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W algebras, cosets and VOAs for 4d $\mathcal{N} = 2$ SCFTs from M5 branes

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Abstract: We identify vertex operator algebras (VOAs) of a class of Argyres-Douglas (AD) matters with two types of non-abelian flavor symmetries. They are the W algebras defined using nilpotent orbit with partition $[q^m, 1^s]$. Gauging above AD matters, we can find VOAs for more general $\mathcal{N} = 2$ SCFTs engineered from 6d $(2, 0)$ theories. For example, the VOA for general (A_{N-1}, A_{k-1}) theory is found as the coset of a collection of above W algebras. Various new interesting properties of 2d VOAs such as level-rank duality, conformal embedding, collapsing levels, coset constructions for known VOAs can be derived from 4d theory.

Keywords: Duality in Gauge Field Theories, Supersymmetric Gauge Theory, Supersymmetry and Duality

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1 Introduction

A remarkable correspondence between a four dimensional $\mathcal{N} = 2$ superconformal field theory (SCFT) and a two dimensional vertex operator algebra (VOA) is found in [1]. which provides a promising organizing principle for the whole landscape of $\mathcal{N} = 2$ theories (see [2–22] for some further developments). Once a 4d/2d pair is found, one can use the 2d theory to learn 4d theory and vice versa. For example, one can compute the Schur index of 4d SCFT by calculating the vacuum character of 2d VOA which is often much easier to work out, meanwhile 4d result also motivates the study of certain 2d VOAs which received little attention before [23].

If our 4d SCFTs can be enginnered from string/M theory, it is possible to gain more insights about 4d/2d pair. In the past few years, People have found a large class of 4d $\mathcal{N}=2$ SCFTs by putting 6d $(2,0)$ theory on a Riemann surface with various defects [24–28], so we should have a map between such 6d configuration and a 2d VOA, which is schematically depicted in figure 1. The 2d VOA for the theory defined using only regular punctures was studied in [2, 3, 29]. The most important step is to understand the VOA for the theory defined by three full punctures (maximal flavor symmetry), since the general cases can be found from following two correspondence between operations in 4d theory and operations in 2d VOAs:

- On the 4d side one can reduce the full puncture to a generic puncture labeled by a nilpotent orbit f. The 2d counterpart of such operation corresponds to the quantum Drinfeld-Sokolov (qDS) reduction of the original VOA [2], as sketched in figure 2.
- If a theory is formed by conformally gauging various matters together, the VOA is formed by performing cosets on those VOAs [1, 2] as in figure 3.

Things become more interesting and complicated if we consider Argyres-Douglas (AD) theories which are engineered using one irregular singularity Φ and one regular singularity f on a sphere as in figure 1. The correspondence between 2d VOAs and certain AD theories were discussed in [5, 10, 11, 20, 28, 30–39]. More generally it was conjectured in [11, 28, 35] that if there is no mass parameter associated with the irregular singularity, the corresponding VOA is just the vacuum module of a W algebra denoted by $W^{k'}(\mathfrak{g}, f)$ which is obtained through the quantum Hamiltonian reduction from the vacuum $\hat{\mathfrak{g}}$-module of level k' [40], where k' is determined by the data Φ. Again, the choice of the generic regular puncture f determines the qDS reduction type.

There remains the question on determining the VOA for remaining cases, and the main purpose of this paper is to partially solve this problem by using following two facts:

- Irregular singularities with mass deformations often have exact marginal deformations and their weakly coupled gauge theory descriptions are found in [41–43]. They are described by gauging AD matters with at least two types of non-abelian flavor symmetries.[1] Therefore once we find the VOA for such AD matter, the VOA of the full theory can be found by the coset construction.
- A crucial observation for this paper is that all AD matters with two non-abelian flavor symmetries studied in [42, 43] can be engineered by a different realization whose VOAs are known as certain W algebra studied in [11, 28, 35]. The W algebra takes the form $W^{k'}(\mathfrak{g}, [q^m, 1^s])$ with k' depends on parameter (q, m, s), see the summary in section 3.2.5.

[1] One can get more non-abelian flavor symmetries from closing regular puncture, but gauge theory description found in [42] has to use the non-abelian flavor symmetry arising from irregular singularity.

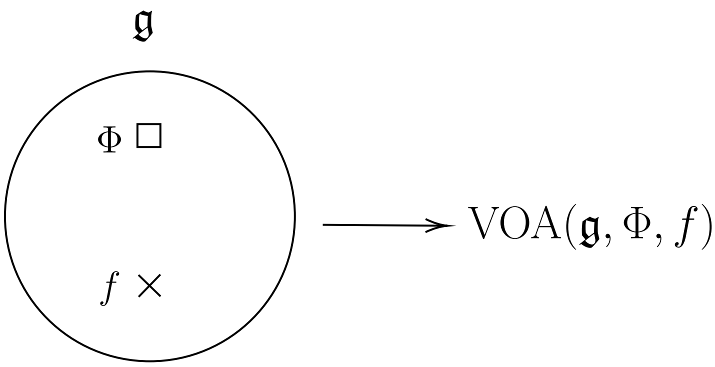

Figure 1. A mapping of a 6d $(2,0)$ configuration to a 2d VOA, here $\mathfrak{g}$ is a simple Lie algebra, Φ is an irregular singularity, and f represents a regular singularity.

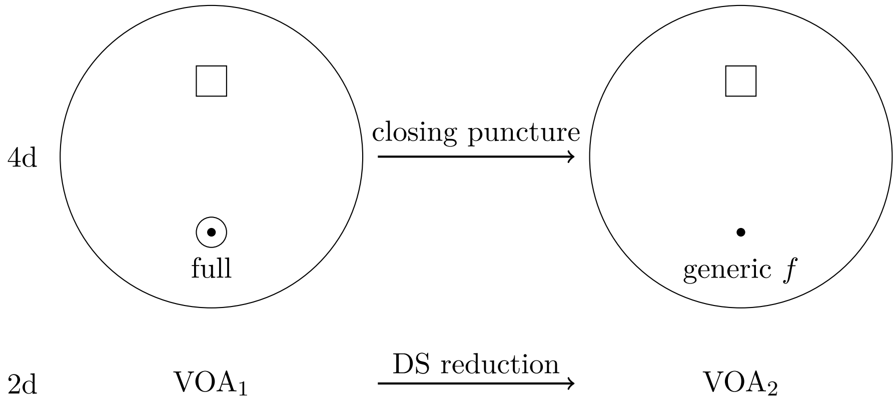

Figure 2. Closing puncture in Class S construction corresponds to qDS reduction of 2d VOA.

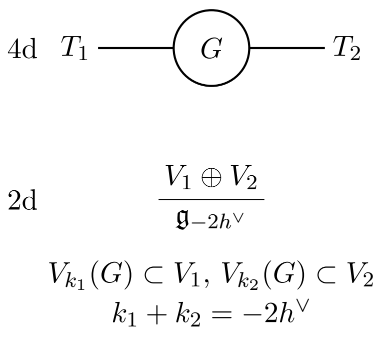

Figure 3. Four dimensional conformal gauging is interpreted as cosets of two dimensional VOAs. T_1 and T_2 are four dimensional matter with non-abelian flavor symmetry G, and one gauge them to get a new conformal field theory with exact marginal deformation. Here V_1 and V_2 are VOAs for matter T_1 and T_2, and each of them has an affine vertex subalgebra $V_{k_i}(G)$. $\mathfrak{g}$ is the Lie algebra of G. Since we are taking the diagonal coset, the generators J^a_{diag} of $\mathfrak{g}_{-2h^\vee}$ is the sum $J^a_{\mathrm{diag}} = J^a_1 + J^a_2$ where $J^a_{1,2}$ are generators of $V_{k_{1,2}}(G)$ respectively.

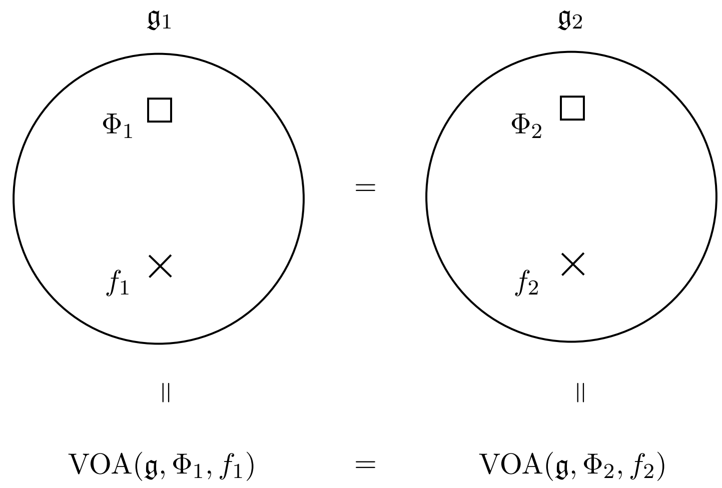

Figure 4. Equivalence of 6d $(2,0)$ configuration implies the equivalence of 2d VOAs.

There are several new interesting features about VOAs of AD matters studied in this paper:

a) The VOA has an affine VOA $V_{k_1}(\mathfrak{g}_1) \oplus V_{k_2}(\mathfrak{g}_2)$ as its subalgebra, where $V_k(\mathfrak{g})$ is the affine Kac-Moody (AKM) vertex algebra[2] of Lie algebra $\mathfrak{g}$ with level k. This affine VOA has the same central charge as the W algebra and therefore we found a large number of new possible conformal embeddings of VOAs.

b) The simple fact that a theory can be engineered in various ways can often tell us interesting properties about VOAs, see figure 4. For instance, we can derive new level-rank type duality.

c) S-duality of 4d theory implies the equivalence between different cosets constructions of a single VOA.

Once VOAs for AD matters are known, we are able to write down VOAs for more general theories engineered from M5 branes. For example, the VOA for (A_{N-1}, A_{k-1}) theory with arbitrary N and k is found by using its weakly coupled gauge theory descriptions in section 5.1.

This paper is organized as the following. Section 2 reviews known results about the mapping between AD theories engineered from M5 branes and VOAs. Section 3 studies VOAs corresponding to AD matters with two distinct non-abelian flavor symmetries. Section 4 focuses on the associated variety of the VOA for a given AD matter, which determines the Higgs branch chiral ring of the theory. Section 5 describes weakly coupled descriptions of AD theories and the coset construction of the corresponding VOA. Section 6 discusses conformal embeddings and VOAs for most general AD theories. Finally, a summary is given in section 7.

[2]By affine Kac-Moody and W algebra, we mean the irreducible vertex operator algebra constructed from the vacua module of AKM and W algebra.

2 Known results

A four dimensional $\mathcal{N} = 2$ SCFT has a bosonic symmetry group $\mathrm{SO}(2,4) \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_R \times G_F$, where $\mathrm{SO}(2,4)$ is the four dimensional conformal group, $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ is the R symmetry group which exists for every $\mathcal{N} = 2$ SCFT, and G_F is the flavor symmetry group which might be absent for some theories. The representation theory of 4d $\mathcal{N} = 2$ superconformal algebra is studied in [44], in which short representations (where some of states in the representation are annihilated by a fraction of supercharges) were completely classified. Important half-BPS operators include primary operators of multiplets $\mathcal{E}_r$ and $\hat{B}_R$.

The moduli space of vacua of a 4d $\mathcal{N} = 2$ SCFT is extremely rich. It consists of a Coulomb branch, whose low energy effective theory involves abelian gauge theory in general. The Coulomb branch is parameterized by expectation values of primary operators of $\mathcal{E}_r$ multiplets, and the low energy effective theory is described by a Seiberg-Witten geometry [45, 46]. The set of rational numbers $[r_1, \ldots, r_s]$ of $\mathrm{U}(1)_r$ charges of $\mathcal{E}_r$ (unitarity implies that $r_i > 1$) is an important set associated to a 4d $\mathcal{N} = 2$ SCFT. The $\mathrm{U}(1)_r$ symmetry acts non-trivially on the Coulomb branch while $\mathrm{SU}(2)_R \times G_F$ symmetry acts trivially.

Some theories also have a Higgs branch where the gauge group is completely broken in general should the SCFT has a gauge theory description. The Higgs branch, being a conical hyperkhaler manifold, is parameterized by expectation values of primary operators of $\hat{B}_R$ multiplets. One of important questions about the Higgs branch is to determine the affine chiral ring of the cone. The $\mathrm{SU}(2)_R \times G_F$ symmetry acts non-trivially on Higgs branch, while $\mathrm{U}(1)_R$ symmetry acts trivially.

Besides the Coulomb branch spectrum (just a set of rational numbers $r_i > 1$) and the Higgs branch chiral ring, one would also like to determine three interesting quantities of a 4d $\mathcal{N} = 2$ SCFT: central charges a_{4d} and c_{4d} which is defined using the energy-momentum tensor, and the flavor central charge k_G.

There is an interesting set of short multiplets called Schur sector [47] which consists of Higgs branch operators $\hat{B}_R$, energy momentum tensors, and etc. Moreover, one can define a Schur index which counts those operators. It was proposed in [1] that one can get a 2d VOA from the Schur sector of a 4d $\mathcal{N} = 2$ SCFT, and the basic 4d/2d correspondence used in current paper is [1]:

- There is an AKM subalgebra ($V_{k_{2d}}(\mathfrak{g})$) in 2d VOA, where $\mathfrak{g}$ is the Lie algebra of four dimensional flavor symmetry G_F.
- The 2d central charge c_{2d} and the level of AKM algebra k_{2d} are related to the 4d central charge c_{4d} and the flavor central charge k_F as

$$c_{2d} = -12 c_{4d}, \quad k_{2d} = -k_F.^{3} \tag{2.1}$$

- The (normalized) vacuum character of 2d VOA is the 4d Schur index $\mathcal{I}(q)$.

[3] Our normalization of k_F is half of that of [1, 2].

Many 4d/2d pairs are found in [2, 3, 10, 11, 20, 28–39]. Various interesting properties of 4d theory using 2d VOA are studied in [4, 8, 9, 11, 13, 15, 30–32, 48–57]. See also [19, 56, 58] for VOAs corresponding to 6d $(2,0)$ theory on a four manifold. Moreover VOAs corresponding to three dimensional $\mathcal{N}=4$ theory are discussed in [59, 60].

VOAs arise from study of the chiral part of two dimensional conformal field theories. A general definition has been given by mathematicians [61], but it seems difficult to construct them abstractly. On the other hand, a VOA is often defined as an irreducible (which is also called simple) vacuum module of a particular algebra, which seems much more tractable. The simplest case is the Virasoro algebra whose elements are the modes of energy momentum tensor $T(z)=\sum L_n z^{-n-2}$:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{2.2}$$

Another important algebra is the AKM algebra associated with a simple Lie algebra $\mathfrak{g}$,

$$[J_n^a, J_m^b] = i f_c^{ab} J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}, \tag{2.3}$$

with f_c^{ab} the structure constant of $\mathfrak{g}$. One can construct an energy-momentum tensor using the Sugawara construction, and the simple vacuum module is indeed an VOA [62]. The representation theory of above two algebras has been studied thoroughly in the literature (see [63] and many others).

Finally, one can have intricated algebras built from a set of higher spin fields $(W_{d_1},\ldots,W_{d_i})$, which are called W algebras [64]. The full algebra content and its representation theory of a W algebra is very complicated, however, if a W algebra can be derived from the qDS reduction of an AKM algebra [40], one can actually derive lots of important information of this W algebra from the representation theory of the AKM algebra.

In general it is difficult to work out the full Schur sector of a four dimensional $\mathcal{N}=2$ SCFT. However, using a surprising fact that almost all the known 2d VOA for the above 4d/2d mapping involves the W algebra derived from qDS reduction [65], one can hope that a lot can be learned about 4d theories by using existing knowledge of 2d VOAs.

2.1 AD theories correspond to $W^{k'}(\mathfrak{g}, f)$ algebras

One can engineer a large class of four dimensional $\mathcal{N}=2$ SCFTs by starting with a 6d $(2,0)$ theory of type $\mathfrak{j}$ = ADE on a sphere with an irregular singularity and a regular singularity [24–28].[4] The Coulomb branch is captured by a Hitchin system with singular boundary conditions near the singularity. The Higgs field of the Hitchin system near the irregular singularity takes the following form,

$$\Phi = \frac{T}{z^{2+\frac{k}{b}}} + \ldots . \tag{2.4}$$

Here T is determined by a positive principle grading of Lie algebra $\mathfrak{j}$ [66], and is a regular semi-simple element of $\mathfrak{j}$. $k > -b$ and is an integer. Subsequent terms are chosen such that

[4]See the appendix A for relations between this construction and other constructions.

