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Vortex, Spin and Triad for Quantum Mechanics of Spinning Particle.I*)

—— *General Theory* ——

Takehiko TAKABAYASI

Department of Physics, Nagoya University, Nagoya 464

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It is shown that a natural extension of the hydrodynamical formalism of quantum mechanics for a Schrödinger particle to include vortical flows leads to the hydrodynamical formalism of quantum mechanics for a spinning particle. This latter formalism is then analysed in regard to its characteristic features, especially the subsidiary condition connecting the vorticity of flow with the inhomogeneity of spin field and the existence of spin stress. Also the formalism is brought to completion by establishing the global condition that quantizes circulation around a singular vortex line. The geometro-hydrodynamical formalism which is equivalent to the above hydrodynamical formalism but introduces a triad structure underlying the classical spin is reconstructed on its own footing. The geometrical property of the triad implies the invariance of theory with respect to the rotation of each triad around its symmetry axis by an arbitrary angle, and this necessitates the introduction of the electromagnetic potential, providing the geometrical interpretation of local gauge invariance. Applications of theory to various special cases and typical examples are deferred to Part II.

§ 1. Introduction and summary

1. 1.

Quantum mechanics of a non-relativistic particle without spin is represented usually by the Schrödinger equation

$$i\hbar\dot{\psi}=H\psi, \qquad H=(\boldsymbol{p}-e\boldsymbol{A})^2/(2\mu)+V\,, \tag{1·1}$$

together with the prescriptions of statistical interpretation for the wave function ψ. In (1·1) $\boldsymbol{A}$ and $A_0=V/e$ are vector and scalar potentials which give the external electromagnetic field by $\boldsymbol{E}=-\nabla A_0-\dot{\boldsymbol{A}}/c$ and $\boldsymbol{H}=\text{rot}\,\boldsymbol{A}$. If there is non-electromagnetic scalar potential V_1 also, V is to be understood as $V=eA_0+V_1$. That this theory is represented equivalently as a hydrodynamics was suggested early by Madelung for the case without vector potential.[1] Later this was extended and completed as a self-contained formalism by the author as follows.[2,3]

A state of flow is described by a density function P (or modulus R with $R^2=P$) and a velocity function $\boldsymbol{v}$. Their time evolutions are governed by**)

$$\dot{P}+\text{div}(P\boldsymbol{v})=0\,, \tag{1·2}$$

$$\frac{Dv_i}{Dt}=\frac{1}{\mu}\left\{e\left(\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\times\boldsymbol{H}]\right)_i-\frac{\partial V_1}{\partial x_i}\right\}-\frac{1}{\rho}\frac{\partial\tau_{ik}}{\partial x_k}\,, \tag{1·3}$$

where $D/Dt=\partial/\partial t+\boldsymbol{v}\cdot\nabla$ is the substantial derivative. In (1·3) the first term on the right side is the external force per unit mass, $\rho\equiv\mu P$ is the mass density, and

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**) The dot denotes $\partial/\partial t$, and we usually understand summation convention for repeated indices.

$$\tau_{ik}=\frac{\hbar^2}{4\mu}\left(\frac{1}{P}\frac{\partial P}{\partial x_i}\frac{\partial P}{\partial x_k}-\delta_{ik}\Delta P\right). \tag{1·4}$$

The above equations are the hydrodynamical equations for a compressible non-viscous charged fluid under the usual external force (Lorentz force) and the internal stress τ_{ik} and thus the general concepts and theorems of the usual hydrodynamics are applicable to this theory even though the pressure term $p\delta_{ik}$ in the usual hydrodynamics is supplanted here by the peculiar τ_{ik} ("quantum stress"). Note that Eq.(1·3) is rewritten as

$$\mu\frac{D\boldsymbol{v}}{Dt}=\boldsymbol{K}+\nabla\left(\frac{\hbar^2}{2\mu}\frac{\Delta R}{R}\right).\quad\left(\boldsymbol{K}=e\boldsymbol{E}+\frac{e}{c}[\boldsymbol{v}\times\boldsymbol{H}]-\nabla V_1\right) \tag{1·5}$$

Besides the equations of motion our hydrodynamics is characterized by the presence of subsidiary conditions which restrict vorticity and circulation; namely

$$\mathrm{rot}\ \boldsymbol{v}=-(e/\mu c)\boldsymbol{H}\ , \tag{1·6}$$

which holds except at nodal points (where P vanishes), and

$$\Gamma=\mu\oint_C\boldsymbol{v}\cdot d\boldsymbol{s}+\frac{e}{c}\Phi=nh,\qquad(n=\text{integer}) \tag{1·7}$$

where C is a closed contour and Φ is the magnetic flux going through C. These conditions are of course compatible with the equations of motion (see § 2). It is interesting to note that Eq. (1·6) has analogy to London's condition in his theory of superconductivity and Eq.(1·7) to the condition of fluxoid quantization for a superconducting ring.

All the above equations are gauge-independent, but the equivalent gauge-dependent form which employs A_μ and the 'canonical' momentum

$$\boldsymbol{\Pi}=\mu\boldsymbol{v}+(e/c)\boldsymbol{A}$$

is sometimes more convenient. Thus, corresponding to (1·5), (1·6) and (1·7) we have

$$\frac{D\Pi_i}{Dt}=\frac{e}{c}v_k\frac{\partial A_k}{\partial x_i}-\frac{\partial}{\partial x_i}\left(V-\frac{\hbar^2}{2\mu}\frac{\Delta R}{R}\right), \tag{1·8}$$

$$\mathrm{rot}\ \boldsymbol{\Pi}=0\ , \tag{1·9}$$

$$\Gamma=\oint_C\boldsymbol{\Pi}\cdot d\boldsymbol{s}=nh\ . \tag{1·10}$$

Owing to (1·9) $\boldsymbol{\Pi}$ is derivable from a potential S as $\boldsymbol{\Pi}=\nabla S$, where S is in general a multivalued function because of (1·10). Then (1·8) can be integrated into the form

$$\dot{S}+\frac{1}{2\mu}\left(\nabla S-\frac{e}{c}\boldsymbol{A}\right)^2+V-\frac{\hbar^2}{2\mu}\frac{\Delta R}{R}=0 \tag{1·11}$$

with an adjustment of arbitrary additive term of S. (This equation corresponds to Bernouilli's theorem.) Also the condition (1·10) is rewritten as

$$\Gamma=\oint_C dS=nh\ . \tag{1·12}$$

The basic quantities in this formalism are related to the wave function ψ in the Schrödinger theory through

$$\psi = Re^{iS/\hbar}, \tag{1·13}$$

or in other words

$$P=\psi^*\psi, \qquad \boldsymbol{\Pi}=\frac{\hbar}{2i}\{\psi^*\nabla\psi-(\nabla\psi^*)\psi\}/\psi^*\psi\,. \tag{1·14}$$

Note that Eq. (1·11) can be obtained from the Schrödinger equation (1·1), by isolating the continuity equation (1·2), as

$$\{\dot{S}+H(\boldsymbol{x},\nabla S)\}R-(\hbar^2/2\mu)\Delta R=0\,, \tag{1·15}$$

and is linearly homogeneous in R.

This hydrodynamical formalism has been restated later[4] and gradually been employed.[5),6)] The formalism is equivalent to the usual quantum mechanics but is represented in terms of classical quantities, which, however, should not be taken as literally real and observable lest it would contradict the uncertainty principle, but this fact does not prevent us from using it for the analysis of various problems with due precaution.[*)] The formalism brings to fore by its own characteristics associated with hydrodynamical picture those aspects and relations which may be difficult to notice in the conventional formalism, e.g., the effects of quantum stress, line vortex with quantized circulation, the relation and analogy between vorticity and magnetic field, etc.

1. 2.

By extending the theory stated in § 1.1 we can obtain hydrodynamical theory of a classical spinning fluid, which just represents quantum mechanics of a non-relativistic particle with $\frac{1}{2}$ spin. This extension of the original hydrodynamics can be made by various methods but they lead to the same theory, which is equivalent to the conventional Schrödinger-Pauli wave mechanics. We have the following three methods.

