Tohoku Math. J.
55 (2003), 311–320

VORTEX FILAMENT EQUATION IN A RIEMANNIAN MANIFOLD

NORIHITO KOISO

(Received May 28, 2001, revised July 23, 2002)

Abstract. We define a riemannian version of the vortex filament equation. Using perturbation to a parabolic equation, we prove the short time unique existence of a solution for any initial closed curve.

1. Introduction and preliminaries. The vortex filament equation is an equation of a curve $\gamma(x, t)$ in the three-dimensional euclidean space:

$$\text{(V)} \qquad \gamma_t = \gamma_x \times \gamma_{xx}\,, \quad |\gamma_x| \equiv 1\,,$$

where $\times$ is the exterior product. Hasimoto [H] showed that this equation can be transformed to a standard nonlinear Schrödinger equation. However, his transformation was not mathematically well-defined.

The existence of a solution of (V) was first proved by Nishiyama and Tani [NT] using a perturbation to a fourth order parabolic equation. The present author gave another proof using a perturbation to a second order parabolic equation, and justified mathematically Hasimoto's transformation [K].

For a solution $\gamma(x, t)$ of (V), $\xi := \gamma_x$ satisfies $\xi_t = \xi \times \xi_{xx}$. Moreover, the norm $|\xi|$ is preserved along time. Therefore, the equation of ξ becomes an equation in the standard sphere S^2 in the euclidean three-space. This is a key point of the proofs in both [NT] and [K]. We can perturb the equation of ξ to a parabolic equation in S^2.

In this paper, we consider the vortex filament equation in a general oriented three-dimensional Riemannian manifold (M, g):

$$\text{(VM)} \qquad \gamma_t = \gamma_x \times \nabla_x \gamma_x\,, \quad |\gamma_x| \equiv 1\,,$$

where ∇ is the covariant differentiation. When (M, g) is homogeneous, we can generalize the above technique, and obtain the existence of a short time solution [K].

Our main interest is the stability of Equation (V) under the most natural generalization from a point of view of Riemannian geometry.

In the euclidean space, Hasimoto's transformation reduces Equation (V) of three unknown functions to an equation of two unknown functions. However, in a general Riemannian manifold, such a transformation converts Equation (V) to an equation of five unknown functions, because the equation contains position variables.

2000 *Mathematics Subject Classification.* Primary 53C44; Secondary 35Q35, 35B25.
Key words and phrases. Vortex filament, Riemannian manifold, singular perturbation.

Therefore, we have to take more direct approach. We perturb the equation of γ itself to a parabolic equation:

$$\text{(P)} \qquad \gamma_t = \gamma_x \times \nabla_x \gamma_x + \varepsilon \nabla_x \gamma_x \, .$$

Nishiyama [N] took this approach in a different setting. He proved the existence of a solution, but did not show its uniqueness. The difficulty is caused by the variation of the norm $|\gamma_x|$ along time. We will overcome this difficulty by estimating γ and $w := |\gamma_x|^2$ simultaneously, and prove

THEOREM 3.1. *The equation* (VM) *has a unique short time solution for any* C^∞ *closed initial curve* $\gamma_0(x)$ *with* $|\nabla_x \gamma_0| \equiv 1$.

We here summarize our notation. We denote by $|*|$ the pointwise norm, by ∇ the covariant differentiation, by R the curvature tensor, and by $\times$ the exterior product on each tangent space $T_p M$, respectively. Partial derivation is denoted by subscript or ∂_x, ∂_t:

$$\eta_u = \partial_u \eta := \partial_u \eta^i \frac{\partial}{\partial x^i} = \frac{\partial \eta^i}{\partial u} \frac{\partial}{\partial x^i} \, .$$

The manifold M, its structure and all functions on M are supposed to be of class C^∞. We may assume that the curvature and its derivatives are bounded on M, because we only consider the short time existence.

For convenience, we recall relevant basic facts from Riemannian geometry. For a map $\eta = \eta(u, v) : \boldsymbol{R}^2 \to M$, η_u is a vector field along the map η. The covariant derivative $\nabla_u X$ of a vector field X along η for u-direction is given by

$$\nabla_u X = (\nabla_u X)^i \frac{\partial}{\partial x^i} = \{\partial_u X^i + \Gamma(\eta)_j{}^i{}_k \cdot \partial_u \eta^j \cdot X^k\} \frac{\partial}{\partial x^i} \, ,$$

where $\Gamma_j{}^i{}_k$ are Christoffel's symbols. We see $\nabla_u \eta_v = \nabla_v \eta_u$ by definition, but higher covariant differentiations do not commute: $\nabla_v \nabla_u X - \nabla_u \nabla_v X = R(\eta_v, \eta_u) X$. The curvature tensor R has many symmetries, but we will not use them. The Riemannian metric g and the exterior product $\times$ are parallel with respect to the covariant differentiation: $\partial_u \{g(X, Y)\} = g(\nabla_u X, Y) + g(X, \nabla_u Y)$, $\nabla_u (X \times Y) = (\nabla_u X) \times Y + X \times (\nabla_u Y)$.

We may assume, by rescaling, that the initial length of the curve is 1. Therefore, we may consider γ as a map from $(\boldsymbol{R}/\boldsymbol{Z}) \times \boldsymbol{R}_{\geq 0}$ to M.