$\mathfrak{j}$	b	Singularity
A_{N-1}	N	$x_1^2+x_2^2+x_3^N+z^k=0$
	$N-1$	$x_1^2+x_2^2+x_3^N+x_3z^k=0$
D_N	$2N-2$	$x_1^2+x_2^{N-1}+x_2x_3^2+z^k=0$
	N	$x_1^2+x_2^{N-1}+x_2x_3^2+z^kx_3=0$
E_6	12	$x_1^2+x_2^3+x_3^4+z^k=0$
	9	$x_1^2+x_2^3+x_3^4+z^kx_3=0$
	8	$x_1^2+x_2^3+x_3^4+z^kx_2=0$
E_7	18	$x_1^2+x_2^3+x_2x_3^3+z^k=0$
	14	$x_1^2+x_2^3+x_2x_3^3+z^kx_3=0$
E_8	30	$x_1^2+x_2^3+x_3^5+z^k=0$
	24	$x_1^2+x_2^3+x_3^5+z^kx_3=0$
	20	$x_1^2+x_2^3+x_3^5+z^kx_2=0$

Table 1. Three-fold isolated quasihomogenous singularities of cDV type corresponding to the $J^{(b)}[k]$ irregular punctures of the regular-semisimple type in [27]. These 3d singularity is very useful in extracting the Coulomb branch spectrum, see [67].

they are compatible with the leading order term (essentially the grading determines the choice of these terms). We call them $J^{(b)}[k]$ type irregular puncture. Theories constructed using only above irregular singularity can also be engineered using a three dimensional singularity in type IIB string theory as summarized in table 1 [67].

One can add another regular singularity which is labeled by a nilpotent orbit f of $\mathfrak{j}$ (We use Nahm labels such that the trivial orbit corresponding to regular puncture with maximal flavor symmetry). A detailed discussion about these defects can be found in [68]. If there is no mass parameter encoded in the irregular singularity Φ (which means that the z^{-1} term in Φ is not allowed, and we also assume that singular parts of Φ are diagonalized near the irregular singularity), the VOA which corresponds to this 4d SCFT is the vacuum module of following W algebra[5] [11, 35],

$$W^{k'}(\mathfrak{j}, f), \qquad k' = -h^\vee + \frac{b}{k+b}. \tag{2.5}$$

Here $h^\vee$ is the dual Coxeter number of $\mathfrak{j}$, and the W algebra is defined as the qDS reduction from the affine VOA [69].

To get non-simply laced flavor groups, we need to consider the outer-automorphism twist of ADE Lie algebra and its Langlands dual. A systematic study of these AD theories was performed in [28]. Denoting the twisted Lie algebra of $\mathfrak{j}$ as $\mathfrak{g}^\vee$ and its Langlands dual as

[5] We always use the irreducible vacuum module as the VOA corresponding to 4d theory.

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j	A_{2N}	A_{2N-1}	D_{N+1}	E_6	D_4
Outer-automorphism o	Z_2	Z_2	Z_2	Z_2	Z_3
Invariant subalgebra $\mathfrak{g}^\vee$	B_N	C_N	B_N	F_4	G_2
Flavor symmetry $\mathfrak{g}$	$C_N^{(1)}$	B_N	$C_N^{(2)}$	F_4	G_2

Table 2. Outer-automorphisms of simple Lie algebras j, its invariant subalgebra $g^\vee$ and flavor symmetry g from the Langlands dual of $g^\vee$.

$\mathfrak{g}$, outer-automorphisms and twisted algebras of $\mathfrak{j}$ are summarized in table 2. The irregular singularity of regular semi-simple type is also classified as in table 3 with the following form,

$$\Phi = \frac{T^t}{z^{2+\frac{k}{b}}} + \dots \tag{2.6}$$

Here T^t is an element of Lie algebra $\mathfrak{g}^\vee$ or other parts of the decomposition of $\mathfrak{j}$ under outer automorphism. $k > -b$, and the novel thing is that k could take half-integer value or in thirds ($\mathfrak{g} = G_2$). One can also represent those irregular singularities by 3-fold singularities as in table 3.

We could again add a twisted regular puncture labeled also by a nilpotent orbit f of $\mathfrak{g}$. If there is no mass parameter in the irregular singularity, the corresponding VOA is given by following W algebra [28],

$$\boxed{W^{k'}(\mathfrak{g}, f), \quad k' = -h^\vee + \frac{1}{n}\frac{b}{k+b},} \tag{2.7}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, n is the number listed in table 4, and k is restricted to the value such that no mass parameter is in the irregular singularity.

The Seiberg-Witten geometry of these theories are identified as the spectral curve of the Hitchin system [70].

$$\det(x - \Phi) = 0, \tag{2.8}$$

and one can read off the Coulomb branch spectrum from an associated Newton polygon [26–28], which is also reviewed in the appendix B.

2.2 AD theories correspond to $B_{p+1}(g)$ and $W_{p+1}(g)$ algebras

Consider the following irregular singularity of $\mathfrak{j} = \mathrm{ADE}$ $(2, 0)$ theory,

$$\Phi = \frac{T}{z^{2+p}} + \dots \tag{2.9}$$

Notice that there are l (the rank of $\mathfrak{j}$) mass parameters in this singularity. We add a trivial regular singularity (f is regular nilpotent orbit), then these theories can be engineered by following three-fold singulariities,

$$f_{\mathrm{ADE}}(x, y, z) + w^{ph^\vee} = 0, \tag{2.10}$$

j with twist	b_t	SW geometry at SCFT point	$\Delta[z]$
A_{2N}/Z_2	$4N+2$	$x_1^2+x_2^2+x^{2N+1}+z^{k+\frac{1}{2}}=0$	$\frac{4N+2}{4N+2k+3}$
	$2N$	$x_1^2+x_2^2+x^{2N+1}+xz^k=0$	$\frac{2N}{k+2N}$
A_{2N-1}/Z_2	$4N-2$	$x_1^2+x_2^2+x^{2N}+xz^{k+\frac{1}{2}}=0$	$\frac{4N-2}{4N+2k-1}$
	$2N$	$x_1^2+x_2^2+x^{2N}+z^k=0$	$\frac{2N}{2N+k}$
D_{N+1}/Z_2	$2N+2$	$x_1^2+x_2^N+x_2x_3^2+x_3z^{k+\frac{1}{2}}=0$	$\frac{2N+2}{2k+2N+3}$
	$2N$	$x_1^2+x_2^N+x_2x_3^2+z^k=0$	$\frac{2N}{k+2N}$
D_4/Z_3	12	$x_1^2+x_2^3+x_2x_3^2+x_3z^{k\pm\frac{1}{3}}=0$	$\frac{12}{12+3k\pm1}$
	6	$x_1^2+x_2^3+x_2x_3^2+z^k=0$	$\frac{6}{6+k}$
E_6/Z_2	18	$x_1^2+x_2^3+x_3^4+x_3z^{k+\frac{1}{2}}=0$	$\frac{18}{18+2k+1}$
	12	$x_1^2+x_2^3+x_3^4+z^k=0$	$\frac{12}{12+k}$
	8	$x_1^2+x_2^3+x_3^4+x_2z^k=0$	$\frac{8}{12+k}$

Table 3. Seiberg-Witten geometry of twisted theories at the SCFT point.

	dimension	h	$h^\vee$	n
A_{N-1}	N^2-1	N	N	1
B_N	$(2N+1)N$	$2N$	$2N-1$	2
$C_N^{(1)}$	$(2N+1)N$	$2N$	$N+1$	4
$C_N^{(2)}$	$(2N+1)N$	$2N$	$N+1$	2
D_N	$N(2N-1)$	$2N-2$	$2N-2$	1
E_6	78	12	12	1
E_7	133	18	18	1
E_8	248	30	30	1
F_4	52	12	9	2
G_2	14	6	4	3

Table 4. Lie algebra data. h is the Coexter number and $h^\vee$ is the dual Coexter number.

where $h^\vee$ is the dual Coxeter number of $\mathfrak{j}$, and $f_{\mathrm{ADE}}(x,y,z)$ is the famous two dimensional ADE singularity. The 4d central charge is computed using following formula [71],

$$c_{4d}=\frac{\mu\alpha_{\max}}{12}+\frac{r}{6}, \tag{2.11}$$

where $\alpha_{\max}$ is the maximal scaling dimension of Coulomb branch spectrum, r is the rank of Coulomb branch, and $\mu=2r+f_0$ with f_0 the number of mass parameters. Using the result found in [35], we have,

$$\mu = l(ph^\vee - 1), \quad \alpha_{\max} = \frac{ph^\vee}{p+1}, \quad f_0 = l, \tag{2.12}$$

then the central charge takes following form,

$$c_{4d}(\mathfrak{j}, p) = \frac{1}{12}\left(-2l + \frac{(h^\vee+1)lh^\vee p^2}{p+1}\right). \tag{2.13}$$

When $\mathfrak{j} = A_{N-1}$, it was propsed in [20] that the corresponding VOA is the $B_{p+1}(A_{N-1})$ algebra constructed in [72]. For general $\mathfrak{j}$ = ADE, the central charge of $B_{p+1}(\mathfrak{j})$ algebra is,

$$c_{2d}(B_{p+1}(\mathfrak{j})) = 2l + h^\vee \dim(\mathfrak{j})\left(2 - (p+1) - \frac{1}{p+1}\right). \tag{2.14}$$

One finds that $c_{2d}(B_{p+1}(\mathfrak{j})) = -12c_{4d}(\mathfrak{j}, p)$ for general $\mathfrak{j}$,[6] therefore we conjecture that the VOA of above 4d SCFT is given by the $B_{p+1}(\mathfrak{j})$ algebra.

We could also consider twisted AD theories with the following Higgs field,

$$\Phi = \frac{T^t}{z^{2+p}} + \dots \tag{2.15}$$

Using the method proposed in [28], the 4d central charge is,

$$c_{4d} = \frac{1}{12}\left(-2l + \frac{(h^\vee l(h^\vee+1))(n(p+1)-1)^2}{n(p+1)}\right). \tag{2.16}$$

$n = 1$ for simply-laced cases, and values of n for non simply-laced cases are summarized in table 4.[7] This implies that it should be possible to generalize the construction in [72] to non-simply laced Lie algebra whose central charge is given by $-12c_{4d}$ with c_{4d} in equation (2.16).

One can also add another full puncture to get a theory with G flavor symmetry whose flavor central charge is,

$$k_G = h^\vee - \frac{1}{n(p+1)}. \tag{2.17}$$

The central charge for these theories are,

$$c_{4d} = \frac{\left(h^\vee - \frac{1}{n(p+1)}\right) l(h^\vee+1)n(p+1)}{12} - \frac{l}{12}. \tag{2.18}$$

This suggests that there should be a VOA $W_{p+1}(\mathfrak{g})$ with an affine vertex operator subalgebra $V_{-k_G}(\mathfrak{g})$. Some suggestions on the construction of this class of VOA are given in [20], and we will give a coset construction in later sections. The qDS reduction of $W_p(\mathfrak{g})$ produces the $B_p(\mathfrak{g})$ VOA.

[6] We used the fact that $\dim(\mathfrak{j}) = l(h^\vee + 1)$, here $h^\vee$ is the dual Coxeter number and l is the rank of $\mathfrak{j}$.
[7] Notice that n can take two values for C_N type Lie algebra.

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3 VOA for AD matter with two non-abelian flavor symmetries

Theories considered in the last section usually carry only one type of non-abelian flavor symmetries, which is determined by the regular puncture. When the order of the irregular singularity considered in the last section is integral, one can consider degenerating cases and could get another type of non-abelian flavor symmetries [26].

It was realized in [42, 43] that besides the theory considered in the last section, we can get new 4d $\mathcal{N}=2$ SCFT by considering more general irregular singularities (taking $\mathfrak{g}=A_{N-1}$ for example),

$$\Phi=\frac{T}{z^{2+\frac{k}{n}}} \tag{3.1}$$

with T being the following diagonal matrix,

$$T=\mathrm{diag}\left(I_{n\times n},\underbrace{0,\ldots,0}_{N-n}\right). \tag{3.2}$$

Here $I_{n\times n}$ being a diagonal matrix with eigenvalues $(1,w,\ldots,w^{n-1})$ with w the nth root of unity. To get a SCFT, coefficients of subsequent terms in the Higgs field have to take the same form as the leading one. Such irregular singularity carries a flavor symmetry $\mathrm{U}(N-n)$. One can also add an extra regular puncture so that there are two distinct types of non-abelian flavor symmetries.

The purpose of this section is to find the VOA for the above class of theories when $(k,n)=1$. The key observation is that the same theory can be realized by a different $(2,0)$ configuration whose VOA is already found in section 2. We first study the A_{N-1} case in detail in section 3.1. The same construction is then generalized to other classical Lie algebras as well.

3.1 $A_{N-1}=\mathfrak{sl}_N$: $\mathrm{SU}(N)\times\mathrm{U}(n_1)$ flavor symmetry

The regular puncture of A_{N-1} theory is classified by a size N Young tableaux (or the partition of N), which also gives a nilpotent orbit of the A_{N-1} Lie algebra. Given a Young tableaux $[h_1^{r_1},h_2^{r_2},\ldots,h_s^{r_s}]$, the flavor symmetry is,

$$G_F=\left(\prod\mathrm{U}(r_i)\right)\Big/\,\mathrm{U}(1). \tag{3.3}$$

We are interested in regular punctures with partitions like $[m^q,1^s]$, whose 4d flavor symmetry is $\mathrm{SU}(q)\times\mathrm{SU}(s)\times\mathrm{U}(1)$ with flavor central charges,

$$k_{\mathrm{SU}(s)}=m+s-[z],\quad k_{\mathrm{SU}(q)}=s+qm-m[z], \tag{3.4}$$

where $[z]$ is the scaling dimension of z coordinate in the spectral curve of A type Hitchin system.

Now consider the following configuration,

$$\mathfrak{g}=\mathfrak{sl}_{n_1+n},\qquad \Phi=\frac{T}{z^{2+\frac{k}{n}}},\qquad f=\left[\underbrace{1,\ldots,1}_{n+n_1}\right]=[1^{n+n_1}], \tag{3.5}$$

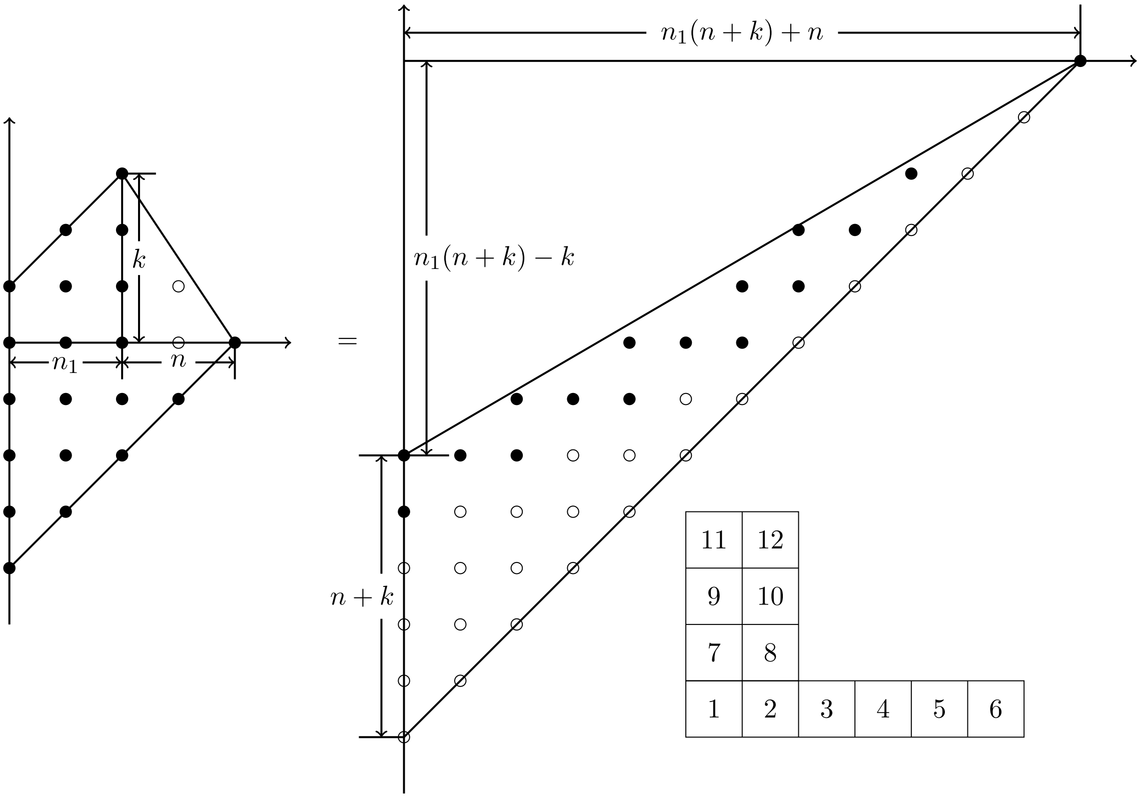

Figure 5. Equivalence of two different $(2, 0)$ configurations of A type theory.

with k and n coprime. Here T takes the following form $T = [I_{n_1\times n_1}, 0, \ldots, 0]$ with $I_{n\times n}$ being a diagonal matrix with eigenvalues $(1, w, \ldots, w^{n-1})$ with w being the nth root of unity. This theory has a $\mathrm{U}(n_1) \times \mathrm{SU}(n_1 + n)$ flavor symmetry with 4d flavor central charges [43],[8]

$$k_{\mathrm{SU}(n_1)} = n_1 + \frac{n}{n+k}, \quad k_{\mathrm{SU}(n_1+n)} = n_1 + n - \frac{n}{n+k}. \tag{3.6}$$

The Newton polygon for this theory is shown on the left of figure 5, from which one can read the Coulomb branch spectrum using the procedure in the appendix B.