(A) To generalize the original hydrodynamics to include vortical flows.

(B) To generalize the original hydrodynamics by endowing the fluid with distribution of intrinsic angular momentum.

(C) Geometro-hydrodynamical viewpoint in which each fluid element is regarded as a 'triad' having degrees of freedom of rotation.

The first method (A) may be the most natural approach. Indeed this is considered in itself as one of the most significant insights[2),3)] which the original hydrodynamical formulation has suggested, and it shows most clearly that the hydrodynamical formalism gives the intrinsic relation between vortex and spin. This method consists of simply dropping the subsidiary condition (1·9). Thus the $\boldsymbol{\Pi}$-field is now expressed as

$$\boldsymbol{\Pi}=\nabla S+\xi\nabla\eta \tag{1·16}$$

with the introduction of Clebsch potentials ξ and η. We denote the $\boldsymbol{\Pi}$-field vorticity as

$$\boldsymbol{w}\equiv\mathrm{rot}\,\boldsymbol{\Pi}=\mu\,\mathrm{rot}\,\boldsymbol{v}+(e/c)\boldsymbol{H}\,, \tag{1·17}$$

which is now given by

$$\boldsymbol{w}=[\nabla\xi\times\nabla\eta]. \tag{1·18}$$

*) Hydrodynamical formalism and its applications to various problems in atomic, molecular and nuclear physics are reviewed recently in Ref.7), which contains a lot of literature.

Next we assume that just as the gradient of the density P gives rise to internal stress (1·4) the gradients of ξ and η also contribute to internal stress to be represented by its energy density

$$W_s=\frac{P}{2\mu}\{(\nabla\xi)^2/\rho+\rho(\nabla\eta)^2\}, \tag{1·19}$$

where ρ is a certain weight function depending on ξ alone. This modifies the Euler equation (1·3) by an additive term. We give details of this approach in § 3 but remark here the following. The introduction of the above W_s implies to fix the gauge of the Clebsch potentials, ξ and η, within a certain small subgroup, and as the result these variables acquire the status of intrinsic degrees of freedom themselves, and in fact imply the spin (intrinsic angular momentum of constant magnitude). It is then convenient to substitute (ξ, η) by the spin vector $\boldsymbol{S}$, or the equivalent 'polarization vector' $\boldsymbol{\Sigma}$, with

$$\boldsymbol{S}=(\hbar/2)\boldsymbol{\Sigma}, \qquad \boldsymbol{\Sigma}^2=1\,, \tag{1·20}$$

and to reformulate the above theory in terms of them. Explicitly the relations above mentioned are

$$\xi=-S_3, \qquad \eta=\tan^{-1}(S_2/S_1). \tag{1·21}$$

This reformulation leads us just to the second method (B).

1. 3.

Now we shall summarize the method (B).*) A state is described by the previous variables, P (or R) and $\boldsymbol{\Pi}$ (or $\boldsymbol{v}$), and the additional variable $\boldsymbol{S}$ which is again a classical quantity. The vorticity equation (1·18) is here reexpressed, through (1. 21), as

$$\boldsymbol{w}=\mathrm{rot}\,\boldsymbol{\Pi}=(\hbar/2)\boldsymbol{T} \tag{1·22}$$

with

$$T_i=\frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\Sigma_l\partial_j\Sigma_m\partial_k\Sigma_n\,. \tag{1·23}$$ **)

Since $\boldsymbol{\Sigma}$ is now regarded as proper internal degrees of freedom, the relation (1·22) appears here as *subsidiary condition* which relates the vorticity of flow with the inhomogeneity of the spin field, and is regarded as the counterpart of (1·6) in the spinless case. The vector $\boldsymbol{T}$, which plays a central role in this hydrodynamics, has various conspicuous properties, besides div $\boldsymbol{T}=0$. We shall call it 'spin-vorticity vector'.

The Euler equation (1·3) is now modified to

$$\mu\frac{Dv_i}{Dt}=K_i+\frac{e}{\mu c}S_k\partial_i H_k-\frac{1}{P}\frac{\partial\tau_{ik}{}^{(\mathrm{tot})}}{\partial x_k}, \tag{1·24}$$

$$\tau_{ik}{}^{(\mathrm{tot})}=\frac{1}{\mu}\left\{\frac{1}{P}\frac{\partial(PS_l)}{\partial x_i}\frac{\partial(PS_l)}{\partial x_k}-\frac{\hbar^2}{4}\delta_{ik}\Delta P\right\}. \tag{1·25}$$

The total internal stress (1·25) is viewed as consisting of τ_{ik} of (1·4) and the additional

*) This was given in Ref. 8). Later it was restated by Janossy.[9] See also Refs. 10) and 11).

**) $\partial_j\equiv\partial/\partial x_j$.

stress due to the spin field such that

$$\tau_{ik}^{(\mathrm{tot})}=\tau_{ik}+\mu^{-1}P\partial_i S_l \partial_k S_l\,. \tag{1·26}$$

Besides, there is the equation of motion for $\boldsymbol{S}$:

$$\frac{D\boldsymbol{S}}{Dt}=\frac{e}{\mu c}[\boldsymbol{S}\times\boldsymbol{H}]+\frac{1}{\mu P}[\boldsymbol{S}\times\partial_k(P\partial_k\boldsymbol{S})]. \tag{1·27}$$

Again the subsidiary condition (1·22) is compatible with the equations of motion, (1.24) and (1·27). (See § 4.)

The above hydrodynamical theory is equivalent to the usual wave mechanics for a spinning particle. [Equations of motion (1·24) and (1·27) correspond to the case when the quantum Hamiltonian is

$$H=\frac{1}{2\mu}(\boldsymbol{p}-e\boldsymbol{A})^2+V-\frac{e\hbar}{2\mu c}\boldsymbol{\sigma H}\,,$$

where the spin-orbit coupling term is neglected for simplicity.] The correspondence between both formulations is given by the former relations (1·14) (where, e.g., $P=\psi^*\psi$ is to be understood now as $P=\sum_{\alpha=1,2}\psi_\alpha^*\psi_\alpha$), and

$$\Sigma_i=(\psi^*\sigma_i\psi)/P\,. \tag{1·28}$$

Naturally the present hydrodynamics is more complicated and richer than the original one stated in § 1. 1; it again exhibits various characteristic aspects which may be difficult to notice in the usual formalism, and gives insights which are helpful to the actual treatment of the problems. Some such examples are given in Part II.

1. 4.

In the third method*) (C) we introduce internal configurational variables, the triad, underlying the classical spin $\boldsymbol{S}$, to reexpress the theory in terms of rotations of the triads. Thus each fluid element is now represented by $\{\boldsymbol{a}^r(\boldsymbol{x},t)\}$ satisfying

$$a_k{}^r a_k{}^s=\delta_{rs}. \qquad (r,s=1,2,3) \tag{1·29}$$

Characteristic dynamical property of this triad is that its motion is such that its angular momentum of rotation, i.e., the spin $\boldsymbol{S}$, is fixed to the body and has the constant magnitude $\hbar/2$. By taking this direction as the triad third axis, this is expressed as

$$\boldsymbol{\Sigma}=\boldsymbol{a}^3=[\boldsymbol{a}^1\times\boldsymbol{a}^2]. \tag{1·30}$$

This distinguishes the triad from a conventional symmetric top, though our triad has also the symmetry around the third axis and (1·30) is consistent with this symmetry.

With the use of $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$, Eq.(1·23) is rewritten as

$$T_i=\varepsilon_{ijk}\partial_j a_l{}^1\partial_k a_l{}^2\,. \tag{1·31}$$

Then it is verified that the variable $\boldsymbol{\Pi}$ satisfying (1·22) is realized as

$$\boldsymbol{\Pi}=-\frac{\hbar}{2}a_k{}^2\nabla a_k{}^1\,. \tag{1·32}$$

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*) This was formerly given in Ref. 12).