We will take function norms only for x-direction. More precisely, we define the L_2 inner product $\langle *, * \rangle$ and the L_2 norm $\|*\|$ as follows.

$$\langle \alpha, \beta \rangle := \int_0^1 g(\alpha, \beta)\, dx \, , \quad \|\alpha\|^2 := \langle \alpha, \alpha \rangle \, , \quad \|\alpha\|_n^2 = \sum_{i=0}^n \|\nabla_x^i \alpha\|^2 \, .$$

Also, $\|\alpha\|_{C^n}$ measures only x-derivatives and is a function in t. By integration by parts, we have $\langle \nabla_x X, Y \rangle = -\langle X, \nabla_x Y \rangle$.

2. Existence. In this section we consider Problem (P) with a closed initial curve $\gamma_0(x)$ such that $|\gamma_{0x}| \equiv 1$. We assume that $0 < \varepsilon \leq 1$. Then (P) becomes parabolic, and a short time solution $\gamma(x,t)$ exists for each ε (see, e.g., [E, Theorem 6.3]. We apply it to periodic functions on $\boldsymbol{R}$). In the following, we denote by C, C_i, K and T positive constants independent of ε.

LEMMA 2.1. $w_t = \varepsilon(w_{xx} - 2|\nabla_x\gamma_x|^2)$.

PROOF. It follows from a simple calculation that

$$\begin{aligned} w_t &= 2g(\gamma_x, \nabla_t\gamma_x) = 2g(\gamma_x, \nabla_x\gamma_t) = 2g(\gamma_x, \nabla_x\gamma_x \times \nabla_x\gamma_x + \gamma_x \times \nabla_x^2\gamma_x + \varepsilon\nabla_x^2\gamma_x) \\ &= 2\varepsilon\{\partial_x(g(\gamma_x, \nabla_x\gamma_x)) - |\nabla_x\gamma_x|^2\} = \varepsilon(w_{xx} - 2|\nabla_x\gamma_x|^2)\,. \end{aligned}$$

□

LEMMA 2.2. *It holds that* $\max w \leq 1$ *and* $\|\nabla_x\gamma_x\|$, $\|w_x\| \leq C$.

PROOF. By the maximum principle, Lemma 2.1 implies that $\max w \leq 1$, i.e.,

$$\begin{aligned} \limsup_{h\downarrow 0}\frac{1}{h}\{\max w(*,t) - \max w(*,t-h)\} &\leq \limsup_{h\downarrow 0}\frac{1}{h}\{w(x,t) - w(x,t-h)\} \\ &= w_t(x,t) \leq \varepsilon w_{xx}(x,t) \leq 0\,, \end{aligned}$$

where x is the maximum point of w at t. For $\|\nabla_x\gamma_x\|$, using integration by parts, we have

$$\begin{aligned} \frac{d}{dt}\|\nabla_x\gamma_x\|^2 &= 2\langle\nabla_x\gamma_x, \nabla_t\nabla_x\gamma_x\rangle = 2\langle\nabla_x\gamma_x, R(\gamma_t,\gamma_x)\gamma_x + \nabla_x^2\gamma_t\rangle \\ &= 2\langle\nabla_x\gamma_x, R(\gamma_x\times\nabla_x\gamma_x + \varepsilon\nabla_x\gamma_x, \gamma_x)\gamma_x\rangle - 2\langle\nabla_x^2\gamma_x, \gamma_x\times\nabla_x^2\gamma_x + \varepsilon\nabla_x^2\gamma_x\rangle \\ &\leq C_1\|\nabla_x\gamma_x\|^2 - 2\varepsilon\|\nabla_x^2\gamma_x\|^2\,, \end{aligned}$$

which means that $\|\nabla_x\gamma_x\|$ increases at most exponentially. Consequently, we see that $\|w_x\| = \|2g(\gamma_x, \nabla_x\gamma_x)\| \leq 2\|\nabla_x\gamma_x\| \leq C_2$. □

LEMMA 2.3. *There exists a positive constant* T *such that* $w \geq 1/2$ *holds for any solution* $\gamma(x,t)$ *defined on a subinterval* $[0,T')$ *of* $[0,T)$.

PROOF. By Lemma 2.2, we have $\|\nabla_x\gamma_x\|$, $\|w_x\| \leq C_1$. Hence, from

$$\begin{aligned} \frac{d}{dt}\|w\|^2 &= 2\langle w, w_t\rangle = 2\varepsilon\langle w, w_{xx} - 2|\nabla_x\gamma_x|^2\rangle = -2\varepsilon\|w_x\|^2 - 4\varepsilon\langle w, |\nabla_x\gamma_x|^2\rangle \\ &\geq -C_2 - 4\|\nabla_x\gamma_x\|^2 \geq -C_3\,, \end{aligned}$$

we see that $\|w\|^2 \geq 1 - C_3T$ holds on $0 \leq t < T$, and that

$$\|1-w\|^2 \leq \|1-w\|^2 + \|w\|^2 - 1 + C_3T \leq 2\langle w, w-1\rangle + C_3T \leq C_3T\,.$$