To find its VOA, we would like to find a different realization of this theory whose VOA is known through results in the last section, which has the following data,

$$\mathfrak{g} = \mathfrak{sl}_{n_1(n+k)+n}, \quad \Phi = \frac{T}{z^{2+\frac{k-n_1(n+k)}{n_1(n+k)+n}}}, \quad f = [(n+k-1)^{n_1}, 1^{n_1+n}]. \tag{3.7}$$

One can show that these two realizations have the same Coulomb branch and flavor symmetries. Now we compute 4d flavor central charges of this new realization of the AD theory (3.7). Notice $[z] = \frac{n_1(n+k)+n}{n+k}$ and using the formula (3.4), we have,

$$\begin{aligned} k_{\mathrm{SU}(n_1+n)} &= n + 2n_1 - \frac{n_1(n+k)+n}{n+k} = n + n_1 - \frac{n}{n+k}, \\ k_{\mathrm{SU}(n_1)} &= n_1(n+k) + n - (n+k-1)\frac{n_1(n+k)+n}{n+k} = n_1 + \frac{n}{n+k}, \end{aligned} \tag{3.8}$$

[8] We ignore the subscript $2d$ and $4d$ when apparenat.

which is exactly the same as flavor central charges (equation (3.6)) of the previous description (3.5). Therefore, we have compelling reasons to believe that (3.5) and (3.7) give the same AD theory and they correspond to the same 2d VOA.

The later realization (3.7) is just the AD theory discussed in 2.1. According to equation (2.7), the corresponding VOA is the following W algebra,

$$\boxed{\mathrm{VOA}_A = W^{-n_1(n+k)-n+\frac{n_1(n+k)+n}{n+k}}(\mathfrak{sl}_{n_1(n+k)+n}, [(n+k-1)^{n_1}, 1^{n_1+n}])\,.} \tag{3.9}$$

Using the central charge formula (C.15) in the appendix C, the central charge of VOA_A is

$$c(\mathrm{VOA}_A) = -\left(n^2-1\right)(k+n-1) - n_1(n+n_1)(3k+3n-2). \tag{3.10}$$

Also by the 4d/2d correspondence, there is an affine $\mathfrak{u}(1) \oplus V_{-\left(n_1+\frac{n}{n+k}\right)}(\mathfrak{su}_{n_1}) \oplus V_{-(n+n_1)+\frac{n}{n+k}}(\mathfrak{su}_{n+n_1})$ subalgebra within VOA_A.

Example. Let us look at the example in figure 5. For the left Newton polygon, we have $n_1 = 2, n = 2, k = 3$. Using the formula (B.5), we have following Coulomb branch spectrum $\left[\frac{8}{5}, \frac{6}{5}, \frac{13}{5}, \frac{11}{5}, \frac{9}{5}, \frac{7}{5}, \frac{18}{5}, \frac{16}{5}, \frac{14}{5}, \frac{12}{5}\right]$. For the right Newton polygon, we have the data $h(l) = [0,1,1,1,1,1,2,2,3,3,4,4]$, and we have $n_1' = 0, n' = 12, k' = -7$. Equation (B.6) gives the spectrum $\left[\frac{8}{5}, \frac{13}{5}, \frac{18}{5}, \frac{6}{5}, \frac{11}{5}, \frac{16}{5}, \frac{9}{5}, \frac{14}{5}, \frac{7}{5}, \frac{12}{5}\right]$. One can see that two Coulomb branch spectra match, although they are encoded in quite different ways.

3.1.1 New level-rank dualitys

Now consider the theory with f being a regular nilpotent orbit (hence no flavor symmetry), and the irregular singularity,

$$\mathfrak{g} = \mathfrak{sl}_{n_1+n}, \qquad \Phi = \frac{T}{z^{2+\frac{k}{n}}}, \quad f = [n_1+n]. \tag{3.11}$$

Again, we take n and k to be coprime. This theory has a $\mathrm{U}(n_1)$ flavor symmetry. Using the above result (equation (3.9)) for f being a full puncture and the fact that closing of a puncture is equivalent to qDS, its VOA is

$$W^{-n_1(n+k)-n+\frac{n_1(n+k)+n}{n+k}}(\mathfrak{sl}_{n_1(n+k)+n}, [(n+k-1)^{n_1}, n+n_1]). \tag{3.12}$$

Here we take $k > n_1$. The same theory can be realized by the following configuration,

$$\mathfrak{g} = \mathfrak{sl}_k, \quad \Phi = \frac{T}{z^{2+\frac{n}{k}}}, \quad f = [k-n_1, 1^{n_1}]. \tag{3.13}$$

Using the correspondence (2.7) in section 2.1, the corresponding VOA is then

$$W^{-k+\frac{k}{n+k}}(\mathfrak{sl}_k, [k-n_1, 1^{n_1}]). \tag{3.14}$$

Therefore, we find the following equivalence of two 2d VOAs from different realizations of the same 4d AD theories,

$$\begin{aligned} &W^{-n_1(n+k)-n+\frac{n_1(n+k)+n}{n+k}}(\mathfrak{sl}_{n_1(n+k)+n}, [(n+k-1)^{n_1}, n+n_1]) \\ &= W^{-k+\frac{k}{n+k}}(\mathfrak{sl}_k, [k-n_1, 1^{n_1}]). \end{aligned} \tag{3.15}$$

We call this the level-rank duality as the rank and the level are sort of exchanged for these two W algebras.

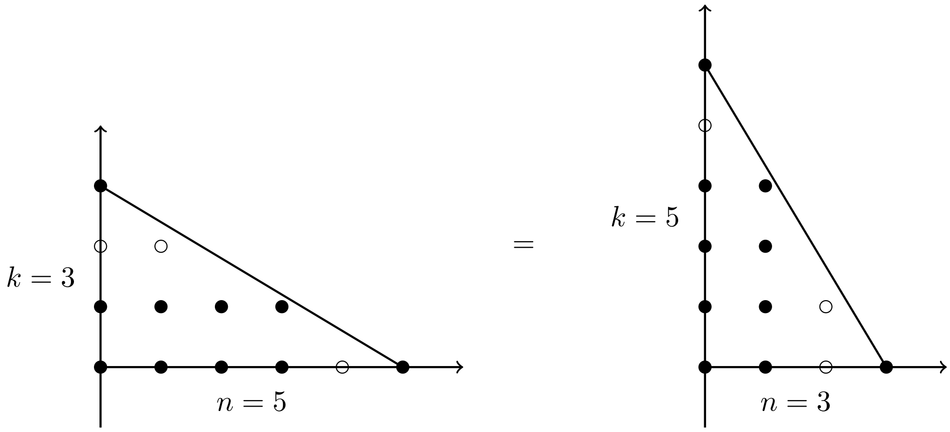

Figure 6. Level rank duality example. The Coulomb branch spectrum of two configurations are the same. Notice that the left hand side uses 6d A_4 $(2,0)$ theory while the right hand side uses 6d A_2 $(2,0)$ theory.

Example. Taking $n_1 = 0$, we have the following equivalence of VOAs:

$$W^{-n+\frac{n}{n+k}}(sl_n,[n]) = W^{-k+\frac{k}{n+k}}(sl_k,[k]). \tag{3.16}$$

This is the familiar level-rank duality discovered in [73]. See figure 6 for an illustration from the four dimensional theory point of view.

The above level-rank duality can be further generalized as follows. Consider two configurations,

$$\begin{aligned} A: \quad & \mathfrak{g} = \mathfrak{sl}_{n_1+n}, \quad & \Phi = \frac{T_1}{z^{2+\frac{k}{n}}}, \quad & f = [n+n_1-n_2, 1^{n_2}], \\ B: \quad & \mathfrak{g} = \mathfrak{sl}_{n_2+k}, \quad & \Phi = \frac{T_2}{z^{2+\frac{n}{k}}}, \quad & f = [k+n_2-n_1, 1^{n_1}]. \end{aligned} \tag{3.17}$$

Choose $n+n_1-n_2 > 1$, $k+n_2-n_1 > 1$ and $T_1 = \mathrm{diag}(I_{n_1\times n_1}, 0_{n\times n})$, $T_2 = \mathrm{diag}(I_{n_2\times n_2}, 0_{k\times k})$. These two configurations give the same Coulomb branch spectrum, therefore we conjecture that they give the same 4d theory. The manifest flavor symmetry is $\mathrm{U}(n_1)\times\mathrm{U}(n_2)$ with following flavor central charges,

$$k_{\mathrm{SU}(n_1)} = n_1 + \frac{n}{n+k}, \quad k_{\mathrm{SU}(n_2)} = n_2 + \frac{k}{n+k}. \tag{3.18}$$

The two seemingly different VOAs should be the same, so we have following (conjectured) equivalence,

$$\boxed{\begin{aligned} & W^{-n_1(n+k)-n+\frac{n_1(n+k)+n}{n+k}}(\mathfrak{sl}_{n_1(n+k)+n}, [(n+k-1)^{n_1}, n+n_1-n_2, 1^{n_2}]) = \\ & W^{-n_2(n+k)-k+\frac{n_2(n+k)+k}{n+k}}(\mathfrak{sl}_{n_2(n+k)+k}, [(n+k-1)^{n_2}, k+n_2-n_1, 1^{n_1}]). \end{aligned}} \tag{3.19}$$

An example is illustrated in figure 7.

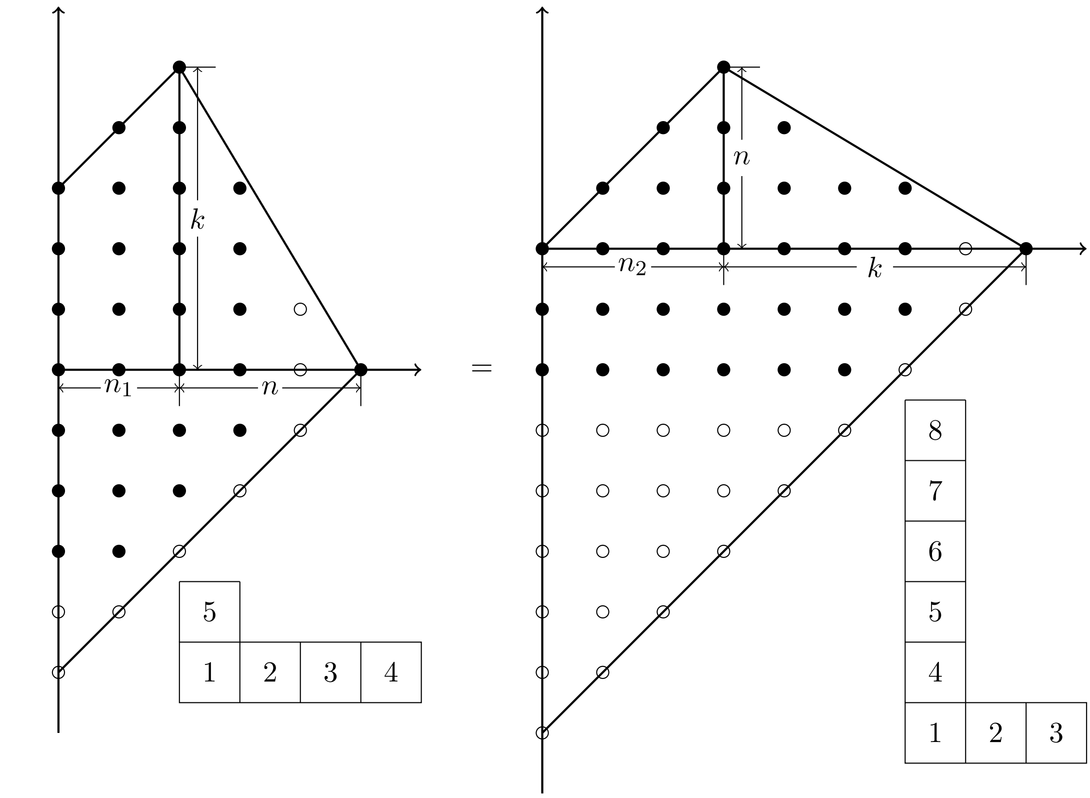

Figure 7. An example of the generalized level rank duality from Newton polygon of four dimensional $\mathcal{N}=2$ theory. One can check that these two configuraitons give the same Coulomb branch spectrum.

3.1.2 Conformal embedding in W algebra

Conformal embedding is defined as the following [74]. Let V be a vertex algebra with a Virasoro (= conformal) vector w_V and let W be a vertex subalgebra of V endowed with a Virasoro vector w_W. The embedding $W \subset V$ is called conformal if $w_W = w_V$. A necessary condition for conformal embedding is that $c_V = c_W$.

For our W algebra VOA_A defined in equation (3.9), there is an affine vertex operator subalgebra $\mathfrak{u}(1) \oplus V_{-\left(n_1+\frac{n}{n+k}\right)}(\mathfrak{sl}_{n_1}) \oplus V_{-(n+n_1)+\frac{n}{n+k}}(\mathfrak{sl}_{n+n_1})$. It is interesting to note that the central charge of VOA_A is equal to the central charge of this affine vertex operator subalgebra. So the necessary condition for conformal embedding is achieved, and it is interesting to check whether the following embedding,

$$\begin{aligned} &u(1) \oplus V_{-\left(n_1+\frac{n}{n+k}\right)}(\mathfrak{sl}_{n_1}) \oplus V_{-(n+n_1)+\frac{n}{n+k}}(\mathfrak{sl}_{n+n_1}) \subset \\ &W^{-n_1(n+k)-n+\frac{n_1(n+k)+n}{n+k}}(\mathfrak{sl}_{n_1(n+k)+n}, [(n+k-1)^{n_1}, 1^{n_1+n}]) \end{aligned} \tag{3.20}$$

is indeed the conformal embedding (see [52] for physical discussions).

Example. Taking $n_1 = 1$ and $n+k-1 = 2$, we have the following embedding:

$$\mathrm{U}(1) \oplus V_{-\frac{2n}{3}+1}(\mathfrak{su}(n+1)) \subset W^{-\frac{2(n+3)}{3}}(\mathfrak{sl}_{n+3}, [2, 1^{n+1}]), \tag{3.21}$$

which is the conformal embedding studied in [74].

3.1.3 Collapsing levels and decoupling of flavor symmetry

We have found VOAs of some AD matters with two distinct non-abelian flavor symmetries by finding an alternative $(2,0)$ construction.[9] Notice that the above construction does not give a new realization for theories with just a $\mathrm{SU}(n)$ flavor symmetry arising only from regular singularity. We would like to find a different realization for those theories too. Such realization is not very good for 4d theory as there appears to be more flavor symmetries which will decouple in the IR, but they do have interesting implications for VOAs.

The basic idea is the following. Start with a theory engineered by the following configuration,

$$\mathfrak{g} = \mathfrak{sl}_n, \quad \Phi = \frac{T}{z^{2+\frac{k}{n}}}, \quad f = [1^n]. \tag{3.22}$$

We take $(k,n) = 1$, and T being the principle type (so no mass term is allowed in the irregular singularity) so that the VOA is the affine vertex operator algebra

$$V^{k'}(\mathfrak{sl}_n), \quad k' = -n + \frac{n}{k+n}. \tag{3.23}$$

Now we would like to find another realization whose assosiated VOA is known by the result of section 2 too, so it should take the following form,

$$\mathfrak{g} = \mathfrak{sl}_N, \quad \Phi = \frac{T}{z^{2+\frac{-N+n+k}{N}}}, \quad f = [q^m, 1^n]. \tag{3.24}$$

Naively, this configuration has flavor symmetry $\mathrm{SU}(n) \times \mathrm{U}(m)$. The flavor central charge for $\mathrm{SU}(n)$ flavor group is $n+m-\frac{N}{n+k}$. The necessary condition for the equivalence is that the flavor central charge for $\mathrm{SU}(n)$ group should be the same,

$$n - \frac{n}{n+k} = n + m - \frac{N}{n+k}, \tag{3.25}$$

therefore we find $N = m(n+k)+k$, so $q = (n+k)$. The flavor central charge of $\mathrm{SU}(m)$ group is

$$m(n+k) + n - (n+k)\frac{m(n+k)+n}{n+k} = 0. \tag{3.26}$$

Physically, we interpret that this result implies that the $\mathrm{U}(m)$ flavor symmetry is decoupled in the IR 4d SCFT. These two configuration defines the same 4d SCFT in the IR (One can check that they give the same Coulomb branch spectrum), and we have the following equivalence of VOAs,

$$V^{-n+\frac{n}{k+n}}(\mathfrak{sl}_n) = W^{-m(n+k)-n+\frac{m(n+k)+n}{n+k}}(\mathfrak{sl}_{m(n+k)+n}, [(n+k)^m, 1^n]). \tag{3.27}$$

Mathematically, this means that the W algebra collapses to its affine subalgebra [74, 75].

