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W_2^1-ESTIMATES ON THE PREY-PREDATOR SYSTEMS WITH CROSS-DIFFUSIONS AND FUNCTIONAL RESPONSES

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Abstract. As a mathematical model proposed to understand the behaviors of interacting species, cross-diffusion systems with functional responses of prey-predator type are considered. In order to obtain W_2^1-estimates of the solutions, we make use of several forms of calculus inequalities and embedding theorems. We consider the quasilinear parabolic systems with the cross-diffusion terms, and without the self-diffusion terms because of the simplicity of computations. As the main result we derive the uniform W_2^1-bound of the solutions and obtain the global existence in time.

1. Introduction

In attempt to understand spatial and temporal behaviors of interacting species in population ecology many types of mathematical models have been introduced and tested theoretically as well as in field works during last fifty years or so. Among those, population models incorporated with cross-diffusion terms and various response functions have been studied in recent papers as [7], [9], [13], [15], [16], [17], [19], [20].

We investigate in this paper the global existence of the solutions to the following cross-diffusion system with Holling type II functional responses;

$$(1.1)\quad \begin{cases} u_t = (d_1 u + \alpha_{12} uv)_{xx} + u(a_1 - b_1 u - \dfrac{c_1 v}{1+qu}) & \text{in } [0,1]\times(0,\infty), \\ v_t = (d_2 v + \alpha_{21} uv)_{xx} + v(a_2 + \dfrac{b_2 u}{1+qu} - c_2 v) & \text{in } [0,1]\times(0,\infty), \\ u_x(x,t) = v_x(x,t) = 0 & \text{at } x = 0,1, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } [0,1], \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. Throughout this paper we assume that the initial functions $u_0(x)$, $v_0(x)$ are not identically zero. The coefficients

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α_{12} and α_{21} are positive constants. And d_i, a_1, b_i, c_i $(i = 1, 2)$, q are positive constants. a_2 is a real number. These parameters are defined as the following. The coefficients d_1 and d_2 are the diffusion rates of the two species, respectively. The positive cross-diffusion rates α_{12} and α_{21} mean that the prey tends to avoid higher density of the predator species and vice versa by diffusing away. For details in the biological background of cross-diffusions, we refer the reader to the monograph of Okubo and Levin [12]. a_1 and a_2 are the growth rates of the functions $u(x,t)$ and $v(x,t)$. The assumption that the individuals of prey species are sharing limited resources is represented by the coefficient b_1. And c_2 the same for the predator species. The coefficient q has the role that the quantity $\frac{1}{q}$ measures the extent to which environment provides protection to both species u and v. $\frac{c_1}{q}$ is the maximum value which *per capita* reduction rate of u can attain. And $\frac{b_2}{q}$ has means similarly to v for the predator species v. More explanations for the response functions of this type are found in [4], [6], [8], [10], [14] and references therein.

In system (1.1) u and v are nonnegative functions which represent the population densities of the prey and predator species, respectively, which are interacting and migrating in the same habitat Ω. By using the strong maximum principle and the Hopf boundary lemma for parabolic equations, it is shown in Theorem 3.1 of [18] that

$$u(x,t) > 0 \ \text{ and } \ v(t,x) > 0 \ \text{ in } [0,1] \times (0,\infty).$$

Referring to the results stated in Theorem 8 in Section 3 by Amann [1], [2], [3] we have the local existence of solutions to (1.1). In that series of papers he deals with more general form of equations :

$$(1.2) \qquad \begin{cases} u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u\, f(x,u,v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v\, g(x,u,v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where f and g are functions in $C^\infty(\overline{\Omega} \times (R^+)^2, R)$. According to his results the system (1.2) has a unique nonnegative solution $u(\cdot,t)$, $v(\cdot,t)$ in $C([0,T), W_p^1(\Omega)) \cap C^\infty((0,T), C^\infty(\Omega))$, where $T \in (0,\infty]$ is the maximal existence time for the solution u, v.

The results in Theorem 8 mean that once we establish the uniform W_p^1-bound, (with $p > n$), independent of the maximal existence time T for the solutions, the global existence of the solutions will follow. And also the uniform L_∞-bound of the solutions will be obtained from the Sobolev embedding theorems.

In this paper we obtain a uniform W_2^1 bound of the solution to (1.1) a cross-diffusion predator-prey system of the Holling type II functional response under the condition $d_1 = d_2 = d$ and without any extra conditions on the constants

q, a_i, b_i, c_i, $i = 1, 2$. In [17] the author obtained a uniform W_2^1 bound of the solution to a cross-diffusion predator-prey system with the Lotka-Volterra type reaction functions

$$f(x, u, v) = a_1 - b_1 u - c_1 v, \qquad g(x, u, v) = a_2 + b_2 u - c_2 v$$

under the condition

$$0 < b_2 < c_1 + 2\min\{b_1, c_2\}$$

that was necessary to obtain L_1-estimates for the solutions.

We look for the contribution of the diffusion coefficients d in each step of estimates of the solution, and derive the uniform bound of the solution independent of d when $d \geq 1$. Here we state the main theorems of this paper.

Theorem 1. *Assume that* $d_1 = d_2 = d$ *and the initial functions* u_0, v_0 *are in* $W_2^2([0,1])$. *And let* $(u(x,t), v(x,t))$ *be the maximal solution of system* (1.1) *obtained as in Theorem 8. Then there exist positive constants* t_0, $M' = M'(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$, *and* $M = M(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ *such that*

$$\max\{\|u(\cdot,t)\|_{1,2}, \|v(\cdot,t)\|_{1,2} : t \in (t_0, T)\} \leq M',$$
$$\max\{u(x,t),\ v(x,t) : (x,t) \in [0,1] \times (t_0, T)\} \leq M,$$

and $T = +\infty$. *In the case* $d \geq 1$, *the constant* M *is independent of* $d \geq 1$, *that is,* $M = M(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$.

This paper is organized as follows. Section 1, Introduction. In Section 2 we introduce a few calculus inequalities which are necessary in the course of driving W_2^1-estimates. In Section 3 we present a proof of Theorem 1 and obtain the global existence of the solutions to the system (1.1).

2. Preliminaries

In the process to obtain related estimates for the cross-diffusion system we make use of various types of calculus inequalities. In this section we collect those inequalities. First let us introduce the notation that are used in the present paper.