Therefore, by the Sobolev imbedding theorem,

$$\max(1-w)^2 \leq \|1-w\|(\|1-w\| + \|w_x\|) \leq C_4\sqrt{T}(\sqrt{T} + C_5)\,,$$

and the result holds for a small T. □

LEMMA 2.4. *Let $\gamma(x,t)$ be as above. There exists a positive constant C such that*

$$\varepsilon^{-1}\frac{d}{dt}\|w_x\|^2 \le -\|w_{xx}\|^2 + C(1+\|\nabla_x^2\gamma_x\|),$$
$$\varepsilon^{-1}\frac{d}{dt}\|w_{xx}\|^2 \le -\|w_{xxx}\|^2 + C(1+\|\nabla_x^2\gamma_x\|^3).$$

PROOF.

$$\begin{aligned}\frac{d}{dt}\|w_x\|^2 &= 2\langle w_x, w_{tx}\rangle = -2\varepsilon\langle w_{xx}, w_{xx} - 2|\nabla_x\gamma_x|^2\rangle \\ &= -2\varepsilon\|w_{xx}\|^2 + 4\varepsilon\langle w_{xx}, |\nabla_x\gamma_x|^2\rangle \le -\varepsilon\|w_{xx}\|^2 + C_1\varepsilon\|\nabla_x\gamma_x\|^2\|\nabla_x\gamma_x\|_{C^0}^2,\end{aligned}$$
$$\begin{aligned}\frac{d}{dt}\|w_{xx}\|^2 &= 2\langle w_{xx}, w_{txx}\rangle = -2\varepsilon\langle w_{xxx}, w_{xxx} - 4g(\nabla_x\gamma_x, \nabla_x^2\gamma_x)\rangle \\ &= -2\varepsilon\|w_{xxx}\|^2 + 8\varepsilon\langle w_{xxx}, g(\nabla_x\gamma_x, \nabla_x^2\gamma_x)\rangle \\ &\le -\varepsilon\|w_{xxx}\|^2 + C_2\varepsilon\|\nabla_x\gamma_x\|_{C^0}^2\|\nabla_x^2\gamma_x\|^2.\end{aligned}$$

□

PROPOSITION 2.5. *There exist positive constants T and C such that $\|\nabla_x^2\gamma_x\| \le C$ and $\|w_x\|_{C^0} \le C\sqrt{\varepsilon}$ hold for any solution defined on a subinterval $[0,T')$ of $[0,T)$.*

PROOF. We consider in a small time interval such that $1/2 \le w \le 1$ holds by Lemma 2.3. We calculate the time derivative of $\|\nabla_x^2\gamma_x\|^2$ to get

$$\begin{aligned}\frac{d}{dt}\|\nabla_x^2\gamma_x\|^2 &= 2\langle\nabla_x^2\gamma_x, \nabla_t\nabla_x^2\gamma_x\rangle \\ &= 2\langle\nabla_x^2\gamma_x, R(\gamma_t,\gamma_x)\nabla_x\gamma_x + \nabla_x(R(\gamma_t,\gamma_x)\gamma_x) + \nabla_x^3\gamma_t\rangle.\end{aligned}$$

The curvature terms are bounded by

$$C_3\|\nabla_x^2\gamma_x\|\||\nabla_x\gamma_x|^2 + |\nabla_x^2\gamma_x|\| \le C_4(1+\|\nabla_x^2\gamma_x\|^2).$$

For the remaining term $2\langle\nabla_x^2\gamma_x, \nabla_x^3\gamma_t\rangle$, we have

$$\begin{aligned}2\langle\nabla_x^2\gamma_x, \nabla_x^3\gamma_t\rangle &= -2\langle\nabla_x^3\gamma_x, \nabla_x^2\gamma_t\rangle \\ &= -2\langle\nabla_x^3\gamma_x, \nabla_x\gamma_x\times\nabla_x^2\gamma_x + \gamma_x\times\nabla_x^3\gamma_x + \varepsilon\nabla_x^3\gamma_x\rangle \\ &= -2\varepsilon\|\nabla_x^3\gamma_x\|^2 - 2\langle\nabla_x^3\gamma_x, \nabla_x\gamma_x\times\nabla_x^2\gamma_x\rangle.\end{aligned}$$

We decompose each factor of $\langle\nabla_x^3\gamma_x, \nabla_x\gamma_x\times\nabla_x^2\gamma_x\rangle$ to the γ_x part and the component perpendicular to γ_x to get

$$\begin{aligned}-2\langle\nabla_x^3\gamma_x, \nabla_x\gamma_x\times\nabla_x^2\gamma_x\rangle &= -2\langle w^{-1}g(\nabla_x^3\gamma_x,\gamma_x)\gamma_x, \nabla_x\gamma_x\times\nabla_x^2\gamma_x\rangle \\ &\quad - 2\langle\nabla_x^3\gamma_x, w^{-1}g(\nabla_x\gamma_x,\gamma_x)\gamma_x\times\nabla_x^2\gamma_x\rangle \\ &\quad - 2\langle\nabla_x^3\gamma_x, w^{-1}g(\nabla_x^2\gamma_x,\gamma_x)\nabla_x\gamma_x\times\gamma_x\rangle.\end{aligned}$$