We could generalize the above collapsing story as follows. Consider more general theory engineered by following data,

$$\mathfrak{g} = \mathfrak{sl}_{n_1+n}, \qquad \Phi = \frac{T}{z^{2+\frac{k}{n}}}, \quad f = [1^{n_1+n}]. \tag{3.28}$$

[9]This part is motivated by a question by T. Arakawa.

Again $T = \mathrm{diag}(I_{n\times n}, 0^{n_1})$. This theory has flavor symmetry $\mathrm{U}(n_1)\times\mathrm{SU}(n+n_1)$. We found that its VOA is a W algebra given by the nilpotent orbit $[(n+k-1)^{n_1}, 1^{n+n_1}]$. However, the theory can be engineered by the following configuration as well,

$$\mathfrak{g} = \mathfrak{sl}_{(m+n_1)(n+k)+n}, \qquad \Phi = \frac{T}{z^{2+\frac{m}{N}}}, \quad f = [(n+k)^m, (n+k-1)^{n_1}, 1^{n+n_1}]. \tag{3.29}$$

We have $N = (m+n_1)(n+k)+n$ and $N+m = n+k$. The VOA for above configuration is

$$W^{k'}(\mathfrak{sl}_N, [(n+k)^m, (n+k-1)^{n_1}, 1^{n+n_1}]), \qquad k' = -N + \frac{N}{n+k}. \tag{3.30}$$

This configuration has the naive flavor symmetry $\mathrm{U}(m)\times\mathrm{U}(n_1)\times\mathrm{SU}(n+n_1)$. The flavor central charge for the $\mathrm{U}(m)$ flavor group is zero, and therefore is decoupled in the IR. The above W algebra (3.30) is therefore collapsed to VOA_A defined in (3.9).

3.2 Classical Lie algebra

We now discuss how to generaize the above construction to other Lie algebras. The idea is similar: we consider AD matters engineered from one 6d realization whose VOA is not known, then we find an equivalent realization whose VOA can be read from results in section 2.

3.2.1 $D_N = \mathfrak{so}_{2N}$: $\mathrm{SO}(2N)\times\mathrm{Sp}(n_1)$ flavor symmetry

We start with AD theories engineered from 6d D_N $(2,0)$ theory. The regular puncture of D_N theory is classified by nilpotent orbits of $\mathfrak{so}_{2N}$ Lie algebra and is labeled by a size $2N$ Young tableaux whose *even* parts has *even* multiplicities. Given a Young tableau $[r_1^{h_1}, r_2^{h_2}, \ldots]$, the flavor symmetry is

$$G_F = \prod_{h_i\ \text{even}} \mathrm{Sp}(r_i) \times \prod_{h_i\ \text{odd}} \mathrm{SO}(r_i). \tag{3.31}$$

Given the puncture $[m^q, 1^s]$ with (q,m,s) all even, the flavor symmetry is $\mathrm{Sp}(q)\times\mathrm{SO}(s)$ whose central charges are

$$k_{\mathrm{SO}(s)} = (s+q-2) - [z], \quad k_{\mathrm{Sp}(q)} = \frac{s+mq-m[z]}{2}. \tag{3.32}$$

Here $[z]$ is the scaling dimension of z coordinate in the spectral curve of D_N type.

Now consider following configuration,

$$\mathfrak{g} = \mathfrak{so}(n+n_1+2), \quad \Phi = \frac{T}{z^{2+\frac{k}{n}}}, \quad f = [1^{n+n_1+2}], \tag{3.33}$$

with $(k,n) = 1$, and T takes a specific form, see [43]. We choose n to be even and consider the irregular singularity with which no flavor symmetry is associated. This also implies that $n+k$ is odd. See figure 8 for its corresponding Newton polygon. The flavor symmetry is $\mathrm{Sp}(n_1)\times\mathrm{SO}(n+n_1+2)$. The flavor central charges of them are

$$k_{\mathrm{SO}(2N)} = n+n_1 - \frac{n}{n+k}, \quad k_{\mathrm{Sp}(n_1)} = \frac{n_1+2}{2} + \frac{n}{2(n+k)}. \tag{3.34}$$

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The same theory can be engineered by following configuration,

$$\mathfrak{g} = \mathfrak{so}_{n_1(n+k)+n+2}, \quad \Phi = \frac{T}{z^{2+\frac{m}{2N-2}}}, \quad f = [(n+k-1)^{n_1}, 1^{n+n_1+2}]. \tag{3.35}$$

We have $2N = n_1(n+k)+n+2$ and $2N-2+m = n+k$. It has the same Coulomb branch with realization (3.33). Using $[z] = \frac{2N-2}{n+k}$ and equation (3.32), flavor central charges of the $\mathrm{SO}(n+n_1+2) \times \mathrm{Sp}(n_1)$ flavor groups are

$$\begin{aligned} k_{\mathrm{SO}(n+n_1+2)} &= n + 2n_1 - \frac{n_1(n+k)+n}{n+k} = n + n_1 - \frac{n}{n+k}, \\ k_{\mathrm{Sp}(n_1)} &= \frac{1}{2}\left[n_1(n+k)+n+2-(n+k-1)\frac{n_1(n+k)+n}{n+k}\right] = \frac{1}{2}\left(n_1+2+\frac{n}{n+k}\right), \end{aligned} \tag{3.36}$$

which is exactly the result found in other description (3.33) (cf. equation (3.34)).

The VOA for the realization (3.35) (See section 2) is

$$\boxed{\begin{aligned} &\mathrm{VOA}_B = W^{k'}(\mathfrak{so}_{n_1(n+k)+n+2}, [(n+k-1)^{n_1}, 1^{n+n_1+2}]), \\ &k' = -[n_1(n+k)+n] + \frac{n_1(n+k)+n}{n+k}. \end{aligned}} \tag{3.37}$$

Following equation (C.20), the central charge of VOA_B is

$$c(VOA_B) = -\frac{1}{2}(n+1)(n+2)(n+k-1) - \frac{1}{2}n_1(n+n_1+2)(3k+3n-2). \tag{3.38}$$

It is then the corresponding VOA of AD theory (3.33).

In previous discussions, we require n to be even. We can also consider the case where n is odd. The AD matter with two non-abelian flavor symmetries are given by following configuration,

$$\mathfrak{g} = \mathfrak{so}_{2n+n_1}, \quad \Phi = \frac{T}{z^{2+\frac{2k}{2n}}}, \quad f = [1^{2n+n_1}]. \tag{3.39}$$

Here n_1 is even and n is odd, and $T = diag(I_{2n\times 2n}, 0^{n_1})$, see [43] for the specific form of diagonal matrix $I_{2n\times 2n}$. We also require $(n,k) = 1$. The flavor symmetry is $\mathrm{SO}(n_1) \times \mathrm{SO}(2n+n_1) \times \mathrm{SO}(2)$. Unlike the previous case, there is an extra $\mathrm{SO}(2)$ flavor symmetry besides two simple flavor groups. The same theory can be described by following configuration,

$$\mathfrak{g} = \mathfrak{so}_{2N}, \quad \Phi = \frac{T}{z^{2+\frac{2k'}{2N}}}, \quad f = [(2n+2k-1)^{n_1}, 1^{2n+n1}]. \tag{3.40}$$

We have $N = n_1(n+k-1)+n+n_1$ and $N+k' = n+k$, hence

$$k' = k - (k+n)n_1\,. \tag{3.41}$$

Here N is odd, and there is a $\mathrm{SO}(2)$ flavor symmetry in irregular singularity. Unfortunately, we do not know the VOA for this configuration yet.

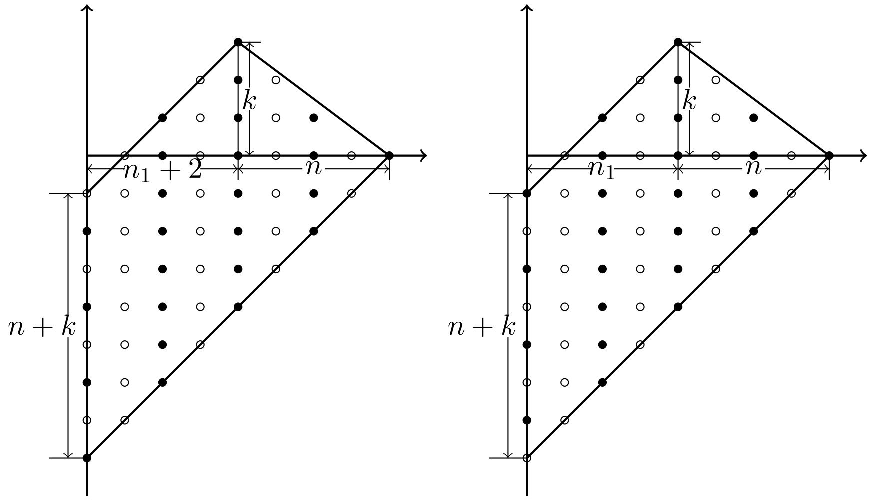

Figure 8. Newton polygon for D-type and twisted D-type theories: there is a Coulomb branch parameter associated with each black lattice point of the Newton polygon. Unlike the A type case, the lattice points on odd x (horizontal) axis is deleted. The difference for untwisted and twisted case is that on $x=0$ axis: even points are kept for D type theory while odd points are kept for twisted D type theory.

3.2.2 Twisted $D_N = \mathfrak{so}_{2N}$ theory: $\mathrm{Sp}(2N-2)\times \mathrm{SO}(n_1)$ flavor symmetry

The regular puncture of twisted D_N theory is classified by nilpotent orbits of $\mathfrak{sp}_{2N-2}$ Lie algebra and is labeled by a $2N-2$ size Young tableaux whose *odd* parts has *even* multiplicities. Given a Young tableau $[r_1^{h_1}, r_2^{h_2}, \ldots]$, the flavor symmetry is

$$G_F = \prod_{h_i \text{ even}} \mathrm{SO}(r_i) \times \prod_{h_i \text{ odd}} \mathrm{Sp}(r_i). \tag{3.42}$$

Given the partition $[m^q, 1^s]$ with (q,m,s) even, the flavor groups are $\mathrm{SO}(q)\times \mathrm{Sp}(s)$ with following central charge:

$$k_{\mathrm{Sp}(s)} = \frac{(s+q+2)}{2} - \frac{[z]}{2}, \quad k_{\mathrm{SO}(q)} = s + qm - m[z]. \tag{3.43}$$

here $[z]$ is the scaling dimension of the coordinate z in spectral curve of twisted D_N theory.

Next consider the twisted theory with the following data,

$$\mathfrak{g} = \mathfrak{so}(n+n_1)_{\mathbb{Z}_2}, \quad \Phi = \frac{T^t}{z^{2+\frac{k}{n}}}, \quad f = [1^{n+n_1-2}]. \tag{3.44}$$

We take $(n,k)=1$ and n is even so that there is no flavor symmetry associated with the irregular part (this implies that $n+k$ is odd), and figure 8 illustrates its Newton polygon. The flavor symmetry is $\mathrm{SO}(n_1)\times \mathrm{Sp}(n+n_1-2)$, and flavor central charges are

$$k_{\mathrm{Sp}(n+n_1-2)} = \frac{n+n_1}{2} - \frac{n}{2(n+k)}, \quad k_{\mathrm{SO}(n_1)} = n_1 - 2 + \frac{n}{n+k}. \tag{3.45}$$

To find its VOA, we realize that there is another equivalent description:

$$\mathfrak{g} = (\mathfrak{so}_{n_1(n+k)+n})_{\mathbb{Z}_2}, \quad \Phi = \frac{T^t}{z^{2+\frac{m}{2N}}}, \quad f = [(n+k-1)^{n_1}, 1^{n+n_1-2}], \tag{3.46}$$

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and we have $2N = n_1(n+k)+n$ and $2N+m = n+k$. The flavor central charge for flavor groups are computed by using the fact $[z] = \frac{2N}{n+k}$ and equation (3.43),

$$\begin{aligned} k_{\mathrm{Sp}(n+n_1-2)} &= \frac{n+2n_1}{2} - \frac{n_1(n+k)+n}{2(n+k)} = n+n_1 - \frac{n}{2(n+k)}, \\ k_{\mathrm{SO}(n_1)} &= n_1(n+k)+n-2-(n+k-1)\frac{n_1(n+k)+n}{n+k} = n_1 - 2 + \frac{n}{n+k}. \end{aligned} \tag{3.47}$$

which are the same as results from other description, see (3.45). One can also check that the two configurations have the same Coulomb branch spectrum. Therefore the corresponding VOA is the W algebra

$$\boxed{\begin{aligned} &\mathrm{VOA}_C = W^{k'}(sp_{n_1(n+k)+n-2}, [(n+k-1)^{n_1}, 1^{n+n_1-2}]), \\ &k' = -\frac{n_1(n+k)+n}{2} + \frac{n_1(n+k)+n}{2(n+k)}\,. \end{aligned}} \tag{3.48}$$

From (C.30) the central charge of VOA_C is

$$c(\mathrm{VOA}_C) = -\frac{1}{2}(n-2)(n-1)(n+k-1) - \frac{1}{2}n_1(n+n_1-2)(3k+3n-2). \tag{3.49}$$

3.2.3 Twisted $\mathfrak{sl}_{2N}$ theories: SO$(2N+1) \times$ Sp(n_1) flavor symmetry

Let's now consider twisted $\mathfrak{sl}_{2N}$ theory from which we can get SO_{2N+1} flavor symmetry. The regular puncture is labeled by a Young tableaux with size $2N+1$, and the constraint is that *even* parts has *even* multiplicities. Given a Young tableaux $[r_1^{h_1}, r_2^{h_2}, \ldots]$, and the flavor symmetry is

$$G_F = \prod_{h_i \text{ even}} \mathrm{Sp}(r_i) \times \prod_{h_i \text{ odd}} \mathrm{SO}(r_i). \tag{3.50}$$

We are interested in punctures like $[m^q, 1^s]$, with m even, and (q,s) odd. The flavor symmetry is SO$(s) \times$ Sp(q) with following central charge:

$$k_{\mathrm{SO}(s)} = s+q-2-[z]/2, \quad k_{\mathrm{Sp}(q)} = \frac{s+mq-m[z]/2}{2}. \tag{3.51}$$

Here $[z]$ is the scaling dimension of z coordinate in spectral curve of twisted sl_{2N} type.