Notation. For $p \geq 1$, $L_p(\Omega)$ denotes the space of all functions with finite $|\cdot|_p$-norm, where

$$|u|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}.$$

When m is a nonnegative integer and $r \geq 1$ is a real number we define the norm

$$\|u\|_{W_r^m(\Omega)} = \sum_{j=0}^{m} |D^j u|_r,$$

and let $W_r^m(\Omega)$ denote the space of all functions with finite $\|\cdot\|_{W_r^m(\Omega)}$-norms.

The following theorem states the well-known Gagliardo-Nirenberg type inequalities. In Section 3 several cases of these inequalities are used to derive appropriate estimates for the solutions to (1.1), the cross-diffusion system with functional responses.

Theorem 2. *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ in C^m. For every function u in $W_r^m(\Omega) \cap L_q(\Omega)$, $1 \le q, r \le \infty$, the derivative $D^j u$, $0 \le j < m$, satisfies the inequality*

$$|D^j u|_p \le C(|D^m u|_r^a |u|_q^{1-a} + |u|_q), \tag{2.1}$$

where $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$, for all a in the interval $\frac{j}{m} \le a < 1$, provided one of the following three conditions :

(i) $r \le q$,

(ii) $0 < \dfrac{n(r-q)}{mrq} < 1$, *or*

(iii) $\dfrac{n(r-q)}{mrq} = 1$ *and* $m - \dfrac{n}{q}$ *is not a nonnegative integer.*

(The positive constant C depends only on n, m, j, q, r, a.)

Proof. We refer the reader to A. Friedman [5] or L. Nirenberg [11] for the proof. □

The following two corollaries are obtained from Theorem 2 by using various values for the parameters m, r and q. They give estimates for the functions in the function space $W_2^1([0,1])$ that their L_p-norms with $p \ge 2$ are bounded by the W_1^2 and L_1-norms.

Corollary 3. *There exist positive constants C, $\tilde{C}$ and $\hat{C}$ such that for every function u in $W_2^1([0,1])$*

$$|u|_2 \le C(|u_x|_2^{\frac{1}{3}} |u|_1^{\frac{2}{3}} + |u|_1), \tag{2.2}$$

$$|u|_{\frac{5}{2}} \le \tilde{C}(|u_x|_2^{\frac{2}{5}} |u|_1^{\frac{3}{5}} + |u|_1), \tag{2.3}$$

$$|u|_3 \le \hat{C}(|u_x|_2^{\frac{4}{9}} |u|_1^{\frac{5}{9}} + |u|_1). \tag{2.4}$$

Proof. $m = 1$, $r = 2$, $q = 1$ satisfy the condition (ii) in Theorem 2. □

Corollary 4. *For every function u in $W_2^2([0,1])$*

$$|u_x|_2 \le C(|u_{xx}|_2^{\frac{3}{5}} |u|_1^{\frac{2}{5}} + |u|_1). \tag{2.5}$$

Proof. $m = 2$, $r = 2$, $q = 1$ satisfy the condition (ii) in Theorem 2. □

Lemmas 5 and 6 below have some estimates that are proved by applying Theorem 2 and the integration by parts to functions in the function spaces $W_2^2([0,1])$ and $W_2^3([0,1])$.

Lemma 5. *For every function u in $W_2^2([0,1])$ with $u_x(0) = u_x(1) = 0$,*

$$|u_x|_2 \le |u_{xx}|_2^{\frac{1}{2}} |u|_2^{\frac{1}{2}}. \tag{2.6}$$

Lemma 6. *For every u in $W_2^3([0,1])$ with $u_x(0) = u_x(1) = 0$,*

$$|u_{xx}|_2 \le |u_{xxx}|_2^{\frac{2}{3}} |u|_2^{\frac{1}{3}}. \tag{2.7}$$

Lemma 7. *If a function f is in the space $W_2^1([0,1])$ then there exists a constant $C > 0$ such that*

$$|f^2|_\infty \le C((1 + \tfrac{1}{\epsilon})|f|_2^2 + \epsilon|f_x|_2^2) \tag{2.8}$$

for every $0 < \epsilon < 1$.

Proof. A proof of the present lemma may be found in [17], Section 5, Lemma 14. □

The a priori estimate in Lemma 7 is used in Section 3 during the derivations of estimates for the L_∞-norms of the functions $u^{\frac{3}{2}}$ and $v^{\frac{3}{2}}$ in **Step 2**, and the function ζ_{xx} with $\zeta = v - u$ in **Step 3**.

3. Existences of solutions

The following result from Amann [2] is regarding the local existence and some conditions that provide the global existence of the solution to the system (1.2) with the general form of reaction functions which includes Holling type II reaction functions in system (1.1).

Theorem 8. *Let u_0 and v_0 be in $W_p^1(\Omega)$. The system (1.2) possesses a unique nonnegative maximal smooth solution*

$$u(x,t), v(x,t) \in C([0,T), W_p^1(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,T)) \text{ for } 0 \le t < T,$$

where $p > n$ and $0 < T \le \infty$. If the solution satisfies the estimates

$$\sup_{0<t<T} \|u(\cdot,t)\|_{W_p^1(\Omega)} < \infty, \quad \sup_{0<t<T} \|v(\cdot,t)\|_{W_p^1(\Omega)} < \infty,$$

then $T = +\infty$. If, in addition, u_0 and v_0 are in $W_p^2(\Omega)$ then

$$u(x,t), v(x,t) \in C([0,\infty), W_p^2(\Omega)),$$

and

$$\sup_{0\le t<\infty} \|u(\cdot,t)\|_{W_p^2(\Omega)} < \infty, \quad \sup_{0\le t<\infty} \|v(\cdot,t)\|_{W_p^2(\Omega)} < \infty.$$

Now we present a proof of our main result Theorem 1. It consists of three steps that are devoted to obtain L_1, L_2 and W_1^2 bounds, respectively, for the solution $(u(x,t), v(x,t))$ to system (1.1), and in its conclusion these estimates are combined and applied to Theorem 8 to derive the global existence.