Thus in this method $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$ work as basic variables while both $\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$ are derived therefrom by (1·30) and (1·32) in satisfying the subsidiary condition (1·22) automatically, and in this way the triad is naturally incorporated into the hydrodynamical picture, bringing about further novel insights to the theory.

The triad contains three independent variables represented by Euler angles (ϕ,θ,χ), where θ and ϕ denote the polar angles of $\boldsymbol{\Sigma}$,

$$\Sigma_1=\sin\theta\cos\phi,\quad \Sigma_2=\sin\theta\sin\phi,\quad \Sigma_3=\cos\theta\,, \tag{1·33}$$

while χ describes the rotational orientation of the orthogonal $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$ axes on the plane normal to $\boldsymbol{\Sigma}$. In fact, the triad is regarded as the covariant method to represent χ together with $\boldsymbol{\Sigma}$. In terms of the Euler variables, (1·31) and (1·32) are expressed as

$$\boldsymbol{T}=\sin\theta[\nabla\theta\times\nabla\phi], \tag{1·34}$$

$$\boldsymbol{\Pi}=-(\hbar/2)(\nabla\chi+\cos\theta\nabla\phi), \tag{1·35}$$

which show that $-(\hbar/2)\cos\theta=-(\hbar/2)\Sigma_3$ and $\phi=\tan^{-1}(\Sigma_2/\Sigma_1)$ correspond to Clebsch variables (cf. Eqs.(1·18) and (1·16)) and in this way the present method is naturally unified with the first method(A) also. The variables (ϕ,θ,χ) have the dual meanings, *viz*. Euler angles for the triad and the velocity potentials describing irrotational and rotational movements of the fluid.

An important feature of the geometro-hydrodynamical formalism is that it necessarily introduces the invariance with respect to rotations of all triads around their respective symmetry axes (directions of which vary from triad to triad) by a common angle λ:

$$\begin{bmatrix}\boldsymbol{a}^1\\ \boldsymbol{a}^2\end{bmatrix}\to\begin{bmatrix}\cos\lambda & \sin\lambda\\ -\sin\lambda & \cos\lambda\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{a}^1\\ \boldsymbol{a}^2\end{bmatrix}, \tag{1·36}$$

i.e.,

$$\chi\to\chi+\lambda,\qquad \theta \text{ and } \phi=\text{invariant}. \tag{1·37}$$

This is evident because $\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$ are invariant there.

The correspondence between this formalism and the conventional one is mediated by the Nullvector

$$\Xi_k=\tilde{\psi}\sigma_k\psi,\qquad \text{where}\quad \tilde{\psi}\equiv(\psi_2,\,-\psi_1), \tag{1·38}$$

such that

$$\Xi_k/P=-(a_k{}^1+ia_k{}^2). \tag{1·39}$$

Equivalently we can give the correspondence as the direct relation between the Euler variables and the spinor wave function:

$$\begin{bmatrix}\psi_1\\ \\ \psi_2\end{bmatrix}=\begin{bmatrix}R\cos\dfrac{\theta}{2}\exp\left[-\dfrac{i}{2}(\chi+\phi)\right]\\ R\sin\dfrac{\theta}{2}\exp\left[-\dfrac{i}{2}(\chi-\phi)\right]\end{bmatrix}. \tag{1·40}$$

1. 5.

In the following sections we analyse our methods stated heretofore further with respect to their kinematical and dynamical aspects, and bring them to completion. In this process we show in particular that our theory implies geometric interpretation of quantum mechanics of a spinning particle.

We shall begin with the verification of the consistency of the subsidiary conditions for the case of the original spinless hydrodynamics (§ 2) and then for the case of spinning hydrodynamics (§ 4) based on the generalized Helmholtz and Kelvin theorems. In §§ 5 and 6 we take the geometro-hydrodynamical viewpoint. We elucidate the meaning of the three angular variables (ϕ, θ, χ) appearing in the relation (1·40) by the concept of 'generalized rotation'. Then we establish the circulation condition for the spinning hydrodynamics, as

$$\mu\oint_c \boldsymbol{v}\cdot d\boldsymbol{s}+\frac{e}{c}\Phi-\frac{\hbar}{2}\oint_\sigma T_n d\sigma=\frac{n}{2}h\,, \tag{1·41}$$

to complete our formalism. This global condition is just the counterpart of (1·7) for the spinless case and is again responsible to the Aharonov-Bohm effect. In § 6 we derive the basic set of equations of geometro-hydrodynamics from its own viewpoint, where the invariance (1·36) is promoted to local one.

§ 2. Vorticity and circulation for the spinless hydrodynamics

We begin with some further general analyses about vorticity and circulation for the original spinless hydrodynamics stated in § 1. 1. In the usual hydrodynamics for non-viscous fluid the equation of motion for the vorticity $\boldsymbol{w}=\text{rot}\,\boldsymbol{\Pi}$ is derived from the Euler equation as

$$\frac{Dw_i}{Dt}=(\partial_k v_i)w_k-(\partial_k v_k)w_i,\quad \text{i.e.,}\quad \frac{\partial \boldsymbol{w}}{\partial t}=\text{rot}[\boldsymbol{v}\times\boldsymbol{w}], \tag{2·1a, b}$$

and it is valid for our hydrodynamics owing to the fact that the internal stress can be represented by a potential (see (1·5)). Equation (2·1) implies the generalized Helmholtz theorem meaning that if $\boldsymbol{w}=0$ initially then $\boldsymbol{w}=0$ at any time; this ensures the compatibility of the subsidiary condition (1·9) with the equations of motion.

Equation (2·1) implies that $D\oint_\sigma w_n d\sigma/Dt=\oint_\sigma(\dot{\boldsymbol{w}}-\text{rot}[\boldsymbol{v}\times\boldsymbol{w}])_n d\sigma=0$, and therefore momentum circulation round a closed 'fluid contour' is conserved. This fact is also verified as follows. First we have

$$D\Big(\oint_c \boldsymbol{\Pi}\cdot d\boldsymbol{s}\Big)/Dt=\oint_c\Big(\boldsymbol{\Pi}\cdot d\boldsymbol{v}+\frac{D\boldsymbol{\Pi}}{Dt}\cdot d\boldsymbol{s}\Big) \tag{2·2}$$

$$=\oint_c\frac{\mu}{2}d(\boldsymbol{v}^2)+\oint_c\Big(\frac{e}{c}A_k\frac{\partial v_k}{\partial x_i}+\frac{D\Pi_i}{Dt}\Big)dx_i\,.$$

Then we insert (1·8) here, to find that this vanishes. This is the generalized Kelvin theorem and ensures the compatibility of (1·10) with the equation of motion.

Clearly the vorticity condition (1·6) and the circulation condition (1·7) are intimately related. Our hydrodynamics allows a flow with nodal line, which is usually a singular vortex line, and the condition (1·7), i.e., (1·10), implies that the circulation around such a vortex line is not only conserved but also quantized.*) Now the condition (1·6), when

*) This fact was noticed first by Dirac.[13] As regards the properties of a line vortex see Refs. 5),6) and 14).

integrated over an arbitrary surface σ encircled by a closed contour C, gives

$$\oint_\sigma (\mathrm{rot}\ \boldsymbol{v})_n d\sigma + (e/\mu c)\Phi = 0 . \qquad (2\cdot3)$$