The defining data for our theory is,

$$\mathfrak{g} = (sl_{n+n_1+1})_{z_2}, \quad \Phi = \frac{T^t}{z^{2+\frac{k+1/2}{n}}}, \quad f = [1^{n+n_1+2}]. \tag{3.52}$$

Here n is odd and n_1 is even (cf. figure 9). The flavor symmetry is SO$(n+n_1+2) \times$ Sp(n_1) with following flavor central charges,

$$k_{\mathrm{SO}(n+n_1+2)} = n+n_1 - \frac{n}{2n+2k+1}, \quad k_{\mathrm{Sp}(n_1)} = \frac{n_1+2}{2} + \frac{n}{2(2n+2k+1)}. \tag{3.53}$$

The VOA is found by identifying following equivalent configuration,

$$\mathfrak{g} = (\mathfrak{sl}_{n_1(2n+2k+1)+n+1})_{z_2}, \quad \Phi = \frac{T^t}{z^{2+\frac{m+\frac{1}{2}}{2N-1}}}, \quad f = [(2n+2k)^{n_1}, 1^{n+n_1+2}]. \tag{3.54}$$

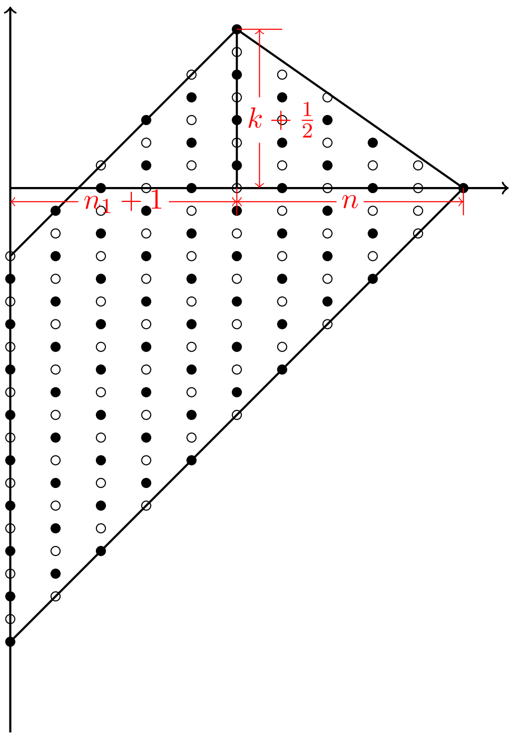

Figure 9. Newton polygon for twisted sl_{2N} theory. The integral points on $x = even$ axis are kept, while the half integral points on odd $x = odd$ axis are kept, here x is the horizonal coordinate.

with $2N = n_1(2n+2k+1)+n+1$ and $2N-1+m+\frac{1}{2} = n+k+\frac{1}{2}$. To compute the flavor central charges for $\mathrm{SO}(n+n_1+2)\times \mathrm{Sp}(n_1)$ flavor symmetry, use the fact that $[z]/2 = \frac{n_1(2n+2k+1)+n}{2n+2k+1}$,

$$\begin{aligned} k_{So(n+n_1+2)} &= 2n_1+n-\frac{n_1(2n+2k+1)+n}{2n+2k+1} = n+n_1-\frac{n}{2n+2k+1}, \\ k_{\mathrm{Sp}(n_1)} &= \frac{1}{2}\left[n_1(2n+2k+1)+n+2-(2n+2k)\frac{n_1(2n+2k+1)+n}{2n+2k+1}\right] = \frac{1}{2}\left(n_1+2+\frac{n}{2n+2k+1}\right), \end{aligned} \tag{3.55}$$

which are exactly the same as that computed in other description (3.53). One can also check that the two configurations give the same Coulomb branch spectrum. Hence the corresponding W algebra is

$$\boxed{\begin{aligned} &\mathrm{VOA}_D = W^{k'}(so_{n_1(2n+2k+1)+n+2}, [(2n+2k)^{n_1}, 1^{n+n_1+2}]), \\ &\quad k' = -(n_1(2n+2k+1)+n)+\frac{n_1(2n+2k+1)+n}{2n+2k+1}. \end{aligned}} \tag{3.56}$$

Using equation (C.25), the central charge is

$$c(\mathrm{VOA}_D) = -(n+1)(n+2)(k+n) - \frac{1}{2}n_1(n+n_1+2)(6k+6n+1). \tag{3.57}$$

3.2.4 Twisted $\mathfrak{sl}_{2N+1}$ theories: Sp$(2N) \times$ SO(n_1+1) flavor symmetry

Now consider twisted $\mathfrak{sl}_{2N+1}$ theory from which we can also get Sp$(2N)$ flavor symmetry, but this time we will also get another B type flavor symmetry, which is different from twisted D type theory. The twisted regular puncture is classified by a Young tableaux with size $2N$. Given a Young tableau $[r_1^{h_1}, r_2^{h_2}, \dots]$, the flavor symmetry is,

$$G_F = \prod_{h_i \text{ even}} \mathrm{SO}(r_i) \times \prod_{h_i \text{ odd}} \mathrm{Sp}(r_i). \tag{3.58}$$

Given the partition $[m^q, 1^s]$ with (q, m, s) even, the flavor groups are $\mathrm{SO}(q) \times \mathrm{Sp}(s)$ with following central charge:

$$k_{\mathrm{Sp}(s)} = \frac{(s+q+2)}{2} - \frac{[z]}{4}, \quad k_{\mathrm{SO}(q)} = s + qm - m[z]/2. \tag{3.59}$$

Here $[z]$ is the scaling dimension of the z coordinate in spectral curve of twisted sl_{2N+1} theory.

Consider a theory defined by following data, here we use the z_2 outerautomorphism of sl_{n+n_1} theory:

$$\mathfrak{g} = (\mathfrak{sl}_{n+n_1})_{z_2}, \quad \Phi = \frac{T}{z^{2+\frac{k+1/2}{n}}}, \quad f^t = [1^{n+n_1-1}]. \tag{3.60}$$

Here n is odd and n_1 is even (cf. figure 10). The flavor symmetry is $\mathrm{Sp}(n+n_1-1)\times\mathrm{SO}(n_1+1)$ and flavor central charges are,

$$k_{\mathrm{Sp}(n+n_1-1)} = \frac{n+n_1+1}{2} - \frac{n}{2(2n+2k+1)}, \quad k_{\mathrm{SO}(n_1+1)} = n_1 - 1 + \frac{n}{2n+2k+1}. \tag{3.61}$$

For above theory, we find following equivalent realization,

$$\mathfrak{g} = (\mathrm{so}_{2N})_{z_2}, \quad \Phi = \frac{T^t}{z^{2+\frac{2m+1}{2N}}} + \dots, \quad f^t = [1^{2N-2}]. \tag{3.62}$$

Here $2N = (n_1+1)(2n+2k+1)+n$, and $2N+2m+1 = 2n+2k+1$. Notice that we found a realization from a *different* type of twisted theory! The flavor central charges for $\mathrm{Sp}(n+n_1-1) \times \mathrm{SO}(n_1+1)$ are (here $\frac{[z]}{2} = \frac{(n_1+1)(2n+2k+1)+n}{2(2n+2k+1)}$)

$$\begin{aligned} k_{\mathrm{Sp}(n+n_1-1)} &= \frac{1}{2}(n+2n_1+2) - \frac{1}{2}\frac{(n_1+1)(2n+2k+1)+n}{2n+2k+1} = \frac{n+n_1+1}{2} - \frac{n}{2(2n+2k+1)}, \\ k_{So(n_1+1)} &= (n_1+1)(2n+2k+1)+n-2-(2n+2k)\frac{(n_1+1)(2n+2k+1)+n}{2n+2k+1} \\ &= n_1 - 1 + \frac{n}{2n+2k+1}, \end{aligned} \tag{3.63}$$

which agrees with the result shown in (3.61). The corresponding VOA is following W algebra:

$$\boxed{\begin{aligned} &\mathrm{VOA}_E = W^{k'}(\mathfrak{sp}_{(n_1+1)(2n+2k+1)+n-2}, [(2n+2k)^{n_1+1}, 1^{n+n_1-1}]), \\ &\quad k' = -\frac{(n_1+1)(2n+2k+1)+n}{2} + \frac{(n_1+1)(2n+2k+1)+n}{2(2n+2k+1)}. \end{aligned}} \tag{3.64}$$

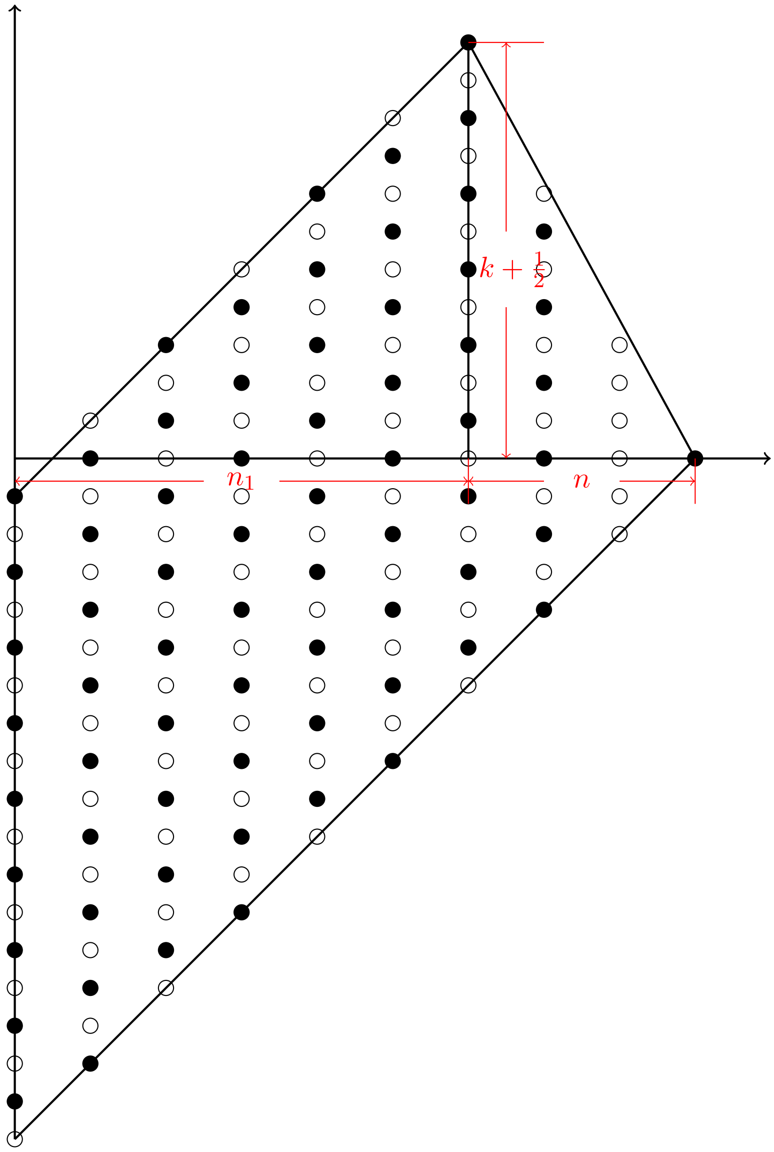

Figure 10. Newton polygons for twisted Sl_{2N+1} theory. The integral points on $x =$ odd axis are kept, while the half integral points on $x =$ even axis are kept, and here x is the horizonal coordinate.

Using equation (C.30), the central charge of VOA_E is

$$c(\mathrm{VOA}_E) = -\frac{1}{2}(n-1)[2n^2+2(k+1)(n+1)-1] - \frac{1}{2}n_1(n+n_1)(6k+6n+1). \quad (3.65)$$

3.2.5 Conformal embedding

In previous discussions, we have found W algebras corresponding to AD matters with two distinct type of non-abelian flavor symmetries. The 4d flavor symmetries give the AKM subalgebras of 2d VOAs with $k_{2d} = -k_{4d}$, so we obtain following embeddings of AKM algebra into W algebra:

$$\begin{aligned}
&A: \ \mathfrak{u}(1)\times V_{-n_1-\frac{n}{n+k}}(\mathfrak{su}_{n_1})\times V_{-n_1-n+\frac{n}{n+k}}(\mathfrak{su}_{n_1+n})\subset \mathrm{VOA}_A,\\
&B: \ V_{-\frac{n_1+2}{2}-\frac{n}{2(n+k)}}(\mathfrak{sp}_{n_1})\times V_{-n-n_1+\frac{n}{n+k}}(\mathfrak{so}_{n+n_1+2})\subset \mathrm{VOA}_B,\\
&C: \ V_{-n_1+2-\frac{n}{n+k}}(\mathfrak{so}_{n_1})\times V_{-\frac{n+n_1}{2}+\frac{n}{2(n+k)}}(\mathfrak{sp}_{n+n_1-2})\subset \mathrm{VOA}_C,\\
&D: \ V_{-\frac{n_1+2}{2}-\frac{n}{2(2n+2k+1)}}(\mathfrak{sp}_{n_1})\times V_{-n-n_1+\frac{n}{2n+2k+1}}(\mathfrak{so}_{n+n_1+2})\subset \mathrm{VOA}_D,\\
&E: \ V_{-n_1+1-\frac{n}{2n+2k+1}}(\mathfrak{so}_{n_1+1})\times V_{-\frac{n+n_1+1}{2}+\frac{n}{2(2n+2k+1)}}(\mathfrak{sp}_{n+n_1-1})\subset \mathrm{VOA}_E.
\end{aligned}\tag{3.66}$$

The righthandside are following W algebras:

$$\begin{aligned}
\mathrm{VOA}_A: \quad & W^{-h^\vee+\frac{h^\vee}{n+k}}(\mathfrak{sl}_{n_1(n+k)+n},[(n+k-1)^{n_1},1^{n+n_1}]),\\
&\qquad\qquad h^\vee=n_1(n+k)+n,\\
\mathrm{VOA}_B: \quad & W^{-h^\vee+\frac{h^\vee}{n+k}}(\mathfrak{so}_{n_1(n+k)+n+2},[(n+k-1)^{n_1},1^{n+n_1+2}]),\\
& n \text{ even},\ n_1 \text{ even},\quad h^\vee=n_1(n+k)+n,\\
\mathrm{VOA}_C: \quad & W^{-h^\vee+\frac{h^\vee}{n+k}}(\mathfrak{sp}_{n_1(n+k)+n-2},[(n+k-1)^{n_1},1^{n+n_1-2}]),\\
& n \text{ even},\ n_1 \text{ even},\quad h^\vee=\frac{n_1(n+k)+n}{2},\\
\mathrm{VOA}_D: \quad & W^{-h^\vee+\frac{h^\vee}{2n+2k+1}}(\mathfrak{so}_{n_1(2n+2k+1)+n+2},[(2n+2k)^{n_1},1^{n+n_1+2}]),\\
& n \text{ odd},\ n_1 \text{ even},\quad h^\vee=n_1(2n+2k+1)+n,\\
\mathrm{VOA}_E: \quad & W^{-h^\vee+\frac{h^\vee}{2n+2k+1}}(\mathfrak{sp}_{(n_1+1)(2n+2k+1)+n-2},[(2n+2k)^{n_1+1},1^{n+n_1-1}]),\\
& n \text{ odd},\ n_1 \text{ even},\quad h^\vee=\frac{(n_1+1)(2n+2k+1)+n}{2},
\end{aligned}\tag{3.67}$$

with explicit 2d central charges given by equations (3.10), (3.38), (3.49), (3.57) and (3.65),

$$\begin{aligned}
c(\mathrm{VOA}_A)&=-\left(n^2-1\right)(k+n-1)-n_1(n+n_1)(3k+3n-2),\\
c(\mathrm{VOA}_B)&=-\frac{1}{2}(n+1)(n+2)(n+k-1)-\frac{1}{2}n_1(n+n_1+2)(3k+3n-2),\\
c(\mathrm{VOA}_C)&=-\frac{1}{2}(n-2)(n-1)(n+k-1)-\frac{1}{2}n_1(n+n_1-2)(3k+3n-2),\\
c(\mathrm{VOA}_D)&=-(n+1)(n+2)(k+n)-\frac{1}{2}n_1(n+n_1+2)(6k+6n+1),\\
c(\mathrm{VOA}_E)&=-\frac{1}{2}(n-1)[2n^2+2(k+1)(n+1)-1]-\frac{1}{2}n_1(n+n_1)(6k+6n+1).
\end{aligned}\tag{3.68}$$

Recalling the central charge of AKM algebra $\mathfrak{g}_k$ is

$$c(\mathfrak{g}_k)=\frac{k\dim\mathfrak{g}}{k+h^\vee},\tag{3.69}$$

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with $h^\vee$ the dual Coxter number of $\mathfrak{g}$, one can show that central charges of AKM subalgebras on the left hand side of equation (3.66) are *equal* to central charges of W algebras on the right hand side. It would be interesting to check whether they are indeed conformal embeddings.

3.3 Exceptional Lie algebra

Now consider AD theories constructed using exceptional 6d $(2,0)$ theory. In general, we have a theory with flavor symmetry $E_n \times G$, where G is some subgroup of E_n. Unfortunately, we do not know how to realize the above theory using the configuration presented in section 2. Instead we analyze its Coulomb branch when a generic regular singularity is present, and check whether one can find the same Coulomb branch spectrum using constructions presented in section 2. We do not attempt to do a general analysis, and only give some examples here:

- Start with $\mathfrak{e}_8$ theory and look at the regular puncture whose Nahm label is $[A_1]$ [68], then the flavor symmetry is E_7. The flavor central charge is,

  $$k_{E_7} = 24 - \frac{30}{30+k}, \tag{3.70}$$

  We would like to realize this theory by using a 6d $(2,0)$ $\mathfrak{e}_7$ theory with a full puncture of E_7 type. The flavor central charge in $\mathfrak{e}_7$ construction is,

  $$k_{E_7} = 18 - \frac{18}{18+k'}, \tag{3.71}$$

  We find a solution with $30+k = 2 = 18+k' = 2$. Both theory has the Coulomb branch spectrum $[9,5,3]$ (these numbers can be derived using Newton polygon of E_8 type theory [43] and the pole structure of E_8 nilpotent orbit $[A_1]$ [76]). However the E_7 description has an extra U(1) flavor symmetry. The VOA is

  $$W^{-15}(\mathfrak{e}_8, [A_1])\,. \tag{3.72}$$

  Here $[A_1]$ denotes minimal nilpotent orbit of $\mathfrak{e}_8$ Lie algebra. The interpretation is that the $\mathfrak{e}_8$ description misses the U(1) flavor symmetry.

- Now look at $\mathfrak{e}_7$ theory with regular puncture whose Nahm label is $[A_1]$. The flavor symmetry is SO(12), and the flavor central charge is,

  $$k_{\mathrm{SO}(12)} = 18 - \frac{18}{18+k}. \tag{3.73}$$

  On the other hand, starting with the $\mathfrak{so}_{12}$ realization, the flavor central charge is,

  $$k_{\mathrm{SO}(12)} = 14 - \frac{10}{10+k'}. \tag{3.74}$$

  Again, we find solution $18+k = 18+k' = 2$. For this value, E_7 configuration has an extra U(1) flavor symmetry. The $\mathfrak{so}_{12}$ descrition also has an extra U(1) flavor symmetry. The Coulomb branch spectrum is [5, 3], which is read from Newton polygon. We can not find its VOA because of this extra U(1) flavor symmetry.