Proof of Theorem 1. **Step 1**. By taking integration on both sides of the first equation in the system (1.1) over the domain $[0,1]$ we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 u(t)\,dx &= \int_0^1 (a_1u - b_1u^2 - \frac{c_1uv}{1+qu})\,dx\\ &\le a_1\int_0^1 u\,dx - b_1\int_0^1 u^2\,dx\\ &\le a_1\int_0^1 u\,dx - b_1(\int_0^1 u\,dx)^2\\ &= b_1(\frac{a_1}{b_1} - \int_0^1 u\,dx)\int_0^1 u\,dx,\end{aligned}$$

since a_1, b_1, c_1, q are positive constants, and $u(x,t) \ge 0$, $v(x,t) \ge 0$ for $(x,t) \in \Omega \times [0,T)$. Now, taking integration of the second equation in the system (1.1) over the domain $[0,1]$ we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 v(t)\,dx &= \int_0^1 (a_2v + \frac{b_2uv}{1+qu} - c_2v^2)\,dx\\ &\le (a_2 + \frac{b_2}{q})\int_0^1 v\,dx - c_2\int_0^1 v^2\,dx\\ &\le (a_2 + \frac{b_2}{q})\int_0^1 v\,dx - c_2(\int_0^1 v\,dx)^2\\ &= c_2(\frac{a_2q+b_2}{c_2q} - \int_0^1 v\,dx)\int_0^1 v\,dx,\end{aligned}$$

since b_2, $c_2 > 0$, $u(x,t) \ge 0$, $v(x,t) \ge 0$, and

$$\frac{u}{1+qu} < \frac{1}{q} \qquad \text{for } u \ge 0,\ q > 0.$$

Note that a_2 can be any real number, positive, zero, and negative as well in prey-predator type reactions. Hence we conclude that there exists a positive constant $M_0 = M_0(|u_0|_1, |v_0|_1, q, a_i, b_i, c_i, i = 1,2)$ such that

$$\int_0^1 u(t)\,dx < M_0, \quad \int_0^1 v(t)\,dx < M_0 \quad \text{for all } t \in [0,\infty).$$

Now, for the convenience of computations in **Step 2** and **3** we reduce the system (1.1) with $d_1 = d_2 = d$ into the following system by using the scaling $u(x,\frac{\tau}{d}) = \frac{d}{\alpha_{21}}\tilde{u}(x,\tau)$, $v(x,\frac{\tau}{d}) = \frac{d}{\alpha_{12}}\tilde{v}(x,\tau)$, $t = \frac{\tau}{d}$ and then use u, v and t instead of $\tilde{u}$, $\tilde{v}$ and τ, respectively :

$$\text{(3.1)}\qquad \begin{cases} u_t = (u+uv)_{xx} + u\tilde{f} & \text{in } [0,1]\times(0,\infty),\\ v_t = (v+uv)_{xx} + v\tilde{g} & \text{in } [0,1]\times(0,\infty),\\ u_x(x,t) = v_x(x,t) = 0 & \text{at } x = 0,1,\\ u(x,0) = \tilde{u}_0(x), \quad v(x,0) = \tilde{v}_0(x) & \text{in } [0,1],\end{cases}$$

where $\tilde{f} = \frac{a_1}{d} - \frac{b_1}{\alpha_{21}} u - \frac{c_1}{\alpha_{12}} \frac{v}{1 + \frac{qd}{\alpha_{21}} u}$, $\tilde{g} = \frac{a_2}{d} + \frac{b_2}{\alpha_{21}} \frac{u}{1 + \frac{qd}{\alpha_{21}} u} - \frac{c_2}{\alpha_{12}} v$. Then the result in **Step 1** is restated for the scaled system (3.1) as follows :
For system (3.1) there exists a positive constant $M_0 = M_0(|u_0|_1, |v_0|_1, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1, 2)$ such that

$$\int_0^1 d\,u(t)\,dx < M_0, \quad \int_0^1 d\,v(t)\,dx < M_0 \quad \text{for all } t \in [0, \infty).$$

Step 2. Making use of the symmetric property between the functions u and v in the reduced system (3.1), we introduce the auxiliary function $\zeta = v - u$ and rewrite the equations in system (3.1) as follows;

$$u_t = (u + u^2 + u\zeta)_{xx} + u\tilde{f}, \tag{3.2}$$

$$v_t = (v + v^2 - v\zeta)_{xx} + v\tilde{g}, \tag{3.3}$$

$$\zeta_t = \zeta_{xx} + G, \tag{3.4}$$

where $G = v\tilde{g} - u\tilde{f}$.

Multiplying u, v, $-\zeta_{xx}$ to the equations (3.2), (3.3), (3.4), respectively and integrating them over the spatial domain $[0, 1]$ we derive the following equations;

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx &= \int_0^1 u(uu^2 + u\zeta)_{xx}\,dx + \int_0^1 u^2\tilde{f}\,dx \\
&= -\int_0^1 u_x(u_x + 2uu_x + u_x\zeta + u\zeta_x)\,dx + \int_0^1 u^2\tilde{f}\,dx \\
&= -\int_0^1 (u_x^2 + 2uu_x^2 + u_x^2\zeta)\,dx - \int_0^1 uu_x\zeta_x\,dx + \int_0^1 u^2\tilde{f}\,dx \\
&= -\int_0^1 (u_x^2 + uu_x^2 + vu_x^2)\,dx + \frac{1}{2}\int_0^1 u^2\zeta_{xx}\,dx + \int_0^1 u^2\tilde{f}\,dx \\
&\le -\int_0^1 (1 + u)u_x^2\,dx + \frac{1}{2}\int_0^1 u^2\zeta_{xx}\,dx + \int_0^1 \frac{a_1}{d}u^2\,dx,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 v^2\,dx &= \int_0^1 v(v + v^2 - v\zeta)_{xx}\,dx + \int_0^1 v^2\tilde{g}\,dx \\
&= -\int_0^1 v_x(v_x + 2vv_x - v_x\zeta - v\zeta_x)\,dx + \int_0^1 v^2\tilde{g}\,dx \\
&= -\int_0^1 (v_x^2 + 2vv_x^2 - v_x^2\zeta)\,dx + \int_0^1 vv_x\zeta_x\,dx + \int_0^1 v^2\tilde{g}\,dx \\
&= -\int_0^1 (v_x^2 + vv_x^2 + uv_x^2)\,dx - \frac{1}{2}\int_0^1 v^2\zeta_{xx}\,dx + \int_0^1 v^2\tilde{g}\,dx \\
&\le -\int_0^1 (1 + v)v_x^2\,dx - \frac{1}{2}\int_0^1 v^2\zeta_{xx}\,dx + \int_0^1 \frac{a_2}{d}v^2\,dx + \int_0^1 \frac{b_2}{qd}v^2\,dx,
\end{aligned}$$

since

$$0 \le \frac{u}{1+\dfrac{qd}{\alpha_{21}}u} < \frac{\alpha_{21}}{qd} \tag{3.5}$$

in estimating $\tilde{g}$, and

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \zeta_x^2\,dx = -\int_0^1 (\zeta_{xx})^2\,dx - \int_0^1 \zeta_{xx}G\,dx$$

from which it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2+\zeta_x^2)\,dx \\ &\le -\int_0^1 (1+u)u_x^2\,dx - \int_0^1 (1+v)v_x^2\,dx - \int_0^1 (\zeta_{xx})^2\,dx \\ &\quad + \frac{1}{2}\int_0^1 \zeta_{xx}(u^2-v^2-2G)\,dx + \frac{C_{1,1}}{d}\int_0^1 (u^2+v^2)\,dx, \end{aligned} \tag{3.6}$$

where $C_{1,1} = \max\{a_1, a_2 + b_2/q\}$. From (3.5) function G is estimated as

$$|G| = |v\tilde{g} - u\tilde{f}| \le \frac{a_1}{d}u + \frac{1}{d}\left(\frac{c_1\alpha_{21}}{q\alpha_{12}} + |a_2| + \frac{b_2}{q}\right)v + \frac{b_1}{\alpha_{21}}u^2 + \frac{c_2}{\alpha_{12}}v^2.$$