For the first term, we use the equality: $2g(\nabla_x^3\gamma_x, \gamma_x) = w_{xxx} - 3\partial_x(|\nabla_x\gamma_x|^2)$. Then

$$\begin{aligned}
&-2\langle w^{-1}g(\nabla_x^3\gamma_x, \gamma_x)\gamma_x, \nabla_x\gamma_x \times \nabla_x^2\gamma_x\rangle \\
&\quad = -\langle w^{-1}\{w_{xxx} - 3\partial_x(|\nabla_x\gamma_x|^2)\}\gamma_x, \nabla_x\gamma_x \times \nabla_x^2\gamma_x\rangle \\
&\quad \le C_5(\|w_{xxx}\| + \|\nabla_x\gamma_x\|_{C^0}\|\nabla_x^2\gamma_x\|)\|\nabla_x\gamma_x\|_{C^0}\|\nabla_x^2\gamma_x\| \\
&\quad \le \|w_{xxx}\|^2 + C_6(1 + \|\nabla_x^2\gamma_x\|^3)\,.
\end{aligned}$$

For the second term, we use the equality: $2g(\nabla_x\gamma_x, \gamma_x) = w_x$. Then

$$\begin{aligned}
-2\langle\nabla_x^3\gamma_x, w^{-1}g(\nabla_x\gamma_x, \gamma_x)\gamma_x \times \nabla_x^2\gamma_x\rangle &= -\langle\nabla_x^3\gamma_x, w^{-1}w_x\gamma_x \times \nabla_x^2\gamma_x\rangle \\
&\le C_7\|w_x\|_{C^0}\|\nabla_x^3\gamma_x\|\|\nabla_x^2\gamma_x\| \le C_8(\|w_x\| + \|w_{xx}\|)\|\nabla_x^3\gamma_x\|\|\nabla_x^2\gamma_x\| \\
&\le \varepsilon\|\nabla_x^3\gamma_x\|^2 + C_9\varepsilon^{-1}(\|w_x\|^2 + \|w_{xx}\|^2)\|\nabla_x^2\gamma_x\|^2\,.
\end{aligned}$$

For the last term, we use the equality: $2g(\nabla_x^2\gamma_x, \gamma_x) = w_{xx} - 2|\nabla_x\gamma_x|^2$. Then

$$-2\langle\nabla_x^3\gamma_x, w^{-1}g(\nabla_x^2\gamma_x, \gamma_x)\nabla_x\gamma_x \times \gamma_x\rangle = 2\langle\nabla_x^2\gamma_x, \partial_x\{w^{-1}g(\nabla_x^2\gamma_x, \gamma_x)\}\nabla_x\gamma_x \times \gamma_x\rangle\,,$$

which has bounds similar to the first term.

Summing up these, we have

$$\begin{aligned}
\frac{d}{dt}\|\nabla_x^2\gamma_x\|^2 &\le -\varepsilon\|\nabla_x^3\gamma_x\|^2 + 2\|w_{xxx}\|^2 \\
&\quad + C_{10}\{1 + \|\nabla_x^2\gamma_x\|^3 + \varepsilon^{-1}(\|w_x\|^2 + \|w_{xx}\|^2)\|\nabla_x^2\gamma_x\|^2\}\,.
\end{aligned}$$

Combining it with Lemma 2.4, we see that $X(t) := \varepsilon^{-1}\|w_x\|^2 + \varepsilon^{-1}\|w_{xx}\|^2 + (1/2)\|\nabla_x^2\gamma_x\|^2$ satisfies $X'(t) \le C_{11}(1 + X(t))^2$. Therefore, $\varepsilon^{-1}\|w_x\|^2$, $\varepsilon^{-1}\|w_{xx}\|^2$ and $\|\nabla_x^2\gamma_x\|$ are uniformly bounded on a certain finite time interval. □

LEMMA 2.6. *Let T be as in Proposition* 2.5 *and n a nonnegative integer. For any positive number K, there exists a positive constant C such that if $\|\gamma_x\|_{n+2} \le K$, then*

$$\varepsilon^{-1}\frac{d}{dt}\|\partial_x^{n+3}w\|^2 \le -\|\partial_x^{n+4}w\|^2 + C(1 + \|\nabla_x^{n+3}\gamma_x\|^2)\,.$$

PROOF.

$$\begin{aligned}
\frac{d}{dt}\|\partial_x^{n+3}w\|^2 &= 2\Big\langle\partial_x^{n+3}w, \partial_x^{n+3}w_t = -2\varepsilon\langle\partial_x^{n+4}w, \partial_x^{n+2}(w_{xx} - 2|\nabla_x\gamma_x|^2)\rangle\Big\rangle \\
&\le -2\varepsilon\|\partial_x^{n+4}w\|^2 + 4\varepsilon\|\partial_x^{n+4}w\|\|\partial_x^{n+2}(|\nabla_x\gamma_x|^2)\|\,.
\end{aligned}$$

Here, we also have

$$\begin{aligned}
\|\partial_x^{n+2}(|\nabla_x\gamma_x|^2)\| &\le 2\|\nabla_x^{n+3}\gamma_x\|\|\nabla_x\gamma_x\|_{C^0} + C_1\|\nabla_x^{n+2}\gamma_x\|\|\nabla_x\gamma_x\|_{C^1} + C_2 \\
&\le C_3(1 + \|\nabla_x^{n+3}\gamma_x\|)\,.
\end{aligned}$$

□

LEMMA 2.7. *Let T be as in Proposition* 2.5 *and n a nonnegative integer. For any positive number K, there exists a positive constant C such that if $\|\gamma_x\|_{n+2} \le K$, then*

$$\frac{d}{dt}\|\nabla_x^{n+3}\gamma_x\|^2 \le -\varepsilon\|\nabla_x^{n+4}\gamma_x\|^2 + C(1 + \|\nabla_x^{n+3}\gamma_x\|^2 + \|\partial_x^{n+4}w\|^2)\,.$$