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- Finally we look at an $\mathfrak{e}_6$ theory with a regular puncture whose Nahm label is $[A_1]$. The flavor symmetry is SU(6), and the flavor central charge is,

$$k_{\mathrm{SO}(10)} = 9 - \frac{12}{12+k}. \tag{3.75}$$

  In an $\mathfrak{su}_6$ realization, the flavor central charge is,

$$k_{\mathrm{SU}(6)} = 6 - \frac{6}{6+k'}. \tag{3.76}$$

  The matching of flavor central charges $k_{\mathrm{SO}(10)} = k_{\mathrm{SU}(6)}$ requires $12 + k = 6 + k' = 2$. Now in the SU(6) description, there is an extra U(1) flavor symmetry, and it is just the flavor symmetry of the $\mathcal{N} = 2$ SU(3) superQCD (SQCD) with six fundamental hypermultiplets. The $\mathfrak{e}_6$ description has just SU(6) manifest flavor symmetry. The VOA is

$$W^{-6}(\mathfrak{e}_6, [A_1])\,. \tag{3.77}$$

  We claim that this W algebra is the VOA for SU(3) SQCD with six fundamental flavors. One simple check is that the central charge of this W algebra is -34 which is equal to $-12c_{4d}$, where $c_{4d} = 34/12$ is the central charge of SU(3) SQCD with six fundamental flavor. It is interesting to notice that there is an emerging U(1) flavor symmetry for the W algebra.

Now we move on to more interesting examples with exceptional flavor symmetries. Some rank one theories with following data were found in [77],

$$\begin{aligned}
&G = B_3, && k_{B_3} = 2, && u = [2], && V_{-2}(B_3),\\
&G = G_2, && k_{G_2} = 2, && u = [2], && V_{-2}(G_2),\\
&G = F_4, && k_{F_4} = 3, && u = [3], && V_{-3}(F_4).
\end{aligned} \tag{3.78}$$

The (a, c) central charges of B_3 and G_2 theory are the same as the $\mathcal{N} = 2$ SU(2) SQCD with four flavors. The F_4 theory also has the same (a, c) central charge as the E_6 Minahan-Nemeschansky theory [78]. There are interesting relations between corresponding VOAs of these 4d theories. In fact, following conformal embeddings were proven in [79],

$$V_{-2}(B_3) \subset V_{-2}(D_4), \quad V_{-2}(G_2) \subset V_{-2}(B_3), \quad V_{-3}(F_4) \subset V_{-3}(E_6). \tag{3.79}$$

It would be interesting to study further the relation between these VOAs and what they imply for 4d theories.

3.4 Comments on collapsing levels

We now make some remarks on collapsing levels of a W algebra into its affine piece, namely finding the proper nilpotent orbit f of the Lie algebra $\mathfrak{g}$ and level k' such that the following equivalence between two VOAs holds,

$$W^{k'}(\mathfrak{g}, f) = V_k(\mathfrak{g}'). \tag{3.80}$$

We determine f and k' by matching the Coulomb branch spectrum and other data of corresponding 4d theories. In practice, important insights can be gained if one first requires that flavor central charges should be equal for two descriptions. First consider some examples in detail. The $(2,0)$ configuration is the following,

$$g', \quad \Phi = \frac{T}{z^{2+k/h^\vee}}, \quad f = f_{\text{trivial}}, \tag{3.81}$$

where k is integer-valued for $g' = \text{ADE}$, and half-integral valued for twisted theories. k is restricted such that there is no flavor symmetry associated with the irregular singularity.

- Consider $\mathfrak{g}' = \mathfrak{so}_{2N}$, and the 4d flavor central charge $-k = (2N-2) - \frac{2N-2}{k+2N-2}$. To find a W algebra whose 4d partner has the same Coulomb branch spectrum, we consider $\mathfrak{so}_{qm+2N}$ theory with the regular singularity $f = [q^m, 1^{2N}]$. Naively, this puncture has the flavor symmetry $\mathrm{SO}(2N) \times \mathrm{SO}(m)$. The 4d flavor central charge $-k'$ of $\mathrm{SO}(2N)$ flavor group in this description is

$$-k' = m + 2N - 2 - \frac{qm + 2N - 2}{2N - 2 + k}. \tag{3.82}$$

Matching $-k'$ with $-k = (2N-2) - \frac{2N-2}{k+2N-2}$ leaves the requriement

$$q = 2N - 2 + k, \tag{3.83}$$

which is odd, so there might be the following equality of VOAs,

$$W^{-h^\vee + \frac{h^\vee}{2N-2+k}}(\mathfrak{so}_{m(2N-2+k)+2N}, [(2N-2+k)^m, 1^{2N}]) = V_{-(2N-2)+\frac{2N-2}{2N-2+k}}, (\mathfrak{so}_{2N}). \tag{3.84}$$

with $h^\vee = m(2N-2+k) + 2N - 2$. The flavor central charge for the $\mathrm{SO}(m)$ flavor symmetry is then,

$$m(2N-2+k) + 2N - 2 - (2N-2+k)\frac{m(2N-2+k) + 2N - 2}{2N - 2 + k} = 0, \tag{3.85}$$

which implies that the $\mathrm{SO}(m)$ flavor symmetry is decoupled in the IR. One can check that above two configurations give the same Coulomb branch spectrum.

- Given $\mathfrak{g}' = \mathfrak{sp}_{2N-2}$, one has the following collapsing levels and nilpotent orbits,

$$W^{-h^\vee + \frac{h^\vee}{2N+2k+1}}(\mathfrak{sp}_{m(2N+2k+1)+2N-2}, [(2N+2k+1)^m, 1^{2N-2}]) = V_{-N+\frac{N}{2N+2k+1}}, (\mathfrak{sp}_{2N-2}) \tag{3.86}$$

and $h^\vee = \frac{m(2N+2k+1)+2N}{2}$.

- Taking $\mathfrak{g}' = \mathfrak{so}_{2N+1}$, one finds the following collapsing levels and nilpotent orbits,

$$W^{-h^\vee + \frac{h^\vee}{4N+2k-1}}(\mathfrak{sp}_{m(4N+2k-1)+2N+1}, [(4N+2k-1)^m, 1^{2N+1}]) = V_{-(2N-1)+\frac{2N-1}{4N+2k-1}}(\mathfrak{so}_{2N+1}), \tag{3.87}$$

with $h^\vee = m(4N + 2k - 1) + 2N - 1$.

In above cases, we can choose more general puncture on AKM side, and we then have collapsing of one W algebra into another W algebra.

We interpret our results as follows. Taking $\mathfrak{g}=\mathfrak{sl}_N$ (other cases are similar) and the irregular singularity,

$$\mathfrak{g}=\mathfrak{sl}_N, \quad \Phi=\frac{T}{z^{2+\frac{k}{N}}}, \quad f \text{ generic}, \tag{3.88}$$

the Seiberg-Witten curve takes the following form

$$x^N+\sum_{i=2}\phi_i(z)x^{N-i}=0. \tag{3.89}$$

If there is no Coulomb branch operators in ϕ_N, it is possible to find a $(2,0)$ realization with a lower rank Lie algebra. Assume we have a generic regular puncture, and denote $h_N(f)$ the height of the Nth box in Young tableaux f with $h_N\leq N$. Denote u as the Coulomb branch operator in ϕ_N, then its scaling dimension is $[u]=N-\frac{h_N N}{k+N}$. To have a reduced theory (no Coulomb branch operators in $\phi_N(z)$), $[u]$ should have scaling dimension less or equal to one, so ϕ_N is zero and the above is factorized as,

$$x\left(x^{N-1}+\sum_{i=2}\phi_i(z)x^{N-1-i}\right)=0. \tag{3.90}$$

Then it is possible to find a description with lower rank $(2,0)$ theory. The constraint on h_N is then,

$$[u]=N-\frac{h_N N}{k+N}\leq 1\rightarrow h_N\geq\frac{(-1+N)(k+N)}{N}. \tag{3.91}$$

Since $h_N\leq N$, the above equation has solution if $k<0$. So if we have following situation

$$W^{k'}(\mathfrak{sl}_N,f), \quad k'=-N+\frac{N}{k+N}, \quad k<0. \tag{3.92}$$

Then if $h_N(f)\geq\frac{(-1+N)(k+N)}{N}$, there is a collapsing of $\mathfrak{sl}_N$ type W algebra into a $\mathfrak{sl}_{N'}$ type W algebra with $N'<N$.

4 The Higgs branch

The Coulomb branch spectrum of theories studied above can be found from Newton polygon, their dimensions are listed here for later uses ($ABCDE$ label theories studied in section 3),

$$\begin{aligned}
A:\ n_C&=\frac{(n+k-1)(2n_1+n-1)}{2},\\
B:\ n_C&=\frac{(n+k-1)(2n_1+2+n)}{4},\\
C:\ n_C&=\frac{(n+k-1)(2n_1-2+n)}{4},\\
D:\ n_C&=\frac{(n+k)(2n_1+n+1)}{2},\\
E:\ n_C&=\frac{(n+k)(2n_1+n-1)}{2}.
\end{aligned} \tag{4.1}$$

We already know the flavor symmetry on the Higgs branch of 4d theories we studied in this paper, in this section we will use the associated variety of their corresponding VOAs to learn the Higgs branch chiral ring of these theories.

4.1 The Higgs branch as the associated variety of the VOA

The W algebras appear in section 3 take following form, $W^{k'}(\mathfrak{g}, f)$ with $f = [m^q, 1^s]$. The associated variety of the above W algebra is given by following formula [80],

$$S_f \cap X_M. \tag{4.2}$$

Here S_f is the Slowdoy slice associated with the nilpotent orbit f, and X_M is the associated variety of the affine vertex operator algebra $\mathfrak{g}$ with the level k'. If the level k' is admissible, the associated variety of AKM is found in [80]. We list the result below (we only show result for $k > 2$ here, interested readers can work out the general case):

$$\begin{aligned}
A: &\quad X_M = [(n+k)^{n_1}, n],\\
B: &\quad X_M = [(n+k)^{n_1}, n+2],\\
C: &\quad X_M = [(n+k)^{n_1}, n-2],\\
D: &\quad X_M = [(2n+2k+1)^{n_1}, n+2],\\
E: &\quad X_M = [(2n+2k+1)^{n_1+1}, n-2].
\end{aligned} \tag{4.3}$$

Here X_M is the nilpotent orbit specified by the listed partition. In the following, we will describe these Higgs branches in some detail.

4.2 The Higgs branch as a quiver variety

Let us compactify our 4d theory on a circle and flow to deep IR to get a 3d $\mathcal{N} = 4$ SCFT. The Higgs branch of the 3d theory is the same as the 4d theory, so it is described by the associated variety of the corresponding VOA. Meanwhile the 3d theory has a Coulomb branch which is also a hyperkahler manifold.

The 3d $\mathcal{N} = 4$ theory has an interesting mirror symmetry: there is a mirror theory B whose Higgs branch is the Coulomb branch of theory A, and vice versa. If the mirror theory has a Lagrangian description, its Higgs branch is described by the classical hyperkahler quotient. Let's look at our class A theory whose VOA is:

$$W^{-n_1(n+k)-n+\frac{n_1(n+k)+n}{n+k}}(\mathfrak{sl}_{n_1(n+k)+n}, [(n+k-1)^{n_1}, 1^{n+n_1}]) \,. \tag{4.4}$$

For the special case $k = 1$, we conjecture that the mirror theory is given by the quiver in figure 11. The simple counting of dimensions of Coulomb and Higgs branch of mirror quiver is:

$$n_H = nn_1 + \frac{1}{2}n(n-1), \quad n_C = \frac{1}{2}n(n-1) + n_1(n+n_1). \tag{4.5}$$

The dimension n_H of 3d mirror is equal to the Coulomb branch dimension of original 4d theory, see (4.1).

In the following, we use the methods proposed in [81] to describe the Higgs branch as a quiver variety. Considering a nilpotent orbit $f = [(n+k-1)^{n_1}, 1^{n+n_1}]$ and X_M with

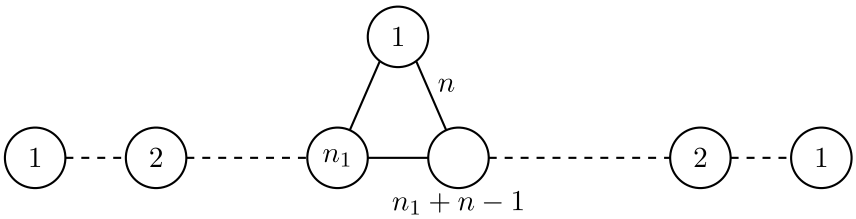

Figure 11. 3d mirror for 4d theory whose associated VOA is (3.9) with $k = 1$. From this quiver, one can actually read another Hitchin system description which uses type III irregular singularity as discussed in [26], and we check the Coulomb branch spectrum which is the same as the one found using the construction in section 3.

$M = [(n+k)^{n_1}, n]$, we would like to find the variety of $S_f \cap X_M$. Firstly we need the transpose of M, $M^t = [(n_1+1)^n, (n_1)^k]$. Using f and M^t, one can form a $D5-NS5-D3$ systems [81] as shown in figure 12. After moving $D5$ branes according to rules in [81], one obtain an equivalent brane configuration shown in figure 13 which leads to the quiver gauge theory of $S_f \cap X_M$ as shown in figure 14. The dimension of $S_f \cap X_M$ can be easily read from the Higgs branch of the quiver 14,

$$\dim(S_f \cap X_M) = n_H = \frac{1}{2}n(n-1) + n_1(n+n_1), \tag{4.6}$$

which is actually independent of k.

5 VOA for theories with exact marginal deformations

If a four dimensional $\mathcal{N} = 2$ SCFT has exact marginal deformations, then it is possible to write down a weakly coupled gauge theory description which typically looks like figure 15. If we know the VOA V_i for each matter (which should have an affine vertex subalgebra $V_{k_i}(G_i)$), the VOA for the parent theory is given by the following coset,

$$V_0 = \frac{V_1 \oplus V_2 \oplus \ldots}{(G_1)_{-2h_1^\vee} \oplus (G_2)_{-2h_2^\vee} \oplus \ldots}. \tag{5.1}$$

We use the fact that the conformal gauging condition implies that the sum of levels from the matter gauged by gauge group G_i is $-2h_i^\vee$. If there are more than one weakly coupled gauge theory descriptions, then we have found equivalence between non-trivial coset constructions.

5.1 Weakly coupled gauge theory descriptions for AD theories

Consider AD theory engineered using following data,

$$\mathfrak{g}, \quad \Phi = \frac{T}{z^{2+\frac{qk}{qn}}}, \quad f, \tag{5.2}$$

where (k, n) is coprime, and T is taken to be the most general matrix allowed by the grading. The corresponding AD theory often has exact marginal deformations, and its weakly coupled gauge theory description is found in [42, 43]. Here we summarize the basic ideas:

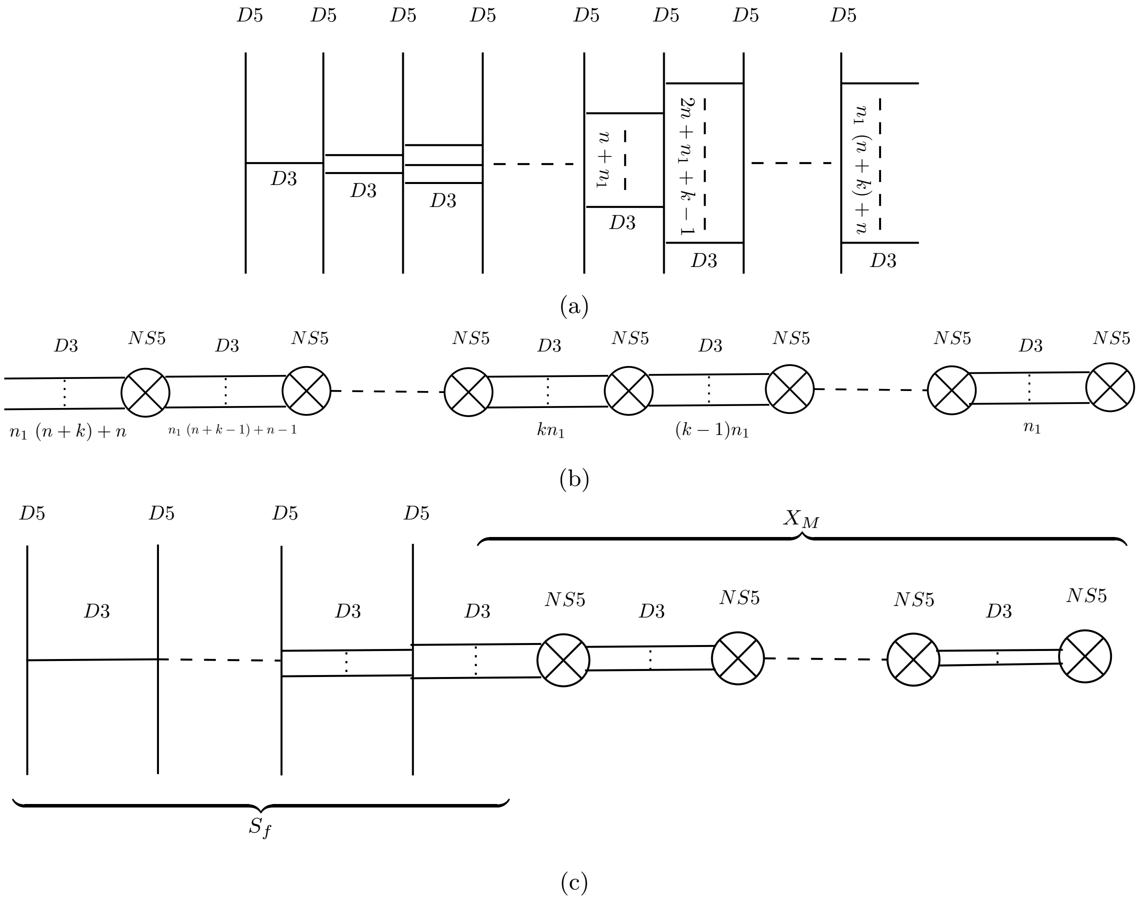

Figure 12. (a): the brane construction of S_f with $f = [(n+k-1)^{n_1}, 1^{n+n_1}]$. The number of $D3$ branes between i-th and $i+1$-th $D5$ branes are $\sum_{j=n+2n_1-i+1}^{n+2n_1} f_j$. (b): the brane construction of X_M with $M^t = [(n_1+1)^n, (n_1)^k]$. The number of $D3$ branes between i-th and $i+1$-th $NS5$ branes is $n_1(n+k)+n-\sum_{j=1}^{i} M_j^t$. (c): schematics of brane construction of $S_f \cap X_M$ which just connects (a) and (b). One can connect these two brane configurations because the total number of boxes of f and M are the same, therefore the same amount of $D3$ branes.