Using this and Young's inequality we notice that

$$\begin{aligned} \frac{1}{2}\int_0^1 \zeta_{xx}(u^2-v^2-2G)\,dx &\le \frac{1}{2}\int_0^1 \left((\zeta_{xx})^2 + \frac{1}{4}(u^2-v^2-2G)^2\right)dx \\ &\le \frac{1}{2}\int_0^1 (\zeta_{xx})^2\,dx + K_{1,1}\int_0^1 (u^4+v^4)\,dx + \frac{C_{1,3}}{d^2}, \end{aligned}$$

where $K_{1,1} = C_{1,2}(1+\frac{1}{d^2})$, and the positive constants $C_{1,2}$, $C_{1,3}$ are depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i = 1, 2$. Hence we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2+\zeta_x^2)\,dx \le &-\int_0^1 (1+u)u_x^2\,dx - \int_0^1 (1+v)v_x^2\,dx \\ &- \frac{1}{2}\int_0^1 (\zeta_{xx})^2\,dx + K_{1,1}\int_0^1 (u^4+v^4)\,dx \\ &+ \frac{C_{1,1}}{d}\int_0^1 (u^2+v^2)\,dx + \frac{C_{1,3}}{d^2}. \end{aligned} \tag{3.7}$$

By using the result in **Step 1** and applying the inequality (2.8) to the function $u^{\frac{3}{2}}$ we find that

$$\begin{aligned} |u^3|_\infty &\le C\left(1+\frac{9}{4\epsilon}\right)\int_0^1 u^3\,dx + C\epsilon\int_0^1 uu_x^2\,dx \\ &\le C\left(1+\frac{9}{4\epsilon}\right)\frac{M_0}{d}|u^2|_\infty + C\epsilon\int_0^1 uu_x^2\,dx \end{aligned}$$

for all $\epsilon > 0$, and hence

$$-\int_0^1 u u_x^2\,dx \le \frac{1}{\epsilon}\left(1+\frac{9}{4\epsilon}\right)\frac{M_0}{d}|u^2|_\infty - \frac{1}{C\epsilon}|u^3|_\infty \tag{3.8}$$

for all $\epsilon > 0$. Similarly for the function v we have that

$$-\int_0^1 v v_x^2\,dx \le \frac{1}{\epsilon}\left(1+\frac{9}{4\epsilon}\right)\frac{M_0}{d}|v^2|_\infty - \frac{1}{C\epsilon}|v^3|_\infty \tag{3.9}$$

for all $\epsilon > 0$. Substituting (3.8) and (3.9) into (3.7) we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2+\zeta_x^2)\,dx \le & -\int_0^1 (u_x^2+v_x^2)\,dx - \frac{1}{2}\int_0^1 (\zeta_{xx})^2\,dx \\ & + \frac{C_{1.1}}{d}\int_0^1 (u^2+v^2)\,dx + K_{1.1}\frac{M_0}{d}(|u|_\infty^3+|v|_\infty^3) \\ & + \frac{1}{\epsilon}\left(1+\frac{9}{4\epsilon}\right)\frac{M_0}{d}(|u|_\infty^2+|v|_\infty^2) \\ & - \frac{1}{C\epsilon}(|u|_\infty^3+|v|_\infty^3) + \frac{C_{1.3}}{d^2},\end{aligned}$$

and after taking $\epsilon = \frac{d}{2CK_{1.1}M_0}$ we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2+\zeta_x^2)\,dx \le & -\int_0^1 (u_x^2+v_x^2)\,dx - \frac{1}{2}\int_0^1 (\zeta_{xx})^2\,dx \\ & + \frac{C_{1.1}}{d}\int_0^1 (u^2+v^2)\,dx - K_{1.1}\frac{M_0}{d}(|u|_\infty^3+|v|_\infty^3) \\ & + 2CK_{1.1}\left(\frac{M_0}{d}\right)^2\left(1+\frac{9}{2}CK_{1.1}\frac{M_0}{d}\right)(|u|_\infty^2+|v|_\infty^2) \\ & + \frac{C_{1.3}}{d^2}.\end{aligned}$$

For every $\gamma \ge 0$ we have that $-K_{1.1}\frac{M_0}{d}\gamma^3 + 2CK_{1.1}(\frac{M_0}{d})^2(1+\frac{9}{2}CK_{1.1}\frac{M_0}{d})\gamma^2 + \frac{C_{1.3}}{d^2} \le K_{1.2}$, where $K_{1.2} = \frac{32}{27}C^3K_{1.1}(\frac{M_0}{d})^4(1+\frac{9}{2}CK_{1.1}\frac{M_0}{d})^3 + \frac{C_{1.3}}{d^2} = \frac{C_{1.4}}{d^4}(1+\frac{1}{d^2})(1+\frac{1}{d}+\frac{1}{d^3})^3 + \frac{C_{1.3}}{d^2}$, $C_{1.4} = C_{1.4}(\alpha_{12},\alpha_{21},a_i,b_i,c_i,i=1,2)$. Thus we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2+\zeta_x^2)\,dx \le & -\int_0^1 (u_x^2+v_x^2)\,dx - \frac{1}{2}\int_0^1 (\zeta_{xx})^2\,dx \\ & + \frac{C_{1.1}}{d}\int_0^1 (u^2+v^2)\,dx + K_{1.2}.\end{aligned} \tag{3.10}$$