PROOF.

$$\begin{aligned}
\frac{d}{dt}\|\nabla_x^{n+3}\gamma_x\|^2 &= 2\langle\nabla_x^{n+3}\gamma_x, \nabla_t\nabla_x^{n+3}\gamma_x\rangle \\
&= 2\Big\langle\nabla_x^{n+3}\gamma_x, \sum_{i=0}^{n+2}\nabla_x^i(R(\gamma_t,\gamma_x)\nabla_x^{n+2-i}\gamma_x) + \nabla_x^{n+4}\gamma_t\Big\rangle \\
&\le C_1\|\gamma_x\|_{n+3}^2 + 2\langle\nabla_x^{n+3}\gamma_x, \nabla_x^{n+4}(\gamma_x \times \nabla_x\gamma_x + \varepsilon\nabla_x\gamma_x)\rangle \\
&= C_1\|\gamma_x\|_{n+3}^2 - 2\varepsilon\|\nabla_x^{n+4}\gamma_x\|^2 + 2\sum_{i=0}^{n+4}\binom{n+4}{i}\langle\nabla_x^{n+3}\gamma_x, \nabla_x^i\gamma_x \times \nabla_x^{n+5-i}\gamma_x\rangle\,.
\end{aligned}$$

In the last summation term, $\|\nabla_x^i\gamma_x \times \nabla_x^j\gamma_x\| \le \|\nabla_x^i\gamma_x\|_{C^0}\|\nabla_x^j\gamma_x\| \le C_2$ if $i < j \le n+2$, and cancels if $i = 2$ or $n+3$. Therefore, we have to measure only terms with $i = 0, 1, n+4$. Moreover, the term with $i = 0$ equals to $-\langle\nabla_x^{n+3}\gamma_x, \nabla_x\gamma_x \times \nabla_x^{n+4}\gamma_x\rangle$, and is reduced to the case $i = 1$.

As in the proof of Proposition 2.5, we decompose each factor of the term with $i = 1$ and $n+4$ to the γ_x part and the component perpendicular to γ_x.

$$\begin{aligned}
\langle\nabla_x^{n+3}\gamma_x, \nabla_x\gamma_x \times \nabla_x^{n+4}\gamma_x\rangle &= \langle w^{-1}g(\nabla_x^{n+3}\gamma_x,\gamma_x)\gamma_x, \nabla_x\gamma_x \times \nabla_x^{n+4}\gamma_x\rangle \\
&\quad + \langle\nabla_x^{n+3}\gamma_x, w^{-1}g(\nabla_x\gamma_x,\gamma_x)\gamma_x \times \nabla_x^{n+4}\gamma_x\rangle \\
&\quad + \langle\nabla_x^{n+3}\gamma_x, w^{-1}g(\nabla_x^{n+4}\gamma_x,\gamma_x)\nabla_x\gamma_x \times \gamma_x\rangle\,.
\end{aligned}$$

We know that $g(\nabla_x^{n+3}\gamma_x,\gamma_x) = (1/2)\partial_x^{n+3}w - C_3 g(\nabla_x^{n+2}\gamma_x, \nabla_x\gamma_x)$+(lower derivatives). The first term is estimated as

$$\begin{aligned}
&\langle w^{-1}g(\nabla_x^{n+3}\gamma_x,\gamma_x)\gamma_x, \nabla_x\gamma_x \times \nabla_x^{n+4}\gamma_x\rangle \\
&\quad = -\langle\partial_x\{w^{-1}g(\nabla_x^{n+3}\gamma_x,\gamma_x)\}\gamma_x, \nabla_x\gamma_x \times \nabla_x^{n+3}\gamma_x\rangle \\
&\qquad - \langle w^{-1}g(\nabla_x^{n+3}\gamma_x,\gamma_x)\gamma_x, \nabla_x^2\gamma_x \times \nabla_x^{n+3}\gamma_x\rangle \\
&\quad \le C_4(\|\partial_x^{n+4}w\| + \|\gamma_x\|_{n+3} + \|\partial_x^{n+3}w\|_{C^0} + \|\gamma_x\|_{C^{n+2}})\|\nabla_x^{n+3}\gamma_x\| \\
&\quad \le C_5(1 + \|\partial_x^{n+4}w\| + \|\nabla_x^{n+3}\gamma_x\|)\|\nabla_x^{n+3}\gamma_x\| \\
&\quad \le C_6(1 + \|\partial_x^{n+4}w\|^2 + \|\nabla_x^{n+3}\gamma_x\|^2)\,.
\end{aligned}$$

The last term $\langle\nabla_x^{n+3}\gamma_x, w^{-1}g(\nabla_x^{n+4}\gamma_x,\gamma_x)\nabla_x\gamma_x \times \gamma_x\rangle$ can be estimated similarly. Since $|g(\nabla_x\gamma_x,\gamma_x)| = (1/2)|w_x| \le C_7\sqrt{\varepsilon}$, the second term is estimated as

$$\langle \nabla_x^{n+3}\gamma_x, w^{-1}g(\nabla_x\gamma_x,\gamma_x)\gamma_x \times \nabla_x^{n+4}\gamma_x\rangle \le C_8\sqrt{\varepsilon}\|\nabla_x^{n+3}\gamma_x\|\|\nabla_x^{n+4}\gamma_x\|$$
$$\le a\varepsilon\|\nabla_x^{n+4}\gamma_x\|^2 + C_9 a^{-1}\|\nabla_x^{n+3}\gamma_x\|^2,$$

where a is an arbitrary positive number.