This is rewritten as $\oint_c \boldsymbol{v}\cdot \boldsymbol{ds} + (e/\mu c)\Phi = 0$ if there is no singularity inside C. But if there is a singular vortex line inside C, we have instead

$$\oint_c \boldsymbol{\Pi}\cdot \boldsymbol{ds} = \mu \oint_c \boldsymbol{v}\cdot \boldsymbol{ds} + (e/c)\Phi = \mathrm{const} = \Gamma , \qquad (2\cdot4)$$

where the value of Γ does not depend on the detailed path of C in so far as it does not pass through a singular line because rot $\boldsymbol{\Pi}=0$ elsewhere. The condition (1·7) means that this Γ of (2·4) must be restricted to the values

$$\Gamma = nh . \qquad (2\cdot5)$$

§ 3. Generalization of hydrodynamics to include vortical flows

In this section we briefly explain the method (A) mentioned in § 1. The first step is to generalize our original hydrodynamics by simply dropping the irrotationality condition (1·9). Then the $\boldsymbol{\Pi}$-field is expressed as (1·16), where the potentials (S, ξ, η) are not unique because (1·16) is invariant under the Clebsch transformation

$$\xi = \frac{\partial F}{\partial \eta}, \quad \xi' = -\frac{\partial F}{\partial \eta'}, \quad S' = S + F . \qquad (3\cdot1)$$

The velocity potentials allow the Lagrangian formalism. We assume the Lagrangian density

$$L_1 = P\left(\frac{\mu}{2}\boldsymbol{v}^2 - \frac{DS}{Dt} - \xi\frac{D\eta}{Dt} - V + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right) - \frac{\hbar^2}{8\mu}\frac{(\nabla P)^2}{P}, \qquad (3\cdot2)$$

which is invariant under (3·1) because $\dot{S}+\xi\dot{\eta}$ is so. This Lagrangian results in (1·2), (1·16) and

$$-(\dot{S}+\xi\dot{\eta}) = \frac{\mu}{2}\boldsymbol{v}^2 + V - \frac{\hbar^2}{2\mu}\frac{\Delta R}{R}, \qquad (3\cdot3)$$

$$D\xi/Dt = 0 , \qquad D\eta/Dt = 0 . \qquad (3\cdot4\mathrm{a,b})$$

Then (1·5) follows from them. Equations (3·4a, b) mean the conservation of ξ and η and result in (2·1), where the vorticity $\boldsymbol{w}$ is represented as (1·18). The theory given above is exactly what we gave in Ref. 3) (see p. 216 thereof).

As stated in § 1. 2 we now make the second step. We introduce the internal stress potential (1·19), to have the new Lagrangian density

$$L = L_1 - W_s . \qquad (3\cdot5)$$

This modifies the equations of motion (3·3) and (3·4) to

$$-(\dot{S}+\xi\dot{\eta}) = \frac{\mu}{2}\boldsymbol{v}^2 + V - \frac{\hbar^2}{2\mu}\frac{\Delta R}{R} + \frac{1}{2\mu}\left\{\frac{(\nabla\xi)^2}{\rho} + \rho(\nabla\eta)^2\right\}, \qquad (3\cdot6)$$

$$\mu\frac{D\xi}{Dt}+\frac{1}{P}\partial_k(\rho P\partial_k\eta)=0\,, \tag{3·7}$$

$$\mu\frac{D\eta}{Dt}+\frac{1}{\rho}\partial_k\left(\frac{1}{\rho}P\partial_k\xi\right)+\frac{1}{2}\frac{d\rho}{d\xi}\left\{\frac{(\nabla\xi)^2}{\rho^2}-(\nabla\eta)^2\right\}=0\,. \tag{3·8}$$

Thus ξ and η are no longer conserved so that (2·1) is modified. Also the Euler equation resulting from (3·6)~(3·8) differs from (1·5) by an extra internal stress term. The important point is that by the introduction of W_s the invariance under (3·1) is lost and the 'gauge' of the Clebsch potentials, ξ and η, becomes essentially restricted. Thus they acquire the property of intrinsic degrees of freedom although they continue to give the vorticity by (1·18). In fact, they are identified with the spin vector $\boldsymbol{S}$ by (1·21), and the weight function ρ is also fixed as

$$\rho(\xi)=1-(4/\hbar^2)\xi^2=1-\Sigma_3^{\,2}\,. \tag{3·9}$$

We can further confirm that with this identification the present theory completely agrees with the hydrodynamics of spinning fluid stated in § 1. 3 (except that in the present treatment we have been neglecting the Zeeman coupling term of the spin). Viewed from a different angle the procedure above taken exhibits a rather striking situation. At first we generalized wave mechanics itself by admitting vortical flows into the hydrodynamical formulation of quantum mechanics, but it is found that the result is restored, by the introduction of the internal potential (1·19), within the wave-mechanical framework based on two-component spinor wave function.

§ 4. Some remarks for the spinning hydrodynamics

In this section we supplement the description of the spinning hydrodynamics (the method (B)) stated in § 1. 3, by deriving some important relations.

(i) To reexpress the equations of motion (1·24) and (1·27), we define

$$\boldsymbol{H}^{\rm in}=(c/e)\partial_k(P\partial_k\boldsymbol{S})/P\,,\qquad \boldsymbol{H}^{\rm eff}=\boldsymbol{H}+\boldsymbol{H}^{\rm in}\,, \tag{4·1}$$

and call $\boldsymbol{H}^{\rm in}$ 'internal magnetic field' (though it does not satisfy div $\boldsymbol{H}^{\rm in}=0$), and $\boldsymbol{H}^{\rm eff}$ 'effective magnetic field'. Then (1·24) and (1·27) are rewritten as[8)]

$$\mu\frac{Dv_i}{Dt}=K_i+F_i\,,\qquad F_i\equiv\frac{e}{\mu c}S_k\partial_i H_k^{\rm eff}+\partial_i\left(\frac{\hbar^2}{2\mu}\frac{\Delta R}{R}+\frac{1}{2\mu}|\nabla\boldsymbol{S}|^2\right), \tag{4·2}$$

$$\frac{D\boldsymbol{S}}{Dt}=\frac{e}{\mu c}[\boldsymbol{S}\times\boldsymbol{H}^{\rm eff}], \tag{4·3}$$

where $|\nabla\boldsymbol{S}|^2\equiv\partial_k S_l\partial_k S_l$. We also employ another form of (1·24):

$$\frac{D\Pi_i}{Dt}=-\frac{\partial V}{\partial x_i}+\frac{e}{c}v_k\frac{\partial A_k}{\partial x_i}+F_i\,. \tag{4·4}$$

(ii) The equation of motion (2·1) for $\boldsymbol{w}$ must be modified in the spinning hydrodynamics to

$$\frac{Dw_i}{Dt}=(\partial_k v_i)w_k-(\partial_k v_k)w_i+\frac{e}{\mu c}\varepsilon_{ijk}\partial_j S_l\partial_k H_l^{\rm eff}\,, \tag{4·5}$$

which is obtained from (4·4). Similarly from (4·3) we get

$$\frac{DT_i}{Dt}=(\partial_k v_i)T_k-(\partial_k v_k)T_i+\frac{e}{\mu c}\varepsilon_{ijk}\partial_j\Sigma_l\partial_k H_l^{\text{eff}}\,. \tag{4·6}$$

Thus, by defining

$$\tilde{\boldsymbol{w}}\equiv\boldsymbol{w}-(\hbar/2)\boldsymbol{T}=\mu\,\text{rot}\;\boldsymbol{v}-(\hbar/2)\boldsymbol{T}-(e/c)\boldsymbol{H},$$

we have

$$\frac{D\tilde{w}_i}{Dt}=(\partial_k v_i)\tilde{w}_k-(\partial_k v_k)\tilde{w}_i\,,\quad\text{i.e.,}\quad\frac{\partial\tilde{\boldsymbol{w}}}{\partial t}=\text{rot}[\boldsymbol{v}\times\tilde{\boldsymbol{w}}], \tag{4·7}$$

which represents the counterpart of Eqs. (2·1a, b) for the spinless case.

Now in the present theory we introduce the subsidiary condition (1·22), i.e., $\tilde{\boldsymbol{w}}=0$ except at nodal points. Equation (4·7) ensures that this condition is compatible with the equations of motion.