Each term on the right-hand side of (3.10) is now analyzed. Applying the inequality (2.2) to the functions u and v and using the uniform boundedness of $|u|_1$ and $|v|_1$ from **Step 1** we have

$$|u|_2 \le C(|u_x|_2^{\frac{1}{3}}|u|_1^{\frac{2}{3}} + |u|_1) \le C_{1.5}d^{-\frac{2}{3}}(|u_x|_2^{\frac{1}{3}} + d^{-\frac{1}{3}}),$$

and similar estimates are done for v. Thus

$$(3.11)\qquad -\int_0^1 (u_x^2+v_x^2)\,dx \le 2d^{-2} - C_{1,6}d^4\left(\int_0^1 (u^2+v^2)\,dx\right)^3,$$

where $C_{1,5}$ and $C_{1,6}$ are positive constants depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i=1,2$. Applying the inequality (2.5) to the function ζ and using the uniform boundedness of $|\zeta|_1$ we have

$$|\zeta_x|_2 \le C(|\zeta_{xx}|_2^{\frac{3}{5}}|\zeta|_1^{\frac{2}{5}} + |\zeta|_1) \le C_{1,7}d^{-\frac{2}{5}}(|\zeta_{xx}|_2^{\frac{3}{5}} + d^{-\frac{3}{5}}),$$

and thus

$$(3.12)\qquad -\frac{1}{2}\int_0^1 (\zeta_{xx})^2\,dx \le \frac{1}{2}d^{-2} - C_{1,8}d^{\frac{4}{3}}\left(\int_0^1 \zeta_x^2\,dx\right)^{\frac{5}{3}},$$

where $C_{1,7}$ and $C_{1,8}$ are positive constants depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i=1,2$ from the result obtained in **Step 1**. Substituting (3.11) and (3.12) into (3.10) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2+\zeta_x^2)\,dx\\
&\le K_{1,3}d^{-2} + \frac{C_{1,1}}{d}\int_0^1 (u^2+v^2)\,dx - C_{1,6}d^4\left(\int_0^1 (u^2+v^2)\,dx\right)^3\\
&\quad - C_{1,8}d^{\frac{4}{3}}\left(\int_0^1 \zeta_x^2\,dx\right)^{\frac{5}{3}},
\end{aligned}$$

where $K_{1,3} = K_{1,2}d^2 + \frac{5}{2} = C_{1,9}(1+\frac{1}{d^2}(1+\frac{1}{d^2})(1+\frac{1}{d}+\frac{1}{d^3})^3)$, and $C_{1,9}$ is a positive constant depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i=1,2$. Thus

(3.13)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 d^2(u^2+v^2+\zeta_x^2)\,dx\\
&\le K_{1,3} + \frac{C_{1,1}}{d}\int_0^1 d^2(u^2+v^2+\zeta_x^2)\,dx\\
&\quad - C_{1,6}\left(\int_0^1 d^2(u^2+v^2)\,dx\right)^3 - C_{1,8}\left(\int_0^1 d^2\zeta_x^2\,dx\right)^{\frac{5}{3}}\\
&\le K_{1,3} + C_{1,6} + \frac{C_{1,1}}{d}\int_0^1 d^2(u^2+v^2+\zeta_x^2)\,dx\\
&\quad - C_{1,6}\left(\int_0^1 d^2(u^2+v^2)\,dx\right)^{\frac{5}{3}} - C_{1,8}\left(\int_0^1 d^2\zeta_x^2\,dx\right)^{\frac{5}{3}}\\
&\le K_{1,4} + \frac{C_{1,1}}{d}\int_0^1 d^2(u^2+v^2+\zeta_x^2)\,dx - C_{1,11}\left(\int_0^1 d^2(u^2+v^2+\zeta_x^2)\,dx\right)^{\frac{5}{3}},
\end{aligned}$$

where $K_{1,4} = K_{1,3} + C_{1,6} = C_{1,10}(1 + \frac{1}{d^2}(1 + \frac{1}{d^2})(1 + \frac{1}{d} + \frac{1}{d^3})^3)$, and $C_{1,10}$, $C_{1,11}$ are positive constants depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i = 1, 2$. Therefore we conclude that there exist positive constant $M_1 = M_1(|u_0|_2, |v_0|_2, |v_0 - u_0|_2^1, d, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1, 2)$ such that

$$\int_0^1 (du(t))^2\,dx < M_1, \quad \int_0^1 (dv(t))^2\,dx < M_1 \quad \text{for all } t \in (\tau_1, \infty). \tag{3.14}$$

For $d \geq 1$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 d^2(u^2 + v^2 + \zeta_x^2)\,dx \leq C_{1,12} + C_{1,1}\int_0^1 d^2(u^2 + v^2 + \zeta_x^2)\,dx \\ - C_{1,11}\left(\int_0^1 d^2(u^2 + v^2 + \zeta_x^2)\,dx\right)^{\frac{5}{3}},\end{aligned} \tag{3.15}$$

where $C_{1,12}$ is a positive constant depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i = 1, 2$. Thus for $d \geq 1$ the positive constant M_1 in (3.14) is independent of $d \geq 1$, that is, $M_1 = M_1(|u_0|_2, |v_0|_2, |v_0 - u_0|_2^1, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1, 2)$.

Step 3. By multiplying $-u_{xx}$, $-v_{xx}$ to the equations (3.2), (3.3), respectively and integrating them over the spatial domain $[0, 1]$ we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2\,dx &= -\int_0^1 u_{xx}(u + u^2 + u\zeta)_{xx}\,dx - \int_0^1 u_{xx}u\tilde{f}\,dx \\ &= -\int_0^1 u_{xx}(u_{xx}2u_x^2 + 2uu_{xx} + \zeta u_{xx} + 2u_x\zeta_x + u\zeta_{xx})\,dx \\ &\quad - \frac{a_1}{d}\int_0^1 uu_{xx}\,dx - \frac{b_1}{\alpha_{21}}\int_0^1 u^2u_{xx}\,dx - \frac{c_1}{\alpha_{12}}\int_0^1 uvu_{xx}\,dx \\ &= -\int_0^1 (u_{xx})^2\,dx - \int_0^1 (u + v)(u_{xx})^2\,dx \\ &\quad - \int_0^1 (u\zeta_{xx} + 2u_x\zeta_x)u_{xx}\,dx - 2\int_0^1 u_x^2u_{xx}\,dx + \frac{a_1}{d}\int_0^1 u_x^2\,dx \\ &\quad + \frac{b_1}{\alpha_{21}}\int_0^1 u^2u_{xx}\,dx + \frac{c_1}{\alpha_{12}}\int_0^1 \frac{uvu_{xx}}{1 + \frac{qd}{\alpha_{21}}u}\,dx,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 v_x^2\,dx &= -\int_0^1 v_{xx}(v + v^2 - v\zeta)_{xx}\,dx - \int_0^1 v_{xx}v\tilde{g}\,dx \\ &= -\int_0^1 (v_{xx})^2\,dx - \int_0^1 (u + v)(v_{xx})^2\,dx \\ &\quad + \int_0^1 (v\zeta_{xx} + 2v_x\zeta_x)u_{xx}\,dx - 2\int_0^1 v_x^2v_{xx}\,dx + \frac{a_2}{d}\int_0^1 v_x^2\,dx \\ &\quad - \frac{b_2}{\alpha_{21}}\int_0^1 \frac{uvv_{xx}}{1 + \frac{qd}{\alpha_{21}}u}\,dx + \frac{c_2}{\alpha_{12}}\int_0^1 v^2v_{xx}\,dx.\end{aligned}$$