Summing up these with sufficiently small a, we get the result. □

THEOREM 2.8. *Let T be as in Proposition* 2.5. *There exists a C^∞ solution γ of* (VM) *on* $0 \le t < T$.

PROOF. By Lemmas 2.6 and 2.7, $X(t) := \|\nabla_x^{n+3}\gamma_x\|^2 + C\varepsilon^{-1}\|\partial_x^{n+3}w\|^2$ satisfies $X'(t) \le C_1(1 + X(t))$, where C is as in Lemma 2.7. Therefore, by induction, each solution is smoothly bounded. Since the bound is uniform with respect to t, we can continue the solution up to T. Moreover, since the bounds are independent of ε, a subsequence of γ^ε ($\varepsilon \downarrow 0$) converges smoothly. The limit is a solution of (VM). □

3. Uniqueness. Once proving the existence of a solution, to show the uniqueness is standard. We take a tubular neighbourhood U of the initial data γ_0, and embed it in $\boldsymbol{R}^3$. In other words, we consider the vortex filament equation in $\boldsymbol{R}^3$ with a curved Riemannian metric g. With the coordinate of $\boldsymbol{R}^3$, we express the covariant differentiation and the exterior product by

$$\nabla_x\alpha = \nabla_x(\alpha^i\partial_i) = (\alpha_x^i + \Gamma_j{}^i{}_k\gamma_x^j\alpha^k)\partial_i, \quad \alpha \times \beta = (\alpha^j\partial_j)\times(\beta^k\partial_k) = \chi_j{}^i{}_k\alpha^j\beta^k\partial_i,$$

where ∂_i are the coordinate vector fields, $\Gamma_j{}^i{}_k$ are the Christoffel symbols, and $\chi_j{}^i{}_k$ are the coordinate expression of the exterior product. Using this, (VM) is written as

$$\gamma_t^i = \chi_j{}^i{}_k\gamma_x^j(\gamma_{xx}^k + \Gamma_l{}^k{}_m\gamma_x^l\gamma_x^m).$$

Let η be another solution with the same initial data. By ignoring ε in Section 2, η satisfies the same estimation as γ. We use $\tilde\chi$ and $\tilde\Gamma$ the corresponding coefficients along η, and put $\zeta^i := \eta^i - \gamma^i$. Then ζ^i satisfies

$$\zeta_t^i = \chi_j{}^i{}_k\gamma_x^j(\zeta_{xx}^k + 2\Gamma_l{}^k{}_m\gamma_x^l\zeta_x^m) + \chi_j{}^i{}_k\zeta_x^j(\gamma_{xx}^k + \Gamma_l{}^k{}_m\gamma_x^l\gamma_x^m) + \chi_j{}^i{}_k\zeta_x^j\zeta_{xx}^k + P, \tag{3.1}$$

where P is a sum of terms that contains $\tilde\chi_j{}^i{}_k - \chi_j{}^i{}_k$, $\tilde\Gamma_j{}^i{}_k - \Gamma_j{}^i{}_k$ or $\zeta_x^i\zeta_x^j$. Since γ and $\zeta = \eta - \gamma$ are smoothly bounded, we know that $|P|, |P_x| \le C_1(|\zeta| + |\zeta_x|)$.

We identify ζ with a vector field $\zeta^i\partial_i$ along γ. Then we obtain

$$\nabla_t\zeta = (\zeta_t^i + \Gamma_j{}^i{}_k\gamma_t^j\zeta^k)\partial_i,$$
$$\nabla_x\zeta = (\zeta_x^i + \Gamma_j{}^i{}_k\gamma_x^j\zeta^k)\partial_i,$$
$$\nabla_x^2\zeta = (\zeta_{xx}^i + 2\Gamma_j{}^i{}_k\gamma_x^j\zeta_x^k + (\Gamma_j{}^i{}_k\gamma_x^j)_x\zeta^k + \Gamma_j{}^i{}_k\gamma_x^j\Gamma_l{}^k{}_m\gamma_x^l\zeta^m)\partial_i.$$

Substituting these to (3.1), we get

$$\nabla_t\zeta = \gamma_x \times \nabla_x^2\zeta + \nabla_x\zeta \times \nabla_x\gamma_x + \nabla_x\zeta \times \nabla_x^2\zeta + Q,$$

where Q is a sum of P and terms that contain ζ^i or $\zeta_x^i\zeta_x^j$. Note that $|Q|, |\nabla_x Q| \le C_2(|\zeta| + |\nabla_x\zeta|)$.