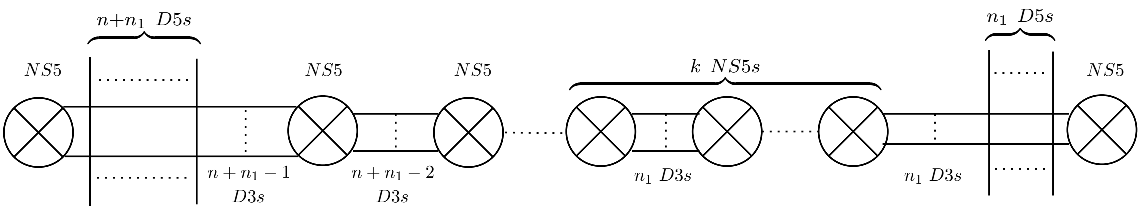

Figure 13. The brane construction after brane moves of $S_f \cap X_M$ with $f = [(n+k-1)^{n_1}, 1^{n+n_1}]$ and $M = [(n+k)^{n_1}, n]$.

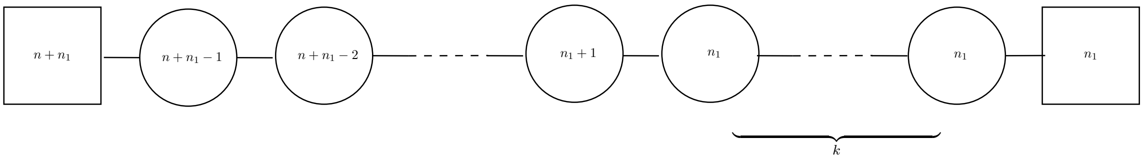

Figure 14. The quiver of $S_f \cap X_M$ with $f = [(n+k-1)^{n_1}, 1^{n+n_1}]$ and $M = [(n+k)^{n_1}, n]$.

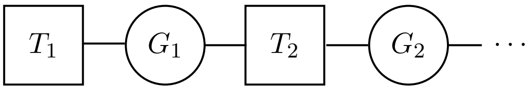

Figure 15. A typical quiver for the weakly coupled gauge theory description of a 4d N=2 SCFT. T_i's are matter systems with non-abelian flavor symmetries, and G_i's are gauge groups.

- We first represent the above theory by an auxiliary punctured sphere Σ with n_a black marked points (n_a is equal to the number of exact marginal deformation plus one), one blue marked point representing the irregular singularity with flavor symmetries, and one red point representing the regular singularity.
- The weakly coupled description is found by finding a pair-of-pants decomposition of Σ with the rules,
  a) In degenerating a tube, one create a pair of blue marked point and red marked point.
  b) Each three punctured sphere in the pants decomposition has one black, one red and one blue puncture.

Now the crucial thing is to determine the puncture type created in the degeneration limit and the matter which is identified with the three punctured sphere. It turns out that the matter appearing in the above degeneration is exactly the matter studied in section 3. Since we have already figured out the VOA for the matter part,[10] we can now describe the VOA for the full theory as a coset.

Now we discuss in more detail about VOA of general A type theory. A general AD theory of A type is represented by following configuration,

$$\mathfrak{g} = \mathfrak{sl}_{nq+n_1}, \quad \Phi = \frac{T}{z^{2+\frac{kq}{nq}}}, \quad f = [h_1^{r_1}, \ldots, h_t^{r_t}]. \tag{5.3}$$

Here $(k, n) = 1$. $T = \mathrm{diag}(I_{nq\times nq}, 0_{n_1})$. This theory has flavor symmetries $\mathrm{U}(n_1) \times \mathrm{U}(1)^{q-1} \times G_F$,[11] where G_F is the flavor symmetry associated to the regular puncture,

$$G_F = \prod \mathrm{U}(r_i)/\mathrm{U}(1). \tag{5.4}$$

Weakly coupled gauge theory descriptions are given in [43]. The idea is to represent our theory by an auxilliary punctured sphere Σ with q black marked points, one red marked

[10]In fact, we only have the full VOA information for following cases: n arbitrary of A type theory; n even of D_N and D_N twisted type theory; and n odd for twisted sl_{2N} and sl_{2N+1} theory.

[11]In general, we can choose a partition of size n_1 such that one can have more general flavor symmetries other than $\mathrm{U}(n_1)$.

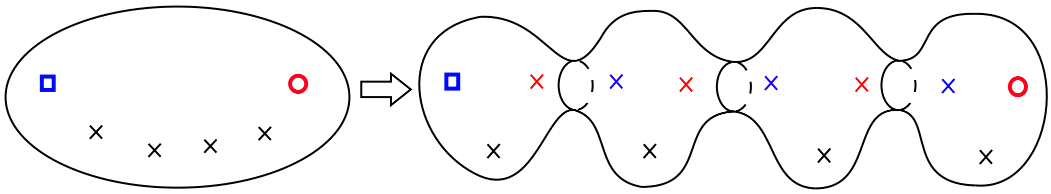

Figure 16. Puncture Riemann surface and its degeneration: each three punctured sphere represents a AD matter, and each tube represents a gauge group.

point reprenting the data of regular singularity, and one blue marked point represent $\mathrm{U}(n_1)$ flavor symmetry for the irregular singularity.[12] In particular, the theory considered in last section is represented by a three puncture sphere, whose VOA is identified as a W algebra.

The weakly coupled gauge theory description is given by finding a pair-of-pants decomposition of Σ such that each tube is connected by a blue and red punctrue. Moreover, each three punctured sphere has to have one blue, one red and one black puncture as shown in figure 16. Now since we have already identified the VOA for each three punctured sphere, the VOA for the original theory is constructed from cosets of the VOA of the matter system.

Example. Taking $n_1 = 0$ and $f = [nq]$ in (5.2), this theory is also called the (A_{nq-1}, A_{kq-1}) theory. To find the weakly coupled gauge theory description, we represent our theory by a sphere Σ with q black points, one trivial blue point and one trivial red point. The weakly coupled description is described by taking a pants decomposition of Σ such that each pant has a black puncture, a red punctrue, and a blue puncture. We assume $k \geq n$ without losing any generality. The weakly coupled gauge theory description is shown in figure 17, where

$$a = \left[\frac{qk}{n+k}\right], \quad b = \left[\frac{qn}{n+k}\right]. \tag{5.5}$$

The square bracket means the integral part of the number inside. The flavor symmetry of the matter content is,

$$T_i: \quad \mathrm{U}((i-1)n) \times \mathrm{SU}(in), \quad L_i: \quad \mathrm{U}(ik) \times \mathrm{U}((i-1)k), \quad T_{a+1}: \quad \mathrm{U}(an) \times \mathrm{U}(bk) \tag{5.6}$$

T_i is the theory studied in section 3, which is engineered by following configuration,

$$\mathfrak{g} = \mathfrak{sl}_{\mathrm{in}}, \quad \Phi = \frac{T}{z^{2+\frac{k}{n}}}, \quad f = [1^{\mathrm{in}}], \tag{5.7}$$

with $T = \mathrm{diag}(I_{n\times n}, 0_{(i-1)n})$. The matter system L_i is engineered by following configuration,

$$\mathfrak{g} = \mathfrak{sl}_{n+ik}, \quad \Phi = \frac{T}{z^{2+\frac{k}{n}}}, \quad f = [n+k, 1^{(i-1)k}], \tag{5.8}$$

with $T = \mathrm{diag}(I_{n\times n}, 0_{ik})$. For L_i's however, a $\mathrm{U}(1)$ flavor symmetry inside $\mathrm{U}(ik)$ is decoupled. The reason is that the central charge of AKM part (with this extra $\mathrm{U}(1)$s) is bigger

[12]For $n = 1,\ k \neq 1$, a blue puncture is equivalent to a red puncture. For $n = 1, k = 1$, all three type of punctures are the same [43].

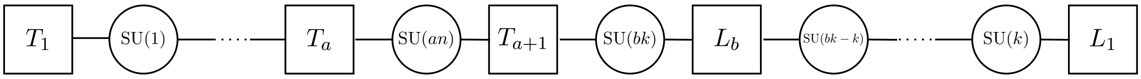

Figure 17. The weakly coupled description of (A_{nq-1}, A_{kq-1}) theory with $a = \left[\frac{qk}{n+k}\right]$ and $b = \left[\frac{qn}{n+k}\right]$.

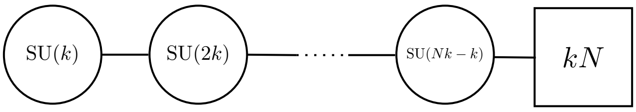

Figure 18. The Lagrangian description of the theory (5.10).

than the full central charge, so some flavor symmetries have to be decoupled. On the other hand, one can find a different realization of L_i where the extra U(1) flavor symmetry is indeed gone (see section 3.1.3).

There is another descrition of the same theory:

$$\mathfrak{g} = \mathfrak{sl}_{qk}, \quad \Phi = \frac{T}{z^{2+\frac{qn}{qk}}}, \quad f = [qk], \tag{5.9}$$

which is just the level-rank dual of the original 4d theory. The weakly coupled gauge theory description is found in similar way, but the VOA of each matter is described differently (which is just the level rank dual discussed in section 3.1.1). So the level rank duality of above theory is the consequence of the level-rank duality of each AD matter.

5.2 Theory with Lagrangian description

Some theories have Lagrangian descriptions, therefore one can find the VOA by cosets of symplectic bosons. Here are some examples,

$$\mathfrak{g} = \mathfrak{sl}_{kN}, \quad \Phi = \frac{T}{z^{2+\frac{-N+1}{N}}}, \quad f = [1^{kN}]. \tag{5.10}$$

This theory has a Lagrangian description given by the quiver in figure 18. Now each matter is just bifundamental hypers and its VOA is a set of symplectic bosons, and the VOA of the full theory is just the coset of symplectic bosons.

6 More on the conformal embedding

VOAs for AD matters considered in this paper have the interesting property that their AKM subalgebras have the same central charges as full VOAs, so they define possible conformal embeddings. In this section, we will show that such possible conformal embeddings are much more general in our theory space. Consider following configuration,

$$\Phi = \frac{T}{z^{2+\frac{k}{b}}}, \quad f = \text{trivial}, \tag{6.1}$$

where T is given by a principle grading, and the classification of b is summarized in section 2. The flavor symmetry group is $\mathrm{U}(1)^{f_0} \times G_F$, where f_0 is the number of mass parameters

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encoded in irregular singularity and G_F is the flavor symmetry from the regular singularity. It was noticed in [35] that the 4d central charge has the following form:

$$c_{4d} = \frac{1}{12}\left(\frac{k_G \dim(G)}{-k_G + h^\vee} - f_0\right). \tag{6.2}$$

Here k_G is the flavor central charge of flavor group G_F. So the corresponding central charge of 2d VOA is

$$c_{2d} = -12c_{4d} = \frac{k_{2d}\dim(G)}{k_{2d} + h^\vee} + f_0. \tag{6.3}$$

Here we use the correspondence $k_{2d} = -k_G$. We know that the 2d VOA has a AKM subalgebra $V_k(G) \times \mathrm{U}(1)^{f_0}$, and the central charge of AKM sector is exactly the central charge of full VOA. Therefore, potentially we have a conformal embedding of AKM subalgebra $V_k(G) \times \mathrm{U}(1)^{f_0}$ into the VOA. If we can indeed prove the above conformal embedding, we could define the full VOA as a reducible module of AKM subalgebra, which would provide us a definition of the full VOA.

The conformal embedding of the AKM algebra into the full VOA is generally not true if we change the regular singularity to a generic one. The conformal embedding is possible only for very special choice of level and nilpotent orbit f, see examples in section 3.

Now consider a theory which has exact marginal deformations, and one can find a weakly coupled gauge theory description. Assuming that each matter has a conformal embedding of AKM, we now prove that the full theory also has a possible conformal embedding (at least the total central charges of AKM pieces and that of the full VOA are the same). Assume that the weakly coupled gauge theory description has the form,

$$T_L - \boxed{G} - T_R, \tag{6.4}$$

and also assume that T_L part has flavor symmetry $G_L \times G$, and T_R part has flavor symmetry $G_R \times G$, here G is a simple factor of flavor symmetry group of two matter systems for simplicity. The full theory then has the flavor symmetry $G_L \times G_R$. The central charge of 4d theory is,

$$c_{4d} = c_{T_L} + \frac{2\dim(G)}{12} + c_{T_R}. \tag{6.5}$$

If each individual piece has a conformal embedding, we have

$$c_{T_L} = c(G_L) + \frac{1}{12}\frac{k_{LG}\dim(G)}{-k_{LG} + h^\vee}, \quad c_{T_R} = c(G_R) + \frac{1}{12}\frac{k_{RG}\dim(G)}{-k_{RG} + h^\vee}, \tag{6.6}$$

where $c(G_L)$ and $c(G_R)$ is the AKM central charge from the flavor symmetry G_L and G_R respectively, and k_{LG} and k_{RG} are flavor central charges. Conformal gauging requires,

$$k_{LG} + k_{RG} = 2h^\vee. \tag{6.7}$$

Substiting (6.6) and (6.7) into (6.5), we find that

$$c_{4d} = c(G_L) + c(G_R). \tag{6.8}$$

So the full theory also has a possible conformal embedding. We have concluded that the gauged system has the conformal embedding if each matter piece has the conformal embedding. On the other hand, if we assume that one piece of matter and the gauged system has the conformal embedding, the other piece of matter would also have the conformal embedding.

The general AD matter has three non-abelian flavor symmetries [41, 42], and the addition of a third non-abelian flavor symmetries would not change the flavor central charge of the other two non-abelian flavor symmetries. We have shown in section 3 that conformal embedding of two non-abelian AKMs into a W algebra $W^{k'}(g, f)$ is possible, one might wonder whether it is possible to have a conformal embedding of general AD theory into a W algebra. However the analysis of section 3.1.3 shows that the conformal embedding of three AKM into a W algebra is not possible (the third non-abelian flavor symmetry would have flavor central charge zero and is decoupled).

7 Conclusion

We have identified VOAs of a class of AD matters with two distinct non-abelian flavor symmetries as W algebras. Using weakly coupled gauge theory descriptions formed by gauging above types of AD matters, we found the VOA for more general AD theories engineered from 6d $(2,0)$ SCFTs, i.e. VOA for general (A_{N-1}, A_{k-1}) theory is found.

One usually learns many interesting properties of 4d theory by using properties of 2d VOA, since 4d theory is strongly coupled and little is known about their spectrum while many aspects of 2d VOA are much more well understood. Therefore it is pleasant that 4d theory can actually predict many interesting features about 2d VOAs. In this paper, we show that the simple fact that a single 4d SCFT can be engineered by different 6d configurations can often teach us very interesting lessons about VOAs. For example, we find new level-rank duality, coset descriptions, possible conformal embeddings, and etc. Although VOA is mainly about the Schur sector which includes the Higgs branch information, the usage of Coulomb branch data is often very useful in telling whether two configurations are the same or not, which in turn teaches us interesting lessons of VOA.