Here we notice that $\int_0^1 u_x^2 u_{xx} dx = \int_0^1 v_x^2 v_{xx} dx = 0$ by using the Neumann boundary conditions. Thus we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2\, dx \le & -\int_0^1 (u_{xx})^2\, dx - \int_0^1 (u\zeta_{xx} + 2u_x\zeta_x)u_{xx}\, dx \\ & + \frac{b_1}{\alpha_{21}}\int_0^1 u^2|u_{xx}|\, dx + \frac{c_1\alpha_{21}}{qd\alpha_{12}}\int_0^1 v|u_{xx}|\, dx \\ & + \frac{a_1}{d}\int_0^1 u_x^2\, dx,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 v_x^2\, dx \le & -\int_0^1 (v_{xx})^2\, dx + \int_0^1 (v\zeta_{xx} + 2v_x\zeta_x)v_{xx}\, dx \\ & + \frac{c_2}{\alpha_{12}}\int_0^1 v^2|v_{xx}|\, dx + \frac{b_2}{qd}\int_0^1 v|v_{xx}|\, dx \\ & + \frac{a_2}{d}\int_0^1 v_x^2\, dx,\end{aligned} \tag{3.17}$$

where (3.5) is used to estimate the term $\frac{u}{1+\frac{qd}{\alpha_{21}}u}$. Taking derivative with respect to x twice on both sides of (3.4), multiplying by ζ_{xx} and integrating over $[0,1]$ we have

$$(\zeta_{xx})_t = \zeta_{xxxx} + G_{xx},$$

$$\int_0^1 \zeta_{xx}(\zeta_{xx})_t\, dx = \int_0^1 \zeta_{xxxx}\zeta_{xx}\, dx + \int_0^1 G_{xx}\zeta_{xx}\, dx,$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (\zeta_{xx})^2\, dx &= -\int_0^1 (\zeta_{xxx})^2\, dx - \int_0^1 G_x(\zeta_{xxx})\, dx \\ &\le -\int_0^1 (\zeta_{xxx})^2\, dx + \int_0^1 (\frac{1}{2}G_x^2 + \frac{1}{2}(\zeta_{xxx})^2)\, dx,\end{aligned}$$

and thus we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\zeta_{xx})^2\, dx = -\frac{1}{2}\int_0^1 (\zeta_{xxx})^2\, dx + \frac{1}{2}\int_0^1 G_x^2\, dx. \tag{3.18}$$

Now we will estimate each term on the right sides of (3.16), (3.17) and (3.18). Using the inequality (2.8) with $\epsilon = \frac{d^2}{32CM_1}$, we have

$$\begin{aligned}\left|\int_0^1 u\zeta_{xx}u_{xx}\, dx\right| &\le |\zeta_{xx}|_\infty |u|_2 |u_{xx}|_2 \le \frac{1}{8}|u_{xx}|_2^2 + 2|\zeta_{xx}|_\infty^2 |u|_2^2 \\ &\le \frac{1}{8}|u_{xx}|_2^2 + 2C\frac{M_1}{d^2}\left\{\left(1+\frac{1}{\epsilon}\right)|\zeta_{xx}|_2^2 + \epsilon|\zeta_{xxx}|_2^2\right\} \\ &\le \frac{1}{8}|u_{xx}|_2^2 + \frac{1}{16}|\zeta_{xxx}|_2^2 + 2C\frac{M_1}{d^2}(1 + 32C\frac{M_1}{d^2})|\zeta_{xx}|_2^2,\end{aligned} \tag{3.19}$$

$$
(3.20)\qquad
\begin{aligned}
\left|\int_0^1 2u_x\zeta_x u_{xx}\,dx\right| &= \left|\int_0^1 \zeta_{xx}u_x^2\,dx\right| \le |\zeta_{xx}|_\infty \int_0^1 u_x^2\,dx \\
&= |\zeta_{xx}|_\infty \int_0^1 (-uu_{xx})\,dx \le |\zeta_{xx}|_\infty |u|_2 |u_{xx}|_2 \\
&\le \frac{1}{8}|u_{xx}|_2^2 + \frac{1}{16}|\zeta_{xxx}|_2^2 + 2C\frac{M_1}{d^2}(1+32C\frac{M_1}{d^2})|\zeta_{xx}|_2^2,
\end{aligned}
$$

$$
\frac{b_1}{\alpha_{21}}\int_0^1 u^2|u_{xx}|\,dx \;\le\; 2\frac{b_1}{\alpha_{21}}\int_0^1 u^4 + \frac{1}{8}\int_0^1 (u_{xx})^2\,dx,
$$

$$
\begin{aligned}
\frac{c_1\alpha_{21}}{qd\alpha_{12}}\int_0^1 v|u_{xx}|\,dx &\le 2(\frac{c_1\alpha_{21}}{qd\alpha_{12}})^2\int_0^1 v^2\,dx + \tfrac{1}{8}\int_0^1 (u_{xx})^2\,dx \\
&\le 2(\frac{c_1\alpha_{21}}{qd\alpha_{12}})^2\frac{M_1}{d^2} + \tfrac{1}{8}\int_0^1 (u_{xx})^2\,dx,
\end{aligned}
$$

$$
\int_0^1 u^4\,dx \;\le\; |u|_\infty^2\int_0^1 u^2\,dx \le C(\frac{M_1}{d^2})^2\,(1+\tfrac{1}{\epsilon}) + C\frac{M_1}{d^2}\epsilon|u_x|_2^2.
$$