Therefore, we have

$$\begin{aligned}\frac{d}{dt}\|\zeta\|^2 &= 2\langle\zeta, \nabla_t\zeta\rangle = 2\langle\zeta, \gamma_x \times \nabla_x^2\zeta + \nabla_x\zeta \times \nabla_x\gamma_x + \nabla_x\zeta \times \nabla_x^2\zeta + Q\rangle \\ &= -2\langle\zeta, \nabla_x\gamma_x \times \nabla_x\zeta\rangle + 2\langle\zeta, \nabla_x\zeta \times \nabla_x\gamma_x\rangle + 2\langle\zeta, \nabla_x\zeta \times \nabla_x^2\zeta\rangle + 2\langle\zeta, Q\rangle \\ &\leq C_3\|\zeta\|(\|\zeta\| + \|\nabla_x\zeta\|),\end{aligned}$$

because $\nabla_x^2\zeta$ is bounded. Also, we have

$$\begin{aligned}\frac{d}{dt}\|\nabla_x\zeta\|^2 &= 2\langle\nabla_x\zeta, \nabla_t\nabla_x\zeta\rangle = 2\langle\nabla_x\zeta, R(\gamma_t, \gamma_x)\zeta + \nabla_x\nabla_t\zeta\rangle \\ &\leq C_4\|\zeta\|\|\nabla_x\zeta\| - 2\langle\nabla_x^2\zeta, \nabla_t\zeta\rangle \\ &= C_4\|\zeta\|\|\nabla_x\zeta\| - 2\langle\nabla_x^2\zeta, \nabla_x\zeta \times \nabla_x\gamma_x + Q\rangle \\ &= C_4\|\zeta\|\|\nabla_x\zeta\| - 2\langle\nabla_x^2\zeta, \nabla_x\zeta \times \nabla_x\gamma_x\rangle + 2\langle\nabla_x\zeta, \nabla_x Q\rangle \\ &\leq C_5\|\zeta\|(\|\zeta\| + \|\nabla_x\zeta\|) - 2\langle\nabla_x^2\zeta, \nabla_x\zeta \times \nabla_x\gamma_x\rangle.\end{aligned}$$

To estimate the remaining term, we use the equality $g(\gamma_x, \gamma_x) = 1$. By the same way as the case of Γ and χ, it implies that $g(\gamma_x, \nabla_x\zeta)$ can be expressed as a sum of terms that contain $\tilde{g}_{ij} - g_{ij}$, ζ^i or $\zeta_x^i\zeta_x^j$, and we have $|g(\gamma_x, \nabla_x\zeta)|$, $|\partial_x(g(\gamma_x, \nabla_x\zeta))|$, $|g(\gamma_x, \nabla_x^2\zeta)| \leq C_6(|\zeta| + |\nabla_x\zeta|)$. Since $\nabla_x\gamma_x$ is perpendicular to γ_x,

$$\begin{aligned}\langle\nabla_x^2\zeta, \nabla_x\zeta \times \nabla_x\gamma_x\rangle &= \langle g(\nabla_x^2\zeta, \gamma_x)\gamma_x, \nabla_x\zeta \times \nabla_x\gamma_x\rangle + \langle\nabla_x^2\zeta, g(\nabla_x\zeta, \gamma_x)\gamma_x \times \nabla_x\gamma_x\rangle \\ &= \langle g(\nabla_x^2\zeta, \gamma_x)\gamma_x, \nabla_x\zeta \times \nabla_x\gamma_x\rangle - \langle\nabla_x\zeta, \nabla_x\{g(\nabla_x\zeta, \gamma_x)\gamma_x \times \nabla_x\gamma_x\}\rangle \\ &\leq C_7\|\nabla_x\zeta\|(\|\zeta\| + \|\nabla_x\zeta\|).\end{aligned}$$

Therefore, $X(t) := \|\zeta\|^2 + \|\nabla_x\zeta\|^2$ satisfies $X'(t) \leq C_8X(t)$, which implies that ζ identically vanishes. We have proved

THEOREM 3.1. (VM) *has a unique short time solution for any closed initial curve* $\gamma_0(x)$ *with* $|\gamma_{0x}| \equiv 1$.

4. Appendix. In Section 2, we heavily used the fact that the time derivative of w is bounded by ε. There is another method of estimation, which we did not use because it is lengthier than the proof given in Section 2. The method uses a weighted norm that has resemblance to [N]. Therefore, there may be some interest to the method. Here, we give its key point.

Since the ε-parts are easy to estimate by usual parabolic equation's argument, we can ignore such terms. Also, we can ignore curvature terms, because they contain only lower derivatives.

Let φ be the part of $\nabla_x^2\gamma_x$ perpendicular to γ_x. Namely,

$$\varphi := \nabla_x^2\gamma_x - u\gamma_x\,; \quad u := w^{-1}g(\nabla_x^2\gamma_x, \gamma_x) = w^{-1}\{(w_{xx}/2) - |\nabla_x\gamma_x|^2\}.$$

By Lemmas 2.2 and 2.3, we know that $\|\nabla_x\gamma_x\|$ and $\|w_x\|$ are bounded from above, and that w is bounded from below. Therefore,

$$\begin{aligned}
&\partial_x|\nabla_x\gamma_x|^2 = 2g(\nabla_x\gamma_x, \nabla_x^2\gamma_x) = 2g(\nabla_x\gamma_x, \varphi + u\gamma_x)\,,\\
&\|\partial_x|\nabla_x\gamma_x|\| \leq \|\varphi\| + \|u\| \leq C_1(\|\varphi\| + \|w_{xx}\| + \|\nabla_x\gamma_x\|_{C^0})\\
&\qquad\qquad \leq \frac{1}{2}\|\partial_x|\nabla_x\gamma_x|\| + C_2(\|\varphi\| + \|w_{xx}\| + 1)\,,\\
&\|\nabla_x\gamma_x\|_{C^0} \leq C_3(\|\varphi\| + \|w_{xx}\| + 1)\,,\\
&\|\nabla_x^2\gamma_x\|^2 = \|\varphi\|^2 + \|u\gamma_x\|^2 \leq C_4(\|\varphi\|^2 + \|w_{xx}\|^2 + \|\nabla_x\gamma_x\|_{C^0})\\
&\qquad\qquad \leq C_5(\|\varphi\|^2 + \|w_{xx}\|^2 + 1)\,,
\end{aligned}$$

