 + m'_3 + m'_2 + l'_2 &= 2q_1 + 2q_2 + 7q_3 \end{cases}$$

- the line segment joining

$$\begin{cases} l'_1 + m'_2 &= q_2 + 4q_3 \\ l'_1 + m'_2 + l'_2 &= q_2 + 7q_3 \end{cases}$$

In this case, we have a bijection

$$\Gamma = \tilde{\Gamma}/(H_{\mathbb{Z}}/L) \xrightarrow{\sim} \pi'(\text{Crystal}(\Gamma))/\pi'(H_{\mathbb{Z}}).$$

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