Combining the three inequalities above we have

$$
(3.21)\qquad
\begin{aligned}
&\frac{b_1}{\alpha_{21}}\int_0^1 u^2|u_{xx}|\,dx + \frac{c_1\alpha_{21}}{qd\alpha_{12}}\int_0^1 v|u_{xx}|\,dx \\
&\le \frac{1}{4}\int_0^1 (u_{xx})^2\,dx + C_{2,1}|u_x|_2^2 + C_{2,2}\frac{(M_1^2+M_1)}{d^4},
\end{aligned}
$$

where $C_{2,1}$, $C_{2,2}$ are positive constants depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i=1,2$. Now substituting (3.19), (3.20) and (3.21) into (3.16) we have

$$
(3.22)\qquad
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2\,dx \le &-\frac{1}{2}\int_0^1 (u_{xx})^2\,dx + \frac{1}{8}\int_0^1 (\zeta_{xxx})^2\,dx \\
&+ K_{2,1}\int_0^1 (u_x^2+(\zeta_{xx})^2)\,dx + C_{2,2}\frac{(M_1^2+M_1)}{d^4},
\end{aligned}
$$

where $K_{2,1}=\max\{\frac{a_1}{d}, C_{2,1}, 8C\frac{M_1}{d^2}(1+32C\frac{M_1}{d^2})\}$. The right-hand side of (3.17) can be estimated analogously to (3.19), (3.20) and (3.21), and thus we have

$$
(3.23)\qquad
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 v_x^2\,dx \le &-\frac{1}{2}\int_0^1 (v_{xx})^2\,dx + \frac{1}{8}\int_0^1 (\zeta_{xxx})^2\,dx \\
&+ K_{2,2}\int_0^1 (v_x^2+(\zeta_{xx})^2)\,dx + C_{2,4}\frac{(M_1^2+M_1)}{d^4},
\end{aligned}
$$

where $K_{2,2}=\max\{\frac{a_2}{d}, C_{2,3}, 8C\frac{M_1}{d^2}(1+32C\frac{M_1}{d^2})\}$, and $C_{2,3}$, $C_{2,4}$ are positive constant depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i=1,2$.

For the right-hand side of (3.18) we observe that

$$\begin{aligned}\left|\left(\frac{uv}{1+\frac{qd}{\alpha_{21}}u}\right)_x\right| &= \left|\frac{(u_xv+uv_x)(1+\frac{qd}{\alpha_{21}}u)-\frac{qd}{\alpha_{21}}uvu_x}{(1+\frac{qd}{\alpha_{21}}u)^2}\right| \\ &= \left|\frac{u_xv+uv_x+\frac{qd}{\alpha_{21}}u^2v_x}{(1+\frac{qd}{\alpha_{21}}u)^2}\right| \\ &\le |u_x|v+u|v_x|+\frac{\alpha_{21}}{qd}|v_x|\end{aligned}$$

by using (3.5) to estimate the term $\frac{u}{1+\frac{qd}{\alpha_{21}}u}$. Thus for

$$G = v\tilde{g}-u\tilde{f} = -\frac{a_1}{d}u+\frac{b_1}{\alpha_{21}}u^2+\frac{c_1}{\alpha_{21}}\frac{uv}{1+\frac{qd}{\alpha_{21}}u}+\frac{a_2}{d}v+\frac{b_2}{\alpha_{21}}\frac{uv}{1+\frac{qd}{\alpha_{21}}u}-\frac{c_2}{\alpha_{21}}v^2$$

we derive that

$$\begin{aligned}\int_0^1 G_x^2\,dx &\le K_{2,3}\int_0^1 (u_x^2+v_x^2)\,dx + C_{2,5}\int_0^1 (u^2+v^2)(u_x^2+v_x^2)\,dx \\ &\le K_{2,3}\int_0^1 (u_x^2+v_x^2)\,dx + 2C_{2,5}\frac{M_1}{d^2}(|u_x|_\infty^2+|v_x|_\infty^2) \\ &\le K_{2,3}\int_0^1 (u_x^2+v_x^2)\,dx + \frac{1}{4}(|u_{xx}|_2^2+|u_{xx}|_2^2) \\ &\quad + K_{2,4}(|u_x|_2^2+|v_x|_2^2),\end{aligned}\tag{3.24}$$

where $K_{2,3}=\frac{C_{2,6}}{d^2}$, $K_{2,4}=C_{2,7}\frac{M_1}{d^2}(1+\frac{M_1}{d^2})$, and $C_{2,5}$, $C_{2,6}$, $C_{2,7}$ are positive constants depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i=1,2$. Substituting (3.24) into (3.18) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (\zeta_{xx})^2\,dx \le & -\frac{1}{2}\int_0^1 (\zeta_{xxx})^2\,dx + \frac{1}{4}\int_0^1 ((u_{xx})^2+(v_{xx})^2)\,dx \\ & + (K_{2,3}+K_{2,4})\int_0^1 (u_x^2+v_x^2)\,dx.\end{aligned}\tag{3.25}$$

By summing up (3.22), (3.23) and (3.25) we find that

(3.26)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u_x^2+v_x^2+(\zeta_{xx})^2)\,dx \le & -\frac{1}{4}\int_0^1 ((u_{xx})^2+(v_{xx})^2+(\zeta_{xxx})^2)\,dx \\ & + K_{2,5}\int_0^1 (u_x^2+v_x^2+(\zeta_{xx})^2)\,dx + K_{2,6},\end{aligned}$$

where $K_{2,5}=K_{2,1}+K_{2,2}+K_{2,3}+K_{2,4}=C_{2,8}(1+\frac{1}{d}+\frac{1}{d^2}+\frac{M_1}{d^2}+(\frac{M_1}{d^2})^2)$, $K_{2,6}=(C_{2,2}+C_{2,4})\frac{(M_1^2+M_1)}{d^4}$, and $C_{2,8}$ is a constant depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i=1,2$. Now we estimate the terms in the first integral of the right-hand side of (3.26). Using the inequality (2.6) and the uniform boundedness of the L_2 norm of u we have

$$|u_x|_2 \le |u_{xx}|_2^{\frac{1}{2}}|u|_2^{\frac{1}{2}} \le (\tfrac{M_1}{d^2})^{\frac{1}{4}}|u_{xx}|_2^{\frac{1}{2}}.$$

Using this inequality we have

$$-\int_0^1 u_{xx}^2\,dx \le -\frac{d^2}{M_1}\left(\int_0^1 u_x^2\,dx\right)^2. \tag{3.27}$$

Similar derivations for v lead that

$$-\int_0^1 v_{xx}^2\,dx \le -\frac{d^2}{M_1}\left(\int_0^1 v_x^2\,dx\right)^2. \tag{3.28}$$