which imply that we can use $\|\varphi\|$ instead of $\|\nabla_x^2\gamma_x\|$. Note also that w_{xx} contains third derivatives of γ, and is comparable to φ. From

$$\begin{aligned}
w_t &= [\varepsilon \text{ terms}]\,,\\
\gamma_t &= \gamma_x \times \nabla_x\gamma_x + \varepsilon\nabla_x\gamma_x = \gamma_x \times \nabla_x\gamma_x + [\varepsilon \text{ terms}]\,,\\
\nabla_x\gamma_t &= \gamma_x \times \nabla_x^2\gamma_x + [\varepsilon \text{ terms}] = \gamma_x \times \varphi + [\varepsilon \text{ terms}]\,,
\end{aligned}$$

we have

$$\begin{aligned}
\nabla_t\varphi &= \nabla_t\nabla_x^2\gamma_x - u_t\gamma_x - u\nabla_x\gamma_t\\
&= \nabla_x^2(\gamma_x \times \varphi) - u_t\gamma_x - u\gamma_x \times \varphi + [\varepsilon, \text{lower terms}]\,.
\end{aligned}$$

For a constant a, we put $X(t) := \|w^a\varphi\|^2 + \|w_{xx}\|^2$. Then we have

$$\begin{aligned}
\frac{d}{dt}\|w^a\varphi\|^2 &= 2\langle w^{2a}\varphi, \nabla_t\varphi\rangle + [\varepsilon \text{ terms}] = 2\langle w^{2a}\varphi, \nabla_x^2(\gamma_x \times \varphi)\rangle + [\varepsilon, \text{lower terms}]\\
&= -2\langle \partial_x(w^{2a})\varphi + w^{2a}\nabla_x\varphi, \nabla_x\gamma_x \times \varphi + \gamma_x \times \nabla_x\varphi\rangle + [\varepsilon, \text{lower terms}]\\
&= -4a\langle w^{2a-1}w_x\varphi, \gamma_x \times \nabla_x\varphi\rangle - 2\langle w^{2a}\nabla_x\varphi, \nabla_x\gamma_x \times \varphi\rangle + [\varepsilon, \text{lower terms}]\,.
\end{aligned}$$

Here,

$$\begin{aligned}
&-2\langle w^{2a}\nabla_x\varphi, \nabla_x\gamma_x \times \varphi\rangle\\
&\quad = -2\langle w^{2a-1}g(\nabla_x\varphi, \gamma_x)\gamma_x, \nabla_x\gamma_x \times \varphi\rangle - 2\langle w^{2a-1}\nabla_x\varphi, g(\nabla_x\gamma_x, \gamma_x)\gamma_x \times \varphi\rangle\\
&\quad = 2\langle w^{2a-1}g(\varphi, \nabla_x\gamma_x)\gamma_x, \nabla_x\gamma_x \times \varphi\rangle - \langle w^{2a-1}w_x\nabla_x\varphi, \gamma_x \times \varphi\rangle\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\|w^a\varphi\|^2 &= (4a-1)\langle w^{2a-1}w_x\nabla_x\varphi, \gamma_x \times \varphi\rangle + 2\|\varphi\|^2\|\nabla_x\gamma_x\|_{C^0}^2 + [\varepsilon, \text{lower terms}]\\
&\leq (4a-1)\langle w^{2a-1}w_x\nabla_x\varphi, \gamma_x \times \varphi\rangle + C_6(1 + X(t)^2) + [\varepsilon \text{ terms}]\,.
\end{aligned}$$

For $a = 1/4$, we have $X'(t) \leq C_7(1 + X(t)^2)$, and $X(t)$ is bounded on a certain finite time interval $[0, T)$. For higher derivatives, we can check by a similar calculation that $X_n(t) := \|w^{(n+1)/4}\nabla_x^n\varphi\|^2 + \|\partial_x^{n+2}w\|^2$ satisfies $X_n'(t) \leq C_8(1 + X_n(t))$, where C_8 depends on X_{n-1}. Thus, by induction, we can estimate all derivatives on $[0, T)$.

REFERENCES

[E] D. E. EIDELMAN, Parabolic systems, North-Holland publ., Amsterdam, 1969.
[H] H. HASIMOTO, A soliton on a vortex filament, J. Fluid Mech. 51 (1972), 477–485.
[K] N. KOISO, The vortex filament equation and a semilinear Schrödinger equation in a hermitian symmetric space, Osaka J. Math. 34 (1997), 199–214.
[N] T. NISHIYAMA, Existence of a solution to the mixed problem for a vector filament equation with an external flow term, J. Math. Sci. Univ. Tokyo 7 (2000), 35–55.
[NT] T. NISHIYAMA AND A. TANI, Initial and initial-boundary value problems for a vortex filament with or without axial flow, SIAM J. Math. Anal. 27 (1996), 1015–1023.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
OSAKA UNIVERSITY
TOYONAKA, OSAKA, 560–0043
JAPAN