(iii) We notice the interesting parallelism between the spinless hydrodynamics and the spinning hydrodynamics. The mathematical analogy between (1·6) and (1·22) and that between (2·1) and (4·7) indicate that the quantities $\mu\boldsymbol{v}$ and $-(e/c)\boldsymbol{H}$ in the spinless case correspond, respectively, to $\boldsymbol{\Pi}$ and $(\hbar/2)\,\boldsymbol{T}$ in the spinning case, mathematically. Thus, just as Eq. (1·6) led to Eq. (2·4) in the former case, Eq. (1·22) in the latter yields the relation

$$\oint_C\boldsymbol{\Pi}\cdot d\boldsymbol{s}-(\hbar/2)\oint_\sigma T_n d\sigma=\Gamma\,. \tag{4·8}$$

Note that $\boldsymbol{T}$ satisfies div $\boldsymbol{T}=0$ (like $\boldsymbol{H}$ satisfies div $\boldsymbol{H}=0$) so that the flux $\oint_\sigma T_n d\sigma$ is fixed when the contour C encircling the surface σ is given. Moreover, Γ of (4·8) does not depend on the details of C in so far as it does not pass through a singular line. (See also § 5.) We have verified in § 2 that for the spinless hydrodynamics the circulation $\Gamma=\oint_C\boldsymbol{\Pi}\cdot d\boldsymbol{s}$ is conserved. By the analogous procedure we can now verify that Γ of (4·8) (where C moves with the fluid) is conserved. On the other hand, for the spinless case Γ was actually quantized as (2·5). We explain in § 5 that the corresponding quantization for Γ of (4·8) is $\Gamma=(n/2)h$, as already presented in Eq. (1·41).

(iv) In this hydrodynamical formalism the spin-vorticity vector $\boldsymbol{T}$ of (1·23) and the energy density of spin-stress $W_S=(P/2\mu)|\nabla\boldsymbol{S}|^2=(\hbar^2P/8\mu)|\nabla\boldsymbol{\Sigma}|^2$ (which is the same as (1·19)) play the central role. These quantities have the important property that they are invariant under an orthogonal transformation of $\boldsymbol{\Sigma}$ (with coordinate frame left unchanged)

$$\Sigma_i\to\Sigma_i'=A_{ik}\Sigma_k\,.\qquad(AA^T=1) \tag{4·9}$$

Indeed, $|\nabla\boldsymbol{\Sigma}'|^2=|\nabla\boldsymbol{\Sigma}|^2$, and

$$T_i'=\frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\Sigma_l'\partial_j\Sigma_m'\partial_k\Sigma_n'=T_i\,. \tag{4·10}$$

Also we note that these quantities are expressed in terms of the wave function as

$$\frac{2}{\hbar}w_i=T_i=\frac{-2i}{P}\varepsilon_{ijk}\Big\{(\partial_j\psi^*\cdot\partial_k\psi)-\frac{1}{P}(\partial_j\psi^*\cdot\psi)(\psi^*\partial_k\psi)\Big\}, \tag{4·11}$$

$$|\nabla \Sigma|^2=\frac{4}{P}\left\{(\partial_k\psi^*\cdot\partial_k\psi)-\frac{1}{P}(\partial_k\psi^*\cdot\psi)(\psi^*\partial_k\psi)\right\}. \tag{4·12}$$

§ 5. Euler variables and circulation condition

5. 1.

In this section we exploit the representation in Euler variables and establish the quantization condition of circulation. We begin with the following derivation of Eq. (1·40) which implies the relation between Euler angles and the spinor, *both depending on* $\boldsymbol{x}$. We are inserting this derivation because of its heuristic significance.*)

We consider rotation in the *active* viewpoint, where we rotate the physical system in keeping the coordinate axes fixed. Then by a rotation $\mathcal{R}$ specified by Euler angles (α, β, γ), $\psi(\boldsymbol{x})$ changes to

$$\psi_\alpha'(\boldsymbol{x})=U_{\alpha\beta}\psi_\beta(\mathcal{R}^{-1}\boldsymbol{x}), \tag{5·1}$$

$$U=\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}, \qquad \begin{aligned} a&=\cos\frac{\beta}{2}e^{-(i/2)(\gamma+\alpha)}, \\ b&=\sin\frac{\beta}{2}e^{-(i/2)(\gamma-\alpha)}. \end{aligned} \tag{5·2}$$

This U is exactly the $\frac{1}{2}$ representation $D^{1/2}$ (α, β, γ) of the rotation group.

Now we take the 'standard' spinor

$$\psi=\begin{bmatrix} R(\boldsymbol{x}) \\ 0 \end{bmatrix}. \tag{5·3}$$

According to (1·39), the triad $\{\boldsymbol{a}^r\}$ corresponding to it is

$$\boldsymbol{a}^1=\hat{\boldsymbol{x}}\,, \quad \boldsymbol{a}^2=\hat{\boldsymbol{y}}\,, \quad \boldsymbol{a}^3=\hat{\boldsymbol{z}}\,, \tag{5·4}$$

so that this 'standard' state is pictured as the distribution with density $R(\boldsymbol{x})^2$ of triads whose axes are everywhere parallel to the coordinate axes. On the other hand, (5·3) is a superposition of the states

$$\psi_{x'}=\begin{bmatrix} R(\boldsymbol{x}') \\ 0 \end{bmatrix}\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \quad \left(\psi(\boldsymbol{x})=\int\psi_{x'}d^3x'\right), \tag{5·5}$$

each of which is a state where triad (5·4) is localized at $\boldsymbol{x}'$. Now we apply to each $\psi_{x'}$ an active rotation (around $\boldsymbol{x}'$) specified by Euler angles $(\phi(\boldsymbol{x}'), \theta(\boldsymbol{x}'), \chi(\boldsymbol{x}'))$ which vary with $\boldsymbol{x}'$, then it changes to

*) Especially, its generalization to the relativistic case leads to a representation of Dirac wave function in terms of the set of Euler angles $(\phi_1, \theta_1, \chi_1)$ and hyperbolic pseudoangles $(\phi_2, \theta_2, \chi_2)$, together with the modulus R and one more angle Θ. This representation supplies a convenient basis for the geometro-hydrodynamical representation of Dirac field,[15] where the counterpart of Eq.(1·35) in the nonrelativistic spinning hydrodynamics is $k_\mu=a_\nu{}^2\partial_\mu a_\nu{}^1=\partial_\mu\chi_1+\cosh\theta_2\cos\theta_1\partial_\mu\phi_1+\sinh\theta_2\sin\theta_1\partial_\mu\phi_2$, which implies two pairs of Clebsch parameters. Details are given in a separate paper.

$$\psi'_{x'}=U\begin{bmatrix}R(\boldsymbol{x}')\\0\end{bmatrix}\delta(\mathcal{R}^{-1}(\boldsymbol{x}-\boldsymbol{x}'))=R(\boldsymbol{x}')\begin{bmatrix}\cos\frac{\theta}{2}e^{-i/2(\chi+\phi)}\\ \sin\frac{\theta}{2}e^{-i/2(\chi-\phi)}\end{bmatrix}\delta(\boldsymbol{x}-\boldsymbol{x}'),\qquad(5\cdot6)$$

where use is made of (5·2). Let us suppose that *we are applying such a local rotation at every point*; then as the result of this operation (which we call '*generalized rotation*') the original $\psi=\int\psi_{x'}d^3x'$ is considered to change to $\psi'=\int\psi'_{x'}d^3x'$. With (5·6) this gives the expression (1·40) for ψ'.