Inequality (2.7) applied to the function ζ and the uniform boundedness of the L_2 norm of ζ give

$$|\zeta_{xx}|_2 \le |\zeta_{xxx}|_2^{\frac{2}{3}}|\zeta|_2^{\frac{1}{3}} \le (\tfrac{2M_1}{d^2})^{\frac{1}{6}}|\zeta_{xxx}|_2^{\frac{2}{3}},$$

and thus

$$-\int_0^1 (\zeta_{xxx})^2\,dx \le -\frac{d}{(2M_1)^{\frac{1}{2}}}\left(\int_0^1 \zeta_{xx}^2\,dx\right)^{\frac{3}{2}}. \tag{3.29}$$

By substituting (3.27), (3.28) and (3.29) into (3.26) we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u_x^2+v_x^2+(\zeta_{xx})^2)\,dx \le K_{2.6} &+ K_{2.5}\int_0^1 (u_x^2+v_x^2+(\zeta_{xx})^2)\,dx\\ &-\frac{d^2}{4M_1}\left\{(\int_0^1 u_x^2\,dx)^2+(\int_0^1 v_x^2\,dx)^2\right\}\\ &-\frac{d}{4(2M_1)^{\frac{1}{2}}}(\int_0^1 (\zeta_{xx})^2\,dx)^{\frac{3}{2}}.\end{aligned}$$

Therefore we arrive at the inequalities that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 d^2(u_x^2+v_x^2+(\zeta_{xx})^2)\,dx\\ &\le C_{2.9}+K_{2.5}\int_0^1 d^2(u_x^2+v_x^2+(\zeta_{xx})^2)\,dx\\ &\quad-\frac{1}{4M_1}\left\{(\int_0^1 d^2u_x^2\,dx)^2+(\int_0^1 d^2v_x^2\,dx)^2\right\}-\frac{1}{4(2M_1)^{\frac{1}{2}}}(\int_0^1 d^2(\zeta_{xx})^2\,dx)^{\frac{3}{2}}\\ &\le C_{2.9}+\frac{1}{2M_1}+K_{2.5}\int_0^1 d^2(u_x^2+v_x^2+(\zeta_{xx})^2)\,dx\\ &\quad-\frac{1}{4M_1}\left\{(\int_0^1 d^2u_x^2\,dx)^{\frac{3}{2}}+(\int_0^1 d^2v_x^2\,dx)^{\frac{3}{2}}\right\}-\frac{1}{4(2M_1)^{\frac{1}{2}}}(\int_0^1 d^2(\zeta_{xx})^2\,dx)^{\frac{3}{2}}\\ &\le K_{2.7}+K_{2.5}\int_0^1 d^2(u_x^2+v_x^2+(\zeta_{xx})^2)\,dx-K_{2.8}\{\int_0^1 d^2(u_x^2+v_x^2+(\zeta_{xx})^2\}^{\frac{3}{2}},\end{aligned}$$

where $K_{2,7} = C_{2,9} + \frac{1}{2M_1}$, $K_{2,8} = C_{2,10}\min\{\frac{1}{4M_1}, \frac{1}{4(2M_1)^{\frac{1}{2}}}\}$, and $C_{2,9}$, $C_{2,10}$ are constants depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i = 1,2$. Hence we conclude that there exist positive constant $M_2 = M_2(||u_0||_2^1, ||v_0||_2^1, ||v_0 - u_0||_2^2, d, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1,2)$ such that

$$\text{(3.30)} \qquad \int_0^1 (d\,u_x(t))^2\,dx < M_2, \quad \int_0^1 (d\,v_x(t))^2\,dx < M_2 \quad \text{for all } t \in [0,\infty).$$

For $d \geq 1$ we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 d^2(u_x^2 + v_x^2 + (\zeta_{xx})^2)\,dx \\ \leq\ & C_{2,11} + C_{2,12}\int_0^1 d^2(u_x^2 + v_x^2 + (\zeta_{xx})^2)\,dx \\ &- C_{2,13}\{\int_0^1 d^2(u_x^2 + v_x^2 + (\zeta_{xx})^2)\,dx\}^{\frac{3}{2}}, \end{aligned}$$

where $C_{2,11}$, $C_{2,12}$, $C_{2,13}$ are positive constants depending only on α_{12}, α_{21}, q, a_i, b_i, c_i, $i = 1,2$. Thus for $d \geq 1$ the positive constant M_2 in (3.30) is independent of $d \geq 1$, that is, $M_2 = M_2(||u_0||_2^1, ||v_0||_2^1, ||v_0 - u_0||_2^2, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1,2)$.

We obtain the following estimate for the maximal solution $(u(x,t), v(x,t))$ to the reduced system (3.1) by combining the results of **Step 1**, **Step 2**, and **Step 3** that there exists a positive constant $\tilde{M} = \tilde{M}(||u_0||_2^1, ||v_0||_2^1, ||v_0 - u_0||_2^2, d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1,2)$ such that

$$\text{(3.31)} \qquad \max\{\|d\,u(\cdot,t)\|_{1,2}, \|d\,v(\cdot,t)\|_{1,2} : t \in [0,\tilde{T})\} \leq \tilde{M}.$$

By scaling back and using the Sobolev embedding inequalities we obtain the desired estimate for the system (1.1) as the following : we have positive constants t_0, $M' = M'(||u_0||_2^1, ||v_0||_2^1, ||v_0 - u_0||_2^2, d, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1,2)$, and $M = M(|\,u_0||_2^1, ||v_0||_2^1, ||v_0 - u_0||_2^2, d, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1,2)$ such that

$$\text{(3.32)} \qquad \begin{aligned} &\max\{\|u(\cdot,t)\|_{1,2}, \|v(\cdot,t)\|_{1,2} : t \in ([0,T)\} \leq M', \\ &\max\{u(x,t),\ v(x,t) : (x,t) \in [0,1] \times [0,T)\} \leq M \end{aligned}$$

for the maximal solution $(u(x,t), v(x,t))$ of (1.1). It is also obtained that $T = +\infty$ from Theorem 8. Hence we conclude that system (1.1) possesses the solution $(u(x,t), v(x,t))$ existing for all time $t > 0$.

For $d \geq 1$ the positive constants M' and M in (3.32) are independent of d, that is, $M' = M'(||u_0||_2^1, ||v_0||_2^1, ||v_0 - u_0||_2^2, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1,2)$, $M = M(||u_0||_2^1, ||v_0||_2^1, ||v_0 - u_0||_2^2, \alpha_{12}, \alpha_{21}, q, a_i, b_i, c_i, i = 1,2)$. □

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