One of interesting lesson we learned in this paper is that the flavor symmetries of SCFTs defined by a 6d $(2,0)$ construction is a subtle issue. There are situations where the naive flavor symmetry is actually decoupled in the IR, which corresponds to collapsing levels of 2d VOA. There are also situations where there is extra flavor symmetries which are not manifest in certain 6d descriptions. Therefore it is interesting to understand the emergency of symmetry from the VOA point of view.

The general AD matter has three non-abelian flavor symmetries, and the remaining task is to identify VOAs for them. Once we find VOAs for these AD matters, we can find the VOA for all SCFTs constructed from 6d $(2,0)$ construction. VOAs defined using junctions of $\mathcal{N} = 4$ boundary conditions are studied in [82, 83]. It appears that they have similar structures involving two or three Lie algebras, and it would be interesting to figure out whether these VOAs have anything to do with VOAs studied in this paper.

We mainly identify the 4d/2d pair in this paper, and a detailed study of characters and its physical implication will be given in a follow-up paper.

Acknowledgments

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A Hitchin system descriptions for (G, G') and $D_p(G)$ theory

There are various class of four dimensional $\mathcal{N} = 2$ AD SCFTs found in the literature, and they have different labels which might cause some confusions. Here we provide a mapping between these labels and our theories. There are three class of theories:

1. Theories with label (G, G') [84]. This class of theories are engineered by following 3-fold singularity:
$$f_G(x, y) + f_{G'}(z, w) = 0. \tag{A.1}$$
Here $G =$ ADE and $f_G(x, y)$ are following polynomials:
$$f_{A_N} = x^2+y^{N+1},\ f_{D_N} = x^{N-1}+xy^2,\ f_{E_6} = x^3+y^4,\ f_{E_7} = x^3+xy^3,\ f_{E_8} = x^3+y^5. \tag{A.2}$$
There is a symmetry exchanging G and G' in the definition of the 3d singularity so that the (G, G') theory is the same as the (G', G) theory. This class of theories include the original AD theory found in [85] (it is the (A_1, A_2) theory), and the later ADE generalizations [86] (They are (A_1, G) type theories) with $G =$ ADE. This class of theories typically do not have any non-abelian flavor symmetries, although they could have abelian flavor symmetries.
2. Theories with label $D_p(G)$ [87], where p is a positive integer and $G =$ ADE. For $G = A_N$, they are called type IV theory in [26]. This class of theories has a flavor symmetry group G and possibly some more abelian flavor symmetry depending on value of p.
3. Theories with label $(J^{(b)}[k], f)$ in [27], with $k > -b$. They were studied in [26, 27] and are defined using 6d $(2, 0)$ SCFT with following data,
$$J = \text{ADE},\quad \Phi = \frac{T}{z^{2+\frac{k}{b}}},\quad f. \tag{A.3}$$
Here f is a nilpotent orbit of $J =$ ADE,[13] and T is a regular semi-simple matrix whose form depending on value b. b takes a finite set of numbers as in table 1, and in particular b can always take the value $h^\vee$ which is the dual Coxeter number. For

[13] Here we use Nahm labels so that a regular nilpotent orbit gives no flavor symmetry, while the trivial nilpotent orbit gives G flavor symmetry with G the Lie group of $\mathfrak{g}$.

$J = A_{N-1}, b = N$, it is called type I theory in [26], and for $b = N-1$, it is called type II theory in [26].

We have the following mapping between the third class of theories and the first two class of theories:

$$(J^{h^\vee}[k], f_{\text{reg}}) = (J, A_{k-1}), \qquad (J^{h^\vee}[k], f_{\text{trivial}}) = D_{k+h^\vee}(J). \tag{A.4}$$

Here $h^\vee$ is the dual Coxeter number.

B Coulomb branch spectrum from the Newton polygon

Let us now briefly review how to find the Coulomb branch spectrum from the Newton polygon:

- The SW curve at SCFT point is

$$x^{n+n_1} + x^{n_1} z^k = 0. \tag{B.1}$$

  The scaling dimension of x and z coordinates can be found as follows. Each term in the above equation has the same scaling dimension, so we have $n[x] = k[z]$. The SW differential $\lambda = xdz$ has scaling dimension one, therefore$[x] + [z] = 1$. We then have,

$$[x] = \frac{k}{n+k}, \quad [z] = \frac{n}{n+k}. \tag{B.2}$$

- The full SW curve takes the following form,

$$x^{n+n_1} + x^{n_1} z^k + \sum_{i,j} u_{ij} x^{n+n_1-i} z^j = 0. \tag{B.3}$$

  We include all monomials in the Newton polygon (including boundary points) and the coefficients u_{ij} are parameters of 4d $\mathcal{N} = 2$ theory including vevs of Coulomb branch operators, coupling constants and mass parameters. The scaling dimension of u_{ij} is computed by the requirement that each term has same scaling dimension, therefore,

$$[u_{ij}] = i[x] - j[z] = \frac{ik}{n+k} - \frac{jn}{n+k}. \tag{B.4}$$

  The Coulomb branch spectrum of a theory is defined as subsets of u_{ij} whose scaling dimension is bigger than one (We only consider the lattice points inside Newton polygon, and the boundary points whose scaling dimensions are bigger than one are actually mass parameters), and we have the following Coulomb branch spectrum,

$$\begin{aligned} &\left\{l - \frac{jn}{n+k}\right\}, \qquad l = 2, \ldots, n, \qquad\qquad j = 1, \ldots, \left[\frac{(l-1)(n+k)}{n}\right], \\ &\left\{l - \frac{jn}{n+k}\right\}, \qquad l = n+1, \ldots, n+n_1, \quad j = 1, \ldots, n+k-1. \end{aligned} \tag{B.5}$$

Now if there is a generic puncture with Young tableaux of size $n+n_1$, we label the boxes of Young tableaux from one to $n+n_1$ row by row, and record the height of the lth box as $h(l)$,[14] then the Coulomb branch spectrum has the following description,

$$\begin{aligned}
&\left\{l-\frac{jn}{n+k}\right\}, \qquad l=2,\dots,n, \qquad &&j=h(l),\dots,\left[\frac{(l-1)(n+k)}{n}\right], \\
&\left\{l-\frac{jn}{n+k}\right\}, \qquad l=n+1,\dots,n+n_1, \qquad &&j=h(l),\dots,n+k-1.
\end{aligned} \tag{B.6}$$

C The central charge of $W^k(\mathfrak{g}, f)$

To any $\mathfrak{sl}_2$-triple $\{f,x,e\}$ in $\mathfrak{g}$, where $[x,f]=-f$, $[x,e]=e$, one associates a W-algebra $W^k(\mathfrak{g},f)$ through the quantum Hamiltonian reduction from the vacuum $\hat{\mathfrak{g}}$-module of level k. The central charge of $W^k(\mathfrak{g},f)$ is

$$c(W^k(\mathfrak{g},f)) = \dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{\frac{1}{2}} - \frac{12}{k+h^\vee}|\rho-(k+h^\vee)x_0|^2, \tag{C.1}$$

where

$$x_0 = \frac{x}{2}, \tag{C.2}$$

and

$$\mathfrak{g}_j = \{g\in\mathfrak{g}|[x_0,g]=jg\}. \tag{C.3}$$

For $\mathfrak{g}=\mathfrak{sl}_n=A_{n-1}$, the Cartan subalgebra $\mathfrak{h}$ is the set of traceless diagonal n by n matrices. Define linear functionals $e_i\in\mathfrak{h}^*$ by $e_i(H)=i^{\text{th}}$ diagonal entry of H where $1\le i\le n$. Then the root system of $\mathfrak{g}$ is

$$\{e_i-e_j\,|\,1\le i,j\le n, i\neq j\}. \tag{C.4}$$

The set of positive roots is $\{e_i-e_j\,|\,i<j\}$. The (e_i-e_j)-root space is spanned by the elementary matrix $E_{i,j}$ with its ij-entry 1 and zeros otherwise.

Nilpotent orbits in A_{n-1} are labelled by partitions of n (or Young tableaux of n boxes). Following the notation and recipe in [88], for the partition $Y=[d_1,\cdots,d_i,\cdots,d_l]$, choose a block of consecutive indices $\{N_i+1,\cdots,N_i+d_i\}$ in such way that disjoint blocks are attached to different d_i's. Then define the set of simple roots for each d_i,

$$\mathcal{C}^+(d_i) = \{e_{N_i+1}-e_{N_i+2},\cdots,e_{N_i+d_i-1}-d_{N_i+d_i}\}, \tag{C.5}$$

with $\mathcal{C}^+$ empty whenever $d_i=1$. One choice of the standard triple $\{H,X,Y\}$ for Y is,

$$H = \sum_{1\le i\le l} H_{\mathcal{C}(d_i)} = \sum_{1\le i\le l}\sum_{1\le j\le d_i}(d_i-2j+1)E_{N_i+j,N_i+j}, \tag{C.6}$$

and

$$\begin{aligned}
X &= \sum_{\alpha\in\cup_i\mathcal{C}^+(d_i)} X_\alpha, \\
Y &= \sum_{\alpha\in\cup_i\mathcal{C}^+(d_i)} X_{-\alpha},
\end{aligned} \tag{C.7}$$

[14] For a full puncture with Young tableaux $[1,\dots,1]$, we have $h(l)=[1,\dots,1]$.

where X_α is the α-root vector. Finally x is the diagonal matrix derived from H via Weyl group of A_{n-1} satisfying the Δ-dominant condition,

$$x_1 \geq x_2 \geq \cdots \geq x_n. \tag{C.8}$$

For example, given the tomahawk tableaux $Y = [2, 2, 1, 1]$ for $\mathfrak{g} = sl_6 = A_5$, H is

$$H = \mathrm{diag}(1, -1, 1, -1, 0, 0), \tag{C.9}$$

and x is

$$x = \mathrm{diag}(1, 1, 0, 0, -1, -1). \tag{C.10}$$

Notice the diagonal entry of x is also its coordinates in orthogonal basis of A_{n-1}, and the coordinates of ρ in orthogonal basis is,

$$\rho = \frac{1}{2}\left(n-1, n-3, \cdots, -n+3, -n+1\right). \tag{C.11}$$

One can then easily compute (ρ, ρ), (ρ, x_0) and (x_0, x_0) because they are just ordinary scalar product in orthogonal basis. For tomahawk tableaux $Y = [q^m, 1^{n-qm}]$, explicit results are,

$$\begin{aligned}
(\rho, \rho) &= \frac{1}{12}(n^3 - n),\\
(x_0, x_0) &= \frac{1}{12} m(q^3 - q),\\
(\rho, x_0) &= \begin{cases} \frac{1}{24} mq(3nq - m(2+q^2)), & q \text{ even},\\ \frac{1}{24} m(3n - mq)(q^2 - 1), & q \text{ odd}.\end{cases}
\end{aligned} \tag{C.12}$$

$\dim \mathfrak{g}_0$ and $\dim \mathfrak{g}_{\frac{1}{2}}$ can be solved easily using the explicit expression of x_0. For tomahawk tablaeux $Y = [q^m, 1^{n+1-qm}]$, the explicit expressions are,

$$\dim \mathfrak{g}_0 = \begin{cases} (n - mq)^2 + m^2 q - 1, & q \text{ even},\\ (n - mq + m)^2 + m^2(q-1) - 1, & q \text{ odd},\end{cases} \tag{C.13}$$

and

$$\dim \mathfrak{g}_{\frac{1}{2}} = \begin{cases} 2m(n - mq), & q \text{ even},\\ 0, & q \text{ odd}.\end{cases} \tag{C.14}$$

Plugging results in the central charge formula (C.1), the central charge for $Y = [q^m, 1^{n-mq}]$ is,

$$c(W^k(\mathfrak{sl}_n, [q^m, 1^{n-mq}])) = mq(k - n + (m+3n)q - (k+m+n)q^2) - \frac{k + k(m-n)n + mn^2}{k+n}. \tag{C.15}$$

For other Lie algebra $\mathfrak{g}$, partitions will be specified to particular cases used in the main context. The general recipe for sl_2-triple can be found in [88]. In the case of $\mathfrak{g} = \mathfrak{so}_{2n} = D_n$ and the partition $Y = [q^m, 1^{2n-qm}]$ with q and m even, one can work out x_0,

$$x_0 = \frac{1}{2}\begin{pmatrix} D & 0\\ 0 & -D\end{pmatrix}, \tag{C.16}$$

with

$$D = \mathrm{diag}(\underbrace{q-1,\ldots,q-1}_{m},\underbrace{q-3,\ldots,q-3}_{m},\ldots,\underbrace{1,\ldots,1}_{m},0,\ldots,0), \tag{C.17}$$

therefore

$$\begin{aligned}(\rho,\rho) &= \frac{1}{6}(n-1)n(2n-1),\\ (x_0,x_0) &= \frac{1}{24}mq(q^2-1),\\ (\rho,x_0) &= -\frac{1}{48}mq(m(q^2+2)+q(-6n+3)),\end{aligned} \tag{C.18}$$

and also

$$\begin{aligned}\dim \mathfrak{g}_0 &= \frac{1}{2}qm^2 + \left(n-\frac{1}{2}mq\right)^2 + \left(n-\frac{1}{2}mq\right)\left(n-\frac{1}{2}mq-1\right),\\ \dim \mathfrak{g}_{\frac{1}{2}} &= 2m\left(n-\frac{1}{2}mq\right).\end{aligned} \tag{C.19}$$

Combining all results, the central charge is,

$$\begin{aligned}c(W^k(\mathfrak{so}_{2n},[q^m,1^{2n-mq}])) = &-\frac{1}{2}mq((k+m+2n-2)q^2-q(6n+m-3)+2n-k+1)\\ &-\frac{kn(m-2n+1)+2mn(n-1)}{k+2n-2}.\end{aligned} \tag{C.20}$$

In the case of $\mathfrak{g} = \mathfrak{so}_{2n+1} = B_n$ and the partition $Y = [q^m, 1^{2n+1-qm}]$ with q and m even, one can work out x_0,

$$x_0 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & D & 0\\ 0 & 0 & -D\end{pmatrix}, \tag{C.21}$$

with

$$D = \mathrm{diag}(\underbrace{q-1,\ldots,q-1}_{m},\underbrace{q-3,\ldots,q-3}_{m},\ldots,\underbrace{1,\ldots,1}_{m},0,\ldots,0), \tag{C.22}$$

therefore

$$\begin{aligned}(\rho,\rho) &= \frac{1}{12}n(2n-1)(2n+1),\\ (x_0,x_0) &= \frac{1}{24}mq(q^2-1),\\ (\rho,x_0) &= -\frac{1}{48}mq(m(q^2+2)-6nq),\end{aligned} \tag{C.23}$$

and also

$$\begin{aligned}\dim \mathfrak{g}_0 &= \frac{1}{2}qm^2 + \left(n-\frac{1}{2}mq\right)^2 + \left(n-\frac{1}{2}mq\right)\left(n-\frac{1}{2}mq-1\right) + 2\left(n-\frac{1}{2}mq\right),\\ \dim \mathfrak{g}_{\frac{1}{2}} &= 2m\left(n-\frac{1}{2}mq\right)+m.\end{aligned} \tag{C.24}$$

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Combining all results, the central charge is,

$$c(W^k(\mathfrak{so}_{2n+1},[q^m,1^{2n+1-mq}])) = -\frac{1}{2}mq((k+m+2n-1)q^2-q(6n+m)+2n-k+2) \\ -\frac{(2n+1)(m(k+2n-1)-2kn)}{2(k+2n-1)}. \tag{C.25}$$

In the case of $\mathfrak{g}=\mathfrak{sp}_{2n}=C_n$ and the partition $Y=[q^m,1^{2n-qm}]$ with q even, one can work out x_0,

$$x_0 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \tag{C.26}$$

with

$$D = \mathrm{diag}(\underbrace{q-1,\dots,q-1}_{m},\underbrace{q-3,\dots,q-3}_{m},\dots,\underbrace{1,\dots,1}_{m},0,\dots,0), \tag{C.27}$$

therefore

$$\begin{aligned} (\rho,\rho) &= \frac{1}{12}n(n+1)(2n+1), \\ (x_0,x_0) &= \frac{1}{12}mq(q^2-1), \\ (\rho,x_0) &= -\frac{1}{48}mq(m(q^2+2)-3nq(2n+1)), \end{aligned} \tag{C.28}$$

and also

$$\begin{aligned} \dim\mathfrak{g}_0 &= \frac{1}{2}qm^2+\left(n-\frac{1}{2}mq\right)^2+\left(n-\frac{1}{2}mq\right)\left(n-\frac{1}{2}mq+1\right), \\ \dim\mathfrak{g}_{\frac{1}{2}} &= 2m\left(n-\frac{1}{2}mq\right). \end{aligned} \tag{C.29}$$

Combining all results, the central charge is,

$$c(W^k(\mathfrak{sp}_{2n},[q^m,1^{2n-mq}])) = -\frac{1}{2}mq((2k+m+2n+2)q^2-q(6n+m+3)+2n-2k-1) \\ -\frac{n(m(n+1)-k(2n-m+1))}{k+n+1}. \tag{C.30}$$

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