Alternatively we may consider the above process directly. Namely by the generalized rotation whose Euler angles (ϕ, θ, χ) vary with $\boldsymbol{x}$ around which each local rotation is applied, the original body axes (5·4) at $\boldsymbol{x}$ rotate to $\{\boldsymbol{a}^r(\boldsymbol{x})\}$ which are clearly

$$\begin{bmatrix}a_1{}^1 & a_2{}^1 & a_3{}^1\\ a_1{}^2 & a_2{}^2 & a_3{}^2\\ a_1{}^3 & a_2{}^3 & a_3{}^3\end{bmatrix}=\begin{bmatrix}c_\phi c_\theta c_\chi-s_\phi s_\chi & s_\phi c_\theta c_\chi+c_\phi s_\chi & -s_\theta c_\chi\\ -c_\phi c_\theta s_\chi-s_\phi c_\chi & -s_\phi c_\theta s_\chi+c_\phi c_\chi & s_\theta s_\chi\\ c_\phi s_\theta & s_\phi s_\theta & c_\theta\end{bmatrix},\qquad(5\cdot7)$$

where $c_\theta\equiv\cos\theta$, $s_\theta\equiv\sin\theta$, etc. This expression (5·7) must agree with what we obtain by inserting (1·40) into (1·38) and (1·39); this is actually confirmed.

5. 2.

The important point in the use of Euler variables is that for specification of a rotation they have certain arbitrariness. Indeed the set of

$$\phi'=\phi+2\pi n_\phi,\quad \theta'=\theta+2\pi n_\theta,\quad \chi'=\chi+2\pi n_\chi\qquad(5\cdot8\text{a})$$

or of

$$\phi''=\phi+2\pi n_\phi+\pi,\quad \theta''=-\theta+2\pi n_\theta,\quad \chi''=\chi+2\pi n_\chi+\pi,\qquad(5\cdot8\text{b})$$

represents the same rotation as (ϕ, θ, χ). In accord with this, the relation (1·40) does not determine Euler variables uniquely for a given ψ; there remains the arbitrariness of (5·8a) and (5·8b), each of which gives the same ψ by (1·40) if $n_\phi+n_\theta+n_\chi=$even. On the other hand, if this is odd they give $-\psi$, but $-\psi$ corresponds to the same physical state, so that the odd case is equally allowed in our formalism.

Corresponding to the above arbitrariness the variables ϕ, θ and χ can be multivalued function of $\boldsymbol{x}$ such that

$$\oint d\chi=2\pi n_1,\quad \oint d\phi=2\pi n_2,\quad \oint d\theta=2\pi n_3,\qquad(5\cdot9\text{a, b, c})$$

where $\oint$ denotes integral along any closed contour.

We remark that (1·40) is viewed as a factorization of ψ into three factors as follows:

$$\psi=R\begin{bmatrix}u_1\\u_2\end{bmatrix}e^{-(i/2)\chi},\qquad \begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\cos\frac{\theta}{2}e^{-(i/2)\phi}\\ \sin\frac{\theta}{2}e^{(i/2)\phi}\end{bmatrix},\qquad(5\cdot10)$$

where R is the modulus and the factor u represents the polarization $\boldsymbol{\Sigma}$ through

$$\Sigma_i=u^*\sigma_i u,\qquad(5\cdot11)$$

reproducing (1·33), while the last factor $e^{-(i/2)\chi}$ represents the common phase (i.e., the average phase) $-\chi/2$ of ψ_1 and ψ_2. Therefore χ corresponds to the phase function S for the spinless case (see Eq. (1·13)) by

$$\chi \leftrightarrow -(2/\hbar)S\,, \tag{5·12}$$

which is clear also from the comparison between (1·35) and (1·16). This suggests the following points. Corresponding to the fact that χ obeys the condition (5·9a), the S function for the spinless case should obey $\oint dS=(n/2)h$, which is just the condition*) (1·12), i.e., (1·10). Thus this circulation condition is considered to have geometric origin rather than it is an ad hoc assumption, in hydrodynamical formalism. Conversely, the global condition (5·9a) on χ in the present case should also have the meaning of circulation quantization. This we see below.

5. 3.

In terms of Euler variables, $\boldsymbol{\Pi}$ is represented as (1·35), which implies that *the velocity field is determined by the gradient of the rotational orientations of the continuously distributed triads.* We rewrite (1·35) as

$$\boldsymbol{\Pi}=\tilde{\boldsymbol{\Pi}}+(\hbar/2)\boldsymbol{G}\,, \tag{5·13}$$

$$\tilde{\boldsymbol{\Pi}}=-(\hbar/2)\nabla\chi\,,\quad \boldsymbol{G}=-\cos\theta\nabla\phi=\frac{-\Sigma_3}{1-\Sigma_3{}^2}(\Sigma_1\nabla\Sigma_2-\Sigma_2\nabla\Sigma_1), \tag{5·14}$$

$$\text{rot}\,\tilde{\boldsymbol{\Pi}}=0,\qquad \text{rot}\,\boldsymbol{G}=\boldsymbol{T}\,. \tag{5·15a, b}$$

Now the condition (5·9a) is rewritten as

$$\Gamma=\oint_C\tilde{\boldsymbol{\Pi}}\cdot d\boldsymbol{s}=(n/2)h\,.\quad (n=\text{integer}) \tag{5·16}$$

With the use of (5·13) and (5·15b) this is reexpressed as (1·41). We repeat that this condition is not an ad hoc postulate but originates from geometrical nature of our theory. We also note that this condition and Eq. (1·22) are put into a single equation

$$\text{rot}\,\boldsymbol{\Pi}=\frac{\hbar}{2}\boldsymbol{T}+\frac{nh}{2}\int_L\delta^3(\boldsymbol{x}-\boldsymbol{x}')d\boldsymbol{x}'\,, \tag{5·17}$$

where $\boldsymbol{x}=\boldsymbol{x}'(\lambda)$ denotes a singular vortex line L. We give some remarks about the consistency of this quantization.

(i) The value of $\oint_C\tilde{\boldsymbol{\Pi}}\cdot d\boldsymbol{s}$ does not depend on the detailed path of C in so far as it does not pass through a singular line, since rot $\tilde{\boldsymbol{\Pi}}=0$. That $\oint_C\tilde{\boldsymbol{\Pi}}\cdot d\boldsymbol{s}$ is conserved with time has been actually proved in § 4.

(ii) The separation of $\boldsymbol{\Pi}$ into its irrotational and rotational parts by (5·13) depends on coordinate frame,**) because ϕ, θ and χ are not scalars whence $\tilde{\boldsymbol{\Pi}}$ and $\boldsymbol{G}$ are not vectors. However, rot $\boldsymbol{G}$ is vector and similarly $\oint_C\tilde{\boldsymbol{\Pi}}\cdot d\boldsymbol{s}$ is scalar.

(iii) By an orthogonal transformation (4·9) of $\boldsymbol{\Sigma}$, $\boldsymbol{G}$ transforms to $\boldsymbol{G}'=-\Sigma_3'\cdot(1-\Sigma_3'^2)^{-1}$

*) In (1·12) n is usually integer but can be half-integer when we allow negative values for R. (See Ref.14).)

**) For frame-independent separation of $\boldsymbol{\Pi}$ into irrotational and rotational parts, $\boldsymbol{\Pi}=\boldsymbol{\Pi}^{\text{ir}}+\boldsymbol{\Pi}^{\text{rot}}$ (rot $\boldsymbol{\Pi}^{\text{ir}}=0$), we should impose the further condition div $\boldsymbol{\Pi}^{\text{rot}}=0$, but then such $\boldsymbol{\Pi}^{\text{rot}}$ is nonlocal with respect to $\boldsymbol{\Sigma}$ and is not a convenient quantity for the present purpose.

$\cdot(\Sigma'_{[1}\nabla\Sigma'_{2]})$, which satisfies rot $\boldsymbol{G}'=\boldsymbol{T}'=\boldsymbol{T}$ owing to (4·10). Therefore for the separation of $\boldsymbol{\Pi}$ into its irrotational and rotational parts we could adopt this $\boldsymbol{G}'$ as well, such that $\boldsymbol{\Pi}=\tilde{\boldsymbol{\Pi}}'+(\hbar/2)\boldsymbol{G}'$. This transformation is really induced by

$$\chi\to\chi'=\chi+\chi_1\,, \tag{5·18}$$

such that

$$\boldsymbol{G}'=\boldsymbol{G}+\nabla\chi_1\,, \qquad \tilde{\boldsymbol{\Pi}}'=-(\hbar/2)\nabla(\chi+\chi_1). \tag{5·19}$$

For illustration, let us take an example where $(\Sigma_1', \Sigma_2', \Sigma_3')$ is a circular permutation of $(\Sigma_1, \Sigma_2, \Sigma_3)$, and therefore $\boldsymbol{G}'=-\Sigma_1\cdot(1-\Sigma_1^2)^{-1}\cdot(\Sigma_{[2}\nabla\Sigma_{3]})$. This is actually induced by (5·18) with $\chi_1=\tan^{-1}(\Sigma_1\Sigma_3/\Sigma_2)$.

5. 4.

Coming back to the pure hydrodynamical formalism (the method (B)) which represents a state by the set of variables $(P, \boldsymbol{\Pi}, \boldsymbol{\Sigma})$ under the subsidiary condition (1·22), we pose the question how one can reconstruct therefrom the wave function in the usual formalism. We proceed as follows. First we determine θ and ϕ from $\boldsymbol{\Sigma}$, and then form $\tilde{\boldsymbol{\Pi}}=\boldsymbol{\Pi}+(\hbar/2)\cos\theta\nabla\phi$, which must be irrotational because of the condition (1·22). Thus we can obtain χ as its potential, where its possible multivalued character is adjusted in accord with (5·16). This determines χ apart from an additive constant (for each instant). Then the two-component wave function ψ_a is fixed apart from a common constant phase.

Once the variable χ, which was originally hidden in the hydrodynamical formalism of the method (B), is constructed this way, it is possible to integrate the equation of motion (1·24) once. The result is essentially the same as Eq. (3·6) in the method (A).

§ 6. Lagrangian and geometro-hydrodynamical formalism

In this section we reproduce geometro-hydrodynamics consistently based on its Lagrangian. For this purpose we pay attention to the angular velocity $\boldsymbol{\omega}$ of the triad, which is given, as in the usual rigid body, by

$$\omega_1=-\dot\theta\sin\phi+\dot\chi\sin\theta\cos\phi\,,\quad \omega_2=\dot\theta\cos\phi+\dot\chi\sin\theta\sin\phi\,,$$
$$\omega_3=\dot\phi+\dot\chi\cos\theta\,. \tag{6·1}$$

Its body-frame components are $\omega^r=a_k{}^r\omega_k=\frac{1}{2}\varepsilon_{rst}\dot{\boldsymbol{a}}^s\boldsymbol{a}^t$, i.e.,

$$\omega^1=\dot{\boldsymbol{a}}^2\boldsymbol{a}^3=\dot\theta\sin\chi-\dot\phi\sin\theta\cos\chi\,,$$
$$\omega^2=\dot{\boldsymbol{a}}^3\boldsymbol{a}^1=\dot\theta\cos\chi+\dot\phi\sin\theta\sin\chi\,, \tag{6·2}$$
$$\omega^3=\dot{\boldsymbol{a}}^1\boldsymbol{a}^2=\dot\chi+\dot\phi\cos\theta\,.$$

These ω_i and ω^r are quantities viewed at each fixed position, while the co-moving angular velocities are given by

$$\Omega_i=\frac{1}{2}\varepsilon_{ijk}a_j{}^r\frac{Da_k{}^r}{Dt}\,,\quad \Omega^{(r)}=\frac{1}{2}\varepsilon_{rst}\frac{D\boldsymbol{a}^s}{Dt}\cdot\boldsymbol{a}^t\,. \tag{6·3}$$

Now our triad is distinct from a customary rigid body, such as a symmetric top, because of its special property (1·30). This distinction shows up in the relation between angular momentum $\boldsymbol{S}$ and angular velocity $\boldsymbol{\omega}$ for our triad

$$S_1=\frac{\hbar}{2\dot{\chi}}(\omega_1+\dot{\theta}\sin\phi),\quad S_2=\frac{\hbar}{2\dot{\chi}}(\omega_2-\dot{\theta}\cos\phi),\quad S_3=\frac{\hbar}{2\dot{\chi}}(\omega_3-\dot{\phi}). \tag{6·4}$$

To find out the Lagrangian we consider first the kinetic term for rotation of a triad. This is considered to be $T^{\rm rot}=\boldsymbol{S\Omega}$, which becomes, owing to (1·30),

$$T^{\rm rot}=\boldsymbol{S\Omega}=\frac{\hbar}{2}\Omega^{(3)}=\frac{\hbar}{2}\frac{D\boldsymbol{a}^1}{Dt}\cdot\boldsymbol{a}^2=\frac{\hbar}{2}\left(\frac{D\chi}{Dt}+\cos\theta\frac{D\phi}{Dt}\right). \tag{6·5}$$

This $T^{\rm rot}$ is distinct from that of a conventional symmetric top: $T^{\rm rot}=\frac{1}{2}\{I(\Omega^{(1)})^2+I(\Omega^{(2)})^2+I_3(\Omega^{(3)})^2\}$. The latter has the degrees of freedom responsible for the isospin-like degeneracy and the spin tower, whereas our $T^{\rm rot}$ has not such degrees of freedom, being linear in $\dot{\chi}$ and $\dot{\phi}$ and lacking $\dot{\theta}$ term. (This point is related to that $T^{\rm rot}$ is $\boldsymbol{S\Omega}$ instead of $\frac{1}{2}\boldsymbol{S\Omega}$.)

The Lagrangian density for free case should be of the form

$$L_0=P\left(\frac{\mu}{2}\boldsymbol{v}^2+\frac{\hbar}{2}\Omega^{(3)}\right)-W+\frac{1}{2}P\lambda_{rs}(\boldsymbol{a}^r\boldsymbol{a}^s-\delta_{rs}), \tag{6·6}$$

where W denotes the energy density due to the internal stress. We require also that the theory is invariant under

$$R\to\alpha R.\quad (P\to\alpha^2P)\quad (\alpha=\text{real const}) \tag{6·7}$$

This invariance is universal characteristics of our hydrodynamics representing quantum mechanics. For this invariance W in (6·6) must be linear in P. Further it is natural to assume that W originates from the inhomogeneity of the spin density $P\boldsymbol{S}$, and is nonnegative. Then its form is essentially determined. We have

$$W=\frac{1}{2\mu P}|\nabla(P\boldsymbol{S})|^2=\frac{\hbar^2}{8\mu}\frac{(\nabla P)^2}{P}+\frac{P}{2\mu}|\nabla\boldsymbol{S}|^2. \tag{6·8}$$

Now from our viewpoint, the invariance of theory under rotation of each triad around its symmetry axis is to be regarded as due to the geometric nature of the triad and therefore this invariance must be a local one, namely the theory should be invariant when each triad rotates at each point around its symmetry axis by an arbitrary angle $\lambda(\boldsymbol{x})$:

$$\begin{bmatrix}\boldsymbol{a}^1(\boldsymbol{x})\\ \boldsymbol{a}^2(\boldsymbol{x})\end{bmatrix}\to\begin{bmatrix}\cos\lambda(\boldsymbol{x}) & \sin\lambda(\boldsymbol{x})\\ -\sin\lambda(\boldsymbol{x}) & \cos\lambda(\boldsymbol{x})\end{bmatrix}\begin{bmatrix}\boldsymbol{a}^1(\boldsymbol{x})\\ \boldsymbol{a}^2(\boldsymbol{x})\end{bmatrix}, \tag{6·9}$$

i.e.,

$$\chi(\boldsymbol{x})\to\chi(\boldsymbol{x})+\lambda(\boldsymbol{x}),\qquad \theta,\phi=\text{inv}. \tag{6·10}$$

But, under this transformation, L_0 is no longer invariant because $(D\boldsymbol{a}^1/Dt)\cdot\boldsymbol{a}^2=\dot{\boldsymbol{a}}^1\boldsymbol{a}^2+(\boldsymbol{v}\cdot\nabla a_k{}^1)a_k{}^2$ changes by

$$\dot{\boldsymbol{a}}^1\boldsymbol{a}^2\to\dot{\boldsymbol{a}}^1\boldsymbol{a}^2+\dot{\lambda},\quad (\nabla\boldsymbol{a}^1)\boldsymbol{a}^2\to(\nabla\boldsymbol{a}^1)\boldsymbol{a}^2+\nabla\lambda. \tag{6·11}$$

Thus, to preserve the invariance of the Lagrangian we need to introduce a gauge field

$(\boldsymbol{A}(\boldsymbol{x}), A_0(\boldsymbol{x}))$ which transforms, simultaneously with (6·9), as

$$\boldsymbol{A}\to\boldsymbol{A}-(\hbar c/2e)\nabla\lambda\,,\quad A_0\to A_0+(\hbar/2e)\dot{\lambda}\,, \tag{6·12}$$

and to replace the kinetic term of rotation in L_0 as

$$\frac{\hbar}{2}\frac{D\boldsymbol{a}^1}{Dt}\boldsymbol{a}^2\to\frac{\hbar}{2}\frac{D\boldsymbol{a}^1}{Dt}\boldsymbol{a}^2-eA_0+\frac{e}{c}\boldsymbol{v}\boldsymbol{A}\,. \tag{6·13}$$

Then with the identification of A_μ as electromagnetic potential the second and third terms in (6·13) represent just the classical electromagnetic interaction. This provides a geometrical interpretation of local gauge invariance. In fact we assume an additional term $(e/\mu c)\boldsymbol{HS}$, which expresses the direct magnetic moment coupling of the classical spin by g-factor 2 and is itself invariant under (6·9). Thus we have the Lagrangian density

$$L=PN-W+\frac{1}{2}P\lambda_{rs}(\boldsymbol{a}^r\boldsymbol{a}^s-\delta_{rs}),$$

$$N=\frac{\hbar}{2}\frac{D\boldsymbol{a}^1}{Dt}\boldsymbol{a}^2+\frac{\mu}{2}\boldsymbol{v}^2-V+\frac{e}{c}\boldsymbol{v}\boldsymbol{A}+\frac{e\hbar}{2\mu c}\boldsymbol{H}\boldsymbol{a}^3 \tag{6·14}$$

which defines our geometro-hydrodynamics by $\delta\int Ld^3xdt=0$.

First the variation of L with respect to v_i gives the relation

$$\mu v_i=-\frac{\hbar}{2}\partial_i\boldsymbol{a}^1\cdot\boldsymbol{a}^2-\frac{e}{c}A_i\,, \tag{6·15}$$

except at nodal points. This is Eq. (1·32), and implies (1·22). Next the variation with respect to P leads to

$$\frac{\hbar}{2}\omega^3=\frac{\mu}{2}\boldsymbol{v}^2+V-\frac{e}{\mu c}\boldsymbol{HS}-\frac{\hbar^2}{2\mu}\frac{\Delta R}{R}+\frac{1}{2\mu}|\nabla\boldsymbol{S}|^2\,, \tag{6·16}$$

which is essentially the same as (3·6). Further the equations resulting from variations with respect to a_i^r give, after the elimination of $\lambda_{rs}=\lambda_{sr}$, Eqs. (1·2) and (1·27). Finally from the gradient of (6·16) we arrive at (1·24). Thus we have reproduced all the basic equations of the spinning hydrodynamics (the method (B)), except the circulation condition (1·41). Because the Lagrangian (6·14) leaves $\boldsymbol{v}$ undetermined at nodal points ($P(\boldsymbol{x})=0$) and also because L is singular at nodal points, we need to supplement the basic set of equations following from this Lagrangian with a certain condition referring to nodal points, and this is just the condition (1·41), i.e., the appearance of the δ-function term in (5·17).

In the Lagrangian (6·14) we could use the Euler variables; then

$$T^{\text{rot}}=\frac{\hbar}{2}\{\dot{\chi}+\dot{\phi}\cos\theta+\boldsymbol{v}\cdot(\nabla\chi+\cos\theta\,\nabla\phi)\},$$

$$|\nabla\boldsymbol{S}|^2=(\hbar^2/4)\{(\nabla\theta)^2+\sin^2\theta(\nabla\phi)^2\}, \tag{6·17}$$

and the term containing λ_{rs} is omitted. This Lagrangian, which is essentially the same as (3·5), leads to the same consequences as above.

Finally we note that the angular velocities, (6·1) and (6·2), are connected to the wave function ψ as

$$\omega_i = i(\psi^* \sigma_i \dot{\psi} - \dot{\psi}^* \sigma_i \psi)/(\psi^* \psi), \quad \omega^3 = i(\psi^* \dot{\psi} - \dot{\psi}^* \psi)/(\psi^* \psi). \tag{6·18}$$

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Note added in proof:

Appendix

—— Different Representation of the Spinning Hydrodynamics ——

From the Lagrangian (6·14) we obtain the total electric current $\boldsymbol{j}$ as

$$j_i = c\delta L/\delta A_i = ePv_i + (e/\mu)\varepsilon_{ijk}\partial_j(PS_k), \tag{A·1}$$

which consists of the convection current $eP\boldsymbol{v}$ and the polarization current rot $(e\mu^{-1}P\boldsymbol{S})$. We define

$$\boldsymbol{v}' = \frac{\boldsymbol{j}}{eP} = \boldsymbol{v} + \frac{\mathrm{rot}(P\boldsymbol{S})}{\mu P}, \tag{A·2}$$

which satisfies $\dot{P} + \mathrm{div}(P\boldsymbol{v}') = 0$ as well, and we can employ it in place of $\boldsymbol{v}$. Then our geometro-hydrodynamics is defined by the Lagrangian density

$$L' = PN' - W' + \frac{1}{2}P\lambda_{rs}(\boldsymbol{a}^r\boldsymbol{a}^s - \delta_{rs}) \tag{A·3}$$

with

$$N' = \frac{\hbar}{2}\frac{D'\boldsymbol{a}^1}{D't}\cdot\boldsymbol{a}^2 + \frac{\mu}{2}(\boldsymbol{v}')^2 - V + \frac{e}{c}\boldsymbol{v}'\cdot\boldsymbol{A}, \quad \left(\frac{D'}{D't} = \frac{\partial}{\partial t} + \boldsymbol{v}'\cdot\nabla\right) \tag{A·4}$$

$$W' = \frac{1}{2\mu P}[\mathrm{div}(P\boldsymbol{S})]^2 + P\boldsymbol{S}\cdot\mathrm{rot}\,\boldsymbol{v}'. \tag{A·5}$$

Indeed we can verify that the Lagrangian (6·14) is transformed to (A·3) (or vice versa) by the aid of Eq. (1·22) and the following useful identities on the spin-vorticity vector $\boldsymbol{T}$:

$$2\boldsymbol{\Sigma}\cdot\boldsymbol{T} = (\mathrm{div}\boldsymbol{\Sigma})^2 - \partial_k\Sigma_l\partial_l\Sigma_k, \tag{A·6}$$

$$[\boldsymbol{\Sigma}\times\boldsymbol{T}]_i = -\varepsilon_{lmn}\partial_i\Sigma_l\partial_m\Sigma_n. \tag{A·7}$$

We see that this transformation, from (6·14) to (A·3), eliminates the direct magnetic-moment coupling term $(e/\mu c)\boldsymbol{H}\cdot\boldsymbol{S}$ in the former. However, at the same time it induces dependence of the internal stress (τ'_{ik}) on the velocity gradient and also its asymmetry $\tau'_{ik} \neq \tau'_{ki}$. Thus, for our spinning hydrodynamics the original representation is much simpler and more adequate.

On the other hand, we know that the Dirac equation is represented equivalently as relativistic geometro-hydrodynamics,[15] which contains electromagnetic coupling as minimum interaction alone, and we can verify that when we take the non-relativistic approximation to the Lagrangian for this hydrodynamical representation of the Dirac field we obtain at first just the form (A·3). This can then be transformed to the form (6·14) through (A·2) as stated above. (Details of these points are given in a separate paper.)